P.N. NATARAJAN
V. SRINIVASAN
Convolution of Nörlund methods in non-archimedean fields


<http://www.numdam.org/item?id=AMBP_1997__4_2_41_0>
Convolution of Nörlund methods in non-archimedean fields

P.N. Natarajan
Department of Mathematics
Ramakrishna Mission Vivekananda College
Chennai - 600 004
India

V. Srinivasan
Department of Mathematics
V. Ramakrishna Polytechnic
Chennai - 600 019
India

Abstract

In the present paper we obtain a few inclusion theorems for the convolution of Nörlund methods in the form \((N, r_n) \subseteq (N, p_n) \ast (N, q_n)\) in complete, non-trivially valued, non-archimedean fields.

Throughout the present paper \(K\) denotes a complete, non-trivially valued, non-archimedean field. Infinite matrices and sequences, which are considered in the sequel, have entries in \(K\). If \(A = (a_{nk})\), \(a_{nk} \in K\), \(n, k = 0, 1, 2, \ldots\) is an infinite matrix, the \(A\)-transform \(A^*\) of the sequence \(x = \{x_k\}\), \(x_k \in K\), \(k = 0, 1, 2, \ldots\) is defined by

\[
(A^*x)_n = \sum_{k=0}^{\infty} a_{nk}x_k, \quad n = 0, 1, 2, \ldots,
\]

where it is assumed that the series on the right converge. If \(\lim_{n \to \infty} (A^*x)_n = s\), we say that \(\{x_k\}\) is \(A\)-summable to \(s\), written as \(x_k \to s(A)\) or \(A\)-lim \(x_k = s\). If \(\lim_{n \to \infty} (A^*x)_n = s\) whenever \(\lim_{k \to \infty} x_k = s\), we say that \(A\) is regular. The following result is well-known (see [2], [4]).
Theorem 1 \( A = (a_{nk}) \) is regular if and only if
\[
\sup_{n,k} |a_{nk}| < \infty; \quad (1)
\]
\[
\lim_{n \to \infty} a_{nk} = 0, \quad \text{for every fixed } k; \quad (2)
\]
and
\[
\lim_{n \to \infty} \sum_{k=0}^{\infty} a_{nk} = 1. \quad (3)
\]

Any matrix \( A \) for which (1) holds is called a \( K_r \)-matrix. If \( A \) and \( B \) are two infinite matrices such that \( x_k \to s(A) \) implies \( x_k \to s(B) \), we say that \( A \) is included in \( B \), written as \( A \subseteq B \). \( A \) is said to be row-finite if for \( n = 0, 1, 2, \ldots \), there exists a positive integer \( k_n \) such that \( a_{nk} = 0, \ k > k_n \).

Given two infinite matrices \( A \) and \( B \), their convolution is defined as the matrix
\[
C = (c_{nk}), \quad c_{nk} = \sum_{i=0}^{k} a_{ni} b_{n,k-i}, \quad n, k = 0, 1, 2, \ldots \quad (4)
\]

In such a case we write \( C = A \ast B \).

The following properties of the convolution can be easily proved.

1. If \( A \) and \( B \) are both row-finite or both \( K_r \), then their convolution \( C \) is row-finite or \( K_r \), respectively and their row sums satisfy
\[
\sum_{k=0}^{\infty} c_{nk} = \left( \sum_{k=0}^{\infty} a_{nk} \right) \left( \sum_{k=0}^{\infty} b_{nk} \right), \quad n = 0, 1, 2, \ldots \quad (5)
\]

2. If \( A, B \) are both regular, then \( C \) is regular too.

The Nörlund method of summability i.e., \( (N, p_n) \) method in \( K \) is defined as follows (see [5]): \( (N, p_n) \) is defined by the infinite matrix \( (a_{nk}) \) where
\[
a_{nk} = \begin{cases} 
p_{n-k} / p_n & k \leq n; \\
0 & k > n,
\end{cases}
\]
where \( p_0 \neq 0, |p_0| > |p_j|, \ j = 1, 2, \ldots \) and \( P_n = \sum_{k=0}^{n} p_k, \ n = 0, 1, 2, \ldots \). It is to be noted that \( |P_n| = |p_0| \neq 0 \) so that \( P_n \neq 0, n = 0, 1, 2, \ldots \). The following result is very useful in the sequel.

