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STRASSEN THEOREM IN HOLDER NORM FOR SOME BROWNIAN FUNCTIONALS

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Abstract

We prove Strassen's law of the iterated logarithm in Hölder norm for some two parameter Brownian functionals. This result is applied to stochastic integrals and the Brownian sheet. The main tool in the proof is large deviations principle for two parameter diffusion processes, in Hölder topology, established in [5]. Our results strengthen Theorem 4.1 of [8] and extend to the two parameter setting the main result of [2].

1 Introduction

The classical space of the study of the m-dimensional Brownian path is the space $\mathcal{C}([0,1], \mathbb{R}^m)$ of all $\mathbb{R}^m$-valued, continuous functions endowed with the uniform norm. In recent years a great interest in the investigation of the Brownian motion in others spaces with stronger topology has been developped(see e.g. [2], [3] and [11]). For instance, in [2], Strassen's law for of the iterated logarithm in Hölder norm with exponent $\alpha \in (0, 1/2)$ was established for the Brownian motion. The main purpose of this paper is to prove the analogue of this result for some two parameter Brownian functionals including the Brownian sheet. The key of the proof is the close connection between large deviations estimates and Strassen's law. More precisely, we use Freidlin-Wentzell type estimates in Hölder norm for solutions of two parameter stochastic differential equations obtained recently in[5].
The paper is organized as follows: the second section is devoted to the proof of the Strassen's law for some so-called selfsimilar Brownian functionals. In the third section, we apply the main result of the previous section to stochastic integrals and the Brownian sheet.
In the sequel, we adopt the following notations: $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space on which are defined all random variables considered. For every $x = (x_1, x_2)$

and $y = (y_1, y_2), \langle x, y\rangle$ denotes the scalar product in $\mathbb{R}^2$ and $|x|$ stands for the Euclidian norm; further, $x + y = (x_1 + y_1, x_2 + y_2), xy = (x_1y_1, x_2y_2), x/y = (x_1/y_1, x_2/y_2)$ and $x^y = (x_1^{y_1}, x_2^{y_2})$.
Let $\mathbb{R}_+ = [0, +\infty)$; for every $a \in \mathbb{R}_+, \tilde{a} = (a, a)$ and $\mathbb{R}_+^2$ is partially ordered by setting $x \le y$ if and only if $x_1 \le y_1$ and $x_2 \le y_2$.
For every $m \in \mathbb{N} = \{1, 2, \cdots\}$, $\mathcal{C}_m$ denotes the Banach space of all continuous functions $f : [0,1]^2 \longrightarrow \mathbb{R}^m$ endowed with the uniform norm $\|f\| = \sup_{x\in[0,1]^2} |f(x)|$ and $\mathcal{C}_m^\alpha$ is the Banach space of all $\alpha-$Hölder functions $f : [0,1]^2 \longrightarrow \mathbb{R}^m$ null on the axes with the $\alpha-$Hölder norm

$$\|f\|_\alpha = \sup_{0\le t_1 \ne s_1 \le 1} \sup_{0 \le t_2 \ne s_2 \le 1} \frac{|f(t_1, t_2) - f(s_1, s_2)|}{(|t_1 - s_1| + |t_2 - s_2|)^\alpha}.$$

We consider $\mathcal{C}_m^{\alpha,0}$ the subspace of $\mathcal{C}_m^\alpha$ formed by all functions such that $|f(t_1, t_2) - f(s_1, s_2)| = o((|t_1 - s_1| + |t_2 - s_2|)^\alpha)$ as $|t_1 - s_1| + |t_2 - s_2|$ goes to zero. It is well known that $\mathcal{C}_m^{\alpha,0}$ is a separable space.
We will also use the L_2 norm defined by

$$\|f\|_{L_2} = \left(\int_0^1 \int_0^1 |f(s_1, s_2)|^2 ds_1 ds_2\right)^{1/2}.$$

