P.N. NATARAJAN

Some Steinhaus type theorems over valued fields


<http://www.numdam.org/item?id=AMBP_1996__3_2_183_0>

© Annales mathématiques Blaise Pascal, 1996, tous droits réservés.


NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
http://www.numdam.org/
1. Preliminaries:

In this paper $K$ denotes $R$ (the field of real numbers) or $C$ (the field of complex numbers) or a complete, non-trivially valued, non-archimedean field as will be explicitly stated depending on the context.

In the sequel, infinite matrices $A = (a_{nk}), n, k = 1, 2, \ldots$ and sequences $x = \{x_k\}, k = 1, 2, \ldots$ have their entries in $K$. If $X, Y$ are two classes of sequences, we write $(X, Y)$ to denote the class of all infinite matrices $A = (a_{nk}), n, k = 1, 2, \ldots$ for which

$$Ax = \{(Ax)_n\} \in Y \text{ whenever } x = \{x_k\} \in X,$$

where $(Ax)_n = \sum_{k=1}^{\infty} a_{nk}x_k, n = 1, 2, \ldots$, it being assumed that the series on the right converge. The sequence $Ax = \{(Ax)_n\}$ is called the $A$-transform of $x = \{x_k\}$. The sequence spaces $\ell_p, p \geq 1, \ell_\infty, c, c_0$ are defined as usual i.e.,

$$\ell_p = \{x = \{x_k\} : \sum_{k=1}^{\infty} |x_k|^p < \infty\}, p \geq 1;$$

$$\ell_\infty = \{x = \{x_k\} : \sup_{k \geq 1} |x^k| < \infty\};$$

$$c = \{x = \{x_k\} : \lim_{k \to \infty} x_k = s \text{ for some } s \in K\};$$

$$c_0 = \{x = \{x_k\} : \lim_{k \to \infty} x_k = 0\}.$$

Note that $\ell_p \subset c_0 \subset c \subset \ell_\infty$ where $p \geq 1$. For convenience we write $\ell_1 = \ell$. $(\ell, c; P')$ denotes the class of all infinite matrices $A \in (\ell, c)$ such that $\lim_{n \to \infty} (Ax)_n = \sum_{k=1}^{\infty} x_k$ whenever $x = \{x_k\} \in \ell$. 
2. The case $K = R$ or $C$

When $K = R$ or $C$, it is known ([11], p. 417) that $A = (a_{nk}) \in (\ell, c)$ if and only if

$$(1) \sup_{n,k} |a_{nk}| < \infty;$$

and

$$(2) \lim_{n \to \infty} a_{nk} = \delta_k \text{ exists, } k = 1, 2, \ldots.$$  

We now prove the following

**THEOREM 2.1**:  

When $K = R$ or $C$, $A \in (\ell, c; P')$ if and only if (1) holds and (2) holds with

$$(3) \delta_k = 1, k = 1, 2, \ldots.$$  

**Proof.**

Let $A \in (\ell, c; P')$. Let $e_k$ be the sequence in which 1 occurs in the $k^{th}$ place and 0 elsewhere, $k = 1, 2, \ldots$ i.e.,

$$e_k = \left\{ x^{(k)}_i \right\}_{i=1}^\infty$$

where

$$x^{(k)}_i = 1, \text{ if } i = k;$$

$$= 0, \text{ otherwise.}$$

Then $e_k \in \ell, k = 1, 2, \ldots, \sum_{i=1}^\infty x^{(k)}_i = 1$ and $(Ae_k)_n = a_{nk}$ so that $\lim_{n \to \infty} a_{nk} = 1$, i.e.,

$$\delta_k = 1, k = 1, 2, \ldots.$$  

Thus (1) and (3) are necessary for $A \in (\ell, c; P')$.

