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Abstract

In the present paper we obtain a few inclusion theorems for the convolution of Nörlund methods in the form $(N, r_n) \subseteq (N, p_n) \ast (N, q_n)$ in complete, non-trivially valued, non-archimedean fields.

Throughout the present paper $K$ denotes a complete, non-trivially valued, non-archimedean field. Infinite matrices and sequences, which are considered in the sequel, have entries in $K$. If $A = (a_{nk})$, $a_{nk} \in K$, $n, k = 0, 1, 2, \ldots$ is an infinite matrix, the $A$-transform $A x = \{(Ax)_n\}$ of the sequence $x = \{x_k\}$, $x_k \in K$, $k = 0, 1, 2, \ldots$ is defined by

$$(Ax)_n = \sum_{k=0}^{\infty} a_{nk} x_k, \quad n = 0, 1, 2, \ldots,$$

where it is assumed that the series on the right converge. If $\lim_{n \to \infty} (Ax)_n = s$, we say that $\{x_k\}$ is $A$-summable to $s$, written as $x_k \rightarrow s(A)$ or $A\lim x_k = s$. If $\lim_{n \to \infty} (Ax)_n = s$ whenever $\lim_{k \to \infty} x_k = s$, we say that $A$ is regular. The following result is well-known (see [2], [4]).
Theorem 1 \(A = (a_{nk})\) is regular if and only if
\[
\sup_{n,k} |a_{nk}| < \infty; \quad (1)
\]
\[
\lim_{n \to \infty} a_{nk} = 0, \quad \text{for every fixed } k; \quad (2)
\]
and
\[
\lim_{n \to \infty} \sum_{k=0}^{\infty} a_{nk} = 1. \quad (3)
\]
Any matrix \(A\) for which (1) holds is called a \(K_r\)-matrix. If \(A\) and \(B\) are two infinite matrices such that \(x_k \to s(A)\) implies \(x_k \to s(B)\), we say that \(A\) is included in \(B\), written as \(A \subseteq B\). \(A\) is said to be row-finite if for \(n = 0, 1, 2, \ldots\), there exists a positive integer \(k_n\) such that \(a_{nk} = 0, \ k > k_n\).

Given two infinite matrices \(A\) and \(B\), their convolution is defined as the matrix \(C = (c_{nk})\), where
\[
c_{nk} = \sum_{i=0}^{k} a_{ni} b_{n,k-i}, \quad n, k = 0, 1, 2, \ldots. \quad (4)
\]
In such a case we write \(C = A \ast B\).

The following properties of the convolution can be easily proved.

1. If \(A\) and \(B\) are both row-finite or both \(K_r\), then their convolution \(C\) is row-finite or \(K_r\) respectively and their row sums satisfy
\[
\sum_{k=0}^{\infty} c_{nk} = \left(\sum_{k=0}^{\infty} a_{nk}\right) \left(\sum_{k=0}^{\infty} b_{nk}\right), \quad n = 0, 1, 2, \ldots. \quad (5)
\]

2. If \(A, B\) are both regular, then \(C\) is regular too.

The Nörlund method of summability i.e., \((N, p_n)\) method in \(K\) is defined as follows (see [5]): \((N, p_n)\) is defined by the infinite matrix \((a_{nk})\) where
\[
a_{nk} = \frac{p_{n-k}}{p^n}, \quad k \leq n;
\]
\[
= 0, \quad k > n,
\]
where \(p_0 \neq 0, \ |p_0| > |p_j|, \ j = 1, 2, \ldots\) and \(P_n = \sum_{k=0}^{n} p_k, \ n = 0, 1, 2, \ldots\). It is to be noted that \(|P_n| = |p_0| \neq 0\) so that \(P_n \neq 0, \ n = 0, 1, 2, \ldots\).

The following result is very useful in the sequel.

Theorem 2 (See [5], Theorem 1.) \((N, p_n)\) is regular if and only if
\[
p_n \to 0, \ n \to \infty. \quad (6)
\]
The purpose of the present paper is to prove a few inclusion theorems for the convolution of Nörlund methods in the form $(N, r_n) \subseteq (N, p_n) * (N, q_n)$.

We need to define $\{\bar{p}_n\}$ by

$$p_0\bar{p}_0 = 1, \quad p_0\bar{p}_n + p_1\bar{p}_{n-1} + \cdots + p_n\bar{p}_0 = 0, \quad n \geq 1 \quad (7)$$

i.e., $\bar{p}(x) = \sum_{n=0}^{\infty} \bar{p}_n x^n = \frac{1}{\sum_{n=0}^{\infty} p_n x^n} = \frac{1}{p(x)}$, assuming that these series converge.

The following result is an easy consequence of Kojima-Schur theorem (see [2], [4]).