Let us denote by H_m the subspace of $\mathcal{C}_m$ made up of all absolutely continuous functions with square integrable derivative.
We consider the map $\mu :\longrightarrow [0, +\infty]$ defined by

$$\mu(f) = \begin{cases} \frac{1}{2}\int_0^1\int_0^1 \left|\frac{\partial^2 f}{\partial s_1 \partial s_2}(s_1, s_2)\right|^2 ds_1 ds_2 & \text{if } f \in H_m \\ +\infty & \text{otherwise,} \end{cases}$$

and an arbitrary map $F : H_m \longrightarrow \mathcal{C}_k, k \in \mathbb{N}$. Let us define the map $\lambda : \mathcal{C}_k \longrightarrow [0, +\infty]$ by

$$\lambda(g) = \begin{cases} \inf\{\mu(f) : F(f) = g\} & \text{if } F^{-1}(\{g\}) \ne \emptyset \\ +\infty & \text{otherwise.} \end{cases}$$

For every Borel set A of $\mathcal{C}_k^{\alpha,0}$, we put $\Lambda(A) = \inf_{g\in A} \lambda(g)$.
We also put for every $a \in \mathbb{R}_+, K_\mu(a) = \{h \in H_m : \mu(h) \le a\}$ and $K_\lambda(a) = \{g \in \mathcal{C}_k : \lambda(g) \le a\}$.
Finally, let $B = \{B(t) = (B_1(t), \cdots, B_m(t)) : t \in \mathbb{R}_+^2\}$ denotes the $m-$dimensionnnal Brownian sheet. For every $\alpha \in (0, 1/2)$, B can be considered as a random variable taking values in $\mathcal{C}_m^{\alpha,0}$(see[6, p. 549] and [9, Theorem 2.1])

2 Strassen's law for Brownian functionals

Let $F(B)$ be the Brownian functional associated to the map F.
We will need the following conditions:

- (H1)(continuity) For every $a \in \mathbb{R}_+$, the restriction of F to $K_\mu(a)$ is continuous;

- (H2)(Exponential estimates) For every $(R,\rho,a) \in (\mathbb{R}_+ \setminus \{0\})^3$ there exists $(\varepsilon_0,\beta) \in (\mathbb{R}_+ \setminus \{0\})^2$ such that, for every $f \in K_\mu(a)$ and every $\varepsilon \in (0,\varepsilon_0]$

$$\mathbb{P}(\|F(\varepsilon B) - F(f)\|_\alpha \geq \rho, \|\varepsilon B - f\| < \beta) \leq \exp(-R/\varepsilon^2);$$

- (H3)(Selfsimilarity) There exists $\delta \in \mathbb{R}_+ \setminus \{0\}$ such that, for every $\varepsilon \in \mathbb{R}_+ \setminus \{0\}$ and every $(u,t) \in \mathbb{R}_+^2 \times \mathbb{R}_+^2$

$$F(\varepsilon B(u\cdot))(t) = \varepsilon^\delta F(B)(ut);$$

- (H4) For every $\varepsilon \in \mathbb{R}_+ \setminus \{0\}$, there exists $c_\varepsilon \in (1,+\infty)$ such that, for every $c \in (1, c_\varepsilon]$

$$\inf(\Lambda(A_{\varepsilon,c}), \Lambda(D_{\varepsilon,c})) > 2,$$

where

$$A_{\varepsilon,c} = \left\{ g \in C_k^{\alpha,0} : \sup_{1 \leq v \leq c} \|g(\cdot,\cdot) - g(\cdot,\cdot/v)\|_\alpha \geq \varepsilon/2c^\delta \right\}$$

and

$$D_{\varepsilon,c} = \left\{ g \in C_k^{\alpha,0} : \sup_{1 \leq v \leq c} \|g(\cdot,\cdot) - g(\cdot/v,\cdot)\|_\alpha \geq \varepsilon/2c^\delta \right\}.$$

It is well known(see e.g. [4, p. 463]) that under (H1) and (H2) the dynamical system $\{F(\varepsilon B) : \varepsilon \in \mathbb{R}_+ \setminus \{0\}\}$ satisfies the large deviations principle, that is:

- (i) λ is lower semicontinuous;
- (ii) For every $a \in \mathbb{R}_+, K_\lambda(a)$ is compact in $C_k^{\alpha,0}$;
- (iii) For every Borel set A of $C_k^{\alpha,0}$,

$$-\Lambda(\mathring{A}) \leq \liminf_{\varepsilon \to 0} \varepsilon^2 \log \mathbb{P}(F(\varepsilon B) \in A) \leq \limsup_{\varepsilon \to 0} \varepsilon^2 \log \mathbb{P}(F(\varepsilon B) \in A) \leq -\Lambda(\overline{A})$$

where $\mathring{A}$ and $\overline{A}$ denote respectively the interior and the adherence of A in the Hölder topology.
For every $u = (u_1,u_2) \in \mathbb{R}_+^2$, let us put

$$LLu = \begin{cases} \log\log u_1u_2 & \text{for } u_1u_2 \geq 3 \\ 1 & \text{otherwise,} \end{cases}$$