Theorem 2 (See [5], Theorem 1.) \( (N, p_n) \) is regular if and only if
\[
p_n \to 0, \ n \to \infty. \quad (6)
\]
The purpose of the present paper is to prove a few inclusion theorems for the convolution of Nörlund methods in the form \((N, r_n) \subseteq (N, p_n) * (N, q_n)\).

We need to define \(\{\bar{p}_n\}\) by

\[ p_0 \bar{p}_0 = 1, \quad p_0 \bar{p}_n + p_1 \bar{p}_{n-1} + \cdots + p_n \bar{p}_0 = 0, \quad n \geq 1 \quad (7) \]

i.e.,

\[ \bar{p}(x) = \sum_{n=0}^{\infty} \bar{p}_n x^n = \frac{1}{\sum_{n=0}^{\infty} p_n x^n} = \frac{1}{p(x)}, \]

assuming that these series converge.

The following result is an easy consequence of Kojima-Schur theorem (see [2], [4]).

**Lemma 1** Let \(A = (a_{nk})\) be a row-finite matrix and \((N, p_n)\) be a regular Nörlund method. Then \(A\)-\(\lim x_k\) exists whenever \((N, p_n)\)-\(\lim x_k\) exists if and only if

\[ \sup_{0 \leq \gamma \leq s_n} |P_\gamma \sum_{k=\gamma}^{k_n} a_{nk} \bar{p}_{k-\gamma}| = O(1), \quad n \to \infty; \quad (8) \]

\[ \lim_{n \to \infty} \sum_{k=\gamma}^{k_n} a_{nk} \bar{p}_{k-\gamma} = \delta, \quad \text{for every fixed } \gamma; \quad (9) \]

and

\[ \lim_{n \to \infty} \sum_{k=0}^{k_n} a_{nk} = \delta. \quad (10) \]

**Corollary 1** \((N, p_n) \subseteq A\) if and only if (8), (9) and (10) hold with \(\delta_\gamma = 0, \gamma = 0, 1, 2, \ldots\) and \(\delta = 1\).

**Corollary 2** If \((N, p_n)\) and \((N, q_n)\) are regular Nörlund methods, then \((N, p_n) \subseteq (N, q_n)\) if and only if \(h_n \to 0, \quad n \to \infty\) where

\[ h(x) = \sum_{n=0}^{\infty} h_n x^n = \frac{\sum_{n=0}^{\infty} q_n x^n}{\sum_{n=0}^{\infty} p_n x^n} = \frac{q(x)}{p(x)} \]

(see [5]).

Let \((N, p_n), (N, q_n), (N, r_n)\) be regular Nörlund methods. Let \(p_n(x) = \sum_{i=n}^{\infty} p_i x^i\), \(p_0(x) = p(x)\) with similar expressions for \(q_n(x), r_n(x)\). Let

\[ \frac{p(x) q(x)}{r(x)} = \sum_{\gamma=0}^{\infty} \theta_{\gamma} x^\gamma; \]

\[ \frac{p_{n+1}(x) q(x)}{r(x)} = \sum_{\gamma=0}^{\infty} \alpha_{n, \gamma} x^\gamma; \quad (11) \]

\[ \frac{p(x) q_{n+1}(x)}{r(x)} = \sum_{\gamma=0}^{\infty} \beta_{n, \gamma} x^\gamma, \]
and
\[
\frac{1}{r(x)} \{ p(x)q(x) - p_{n+1}(x)q(x) - p(x)q_{n+1}(x) \} = \sum_{\gamma=0}^{\infty} \varphi_{n\gamma} x^\gamma. \tag{12}
\]