**Lemma 1** Let $A = (a_{nk})$ be a row-finite matrix and $(N, p_n)$ be a regular Nörlund method. Then $A$-lim $x_k$ exists whenever $(N, p_n)$-lim $x_k$ exists if and only if

$$\sup_{0 \leq \gamma \leq k_n} |P_\gamma \sum_{k=\gamma}^{k_n} a_{nk}\bar{p}_{k-\gamma}| = O(1), \quad n \to \infty; \quad (8)$$

$$\lim_{n \to \infty} \sum_{k=\gamma}^{k_n} a_{nk}\bar{p}_{k-\gamma} = \delta_{\gamma}, \quad \text{for every fixed } \gamma; \quad (9)$$

and

$$\lim_{n \to \infty} \sum_{k=0}^{k_n} a_{nk} = \delta. \quad (10)$$

**Corollary 1** $(N, p_n) \subseteq A$ if and only if $(8), (9)$ and $(10)$ hold with $\delta_{\gamma} = 0, \gamma = 0, 1, 2, \ldots$ and $\delta = 1$.

**Corollary 2** If $(N, p_n)$ and $(N, q_n)$ are regular Nörlund methods, then $(N, p_n) \subseteq (N, q_n)$ if and only if $h_n \to 0, \ n \to \infty$ where

$$h(x) = \sum_{n=0}^{\infty} h_n x^n = \frac{\sum_{n=0}^{\infty} q_n x^n}{\sum_{n=0}^{\infty} p_n x^n} = \frac{q(x)}{p(x)}$$

(see [5]).

Let $(N, p_n), (N, q_n), (N, r_n)$ be regular Nörlund methods. Let $p_n(x) = \sum_{i=n}^{\infty} p_i x^i, \quad p_0(x) = p(x)$ with similar expressions for $q_n(x), r_n(x)$. Let

$$\frac{p(x)q(x)}{r(x)} = \sum_{\gamma=0}^{\infty} \theta_{\gamma} x^\gamma; \quad (11)$$

$$\frac{p_{n+1}(x)q(x)}{r(x)} = \sum_{\gamma=0}^{\infty} \alpha_{n\gamma} x^\gamma;$$

$$\frac{p(x)q_{n+1}(x)}{r(x)} = \sum_{\gamma=0}^{\infty} \beta_{n\gamma} x^\gamma,$$
and
\[
\frac{1}{r(x)} \{ \varphi(x) q(x) - p_{n+1}(x) q(x) - p(x) q_{n+1}(x) \} = \sum_{\gamma=0}^{\infty} \varphi_{n\gamma} x^{\gamma}. \tag{12}
\]

It now follows that
\[
\varphi_{n\gamma} = \theta_{\gamma} - \alpha_{n\gamma} - \beta_{n\gamma} \tag{13}
\]
and
\[
\alpha_{n\gamma} = \beta_{n\gamma} = 0, \quad 0 \leq \gamma \leq n. \tag{14}
\]

Taking \( C = (N, p_n) * (N, q_n) \), \( C \) is a row-finite matrix with
\[
c_{nk} = \frac{1}{P_n Q_n} \sum_{i=\max(0,k-n)}^{\min(k,n)} p_{n-i} q_{n-k+i} \quad \text{with} \quad k_n = 2n. \tag{15}
\]

We write
\[
f_{n\gamma} = \sum_{k=\max(0,2n-\gamma)}^{2n} c_{nk} \tilde{\varphi}_{k+\gamma-2n}, \quad n, \gamma \geq 0. \tag{16}
\]

Lemma 2
\[
P_n Q_n f_{n\gamma} = \varphi_{n\gamma}, \quad 0 \leq \gamma \leq 2n + 1. \tag{17}
\]

Proof. The result follows as in [6].

Theorem 3 \((N, p_n) * (N, q_n)\)-lim \( x_k \) exists whenever \((N, r_n)\)-lim \( x_k \) exists if and only if
\[
\sup_{0 \leq \gamma \leq 2n} |R_{2n-\gamma} \varphi_{n\gamma}| = O(1), \quad n \to \infty; \tag{18}
\]
and
\[
\lim_{n \to \infty} \frac{\varphi_{n,2n-\gamma}}{P_n Q_n} = \delta_{\gamma}, \quad \text{for every fixed} \quad \gamma. \tag{19}
\]

Proof. Let \((N, p_n) * (N, q_n)\)-lim \( x_k \) exist whenever \((N, r_n)\)-lim \( x_k \) exists. Applying Lemma 1 with \((N, p_n) = (N, r_n)\) and \( A = (N, p_n) * (N, q_n) = (c_{nk}) \), we have,
\[
\sup_{0 \leq \gamma \leq 2n} |R_{\gamma} \sum_{k=\gamma}^{2n} c_{nk} \tilde{\varphi}_{k-\gamma}| = O(1), \quad n \to \infty \tag{20}
\]
and
\[
\lim_{n \to \infty} \sum_{k=\gamma}^{2n} c_{nk} \tilde{\varphi}_{k-\gamma} = \delta_{\gamma}, \quad \text{for every fixed} \quad \gamma. \tag{21}
\]