$\Phi(u) = u_1u_2LLu$ and $Z_u = F\left(\frac{B(u\cdot)}{\sqrt{\Phi(u)}}\right)$.
Now, we state the Strassen's functional law of the iterated logarithm for the Brownian functional $F(B)$ i.e. for the two parameter process $Z = \{Z_u : u \in \mathbb{R}_+^2\}$. For every $u = (u_1,u_2) \in \mathbb{R}_+^2, u \to +\infty$ means that $\min(u_1,u_2) \to +\infty$.

Theorem 2.1 *Assume (H1)-(H4). Then the two parameter process $Z = \{Z_u : u \in \mathbb{R}_+^2\}$ is $\mathbb{P}-$a.s. relatively compact as $u \to +\infty$ and has $K_\lambda(2)$ as set of limit points in the Hölder topology.*

The proof of Theorem 2.1 is based on a two parameter analogue of techniques in [1] and [2] which were previously used in [7] and [8].
We begin by state some preliminaries Lemmas which lead to Propositions 2.5 and Proposition 2.6 from which follows easily Theorem 2.1. We will use repeatedly the following

Remark 2.2 *For every* $\gamma \in \mathbb{R}$ *and every* $j = (j_1, j_2) \in \mathbb{N}^2, \exp(-\gamma LL\bar{c}^j) = const/(j_1 + j_2)^\gamma$ *which is summable if and only if* $\gamma > 2$.

Lemma 2.3 *Assume (H1) and (H2). Then, for every* $c \in (1, +\infty)$ *and every* $\varepsilon \in \mathbb{R} \setminus \{0\}$, *there exists* $j^0 \in \mathbb{N}^2$ *such that, if* $j \geq j^0$ *then* $d(Z_{\bar{c}^j}, K_\lambda(2)) < \varepsilon$, *where for every* $A \subset C_k^{\alpha,0}$ *and every* $f \in C_k^{\alpha,0}, d(f, A) = \inf_{g \in A} \|f - g\|_\alpha$.

Proof. It follows from a straightforward modification of the proof of Lemma 4.3 in [8] so it is omitted.□

Lemma 2.4 *Assume (H1)-(H4). Then, for every* $\varepsilon \in \mathbb{R} \setminus \{0\}$ *there exists* $c_\varepsilon \in (1, +\infty)$ *such that, if* $c \in (1, c_\varepsilon]$ *then*

$$\mathbb{P}\left(\sup_{\bar{c}^j \leq u \leq \bar{c}^{j+\bar{1}}} \frac{1}{(\Phi(u))^{\delta/2}} \|F(B(u\cdot)) - F(B(\bar{c}^j\cdot))\|_\alpha \geq \varepsilon \ i.o.\right).$$

Proof. For every $j \in \mathbb{N}^2$, let us put

$$Y_j = \sup_{\bar{c}^j \leq u \leq \bar{c}^{j+\bar{1}}} \frac{1}{(\Phi(u))^{\delta/2}} \|F(B(u\cdot)) - F(B(\bar{c}^j\cdot))\|_\alpha.$$

$$\begin{aligned}
\mathbb{P}(Y_j \geq \varepsilon) &\leq \mathbb{P}\left(\sup_{\bar{c}^j \leq u \leq \bar{c}^{j+\bar{1}}} \frac{1}{(\Phi(\bar{c}^j))^{\delta/2}} \|F(B(u\cdot)) - F(B(u_1\cdot, c^{j_2}\cdot)\|_\alpha \geq \varepsilon/2\right) \\
&\quad + \mathbb{P}\left(\sup_{\bar{c}^j \leq u \leq \bar{c}^{j+\bar{1}}} \frac{1}{(\Phi(\bar{c}^j))^{\delta/2}} \|F(B(u_1\cdot, c^{j_2}\cdot) - F(B(\bar{c}^j\cdot))\|_\alpha \geq \varepsilon/2\right) \\
&= A_1 + A_2.
\end{aligned}$$

