ANNALES MATHÉMATIQUES BLAISE PASCAL

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Annales mathématiques Blaise Pascal, tome 2, nº 2 (1995), p. 25-33

<http://www.numdam.org/item?id=AMBP_1995__2_2_25_0>

Ann. Math. Blaise Pascal, Vol. 2, N° 2, 1995, pp.25-33

ON THE THICKNESS OF TOPOLOGICAL SPACES

by Bernard BRUNET

We recall there are three classical definitions of the topological dimension : the small inductive dimension, denoted by *ind*, the large inductive dimension, denoted by *Ind* and the covering dimension, denoted by *dim*. (For the definitions, one can see (2).)

In this paper, coming back on a idea of J.P. REVEILLES (7), we give a nonstandard definition of the topological dimension - the thickness, denoted by *ep* (for épaisseur), - and we prove this definition coincides with the classical definitions in the class of separable metric spaces.

A.M.S. Mathematic Subject Classification (1991) : Primary 54 J 05
Secondary 54 F 45.

1 : Preliminary.

In the sequel, we consider a topological space X and an enlargment $\mathcal{E}$ (see, for example, (4)) containing X.

1) Definition 1.1 :

Let us consider a base $\mathcal{B}$ of X, a point a of $^*X$ and put $\mathcal{B}_a = \{B \in \mathcal{B} : a \in {}^*B\}$.
(In the special case where $a = {}^*x$, $\mathcal{B}_a = \{B \in \mathcal{B} : x \in B\}$.)
Then, we call *halo in base $\mathcal{B}$ of a*, the set $h_{\mathcal{B}}(a) = \bigcap_{B \in \mathcal{B}_a} {}^*B$.

Remark :

If $\mathcal{B}'$ is the base consisting of finite intersections of elements of $\mathcal{B}$, we have, for every point a of $^*X$, $h_{\mathcal{B}'}(a) = h_{\mathcal{B}}(a)$, whence the convention : we will call base of X only these bases of X satured by finite intersections.

Proposition 1.2 :

For every base $\mathcal{B}$ of X and for every point a of $^*X$, there exists an element Ω of $^*\mathcal{B}_a$ such that $\Omega \subset h_{\mathcal{B}}(a)$.
Indeed, the relation $\mathcal{R} \subset \mathcal{B}_a \times \mathcal{B}_a$ defined by «$A\mathcal{R}B \iff A \subset B$» is concurrent on $\mathcal{B}_a$.

Corollary 1.3 :

For every base $\mathcal{B}$ of X, every subset A of X and every $a \in {}^*X$, if $a \in {}^*\overline{A}$ (with $\overline{A}$ the closure of A in the space X), then $h_{\mathcal{B}}(a) \cap {}^*A \neq \emptyset$.
Note that, in the special case where $a = {}^*x$, $x \in \overline{A}$ if and only if $h_{\mathcal{B}}({}^*x) \cap {}^*A \neq \emptyset$.

2) Definition 1.4 :

Let us considerer a base $\mathcal{B}$ of X and a and b two elements of $^*X$.
Since a belongs to $h_{\mathcal{B}}(b)$ if and only if $h_{\mathcal{B}}(a)$ is contained in $h_{\mathcal{B}}(b)$, the relation $\leq$ defined by « $a \leq b \iff a \in h_{\mathcal{B}}(b)$ » is a preorder on $^*X$, called the *preorder associated to $\mathcal{B}$.*
Note this relation is not necessarily symmetric.
If we have $a \leq b$ and $b \leq a$, we will say that a and b are *equivalent modulo $\mathcal{B}$* and we will write $a \equiv b$.
Moreover, we will write $a < b$ if and only if $a \leq b$ but not $b \leq a$.

Proposition 1.5 :

For every base $\mathcal{B}$ of X and for every element a of $^*X$, there exists an element b of $^*X$ such that $b \leq a$ and b be minimal for the preorder associated to $\mathcal{B}$.

Indeed, the set $I = \{b \in {}^*X : b \leq a\}$ is inductive.

Proposition 1.6 :

Let $\mathcal{B}$ be a base of X and a an element of $^*X$. If there exists $B \in \mathcal{B}$ such that $a \in {}^*FrB$ (with FrB the boundary of B in the space X), then a is not minimal for the preorder associated to $\mathcal{B}$.

Since $a \in {}^*FrB$, it follows from 1.3 that $h_{\mathcal{B}}(a) \cap {}^*B \neq \emptyset$ and $h_{\mathcal{B}}(a) \cap {}^*(X \setminus B) \neq \emptyset$. There exists then an element b of $^*B$ such that $b \leq a$. If $a \equiv b$, we would have $h_{\mathcal{B}}(a) = h_{\mathcal{B}}(b) \subset {}^*B$ and consequently, ${}^*B \cap {}^*(X \setminus B) \neq \emptyset$ which is impossible. It follows that we have $b < a$, so that a is not minimal.