It now follows that
\[
\varphi_{n\gamma} = \theta_\gamma - \alpha_{n\gamma} - \beta_{n\gamma} \tag{13}
\]
and
\[
\alpha_{n\gamma} = \beta_{n\gamma} = 0, \quad 0 \leq \gamma \leq n. \tag{14}
\]

Taking \( C = (N, p_n) \ast (N, q_n) \), \( C \) is a row-finite matrix with
\[
c_{nk} = \frac{1}{p_n q_n} \sum_{i=\max(0, k-n)}^{\min(k, n)} p_{n-i} q_{n+k+i} \quad \text{with} \quad k_n = 2n. \tag{15}
\]

We write
\[
f_{n\gamma} = \sum_{k=\max(0, 2n-\gamma)}^{2n} c_{nk} \overline{r}_{k+\gamma-2n}, \quad n, \gamma \geq 0. \tag{16}
\]

**Lemma 2**
\[
P_n Q_n f_{n\gamma} = \varphi_{n\gamma}, \quad 0 \leq \gamma \leq 2n + 1. \tag{17}
\]

**Proof.** The result follows as in [6].

**Theorem 3** \((N, p_n) \ast (N, q_n)\)-lim \( x_k \) exists whenever \((N, r_n)\)-lim \( x_k \) exists if and only if
\[
\sup_{0 \leq \gamma \leq 2n} |R_{2n-\gamma} \varphi_{n\gamma}| = O(1), \quad n \to \infty; \tag{18}
\]
and
\[
\lim_{n \to \infty} \frac{\varphi_{n, 2n-\gamma}}{P_n Q_n} = \delta_\gamma, \quad \text{for every fixed } \gamma. \tag{19}
\]

**Proof.** Let \((N, p_n) \ast (N, q_n)\)-lim \( x_k \) exist whenever \((N, r_n)\)-lim \( x_k \) exists. Applying Lemma 1 with \((N, p_n) = (N, r_n)\) and \( A = (N, p_n) \ast (N, q_n) = (c_{nk}) \), we have,
\[
\sup_{0 \leq \gamma \leq 2n} |R_{\gamma} \sum_{k=\gamma}^{2n} c_{nk} \overline{r}_{k-\gamma}| = O(1), \quad n \to \infty \tag{20}
\]
and
\[
\lim_{n \to \infty} \sum_{k=\gamma}^{2n} c_{nk} \overline{r}_{k-\gamma} = \delta_\gamma, \quad \text{for every fixed } \gamma. \tag{21}
\]

Using (16) and (20), we get
\[
\sup_{0 \leq \gamma \leq 2n} |R_{2n-\gamma} f_{n\gamma}| = O(1), \quad n \to \infty. \tag{22}
\]
Using (17), we note that \(|f_{n\gamma}| = \frac{|\varphi_{n\gamma}|}{|p_0| |q_0|}\) since \(|P_n| = |p_0|\) and \(|Q_n| = |q_0|\). Consequently, in view of (22), we get

\[
\sup_{0 \leq \gamma \leq 2n} |R_{2n-\gamma} \varphi_{n\gamma}| = O(1), \quad n \to \infty.
\]

In view of (16) and (21), we have

\[
\lim_{n \to \infty} f_{n,2n-\gamma} = \delta_{\gamma}, \quad \text{for every fixed } \gamma.
\]

Now, using (17), we get

\[
\lim_{n \to \infty} \frac{\varphi_{n,2n-\gamma}}{P_n Q_n} = \delta_{\gamma}, \quad \text{for every fixed } \gamma.
\]

Thus (18) and (19) hold. Conversely (18) and (19) imply (20) and (21) respectively. Using (5), we have, \(\lim_{n \to \infty} \sum_{k=0}^{2n} c_{nk} = 1\). Using Lemma 1, the result follows, completing the proof of the theorem.