Let us deal with A_1. In view of the scaling property of the Brownian sheet and (H3), we have

$$\begin{aligned}
A_1 &\leq \mathbb{P}\left(\sup_{\bar{c}^j \leq u \leq \bar{c}^{j+\bar{1}}} \left(\frac{u_1 u_2}{\Phi(\bar{c}^j)}\right)^{\delta/2} \|F(B)(\cdot) - F(B)(\cdot, c^{j_2} \cdot /u_2)\|_\alpha \geq \varepsilon/2\right) \\
&\leq \mathbb{P}\left(\sup_{c^{j_2} \leq u_2 \leq c^{j_2+1}} \frac{1}{(LL\bar{c}^j)^{\delta/2}} \|F(B)(\cdot) - F(B)(\cdot, c^{j_2} \cdot /u_2)\|_\alpha \geq \varepsilon/2c^\delta\right) \\
&= \mathbb{P}\left(\sup_{1 \leq v \leq c} \frac{1}{(LL\bar{c}^j)^{\delta/2}} \|F(B)(\cdot) - F(B)(\cdot, \cdot/v)\|_\alpha \geq \varepsilon/2c^\delta\right) \\
&= \mathbb{P}\left(F\left(\frac{1}{\sqrt{LL\bar{c}^j}} B\right) \in A_{\varepsilon,c}\right).
\end{aligned}$$

By replacing $u_1 u_2$ by $u_1 c^{j_2}$, one can also prove that

$$A_2 \leq \mathbb{P}\left(F\left(\frac{1}{\sqrt{LL\tilde{c}^j}}B\right) \in D_{\varepsilon,c}\right).$$

By virtue of the large deviations principle(iii) and since $A_{\varepsilon,c}$ is a closed subset of $C_k^{\alpha,0}$, for every $\gamma \in \mathbb{R}_+ \setminus \{0\}$ and j sufficiently large we have

$$\mathbb{P}\left(F\left(\frac{1}{\sqrt{LL\tilde{c}^j}}B\right) \in A_{\varepsilon,c}\right) \leq \exp(-(\Lambda(A_{\varepsilon,c}) - \gamma)LL\tilde{c}^j). \tag{2.1}$$

Now, in view of (H4), one may choose γ such that $\Lambda(A_{\varepsilon,c}) > \gamma + 2$. For such a γ, Remark 2.2 implies that the right member of (2.1) is summable. Therefore, the Borel-Cantelli lemma leads to

$$\mathbb{P}\left(F\left(\frac{1}{\sqrt{LL\tilde{c}^j}}B\right) \in A_{\varepsilon,c} \ i.o.\right) = 0.$$

One can also prove that

$$\mathbb{P}\left(F\left(\frac{1}{\sqrt{LL\tilde{c}^j}}B\right) \in D_{\varepsilon,c} \ i.o.\right) = 0.$$

The end of the proof follows immediately.□

Proposition 2.5 *Assume (H1)-(H4). Then, for every $\varepsilon \in \mathbb{R}_+ \setminus \{0\}$ there exists $\mathbb{P}$−a.s. $u^0 \in \mathbb{R}_+^2$ such that, if $u > u^0$ then $d(Z_u, K_\lambda(2)) < \varepsilon$.*

Proof. We have

$$\begin{aligned} d(Z_u, K_\lambda(2)) &\leq \left\| Z_u - \left(\frac{\Phi(\tilde{c}^j)}{\Phi(u)}\right)^{\delta/2} Z_{\tilde{c}^j}\right\|_\alpha + \left|1 - \left(\frac{\Phi(\tilde{c}^j)}{\Phi(u)}\right)^{\delta/2}\right| \|Z_{\tilde{c}^j}\|_\alpha \\ &\quad + d(Z_{\tilde{c}^j}, K_\lambda(2)). \end{aligned} \tag{2.2}$$

