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HARMONIC MAPS ON CONTACT METRIC MANIFOLDS

S.IANUS and A.M.PASTORE

Abstract In this paper, we study some harmonic or φ-pluriharmonic maps on contact metric manifolds.

1 Introduction

The theory of harmonic maps on Riemannian and Kahler manifolds has been developed in the last thirty years by many authors (see [E-L],[E-R],[LI], [M-R-S] and their references).

In odd dimension, the almost contact metric manifolds represent the analogue of almost hermitian manifolds (see [BL]).The first geometers to consider almost contact manifolds where W.Boothby - H.Wang [B-W], J.Gray [GR] and P.Libermann [LB]. A systematic study of them with adapted Riemannian metrics was initiated by Sasaki and his school.

A reference book for this subject is the one of D.E.Blair ([BL_2]). In the seventies there have been introduced interesting generalizations of almost contact metric structures by Blair [BL_1], S.Goldberg and K.Yano [G-Y], R.Lutz [LU]. There exists now a rich literature concerning the theory of harmonic maps in Kahler manifolds and more generally in almost hermitian manifolds. The purpose of this paper is to initiate the study of harmonic maps into almost contact metric manifolds. Moreover we introduce the concept of φ-pluriharmonicity in analogy with the known one from the geometry of almost hermitian manifolds.

In section 2 we consider φ-holomorphic maps between two contact metric manifolds and we prove that they are harmonic maps.

The third section is devoted to the study of φ-pluriharmonicity on almost contact metric manifolds. If $f : M \to M'$ is a φ-holomorphic map. between Sasaki manifolds, then f is φ-pluriharmonic if and only if it is an isometric immersion.

In the fourth section we study the harmonicity and φ-pluriharmonicity on globally φ-symmetric Sasaki manifolds.

Finally, in section 5 we give some examples.

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2 Harmonic maps on contact metric manifolds

Let (M, g) and (M', g') be two Riemannian manifolds and $f : M \to M'$ a differentiable map. All manifolds and maps are supposed to be of class C^∞.
We denote by ∇, ∇' the Levi-Civita connection on M and M' respectively and by $\tilde{\nabla}$ the connection induced by the map f on the bundle $f^{-1}(TM')$. Then the *second fundamental form* α_f of f is defined as follows:

$$(1) \qquad \alpha_f(X, Y) = \tilde{\nabla}_X df(Y) - df(\nabla_X Y)$$

for any $X, Y \in \mathcal{X}(\mathrm{M})$, where $\mathcal{X}(\mathrm{M})$ denotes the space of differentiable vector fields on M.
We often will write α and $f\star$ instead of α_f and df respectively.
The *tension field* $\tau(f)$ of f is defined as the trace of the second fundamental form α_f, i.e.:

$$(2) \qquad \tau(f)_x = \sum_{i=1}^{m} \alpha_f(e_i, e_i)(x)$$

where $e_i, \ldots, e_m$ is an orthonormal basis for the tangent space T_xM at $x \in M$.
We say that a map $f : M \to M'$ between Riemannian manifolds M and M' is a *harmonic map* iff $\tau(f) = 0$.
Let M be a differentiable manifold of dimension $2n + 1$.
Recall that an almost contact structure on M is a triple (φ, ξ, η), where φ is a tensor field of type (1,1), ξ is a vector field and η is a 1-form which satisfy:

$$(3) \qquad \varphi^2 = -I + \eta \otimes \xi \quad \text{and} \quad \eta(\xi) = 1$$

where I is the identity endomorphism on TM.
These conditions imply

$$(4) \qquad \varphi\xi = 0 \quad \text{and} \quad \eta \circ \varphi = 0$$

Furthemore, if g is an *associated Riemannian metric* on M , that is a metric that satisfies

$$(5) \qquad g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y) \qquad X, Y \in X(M)$$

