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Some Steinhaus type theorems over valued fields


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1. Preliminaries:

In this paper $K$ denotes $R$ (the field of real numbers) or $C$ (the field of complex numbers) or a complete, non-trivially valued, non-archimedean field as will be explicitly stated depending on the context.

In the sequel, infinite matrices $A = (a_{nk}), n, k = 1, 2, \ldots$ and sequences $x = \{x_k\}, k = 1, 2, \ldots$ have their entries in $K$. If $X, Y$ are two classes of sequences, we write $(X, Y)$ to denote the class of all infinite matrices $A = (a_{nk}), n, k = 1, 2, \ldots$ for which

$$Ax = \{(Ax)_n\} \in Y \text{ whenever } x = \{x_k\} \in X,$$

where $(Ax)_n = \sum_{k=1}^{\infty} a_{nk} x_k, n = 1, 2, \ldots,$

it being assumed that the series on the right converge. The sequence $Ax = \{(Ax)_n\}$ is called the $A$-transform of $x = \{x_k\}$. The sequence spaces $\ell_p, p \geq 1, \ell_\infty, c, c_0$ are defined as usual i.e.,

$$\ell_p = \{x = \{x_k\} : \sum_{k=1}^{\infty} |x_k|^p < \infty, p \geq 1\};$$

$$\ell_\infty = \{x = \{x_k\} : \sup_{k \geq 1} |x^k| < \infty\};$$

$$c = \{x = \{x_k\} : \lim_{k \to \infty} x_k = s \text{ for some } s \in K\};$$

$$c_0 = \{x = \{x_k\} : \lim_{k \to \infty} x_k = 0\}.$$

Note that $\ell_p \subset c_0 \subset c \subset \ell_\infty$ where $p \geq 1$. For convenience we write $\ell_1 = \ell, (\ell, c; P')$ denotes the class of all infinite matrices $A \in (\ell, c)$ such that $\lim_{n \to \infty} (Ax)_n = \sum_{k=1}^{\infty} x_k$ whenever $x = \{x_k\} \in \ell$. 
2. The case \( K = R \) or \( C \)

When \( K = R \) or \( C \), it is known ([11], p. 4,17) that \( A = (a_{nk}) \in (\ell, c) \) if and only if

\[
\text{(1)} \quad \sup_{n,k} |a_{nk}| < \infty ;
\]

and

\[
\text{(2)} \quad \lim_{n \to \infty} a_{nk} = \delta_k \text{ exists, } k = 1, 2, \ldots.
\]

We now prove the following

**THEOREM 2.1:**

When \( K = R \) or \( C \), \( A \in (\ell, c; P') \) if and only if (1) holds and (2) holds with

\[\delta_k = 1, k = 1, 2, \ldots.\]

**Proof.**

Let \( A \in (\ell, c ; P') \). Let \( e_k \) be the sequence in which 1 occurs in the \( k^{th} \) place and 0 elsewhere, \( k = 1, 2, \ldots \) i.e.,

\[e_k = \left\{x^{(k)}_i\right\}_{i=1}^\infty\]

where

\[x^{(k)}_i = 1, \text{ if } i = k;\]

\[= 0, \text{ otherwise.}\]

Then \( e_k \in \ell, k = 1, 2, \ldots, \sum_{i=1}^\infty x^{(k)}_i = 1 \) and \( (Ae_k)_n = a_{nk} \) so that \( \lim_{n \to \infty} a_{nk} = 1 \), i.e.,

\[\delta_k = 1, k = 1, 2, \ldots.\]

Thus (1) and (3) are necessary for \( A \in (\ell, c ; P') \).

Conversely, let (1) and (3) hold. Let \( x = \{x_k\} \in \ell \). In view of (1), \( \sum_{k=1}^\infty a_{nk}x_k \) converges,

\[n = 1, 2, \ldots.\]