Let us deal with the right member of (2.2). Lemma 2.4 implies that the first term is $\leq \frac{1}{3}\varepsilon$. Now, by virtue of Lemma 2.3, $\|Z_{\tilde{c}^j}\|_\alpha$ is bounded for j large enough. Therefore, since

$$\lim_{j\to+\infty}\left|1 - \left(\frac{\Phi(\tilde{c}^j)}{\Phi(u)}\right)^{\delta/2}\right| \leq 1 - c^{-\delta},$$

we obtain that for c sufficiently close to 1 and j large enough, the second term is $\leq \frac{1}{3}\varepsilon$.
Finally, by virtue of Lemma 2.3, for c sufficiently close to 1 and j sufficiently large the third term is $\leq \frac{1}{3}\varepsilon$. We conclude that for u sufficiently large, we have $d(Z_u, K_\lambda(2)) < \varepsilon$.□

Proposition 2.6 *Let $g \in K_\lambda(2)$ and assume (H1)-(H4). Then, for every $\varepsilon \in \mathbb{R}_+ \setminus \{0\}$, there exists $c = c_\varepsilon \in (1, +\infty)$ such that $\mathbb{P}(\|Z_{\tilde{c}^j} - g\|_\alpha \leq \varepsilon \ i.o.) = 1$.*

Proof. Let $g \in K_\lambda(2)$ and $f \in K_\mu(2)$ be such that $\mu(f) = \lambda(g)$. For every $(\varepsilon, \beta) \in (I\!R_+ \setminus \{0\})^2$, put

$$A_j = \left\{ \left\| \frac{1}{\sqrt{LL\tilde{c}^j}} B^{(\tilde{c}^j)} - f \right\| < \beta \right\} \text{ and } B_j = \{ Z_{\tilde{c}^j} - g\|_\alpha < \varepsilon \},$$

where

$$B^{(u)} = \frac{B(u\cdot)}{\sqrt{u_1 u_2}}.$$

By virtue of (H2) and the scaling property of the Brownian sheet, for β sufficiently small and j sufficiently large, we have

$$I\!P(A_j \cap B_j^c) \le \exp(-3LL\tilde{c}^j),$$

where $B_j^c = \Omega \setminus B_j$.
In view of Remark 2.2, $\sum_{j \in I\!N^2} I\!P(A_j \cap B_j^c) < +\infty$. Therefore, since there exists $c = c_\varepsilon \in (1, +\infty)$ such that $I\!P(A_j \ i.o.) = 1$(see[10, p. 484]), we conclude that $I\!P(B_j \ i.o.) = 1$ for $c = c_\varepsilon$.□

3 Applications

We will use the following elementary result:

Lemma 3.1 *For every $s = (s_1, s_2) \in I\!R_+^2$ and every $t = (t_1, t_2) \in I\!R_+^2$, let us put*

$$F(v,s,t) = \frac{(|t_1 - s_1| + |t_2 - s_2|)^\alpha}{s_1^{1/2}|t_2 - s_2 \vee (t_2/v)|^{1/2} + s_1^{1/2}|s_2 \wedge (t_2/v) - (s_2/v)|^{1/2} + t_2^{1/2}|t_1 - s_1|^{1/2}|1 - \frac{1}{v}|^{1/2}}.$$

Then, for every $(s,t) \in [\tilde{0}, \tilde{1}]^2$ such that $s \le t$ and for every $v \in [1, c]$, we have

$$F(v,s,t) \ge \frac{2^{\alpha - 1/2}}{2 + c^\alpha}(c-1)^{\alpha - 1/2}.$$

Proof. We will consider the different cases below:

- $\underline{t_2/v \le s_2 \text{ and } t_1/v \le s_1}$

We have

$$F(v,s,t) = \frac{(|t_1 - s_1| + |t_2 - s_2|)^\alpha}{s_1^{1/2}(1 + \frac{1}{v^{1/2}})|t_2 - s_2|^{1/2} + t_2^{1/2}|1 - \frac{1}{v}|^{1/2}|t_1 - s_1|^{1/2}}. \tag{3.1}$$

We deduce that

$$\begin{aligned} F(v,s,t) &\ge \frac{(|t_1 - s_1| + |t_2 - s_2|)^{\alpha - 1/2}}{s_1^{1/2}(1 + \frac{1}{v^{1/2}}) + t_2^{1/2}|1 - \frac{1}{v}|^{1/2}} \\ &\ge \frac{1}{3}(|1 - t_1/s_1| + |1 - t_2/s_2|)^{\alpha - 1/2} \\ &\ge \frac{2^{\alpha - 1/2}}{3}(v-1)^{\alpha - 1/2} \\ &\ge \frac{2^{\alpha - 1/2}}{3}(c-1)^{\alpha - 1/2}. \end{aligned}$$