then we say that (φ, ξ, η, g) is an *almost contact metric* structure. In such a way we obtain an *almost contact metric manifold* which is the analougue of an *almost hermitian manifold* .
We denote by Φ the fundamental 2-form on M defined by $\Phi(X, Y) = g(X, \varphi Y)$.
An almost contact metric structure is called a *contact metric structure* iff $\Phi = d\eta$.
An odd-dimensional manifold is called a *contact manifold* if it carries a contact metric structure. In [MA] J.Martinet proved that every compact orientable 3-manifold carries a contact structure. An almost contact structure (φ, ξ, η) is *normal* if

$$N_\varphi(X, Y) + 2d\eta(X, Y) \otimes \xi = 0, \quad X, Y \in \mathcal{X}(M)$$

where N_φ is the Nijenhuis torsion tensor of φ.
We denote by D the distribution orthogonal to ξ and by $\Gamma(D)$ the space of differentiable sections of D. Using the Levi-Civita connection determined by g we can define a Sasaki manifold as an almost contact metric manifold such that

$$(6) \qquad (\nabla_X\varphi)Y = g(X,Y)\xi - \eta(Y)X$$

for any $X, Y \in T(M)$, (see [BL]).
The condition (2.6) is analogous to what characterizes the Kahler manifolds.
We say that a map $f: M \to M'$ between two almost contact metric manifolds is φ - *holomorphic* (or $+\varphi$ *-holomorphic*) if $f_* \circ \varphi = \varphi' \circ f_*$ and f is φ *-antiholomorphic* (or $-\varphi$ *- holomorphic*) if $f_* \circ \varphi = -\varphi' \circ f_*$.
The following result is suggested by a paper of Tanno [TN].

Theorem 2.1 *Let M and M' be contact metric manifolds with structures (φ, ξ, η, g) and $(\varphi', \xi', \eta', g')$ respectively and let $f: M \to M'$ be a non constant map. Then we have: 1) If f is φ-holomorphic then there exists $a \in R$, $a > 0$ such that $df_x(\xi_x) = a\xi'_{f(x)}$ for any point $x \in M$. Furthermore, f is a D-homothetic immersion and an isometric immersion if and only if $a = 1$. 2) If f is φ-antiholomorphic then there exists $a \in R.a < 0$ such that $df_x(\xi_x) = a\xi'_{f(x)}$ for any point $x \in M$. Furthermore. f is a D-homothetic immersion and an isometric immersion if and only if $a = -1$.*

Proof Suppose that $f_* \circ \varphi = \pm\varphi' \circ f_*$. Then $\varphi'(f_*\xi) = \pm f_*(\varphi\xi) = 0$ implies that there exists a function a on M such that $f_{*x}(\xi_x) = a(x)\xi'_{f(x)}$ for any $x \in M$.
We consider the 1-form $f^*\eta'$ on M defined by $(f^*\eta')(X_x) = \eta'_{f(x)}(f_{*x}X_x))$ for any $x \in M$ and $X_x \in T_xM$. It follows that $f^*\eta' = a\eta$ and by differentiation. since d and f_* commute, we have

$$(7) \qquad f^*(d\eta') = da \wedge \eta + a \wedge d\eta$$

Computing (2.7) in (ξ, X) with $X \in \mathcal{X}$(M), using $d\eta = \Phi, d\eta' = \Phi'$ we obtain:

$$(8) \qquad (da \wedge \eta)(\xi, X) = 0$$

or, equivalently

$$(9) \qquad (L_\xi a)\eta = da$$

where L_ξ denotes the Lie derivative with respect to ξ.
From (2.9), since $(L_\xi a)\eta \wedge \eta = 0$, we have $da \wedge \eta = 0$ and by differentiation $da \wedge d\eta = 0$, so that (2.9) implies $(L_\xi a)\eta \wedge d\eta = 0$ and then $L_\xi a = 0$, since $\eta \wedge d\eta \neq 0$ everywhere.
Again, condition (2.9) gives $da = 0$, i.e.a is a constant.
Now, from $f^*\eta' = a\eta, \Phi = d\eta, \Phi' = d\eta'$, we have $f^*\Phi' = a\Phi$ and then

$$(10) \qquad g'(f_*X, \varphi' f_*Y) = ag(X, \varphi Y), \qquad X, Y \in \mathcal{X}(M)$$