2 : Thickness of a topological space X.

1) Definition 2.1 :

Let $x \in X$ and $\mathcal{B}$ be a base of X. We will call *chain of length* p ($p \in \mathbb{N}$) of $h_{\mathcal{B}}({}^*x)$ every finite subset $\{a_p, \ldots, a_1\}$ of $h_{\mathcal{B}}({}^*x)$ such that $a_p < \ldots < a_1 < {}^*x$ and we will say that :

i) the *thickness in* x of $\mathcal{B}$ is less than n (and we will write $ep\,(x, \mathcal{B}) \leq n$) if and only if , for every chain $\{a_p, \ldots, a_1\}$ of $h_{\mathcal{B}}({}^*x)$, we have $p \leq n$.

ii) the *thickness in* x of $\mathcal{B}$ is equal to n if and only if $ep\,(x, \mathcal{B}) \leq n$ and $ep\,(x, \mathcal{B}) > n-1$.

Note our definition of thickness is the same as the « intended » definition in (7), provided the notion of « consecutive halos » is corrected therein p. 707.

2) Definition 2.2 :

Let $\mathcal{B}$ be a base of X. We will call *thickness of* $\mathcal{B}$, the element of $\mathrm{D} = \{n \in \mathbb{Z} : n \geq -1\} \cup \{+\infty\}$, denoted by $ep\ \mathcal{B}$, defined by $ep\ \mathcal{B} = sup\{ep\,(x, \mathcal{B}) : x \in X\}$.

Remark :

Note that one can give another definition of the thickness of a base $\mathcal{B}$, using the thickness of $\mathcal{B}$ in all the points of $^*X$, standard or not. This thickness, denoted $Ep\ \mathcal{B}$ ($= sup\{ep\,(a, \mathcal{B}) : a \in {}^*X\}$), is such of course that $ep\ \mathcal{B} \leq Ep\ \mathcal{B}$ and it might happen that $ep\ \mathcal{B} < Ep\ \mathcal{B}$. However, one can prove that for the « complemented » bases $\mathcal{B}$, one has $ep\ \mathcal{B} = Ep\ \mathcal{B}$ and that, for every base $\mathcal{B}$, there exists a « complemented » base $\mathcal{C}$ such that $ep\ \mathcal{C} \leq ep\ \mathcal{B}$, so that, if necessary, one only considers « complemented » bases of X. All these results will be proved in another paper of the author.

We now discuss some examples.

Proposition 2.3 :

Let us suppose X non empty and let $\mathcal{B}$ be a base of X. Then ep $\mathcal{B} = 0$ if and only if $\mathcal{B}$ consists of open-closed subsets of X.

i) Suppose *ep* $\mathcal{B} = 0$. Let us consider an element B of $\mathcal{B}$ and x an element of $\overline{B}$. Then, we have $h_{\mathcal{B}}({}^*x) \cap {}^*B \neq \emptyset$. Let $a \in {}^*B$ such that $a \leq {}^*x$. Since *ep* $\mathcal{B} = 0$, we have $a \equiv {}^*x$ and therefore $x \in B$, so that B is closed.

ii) Suppose all the elements of $\mathcal{B}$ are open-closed. Let $x \in X$ and $a \leq {}^*x$. Let us prove that we have ${}^*x \leq a$. Let $B \in \mathcal{B}$ such that $a \in {}^*B$. Then, we have $h_{\mathcal{B}}(a) \cap {}^*B \neq \emptyset$ and therefore $h_{\mathcal{B}}({}^*x) \cap {}^*B \neq \emptyset$, so that $x \in \overline{B}$. Since B is closed, we have $x \in B$ and therefore ${}^*x \leq a$.

Proposition 2.4 :

i) *For every totally ordered space X (totally ordered set X with its order topology), if we denote by $\mathcal{B}_o$ the base of ${}^*X$ consisting of all open intervals, we have ep $\mathcal{B}_o \leq 1$.*

ii) *In the special case where $X = \mathbb{R}$, we have ep $\mathcal{B}_o = 1$.*

Proof :

i) For every $x \in X$ and every $a \in h_{\mathcal{B}}({}^*x)$, we have $h_{\mathcal{B}_o}(a) = h_{\mathcal{B}_o}({}^*x)$ or $h_{\mathcal{B}_o}(a) = h_{\mathcal{B}_o}({}^*x) \cap]{}^*x, \rightarrow [$ or $h_{\mathcal{B}_o}(a) = h_{\mathcal{B}_o}({}^*x) \cap] \leftarrow, {}^*x[$.

ii) If $X = \mathbb{R}$, since $\mathcal{B}_o$ is not a base consisting of open-closed subsets of $\mathbb{R}$, we have *ep* $\mathcal{B}_o > 0$ and therefore *ep* $\mathcal{B}_o = 1$.