Now, since $1 \leq c$ and $\alpha > 0$, we have

$$\frac{2^{\alpha-1/2}}{3} \geq \frac{2^{\alpha-1/2}}{2+c^\alpha}.$$

Therefore,

$$F(v,s,t) \geq \frac{2^{\alpha-1/2}}{2+c^\alpha}(c-1)^{\alpha-1/2}.$$

• $\underline{t_2/v \leq s_2 \text{ and } t_1/v \geq s_1}$

$F(v,s,t)$ is equal to the right member of (3.1).

$$\begin{aligned} F(v,s,t) &= \frac{(t_1|1-s_1/t_1|+s_2|1-t_2/s_2)^\alpha}{s_1^{1/2}s_2^{1/2}(1+\frac{1}{v^{1/2}})|1-t_2/s_2|^{1/2}+t_1^{1/2}t_2^{1/2}|1-\frac{1}{v}|^{1/2}|1-s_1/t_1|^{1/2}} \\ &\geq \frac{t_1^\alpha s_2^\alpha(|1-s_1/t_1|+|1-t_2/s_2|)^\alpha}{s_1^{1/2}s_2^{1/2}(1+\frac{1}{v^{1/2}})|1-t_2/s_2|^{1/2}+t_1^{1/2}t_2^{1/2}|1-\frac{1}{v}|^{1/2}} \\ &\geq \frac{t_1^\alpha s_2^\alpha(|1-\frac{1}{v}|+|1-t_2/s_2|)^{\alpha-1/2}}{t_1^{1/2}s_2^{1/2}(1+\frac{1}{v^{1/2}})+t_1^{1/2}t_2^{1/2}}. \end{aligned}$$

Since

$$t_1^{1/2-\alpha}s_2^{1/2-\alpha}(1+\frac{1}{v^{1/2}})+t_1^{1/2-\alpha}t_2^{1/2-\alpha}(t_2/s_2)^\alpha \leq 2+c^\alpha$$

$$|1-t_2/s_2| \leq c-1 \text{ and } |1-\frac{1}{v}| \leq |1-\frac{1}{c}|,$$

we obtain

$$\begin{aligned} F(v,s,t) &\geq \frac{((1-\frac{1}{c})+(c-1))^{\alpha-1/2}}{2+c^\alpha} \\ &\geq \frac{(1+\frac{1}{c})^{\alpha-1/2}(c-1)^{\alpha-1/2}}{2+c^\alpha} \\ &\geq \frac{2^{\alpha-1/2}(c-1)^{\alpha-1/2}}{2+c^\alpha} \end{aligned}$$

•$\underline{t_2/v \geq s_2}$

$$\begin{aligned} F(v,s,t) &= \frac{(|t_1-s_1|+|t_2-s_2|)^\alpha}{(s_1^{1/2}t_2^{1/2}+s_1^{1/2}s_2^{1/2}+t_2^{1/2}|t_1-s_1|^{1/2})|1-\frac{1}{v}|^{1/2}} \\ &\geq \frac{|t_2-s_2|^\alpha}{(s_1^{1/2}t_2^{1/2}+s_1^{1/2}s_2^{1/2}+t_2^{1/2})|1-\frac{1}{v}|^{1/2}} \\ &\geq \frac{t_2^\alpha|1-s_2/t_2|^\alpha}{3t_2^{1/2}|1-\frac{1}{v}|^{1/2}} \\ &\geq \frac{|1-\frac{1}{v}|^{\alpha-1/2}}{3t_2^{1/2-\alpha}} \end{aligned}$$

$$\geq \frac{1}{3}(c-1)^{\alpha-1/2}$$
$$\geq \frac{2^{\alpha-1/2}}{2+c^{\alpha}}(c-1)^{\alpha-1/2}. \square$$

A. Stochastic integrals

Let $A = (a_{ij})_{1\leq i,j\leq m}$ be a nonrandom $m \times m$ matrix and consider the functional $F : H_m \longrightarrow \mathcal{C}_1$ defined by

$$F(f)(t_1,t_2) = \int_0^{t_2}\int_0^{t_1} \langle Af(s_1,s_2), \frac{\partial^2 f}{\partial s_1 \partial s_2}(s_1,s_2)\rangle ds_1 ds_2.$$

Let $F(B)$ be the Brownian functional associated to F defined by the Itô integral

$$F(B)(t_1,t_2) = \int_0^1\int_0^1 \langle AW(s_1,s_2), W(ds_1,ds_2)\rangle.$$

We put $|A| = (\sum_{i,j=1}^m (a_{ij})^2)^{1/2}$.