Now, suppose that f is φ-holomorphic. Applying (2.10) to X and φY, we obtain

$$(11) \qquad (f^*g')(X,Y) = ag(X,Y) + a(a-1)\eta(X)\eta(Y)$$

Obviously, for $X, Y \in \Gamma(D)$ we have

$$(12) \qquad (f^*g')(X,Y) = ag(X,Y)$$

and $(f^*g')(X,X) = ag(X,X)$ which implies $a \geq 0$.
On the other hand $a = 0$ would imply $f_*\xi = 0$ and $f_*X = 0$ for any $X \in D$, i.e. $f_* = 0$, a contraddiction. So, we have $a > 0$ and f is D-homothetic.
Furthermore, f is an injection. Namely ker $f_* \subset D$ and for $X \in ker(f_*)$ we have $X = 0$ using (2.12). Hence f is an immersion. Obviously, (2.11) implies that f is an isometric immersion if and only if a=1.
Finally, suppose that f is φ-antiholomorphic. Then, applying (2.10) to X and φY we have

(13) $$(f^*g')(X,Y) = -ag(X,Y) + a(a+1)\eta(X)\eta(Y)$$

and in the same way we obtain $a < 0$, f is a D-homothetic immersion and it is isometric if and only if $a = -1$.

REMARK In the hypotheses of the previous proposition we have $dimM \leq dimM'$.
A.Lichnerowicz proved that if φ is a $\pm$ holomorphic map between almost Kahler manifolds, then it is a harmonic map (cf. [LI]). Now we give an analogous result for contact metric manifolds.
Theorem 2.2 *Let M and M' be contact metric manifolds. Any $\pm\varphi$- holomorphic map $f : M \to M'$ is a harmonic map.*
Proof. For any X and Y we have Olszak's formula

(14) $$(\nabla_{\varphi X}\varphi)\varphi Y + (\nabla_X\varphi)Y = 2g(X,Y)\xi - \eta(Y)(X + hX + \eta(X)\xi),$$

where $h = (1/2)L_\xi\varphi$ and L denotes Lie differentiation (cf. [OL], Lemma 3.1).
If we put $Y = \varphi X$, for any $X \in \Gamma(D)$, we obtain from (2.7),

(15) $$(\nabla_X\varphi)\varphi X = (\nabla_{\varphi X}\varphi)X$$

or, equivalently,

(16) $$\nabla_X X + \nabla_{\varphi X}\varphi X = \varphi[\varphi X, X] \quad \text{for any} \quad X \in \Gamma(D).$$

Using (2.15) on M' and since f is $\pm\varphi$ - holomorphic, we obtain

(17) $$\tilde{\nabla}_X f_*X + \tilde{\nabla}_{\varphi X} f_*\varphi X = \varphi'[\varphi' f_*X, f_*X]$$

From (2.16) and (2.17) it follows that for any $X \in D_x$, $x \in M$, we have

(18) $$\alpha(X,X) + \alpha(\varphi X, \varphi X) = 0$$

Let $(e_1, ..., e_n; \varphi e_1, ..., \varphi e_n, \xi)$ be a local orthonornal φ - basis of vector fields tangent to M. We obtain:

(19) $$\alpha(e_i, e_i) + \alpha(\varphi e_i, \varphi e_i) = 0, \quad i \in \{1, ..., n\}$$

Then we have

(20) $$\tau(f) = \sum_{i=1}^{n} \alpha(e_i, e_i) + \sum_{i=1}^{n} \alpha(\varphi e_i, \varphi e_i) + \alpha(\xi,\xi) = \alpha(\xi,\xi)$$