Using (16) and (20), we get
\[
\sup_{0 \leq \gamma \leq 2n} |R_{2n-\gamma} f_{n\gamma}| = O(1), \quad n \to \infty. \tag{22}
\]
Using (17), we note that $|f_{n}\gamma| = \left|\phi_{n}\gamma\right| \frac{|p_{0}|}{|q_{0}|}$ since $|P_{n}| = |p_{0}|$ and $|Q_{n}| = |q_{0}|$. Consequently, in view of (22), we get

$$\sup_{0 \leq \gamma \leq 2n} |R_{2n-\gamma}\phi_{n}\gamma| = O(1), \quad n \to \infty.$$  

In view of (16) and (21), we have

$$\lim_{n \to \infty} f_{n,2n-\gamma} = \delta_{\gamma}, \quad \text{for every fixed } \gamma.$$  

Now, using (17), we get

$$\lim_{n \to \infty} \frac{\phi_{n,2n-\gamma}}{P_{n}Q_{n}} = \delta_{\gamma}, \quad \text{for every fixed } \gamma.$$  

Thus (18) and (19) hold. Conversely (18) and (19) imply (20) and (21) respectively. Using (5), we have $\lim_{n \to \infty} \sum_{k=0}^{2n} c_{nk} = 1$. Using Lemma 1, the result follows, completing the proof of the theorem.

**Corollary 3** $(N, r_{n}) \subseteq (N, p_{n}) * (N, q_{n})$ if and only if (18) and (19) hold with $\delta_{\gamma} = 0$.

**Corollary 4** If $\lim_{n \to \infty} r_{n} = 0$, then $(N, r_{n}) \subseteq (N, p_{n}) * (N, q_{n})$ if and only if (18) holds.

**Proof.** The result follows using (9) and the fact that $(N, p_{n}) * (N, q_{n})$ is regular.

**Theorem 4** If

$$\phi_{n,2n-\gamma} = o(1), \quad n \to \infty, \quad \text{for every fixed } \gamma,$$  

and either

$$\phi_{n}\gamma = O(1), \quad n, \gamma \to \infty,$$  

or

$$\theta_{\gamma}, \alpha_{n}\gamma, \beta_{n}\gamma = O(1), \quad n, \gamma \to \infty,$$  

then

$$(N, r_{n}) \subseteq (N, p_{n}) * (N, q_{n}).$$

**Proof.** Using (23), (19) follows with $\delta_{\gamma} = 0$ since $|P_{n}| = |p_{0}|$ and $|Q_{n}| = |q_{0}|$. Because of (13) and (25), (24) holds. So if (24) or (25) holds, (18) holds since $R_{n} = O(1)$, $n \to \infty$, $(N, r_{n})$ being a regular method. The result now follows from Corollary 3.

We shall now take up an application of Theorem 4.
Theorem 5 Let $\overline{p}_n, \overline{q}_n \to 0$, $n \to \infty$ and $t_n = p_0q_n + p_1q_{n-1} + \cdots + p_nq_0$, $n = 0, 1, 2, \ldots$. Then

$$(N, t_n) \subseteq (N, p_n) \ast (N, q_n)$$

and

$$(N, p_n) \subseteq (N, t_n), (N, q_n) \subseteq (N, t_n).$$

Proof. We apply Theorem 4 with $r_n = t_n$. With the usual notation we have $t(x) = p(x)q(x)$ and $\overline{t}(x) = \overline{p}(x)\overline{q}(x)$. Since $\overline{p}_n, \overline{q}_n \to 0$, $n \to \infty$, $\overline{t}_n \to 0$, $n \to \infty$ (see [3], Theorem 1). Consequently (23) follows using (9). In view of (11), we have,

$$\sum_{\gamma=0}^{\infty} \theta_\gamma x^\gamma = \frac{p(x)q(x)}{t(x)} = 1,$$

so that

$$\theta_0 = 1 \text{ and } \theta_\gamma = 0, \gamma \geq 1;$$

$$\sum_{\gamma=0}^{\infty} \alpha_{n\gamma} x^\gamma = \frac{p_{n+1}(x)q(x)}{t(x)} = p_{n+1}(x)p(x),$$

so that

$$\alpha_{n\gamma} = \sum_{\lambda=0}^{\gamma-(n+1)} \overline{p}_\lambda p_{\gamma-\lambda}, \gamma \geq n+1;$$

$$= 0, \ 0 \leq \gamma \leq n.$$

Consequently $\alpha_{n\gamma} = O(1)$, $n, \gamma \to \infty$. Similarly $\beta_{n\gamma} = O(1)$, $n, \gamma \to \infty$. In view of Theorem 4, $(N, t_n) \subseteq (N, p_n) \ast (N, q_n)$. Now $\frac{t(x)}{p(x)} = q(x)$ and $q_n \to 0$, $n \to \infty$, $(N, q_n)$ being regular, by (6). So by Corollary 2, $(N, p_n) \subseteq (N, t_n)$. Similarly $(N, q_n) \subseteq (N, t_n)$. The proof of the theorem is now complete.

References


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