It is easy to see that (H1) and (H3) are satisfied. Further, in view of [5, Proposition 3.9], (H2) is also satisfied. It remains to show that (H4) is fulfilled. We will only treat the case $A_{\varepsilon,c}$. That of $D_{\varepsilon,c}$ is similar by using an analogue of Lemma 3.1.

To this end, let $g \in A_{\varepsilon,c}$ be such that $\lambda(g) < +\infty$. There exists $f \in H$ with $\mu(f) = \lambda(g)$ and $F(f) = g$. Since $g \in A_{\varepsilon,c}$, there exist $v \in [1,c], s = (s_1,s_2) \in [\tilde{0},\tilde{1}]$ and $t = (t_1,t_2) \in [\tilde{0},\tilde{1}]$, $s \leq t$, such that

$$\frac{\varepsilon}{2c^{\delta}}(|t_1-s_1|+|t_2-s_2|)^{\alpha} \leq |(g(t)-g(t_1,t_2/v)) - (g(s)-g(s_1,s_2/v))|.$$

We have

$$\begin{aligned}
&|(g(t)-g(t_1,t_2/v)) - (g(s)-g(s_1,s_2/v))| \\
&= \left|\int_0^{t_1}\int_{t_2/v}^{t_2} \frac{\partial^2 g}{\partial u_1\partial u_2} g(u_1,u_2)du_1du_2 - \int_0^{s_1}\int_{s_2/v}^{s_2} \frac{\partial^2 g}{\partial u_1\partial u_2}(u_1,u_2)du_1du_2\right| \\
&= \left|\int_{t_2/v}^{t_2} \frac{\partial g}{\partial u_2}(s_1,u_2)du_2 - \int_{s_2/v}^{s_2} \frac{\partial g}{\partial u_2}(s_1,u_2)du_2 + \int_{s_1}^{t_1}\int_{t_2/v}^{t_2} \frac{\partial^2 g}{\partial u_1\partial u_2} g(u)du_1du_2\right| \\
&\leq (|t_2 - s_2 \vee (t_2/v)|^{1/2} + |s_2 \wedge (t_2/v) - s_2/v|^{1/2}) \left\|\frac{\partial g}{\partial u_2}(s_1,\cdot)\right\|_{L_2} \\
&+ (t_2^{1/2}|1-\frac{1}{v}|^{1/2}|t_1-s_1|^{1/2}) \left\|\frac{\partial^2 g}{\partial u_1\partial u_2}(\cdot)\right\|_{L_2} \\
&\leq \left[s_1^{1/2}(|t_2 - s_2 \vee (t_2/v)|^{1/2} + |s_2 \wedge (t_2/v) - s_2/v|^{1/2}) + t_2^{1/2}|1-\frac{1}{v}|^{1/2}|t_1-s_1|^{1/2}\right] \\
&\times \left\|\frac{\partial^2 g}{\partial u_1\partial u_2}(\cdot)\right\|_{L_2}
\end{aligned}$$

Now, let us show that

$$\left\|\frac{\partial^2 g}{\partial u_1\partial u_2}(\cdot)\right\|_{L_2} \leq 2|A|\mu(f)\cdot \tag{3.2}$$

We have

$$\left\|\frac{\partial^2 g}{\partial u_1 \partial u_2}(\cdot)\right\|_{L_2} = \left(\int_0^1\int_0^1 \left|\langle Af(s_1,s_2), \frac{\partial^2 f}{\partial s_1 \partial s_2}(s_1,s_2)\right|^2 ds_1 ds_2\right)^{1/2}.$$