On M and M' we have respectively

$$\nabla_{\xi}\xi = 0 \text{ and } \nabla_{\xi'}\xi' = 0 \tag{21}$$

Then it follows $\alpha(\xi,\xi) = 0$, so that $\tau(f) = 0$. □

3 φ-pluriharmonicity on almost contact metric manifolds

Recently, several authors studied the concept of pluriharmonic map from a Kahler manifold in a Riemannian manifold (see [E-L] [DA] [OH] and their references).
Let M be a Kahler manifold with complex structure J and let M' be a Riemannian manifold. A map $f : M \to M'$ is called *pluriharmonic* if the second fundamental form α of the map f satisfies

$$\alpha(X,Y) + \alpha(JX,JY) = 0 \qquad X,Y \in \mathcal{X}(M)$$

Now we consider an analogous concept for the almost contact metric manifolds

Definition 3.1 *Let M be an almost contact metric manifold with the structure (φ,ξ,η,g) and (M',g') a Riemannian manifold. A map $f : M \to M'$ is said to be φ-pluriharmonic if,*

$$\alpha(X,Y) + \alpha(\varphi X,\varphi Y) = 0, \qquad X,Y \in \mathcal{X}(M) \tag{1}$$

where α is the second fundamental form of f. Furthermore, f is said to be D-pluriharmonic if (3.1) holds for any $X,Y \in \Gamma(D)$.
Obviously, φ - pluriharmonicity implies D - pluriharmonicity.

Proposition 3.2 *Let $f : M \to M'$ be a φ-pluriharmonic map from an almost contact metric manifold into a Riemannian manifold. Then f is a harmonic map.*

Proof. Fixed a local orthonormal φ-basis $(e_1,...,e_n;\varphi(e_1),...,\varphi(e_n),\xi)$ in M. the φ - pluriharmonicity hypothesis implies:

$$\alpha(\xi,\xi) = 0 \text{ and } \alpha(e_i,e_i) + \alpha(\varphi e_i,\varphi e_i) = 0,$$

for $i \in \{1,\dots,n\}$. Then we have,

$$\tau(f) = \sum_{i=1}^{n}\alpha(e_i,e_i) + \sum_{i=1}^{n}\alpha(\varphi e_i,\varphi e_i) + \alpha(\xi,\xi) = 0.$$

so that f is harmonic □.
Let T^cM be the complexification of TM. Then φ can be uniquely extended to a complex linear endomorphism of T^cM, denoted also by φ which satisfies (2.4). The eigenvalues of φ are therefore i, 0, and -i. We consider the usual decomposition

$$T^cM = T^+M \oplus T^\circ M \oplus T^- M$$

of T^cM in the eigenbundles corresponding to the eigenvalues i, 0, -i of φ.

Proposition 3.3 *Let M be an almost contact metric manifold with the structure (φ, ξ, η, g) and M' a Riemannian manifold. Then, for a map $f : M \to M'$ we have that f is φ-pluriharmonic if and only if the following conditions hold:*

i) $\alpha(Z, \overline{Z}) = 0$ for any $Z \in \Gamma(T^+M)$

ii) $\alpha(X, \xi) = 0$ for any $X \in \Gamma(D)$

iii) $\alpha(\xi, \xi) = 0$

Proof If $Z \in \Gamma(T^+M)$ we have $Z = X - i\varphi X$ with $X \in \Gamma(D)$ and $\overline{Z} = X + i\varphi X$, so that

$$\alpha(Z, \overline{Z}) = \alpha(X, X) + \alpha(\varphi X, \varphi X) = 0$$

by the φ-pluriharmonicity of f. Using (2.4) we obtain ii) and iii).
Conversely, the condition (3.1) is satisfied by $X = Y = \xi$ and by $Y = \xi$ and $X \in \Gamma(D)$ because of i) and ii). Finally, for any $X, Y \in \Gamma(D)$ we put $Z_1 = X - i\varphi X$, $Z_2 = Y - i\varphi Y$ and, applying i) to Z_1, Z_2 and to $Z = Z_1 + Z_2$, we obtain $0 = \alpha(Z, \overline{Z}) = 2\alpha(X, Y) + 2\alpha(\varphi X, \varphi Y)$.
Thus, condition (3.1) is completely satisfied and f is a φ-pluriharmonic map □.
Let M , M' be Kahler manifolds. It is well known that any holomorphic map $f : M \to M'$ is pluriharmonic. This property does not hold always for the φ-holomorphic maps between Sasaki manifolds; namely we have:

Theorem 3.4 *Let M, M' be Sasaki manifolds and $f : M \to M'$ a $\pm\varphi$-holomorphic map. Then f is D-pluriharmonic. Moreover, f, is φ - pluriharmonic if and only if it is an isometric immersion.*

Proof. On a Sasaki manifold, for any $X, Y \in \Gamma(D)$ we have

$$(\nabla_X \varphi)Y = g(X, Y)\xi \tag{2}$$

$$\eta(\nabla_Y X) = g(\nabla_Y X, \xi) = -g(X, \nabla_Y \xi) = g(X, \varphi Y) \tag{3}$$

It follows by a direct computation that:

$$\nabla_{\varphi X} \varphi Y = -\nabla_X Y - [Y, X] + \varphi[\varphi X, Y] \tag{4}$$

(5) $\tilde{\nabla}_{\varphi X} f_*(\varphi Y) = -\tilde{\nabla}_X f_* Y - [f_* Y, f_* X] + \varphi'[\varphi' f_* X, f_* Y]$ for any $X, Y \in \Gamma(D)$.

Therefore, for any $X, Y \in \Gamma(D)$, we have:

$$\begin{aligned} \alpha(X, Y) + \alpha(\varphi X, \varphi Y) = {} & \tilde{\nabla}_X f_* Y - f_*(\nabla_X Y) \\ & + \tilde{\nabla}_{\varphi X} f_*(\varphi Y) - f_*(\nabla_{\varphi X} \varphi Y) + \\ & = \varphi'[\varphi' f_* X, f_* Y] - f_* \varphi[\varphi X, Y] = \\ & = \pm\varphi[f_* \varphi X, f_* Y] - f_* \varphi[\varphi X, Y] = \\ & = f_* \varphi[\varphi X, Y] - f_* \varphi[\varphi X, Y] = 0 \end{aligned} \tag{6}$$

Consequently, f is D-pluriharmonic.
On the other hand there exists $a \in R, a \neq 0$, such that $df_x(\xi_x) = a\xi'_{f(x)}$, so that $\alpha(\xi, \xi) = 0$ and for any $X \in \Gamma(D)$ we hawe:

$$\alpha(X, \xi) = \tilde{\nabla}_X f_* \xi - f_* \nabla_X \xi =$$

$$= \nabla'_{f_*X} f_*\xi + f_*\varphi X =$$
$$-a\varphi' f_*X + f_*\varphi X =$$
$$= (\mp a + 1) f_*\varphi X.$$

By Theorem 2.1 it follows that that f is φ - pluriharmonic if and only if it is an isometric immersion. □

4 Harmonic maps on globally φ - symmetric Sasaki manifolds

Let M be a (2n+1) - dimensional manifold with an almost contact metric structure, (φ, ξ, η, g) and let B be a 2m - dimensional manifold ($n \geq m$) with an almost hermitian structure (J, h). A Riemannian submersion $\pi : M \to B$ is said to be φ - holomorphic if $\pi_*\varphi = J\pi_*$.
We have $J\pi_*\xi = \pi_*\varphi\xi = 0$, so that $\pi_*\xi = 0$. Consequently, the vector field ξ is a vertical vector field.
It is well known that a Riemannian submersion is a harmonic map if and only if its fibres are minimal submanifolds ([ER]). If $m = n$ and M is a Sasaki manifold, then ξ is a geodesic vector field, so that π is a harmonic map. It is easy to proof the following proposition.

Proposition 4.1 *Let $\pi : M \to B$ be a φ - holomorphic Riemannian submersion, where M is a (2n+1) - dimensional almost contact metric manifold and B is a 2n-dimensional almost hermitian manifold. The following conditions are equivalent:*

a) π is φ - *pluriharmonic*

b) $\nabla_X\xi = 0$ *for any* $X \in \mathcal{X}(M)$

c) ξ *is a Killing vector field and* $d\eta = 0$.