Conversely, let (1) and (3) hold. Let $x = \{x_k\} \in \ell$. In view of (1), $\sum_{k=1}^\infty a_{nk} x_k$ converges, $n = 1, 2, \ldots$ Now,

$$(Ax)_n = \sum_{k=1}^\infty a_{nk} x_k$$

$$= \sum_{k=1}^\infty (a_{nk} - 1)x_k + \sum_{k=1}^\infty x_k,$$

this being true since $\sum_{k=1}^\infty a_{nk} x_k$ and $\sum_{k=1}^\infty x_k$ both converge.
Since $\sum_{k=1}^{\infty} |x_k| < \infty$, given $\varepsilon > 0$, there exists a positive integer $N$ such that

$$\sum_{k=N+1}^{\infty} |x_k| < \frac{\varepsilon}{2A},$$

where $A = \sup_{n,k} |a_{nk} - 1|$. Since $\lim_{n\to\infty} a_{nk} = 1$, $k = 1, 2, ..., N$, we can choose a positive integer $N' > N$ such that

$$|a_{nk} - 1| < \frac{\varepsilon}{2NM}, \quad n \geq N', k = 1, 2, ..., N,$$

where $M > 0$ is such that $|x_k| \leq M$, $k = 1, 2, ...$. Now, for $n \geq N'$,

$$|\sum_{k=1}^{N}(a_{nk} - 1)x_k| \leq \sum_{k=1}^{N} |a_{nk} - 1||x_k| + \sum_{k=N+1}^{\infty} |a_{nk} - 1||x_k|$$

$$< N\frac{\varepsilon}{2NM}M + A\frac{\varepsilon}{2A},$$

in view of (4) and (5)

$$= \varepsilon,$$

so that $\lim_{n\to\infty} \sum_{k=1}^{\infty} (a_{nk} - 1)x_k = 0$. Thus $\lim_{n\to\infty} (Ax)_n = \sum_{k=1}^{\infty} x_k$.

Consequently $A \in (\ell, c ; P')$ which completes the proof of the theorem.

When $K = R$ or $C$, the Steinhaus theorem ([4], p. 187, Theorem 14) can be written conveniently in the form $(c, c ; P') \cap (\ell, c) = \emptyset$, where $(c, c ; P)$ denotes the class of all infinite matrices $A \in (c, c)$ such that $\lim_{n\to\infty} (Ax)_n = \lim_{k\to\infty} x_k$.

We shall call such type of theorems as "Steinhaus type theorems". Such theorems were considered in [2], [3], [8]. Using Theorem 1, we shall deduce one such theorem.

**THEOREM 2.2**:

$$(\ell, c ; P') \cap (\ell, c) = \emptyset \text{ whenever } p > 1.$$

**Proof.**

Suppose $A = (a_{nk}) \in (\ell, c ; P') \cap (\ell, c)$ where $p > 1$. It is known ([11], p. 4, 16) that $A \in (\ell, c)$ whenever $p > 1$, if and only if (2) holds and

$$\sup_{n} \sum_{k=1}^{\infty} |a_{nk}|^q < \infty,$$

where $\frac{1}{p} + \frac{1}{q} = 1$. It now follows that $\sum_{k=1}^{\infty} |\delta_k|^q < \infty$, which contradicts the fact that
\( \delta_k = 1, k = 1, 2, \ldots \), since \( A \in (\ell, c ; P') \) and consequently \( \sum_{k=1}^{\infty} |\delta_k|^q \) diverges. This establishes our claim.

**Remark 2.3.**

Since \( (\ell_\infty, c) \subset (c, c) \subset (c_0, c) \subset (\ell_p, c) \) where \( p > 1 \), we have \( (\ell, c ; P') \cap (X, c) = \emptyset \), when \( X = \ell_\infty, c, c_0, \ell_p \) where \( p > 1 \).

3. The case when \( K \) is a complete, non-trivially valued, non-archimedean field.

For concepts and results in Analysis over complete, non-trivially valued, non-archimedean fields, we refer to [1]. In this case, Steinhaus type theorems were considered in [6], [7], [8], [10].

When \( K \) is a complete, non-trivially valued, non-archimedean field, one can prove that Theorem 2.1 continues to hold. In this case, if \( A = (a_{nk}) \in (\ell, c ; P') \cap (\ell_\infty, c) \), then \( \limsup_{n \to \infty} |a_{nk} - 1| = 0 \) (see [6], Theorem 2). So for any \( \varepsilon, 0 < \varepsilon < 1 \), there exists a positive integer \( N \) such that

\[ |a_{nk} - 1| < \varepsilon, n \geq N, k = 1, 2, \ldots \]

In particular, \( |a_{Nk} - 1| < \varepsilon, k = 1, 2, \ldots \)