Therefore, in view of Cauchy-Schwarz inequality, we obtain

$$\left\|\frac{\partial^2 g}{\partial u_1 \partial u_2}(\cdot)\right\|_{L_2} \leq \left(\int_0^1\int_0^1 |Af(s_1,s_2)|^2 \left|\frac{\partial^2 f}{\partial s_1 \partial s_2}(s_1,s_2)\right|^2 ds_1 ds_2\right)^{1/2}.$$

Since for every $s \in [\tilde{0}, \tilde{1}]$, by a classical inequality on the norm of linear operators $|Af(s)| \leq |A||f(s)|$, it follows that

$$\left\|\frac{\partial^2 g}{\partial u_1 \partial u_2}(\cdot)\right\|_{L_2} \leq |A|\|f\| \left(\int_0^1\int_0^1 \left|\frac{\partial^2 f}{\partial s_1 \partial s_2}(u_1,u_2)\right|^2 ds_1 ds_2\right)^{1/2}. \tag{3.3}$$

Now, for every $s \in [\tilde{0}, \tilde{1}]$,

$$f(s) = \int_0^{s_1}\int_0^{s_2} \frac{\partial^2 f}{\partial u_1 \partial u_2}(u_1,u_2) du_1 du_2.$$

Therefore, from Hölder's inequality, we have

$$|f(s)| \leq s_1^{1/2} s_2^{1/2} \left(\int_0^{s_1}\int_0^{s_2} \left|\frac{\partial^2 f}{\partial u_1 \partial u_2}(u_1,u_2)\right|^2 du_1 du_2\right)^{1/2}.$$

We deduce that

$$\|f\| \leq \left(\int_0^1\int_0^1 \left|\frac{\partial^2 f}{\partial u_1 \partial u_2}(u_1,u_2)\right|^2 du_1 du_2\right)^{1/2}. \tag{3.4}$$

Now, (3.3) and (3.4) lead to (3.2).
By using (3.2), we obtain

$$\mu(f) \geq \frac{\varepsilon}{4c^\delta |A|} F(v,s,t).$$

By virtue of Lemma 3.1, we have

$$\mu(f) \geq \frac{2^{\alpha-1/2}\varepsilon}{4c^\delta |A|(2+c^\alpha)}(c-1)^{\alpha-1/2}.$$

Since $\alpha < 1/2$, for c sufficiently close to 1 we obtain $\Lambda(A_{\varepsilon,c}) > 2$.

B. Brownian sheet

We put $F(f) = f$ for every $f \in H$. Therefore the Brownian functional associated to F is the Brownian sheet itself. Let us note that in this case we have $\lambda = \mu$.

(H1)-(H3) are trivialy satisfied. By the same arguments as above one can prove that

$$\frac{\varepsilon(|t_1 - s_1| + |t_2 - s_2|)^\alpha}{2c^\delta} \leq$$
$$\left[s_1^{1/2}(|t_2 - s_2 \vee (t_2/v)|^{1/2} + |s_2 \wedge (t_2/v) - s_2/v|^{1/2}) + t_2^{1/2}|1 - \frac{1}{v}|^{1/2}|t_1 - s_1|^{1/2}\right]$$
$$\times \left\| \frac{\partial^2 f}{\partial u_1 \partial u_2}(\cdot) \right\|_{L_2}.$$

In view of Lemma 3.1, we deduce that

$$\begin{aligned} \mu(f) &\geq \frac{\varepsilon^2}{4c^{2\delta}}(F(v,s,t))^2 \\ &\geq \frac{2^{2\alpha-1}\varepsilon^2}{4c^{2\delta}(2+c^\alpha)^2}(c-1)^{2\alpha-1}. \end{aligned}$$

Since $\alpha < 1/2$, for c sufficiently close to 1 we have $\Lambda(A_{\varepsilon,c}) > 2$.
Therefore, in view of Theorem 2.1 we can state the Strassen's law of the iterated logarithm in Hölder norm for the Brownian sheet.

Theorem 3.2 *Let $\xi_u = B(u\cdot)/\sqrt{\Phi(u)}$. The two parameter process $\xi = \{\xi_u : u \in \mathbb{R}_+^2\}$ is $\mathbb{P}-a.s.$ relatively compact and has $K_\mu(2)$ as limit set in the Hölder topology.*

Remark 3.3 *For $m = 1$, Theorem 3.2 can be deduced from [6, Theorem 5.1]*

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