Now,

\[(Ax)_n = \sum_{k=1}^\infty a_{nk}x_k\]

\[= \sum_{k=1}^\infty (a_{nk} - 1)x_k + \sum_{k=1}^\infty x_k,\]

this being true since \( \sum_{k=1}^\infty a_{nk}x_k \) and \( \sum_{k=1}^\infty x_k \) both converge.
Since \( \sum_{k=1}^{\infty} |x_k| < \infty \), given \( \varepsilon > 0 \), there exists a positive integer \( N \) such that

\[
(4) \quad \sum_{k=N+1}^{\infty} |x_k| < \frac{\varepsilon}{2A},
\]

where \( A = \sup_{n,k} |a_{nk} - 1| \). Since \( \lim_{n \to \infty} a_{nk} = 1 \), \( k = 1, 2, \ldots, N \), we can choose a positive integer \( N' > N \) such that

\[
(5) \quad |a_{nk} - 1| < \frac{\varepsilon}{2NM}, \quad n \geq N', \, k = 1, 2, \ldots, N,
\]

where \( M > 0 \) is such that \( |x_k| \leq M \), \( k = 1, 2, \ldots \). Now, for \( n \geq N' \),

\[
|\sum_{k=1}^{N} (a_{nk} - 1)x_k| \leq \sum_{k=1}^{N} |a_{nk} - 1||x_k| + \sum_{k=N+1}^{\infty} |a_{nk} - 1||x_k|
\]

\[
< N \cdot \frac{\varepsilon}{2NM} \cdot M + A \cdot \frac{\varepsilon}{2A} = \varepsilon,
\]

so that \( \lim_{n \to \infty} \sum_{k=1}^{\infty} (a_{nk} - 1)x_k = 0 \). Thus \( \lim_{n \to \infty} (Ax)_n = \sum_{k=1}^{\infty} x_k \).

Consequently \( A \in (\ell, c ; P') \) which completes the proof of the theorem.

When \( K = R \) or \( C \), the Steinhaus theorem ([4], p. 187, Theorem 14) can be written conveniently in the form \( (c, c ; P) \cap (\ell_\infty, c) = \emptyset \), where \( (c, c ; P) \) denotes the class of all infinite matrices \( A \in (c, c) \) such that \( \lim_{n \to \infty} (Ax)_n = \lim_{k \to \infty} x_k \).

We shall call such type of theorems as "Steinhaus type theorems". Such theorems were considered in [2], [3], [8]. Using Theorem 1, we shall deduce one such theorem.

**THEOREM 2.2 :**

\( (\ell, c ; P') \cap (\ell_p, c) = \emptyset \) whenever \( p > 1 \).

**Proof.**

Suppose \( A = (a_{nk}) \in (\ell, c ; P') \cap (\ell_p, c) \) where \( p > 1 \). It is known ([11], p. 4, 16) that \( A \in (\ell_p, c) \) whenever \( p > 1 \), if and only if (2) holds and

\[
(6) \quad \sup_n \sum_{k=1}^{\infty} |a_{nk}|^q < \infty,
\]

where \( \frac{1}{p} + \frac{1}{q} = 1 \). It now follows that \( \sum_{k=1}^{\infty} |\delta_k|^q < \infty \), which contradicts the fact that
\[ \delta_k = 1, k = 1, 2, \ldots, \] since \( A \in (\ell, c ; P') \) and consequently \( \sum_{k=1}^{\infty} |\delta_k|^q \) diverges. This establishes our claim.

**Remark 2.3.**
Since \((\ell_\infty, c) \subset (c, c) \subset (c_0, c) \subset (\ell_p, c)\) where \( p > 1 \), we have \((\ell, c ; P') \cap (X, c) = \emptyset\), when \( X = \ell_\infty, c, c_0, \ell_p \) where \( p > 1 \).

3. The case when \( K \) is a complete, non-trivially valued, non-archimedean field.

For concepts and results in Analysis over complete, non-trivially valued, non-archimedean fields, we refer to [1]. In this case, Steinhaus type theorems were considered in [6], [7], [8], [10].

When \( K \) is a complete, non-trivially valued, non-archimedean field, one can prove that Theorem 2.1 continues to hold. In this case, if \( A = (a_{nk}) \in (\ell, c ; P') \cap (\ell_\infty, c) \), then \( \limsup_{n \to \infty} |a_{nk} - 1| = 0 \) (see [6], Theorem 2). So for any \( \varepsilon, 0 < \varepsilon < 1 \), there exists a positive integer \( N \) such that
\[ |a_{nk} - 1| < \varepsilon, n \geq N, k = 1, 2, \ldots. \]

In particular, \( |a_{nk} - 1| < \varepsilon, k = 1, 2, \ldots. \)