Thus \( \lim_{k \to \infty} |a_{Nk} - 1| \leq \varepsilon \) i.e., \( |0 - 1| \leq \varepsilon \) (since \( A \in (\ell_\infty, c) \), \( \lim_{k \to \infty} a_{nk} = 0, n = 1, 2, \ldots \)), by Theorem 2 of [6]) i.e., \( 1, \leq \varepsilon \), a contradiction on the choice of \( \varepsilon \). Consequently we have:

**Theorem 3.1**

*When \( K \) is a complete, non-trivially valued, non-archimedean field, \((\ell, c ; P') \cap (\ell_\infty, c) = \emptyset. \)*

**Remark 3.2:**

However, \((\ell, c ; P') \cap (c, c) \neq \emptyset \) when \( K \) is a complete, non-trivially valued, non-archimedean field, as the following example illustrates.

Consider the infinite matrix

\[
A = (a_{nk}) = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & \cdots \\
1 & -1 & 0 & 0 & 0 & 0 & \cdots \\
1 & 1 & -2 & 0 & 0 & 0 & \cdots \\
1 & 1 & 1 & -3 & 0 & 0 & \cdots \\
1 & 1 & 1 & 1 & -4 & 0 & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots 
\end{bmatrix}
\]
i.e., \( a_{nk} = 1, k \leq n - 1 \);
\( = -(n-1), \ k = n \);
\( = 0, \) otherwise.

Then \( \sup_{n,k} |a_{nk}| \leq 1 < \infty, \) \( \lim_{n} a_{nk} = 1, k = 1, 2, ... \) and \( \lim_{n-\infty} \sum_{k=1}^{\infty} a_{nk} = 0 \) so that \( A \in (\ell, c ; P') \cap (c, c) \) (for criterion for \( A \in (c, c) \), see \([5, 9]\)). Since \((c, c) \subset (c_0, c) \subset (\ell_p, c)\) where \( p > 1 \), it follows that \((\ell, c ; P') \cap (X, c) \neq \emptyset \), when \( X = c, c_0, \ell_p \) where \( p > 1 \).

This indicates a violent departure in when \( K \) is a non-archimedean valued field from the case \( K = R \) or \( C \).

\((c_0, c ; P')\) denotes the class of all infinite matrices \( A \in (c_0, c) \) such that \( \lim_{n-\infty} (Ax)_n = \sum_{k=1}^{\infty} x_k \) whenever \( x = \{ x_k \} \in c_0 \). In this context it is worthwhile to note that \( \sum_{k=1}^{\infty} x_k \) converges if and only if \( \{ x_k \} \in c_0 \).

**Remark 3.3 :**
\((c_0, c ; P') = (\ell, c ; P')\).

**Proof.**
Adapting the proof of Theorem 2.1, with suitable modifications for the non-archimedean case, we have, \( A \in (c_0, c ; P') \) if and only if (1) and (3) hold. The result now follows.

### 4. General remarks

It is to be noted that \( \ell_p, p \geq 1, c_0, c, \ell_{\infty} \) are linear spaces with respect to coordinatewise addition and scalar multiplication irrespective of how \( K \) is chosen. When \( K = R \) or \( C \), \( c_0, c, \ell_{\infty} \) are Banach spaces with respect to the norm \( \|x\| = \sup_{k \geq 1} |x_k| \) where \( x = \{ x_k \} \in c_0, c \) or \( \ell_{\infty} \), while they are non-archimedean Banach spaces under the above norm when \( K \) is a complete, non-trivially valued, non-archimedean field.

Whatever be \( K, \ell_p \) is a Banach space with respect to the norm

\[ \|x\| = \left( \sum_{k=1}^{\infty} |x_k|^p \right)^{1/p}, \ x = \{ x_k \} \in \ell_p. \]

Whatever be \( K, \) if \( A = (a_{nk}) \in (\ell, c ; P') \), then \( A \) is bounded and \( \|A\| = \sup_{n,k} |a_{nk}|. \)

However, \((\ell, c ; P')\) is not a subspace of \( BL(\ell, c) \), i.e., the space of all bounded linear mappings of \( \ell \) into \( c \), since \( \lim_{n-\infty} 2a_{nk} = 2, k = 1, 2, ... \) and consequently \( 2A \notin (\ell, c ; P') \) when \( A \in (\ell, c ; P') \).

I thank the referee for his helpful comments which enabled me to present the material in a better form.
REFERENCES