Remark If M is a contact metric manifold, $d\eta \neq 0$ and π can not be φ-pluriharmonic.

Proposition 4.2 Let $\pi : M \to B$ *be a φ-holomorphic Riemannian submersion, where M is a (2n+1)-dimensional almost contact metric manifold and B is a 2n-dimensional almost hermitian manifold. If M' is a Riemannian manifold and $f : M \to M'$ a map such that $f_*\xi = 0$, then there exists a uniquely determined map $\overline{f} : B \to M'$ such that $\overline{f} \circ \pi = f$. Furthermore f is D-pluriharmonic if and only if $\overline{f}$ is pluriharmonic.*

Proof Since the condition $f_*\xi = 0$ implies that f is constant on the fibres, it follows that $\overline{f}$ is uniquely determined.
We have the following well known relation for the second fundamental forms:

(1) $$\alpha_f(X, Y) = \alpha_{\overline{f}}(\pi_*X, \pi_*Y) + \overline{f}_*\alpha_\pi(X, Y) \text{ for any } X, Y \in \mathcal{X}(M)$$

Furthermore, since π is a Riemannian submersion we have

(2) $$\alpha_\pi(X, Y) = 0 \text{ for } X, Y \in \Gamma(D)$$

(cf. [ER], ch.IV, Lemma 1.5).
Since π is φ-holomorphic, for any $X, Y \in \Gamma(D)$ we obtain from (4.1),

$$\alpha_f(X,Y) + \alpha_f(\varphi X, \varphi Y) = \alpha_{\overline{f}}(\pi_* X, \pi_* Y) + \alpha_{\overline{f}}(J\pi_* X, J\pi_* Y) \tag{3}$$

On the other hand, $\pi_* : D_x \to T_{\pi(x)}B$ is surjective at any point $x \in M$.
Therefore. f is D-pluriharmonic if and only if $\overline{f}$ is pluriharmonic. □.

Now, let M be a (2n+1)-dimensional Sasaki manifold with structure (φ, ξ, η, g) and suppose that M is a globally φ-symmetric space [TA].
Then we know that M is a principal bundle over a 2n-dimensional hermitian globally symmetric space B with a Lie group G of dimension 1, isomorphic to the 1-parameter group of global transformation spanned by ξ. The projection $\pi : M \to B$ is a Riemannian submersion and the connection 1-form on M is determined by η.
Furthermore, π is a φ-holomorphic and harmonic map, since ξ is a geodesic vector field.
In this section we suppose that $M(c)$ is a $(2n+1$-dimensional globally φ-symmetric Sasaki manifold with constant φ-sectional curvature c. Then, we know that $M(c)$ is a principal bundle over a globally symmetric hermitian space of constant holomorphic sectional curvature c'. Furthermore, if we suppose $c > 0$ then B has to be bi-holomorphic to the complex projective space CP^n with the Fubini-Study metric (since $c' > 0$.)

Theorem 4.3 *Let $M(c)$ be a $(2n+1)$-dimensional globally φ- symmetric Sasaki manifold with constant φ-sectional curvature $c > 0$ and $\pi : M \to CP^n$ the principal bundle associated. Then for any nonconstant map $f : M \to CP^m$ such that $f_*\xi = 0$ there exists a unique map $\overline{f}$ such that $f = \overline{f} \circ \pi$ and f is $\pm\varphi$-holomorphic if and only if $\overline{f}$ is $\pm$ holomorphic. If f is a non constant D-pluriharmonic map such that $f_*(\xi) = 0$, then $\overline{f}$ is a $\pm$ holomorphic map.*

Proof Since $f_*\xi = 0$, the previous proposition implies that $\overline{f}$ is uniquely determined.
Denote by J and J' the complex structures on CP^n and CP^m respectively. Suppose that f is $\pm\varphi$ -holomorphic.