Thus \( \lim_{k \to \infty} |a_{nk} - 1| \leq \varepsilon \) i.e., \( |0 - 1| \leq \varepsilon \) (since \( A \in (\ell_\infty, c) \)), \( \lim_{k \to \infty} a_{nk} = 0, n = 1, 2, \ldots \), by Theorem 2 of [6]) i.e., \( 1, \leq \varepsilon \), a contradiction on the choice of \( \varepsilon \). Consequently we have:

**Theorem 3.1**
When \( K \) is a complete, non-trivially valued, non-archimedean field, \((\ell, c ; P') \cap (\ell_\infty, c) = \emptyset.\)

**Remark 3.2:**
However, \((\ell, c ; P') \cap (c, c) \neq \emptyset\) when \( K \) is a complete, non-trivially valued, non-archimedean field, as the following example illustrates.

Consider the infinite matrix
\[
A = (a_{nk}) = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & \cdots \\
1 & -1 & 0 & 0 & 0 & 0 & \cdots \\
1 & 1 & -2 & 0 & 0 & 0 & \cdots \\
1 & 1 & 1 & -3 & 0 & 0 & \cdots \\
1 & 1 & 1 & 1 & -4 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots
\end{bmatrix}
\]
i.e., \( a_{nk} = \begin{cases} 1, & k \leq n - 1 \\ -(n-1), & k = n \\ 0, & \text{otherwise.} \end{cases} \)

Then \( \sup_{n,k} |a_{nk}| \leq 1 < \infty \), \( \lim_{n \to \infty} a_{nk} = 1, k = 1, 2, ... \) and \( \lim_{n \to \infty} \sum_{k=1}^{\infty} a_{nk} = 0 \) so that \( A \in (\ell, c ; P') \cap (c, c) \) (for criterion for \( A \in (c, c) \), see [5], [9]). Since \( (c, c) \subset (c_0, c) \subset (\ell_p, c) \) where \( p > 1 \), it follows that \( (\ell, c ; P') \cap (X, c) \neq \emptyset \), when \( X = c, c_0, \ell_p \) where \( p > 1 \). This indicates a violent departure in when \( K \) is a non-archimedean valued field from the case \( K = R \) or \( C \).

\((c_0, c ; P')\) denotes the class of all infinite matrices \( A \in (c_0, c) \) such that \( \lim_{n \to \infty} (Ax)_n = \sum_{k=1}^{\infty} x_k \) whenever \( x = \{x_k\} \in c_0 \). In this context it is worthwhile to note that \( \sum_{k=1}^{\infty} x_k \) converges if and only if \( \{x_k\} \in c_0 \).

**Remark 3.3:**

\((c_0, c ; P') = (\ell, c ; P')\).

**Proof.**

Adapting the proof of Theorem 2.1, with suitable modifications for the non-archimedean case, we have, \( A \in (c_0, c ; P') \) if and only if (1) and (3) hold. The result now follows.

4. General remarks

It is to be noted that \( \ell_p, p \geq 1, c_0, c, \ell_\infty \) are linear spaces with respect to coordinatewise addition and scalar multiplication irrespective of how \( K \) is chosen. When \( K = R \) or \( C \), \( c_0, c, \ell_\infty \) are Banach spaces with respect to the norm \( \|x\| = \sup_{k \geq 1} |x_k| \) where \( x = \{x_k\} \in c_0, c \) or \( \ell_\infty \), while they are non-archimedean Banach spaces under the above norm when \( K \) is a complete, non-trivially valued, non-archimedean field.

Whatever be \( K, \ell_p \) is a Banach space with respect to the norm

\[ \|x\| = \left( \sum_{k=1}^{\infty} |x_k|^p \right)^{1/p}, \quad x = \{x_k\} \in \ell_p. \]

Whatever be \( K \), if \( A = (a_{nk}) \in (\ell, c ; P') \), then \( A \) is bounded and \( \|A\| = \sup_{n,k} |a_{nk}|. \)

However, \( (\ell, c ; P') \) is not a subspace of \( BL(\ell, c) \), i.e., the space of all bounded linear mappings of \( \ell \) into \( c \), since \( \lim_{n \to \infty} 2a_{nk} = 2, k = 1, 2, ... \) and consequently \( 2A \notin (\ell, c ; P') \) when \( A \in (\ell, c ; P') \).

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