**Corollary 3** \((N, r_n) \subseteq (N, p_n) * (N, q_n)\) if and only if (18) and (19) hold with \(\delta_{\gamma} = 0\).

**Corollary 4** If \(\lim_{n \to \infty} r_n = 0\), then \((N, r_n) \subseteq (N, p_n) * (N, q_n)\) if and only if (18) holds.

**Proof.** The result follows using (9) and the fact that \((N, p_n) * (N, q_n)\) is regular.

**Theorem 4** If

\[
\varphi_{n,2n-\gamma} = o(1), \quad n \to \infty, \quad \text{for every fixed } \gamma,
\]

and either

\[
\varphi_{n\gamma} = O(1), \quad n, \gamma \to \infty,
\]

or

\[
\theta_{\gamma}, \alpha_{n\gamma}, \beta_{n\gamma} = O(1), \quad n, \gamma \to \infty,
\]

then

\((N, r_n) \subseteq (N, p_n) * (N, q_n)\).

**Proof.** Using (23), (19) follows with \(\delta_{\gamma} = 0\) since \(|P_n| = |p_0|\) and \(|Q_n| = |q_0|\). Because of (13) and (25), (24) holds. So if (24) or (25) holds, (18) holds since \(R_n = O(1), \ n \to \infty, \ (N, r_n)\) being a regular method. The result now follows from Corollary 3.

We shall now take up an application of Theorem 4.
Theorem 5 Let \( p_n, q_n \rightarrow 0 \), \( n \rightarrow \infty \) and \( t_n = p_0q_n + p_1q_{n-1} + \cdots + p_nq_0 \), \( n = 0, 1, 2, \ldots \). Then

\[(N, t_n) \subseteq (N, p_n) \ast (N, q_n)\]

and

\[(N, p_n) \subseteq (N, t_n), (N, q_n) \subseteq (N, t_n).\]

Proof. We apply Theorem 4 with \( r_n = t_n \). With the usual notation we have

\[t(x) = p(x)q(x)\] and \( \bar{t}(x) = \bar{p}(x)\bar{q}(x) \). Since \( p_n, q_n \rightarrow 0 \), \( n \rightarrow \infty \), \( \bar{t}_n \rightarrow 0 \), \( n \rightarrow \infty \) (see [3], Theorem 1). Consequently (23) follows using (9). In view of (11), we have,

\[
\sum_{\gamma=0}^{\infty} \theta_{\gamma}x^\gamma = \frac{p(x)q(x)}{t(x)} = 1,
\]

so that

\[\theta_0 = 1 \text{ and } \theta_{\gamma} = 0, \gamma \geq 1;\]

\[
\sum_{\gamma=0}^{\infty} \alpha_{n\gamma}x^\gamma = \frac{p_{n+1}(x)q(x)}{t(x)} = p_{n+1}(x)p(x),
\]

so that

\[
\alpha_{n\gamma} = \sum_{\lambda=0}^{\gamma-(n+1)} \bar{p}_\lambda p_{\gamma-\lambda}, \gamma \geq n+1;\]

\[= 0, \ 0 \leq \gamma \leq n.\]

Consequently \( \alpha_{n\gamma} = O(1), n, \gamma \rightarrow \infty \). Similarly \( \beta_{n\gamma} = O(1), n, \gamma \rightarrow \infty \). In view of Theorem 4, \( (N, t_n) \subseteq (N, p_n) \ast (N, q_n) \). Now \( \frac{t(x)}{p(x)} = q(x) \) and \( q_n \rightarrow 0 \), \( n \rightarrow \infty \), \( (N, q_n) \) being regular, by (6). So by Corollary 2, \( (N, p_n) \subseteq (N, t_n) \). Similarly \( (N, q_n) \subseteq (N, t_n) \). The proof of the theorem is now complete.

References