For any $X', Y' \in \mathcal{X}\ (CP^n)$, let X,Y $\in \mathcal{X}$ (M) be such that $X' = \pi_* X, Y' = \pi_* Y$.
Since π is φ - holomorphic, we have:

$$\overline{f}_*(JX') = \overline{f}_*(\pi_*\varphi X) = f_*(\varphi X) = \pm J'(f_* X) = \pm J'(\overline{f}_* X') \tag{4}$$

so that $\overline{f}$ is $\pm$ holomorphic. Conversely, suppose that $\overline{f}$ is $\pm$ holomorphic. We have $f_*(\varphi\xi) = 0$ and $J' f_*\xi = 0$.
For any $X \in \Gamma(D)$ we have

$$f_*\varphi(X) = \overline{f}_*\pi_*(\varphi X) = \overline{f}_*(J\pi_* X) = \pm J' f_*(X), \tag{5}$$

so that f is $\pm\varphi$ -holomorphic.
If f is a D-pluriharmonic map, then $\overline{f}$ is pluriharmonic. It is well known that a pluriharmonic map $\overline{f} : CP^n \to CP^m$ is $\pm$ holomorphic map (see, for example [OH]).□

Remark Obviously, $\overline{f}$ and f are harmonic map. However, f is not φ-pluriharmonic, since, otherwise, we have $\alpha_f(X,\xi) = f_\star\varphi X = 0$ and $f_\star = 0$. But, by hypothesis f is a non constant map.

5 Some examples

A vector field ξ on M is said to be regular if every point $x \in M$ has a cubical coordinate neighborhood U such that the integral curves of ξ passing through U pass trough the neighborhood only once. If (φ,ξ,η) is an almost contact structure with ξ regular on a compact manifold M, then M is a principal circle bundle over a manifold B (the set of maximal integral curves with the quotient topology) and we denote the projection by π. Furthermore, η is a connection form on M. Let g be a adapted metric on M. We define an almost complex structure J and a Riemannian metric h on B as follows:

(1) $$h(X,Y) = g(X^\star,Y^\star)\circ\pi \qquad X,Y \in \mathcal{X}(B)$$

(2) $$JX = \pi_\star(\varphi X^\star)$$

where $X^\star, Y^\star$ denote lifts of X and Y respectively with respect to the connection η (see [B-W], [OG].
It is easy to see that π is a Riemannian submersion satisfying $\pi_\star\varphi = J\pi_\star$.
If M is Sasaki, then π is a harmonic map.
A particulary well-known example of this fibration is the S^{2n+1} circle bundle over CP^n.

Odd-dimensional Lie groups and, more generally, odd-dimensional parallelizable manifolds admit almost contact metric structure (see $[BL_2]$). Now we generalize the above examples.

Let $N(J,h)$ be an almost hermitian manifold and denote by G a Lie group with left-invariant metric $<,>$. Let $\xi_1,...,\xi_s$ be an orthonormal basis of the Lie algebra g and $(\eta^1,...,\eta^s)$ the dual 1-forms. We denote by P a principal bundle on N with projection π and connection 1-form $\omega = \sum_{i=1}^s \tilde{\eta}^i \otimes \xi_i$ which takes values in g. For any $X \in \mathcal{X}$ (N) let X^H be its horizontal lift and denote by $A^\star$ the fundamental vertical vector field corresponding to $A \in g$. We define a tensor field φ of type (1,1) on P, putting

(3) $$\varphi(X^H) = (JX)^H \quad and \quad \varphi(A^\star) = 0$$

and we consider a metric g on P defined by

(4) $$g(X^H,Y^H) = h(X,Y)\circ\pi$$

(5) $$g(A^\star,B^\star) = < A,B >$$

(6) $$g(A^\star,X^H) = g(X^H,A^\star) = 0$$

for any $X,Y \in \mathcal{X}$ (N) and $A,B \in g$.
Obiviously $\pi : P \to N$ becomes a Riemannian submersion and π is a φ-holomorphic map according to J.Rawnsley [M-R-S] so that

(7) $$\pi_\star\circ\varphi = J\circ\pi_\star$$

By a straightforward computation we obtain $\tau(\pi) = 0$, so that π is a harmonic map. Furthermore π is φ-pluriharmonic if and only if ω is flat.

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