Proposition 2.5 :

Let X a topological space, $\mathcal{B}$ a base of X and A a subset of X. If we denote by $\mathcal{C}$ the trace of $\mathcal{B}$ on X, we have ep $\mathcal{C} \leq$ ep $\mathcal{B}$.

Indeed, for every couple $(a,b) \in {}^*A \times {}^*A$, the relations « $a < b$ *modulo* $\mathcal{C}$ » and « $a < b$ *modulo* $\mathcal{B}$ » are equivalent.

Proposition 2.6 :

Let X and Y be two topological spaces. For every base $\mathcal{B}$ of X and every base $\mathcal{C}$ of Y, we have ep $(\mathcal{B} \times \mathcal{C}) \leq$ ep $\mathcal{B}$ + ep $\mathcal{C}$.

Indeed, for every $(a,b) \in {}^*X \times {}^*Y$, we have $h_{\mathcal{B}\times\mathcal{C}}(a,b) = h_{\mathcal{B}}(a) \times h_{\mathcal{B}}(b)$.

3) Definition 2.7 :

Let X a topological space. We will call *thickness of* X the element of D, denoted by *ep* X, defined by *ep* $X = inf\{ep\ \mathcal{B} : \mathcal{B} \in \mathcal{B}(X)\}$, where $\mathcal{B}(X)$ is the set of all bases of X.

It follows from this definition and the previous results that :

2.8 : 1) If X is non empty, *ep* $X = 0$ if and only if X has a base consisting of open-closed susbsets.

2) For every totally ordered space X, we have *ep* $X \leq 1$.
In particular, since $\mathbb{R}$ is connected, we have *ep* $\mathbb{R} = 1$.

3) For every topological space X and every subset A of X, we have *ep* $A \leq$ *ep* X.

4) For every topological spaces X and Y, we have *ep* $(X \times Y) \leq$ *ep* $X +$ *ep* Y.

2.9 : Remarks.

1) It follows from 2.8.2) and 2.8.4) that, for every $n \geq 1$, *ep* $\mathbb{R}^n \leq n$. (In the sequel, we will prove that *ep* $\mathbb{R}^n = n$).

2) In contrast to the classical definitions, there is no need for any special hypothesis for 2.8.3) and 2.8.4) to be true : recall, for example, there exists (3) two compact spaces X and Y such that $ind(X \times Y) > indX + indY$.

3 : Comparison between thickness and classical dimensions.

1) Theorem 3.1 :

For every topological space X*, we have :*

a) $epX = 0$ if and only if $indX = 0$,

b) $indX \leq epX$.

Proof :

a) is immediate since these two assertions are equivalents to « there exists a base of X consisting of open-closed subsets ».

b) The theorem is obvious if *ep* $X = +\infty$, so that we can suppose *ep* $X < +\infty$.
Let us prove the theorem by induction on $n =$ *ep* X.
It follows from a) that the statment holds for $n = 0$.
Suppose it holds for every space Y such that *ep* $Y \leq n - 1$ and let us prove then that $indX \leq n$, i.e., that, for every point x of X and every neighbourhood V of x, there exists an open subset 0 such that $x \in 0 \subset V$ and $ind(Fr0) \leq n - 1$.
Since *ep* $X = n$, there exists a base $\mathcal{B}$ of X such that *ep* $\mathcal{B} = n$. Let us prove then that, for every $B \in \mathcal{B}$, we have *ep* $(FrB) \leq n - 1$, which by the induction

hypothesis, implies $indX \leq n$.

Let $B \in \mathcal{B}$. Put $F = FrB$ and call $\mathcal{C}$ the trace of $\mathcal{B}$ on F.

Let us prove that $ep\ \mathcal{C} \leq n-1$. Let $x \in F$ and $\{a_p, \ldots, a_1\}$ be a chain of $h_{\mathcal{C}}({}^*x)$. Since $h_{\mathcal{C}}({}^*x) = h_{\mathcal{B}}({}^*x) \cap {}^*F$, it follows from 1.6 that a_p is not minimal for the preorder associated to $\mathcal{B}$. Consequently, there exists an element a_{p+1} of ${}^*X$ such that $\{a_{p+1}, a_p, \ldots, a_1\}$ is a chain of $h_{\mathcal{B}}({}^*x)$. Since $ep\ \mathcal{B} = n$, we have necessarily $p \leq n-1$, which implies $ep\ (x, \mathcal{C}) \leq n-1$ and therefore $ep\ \mathcal{C} \leq n-1$. Since $ep\ F \leq ep\ \mathcal{C}$, we conclude $ep\ F \leq n-1$.

Corollary 3.2 :

For every $n \geq 1$, *we have* $ep\ \mathbb{R}^n = n$.

Indeed, we know that $ind\ \mathbb{R}^n = n$ (see for example (2)) and $ep\ \mathbb{R}^n \leq n$.

Corollary 3.3 :

For every totally ordered space X, *we have* $indX = epX \leq 1$

This assertion follows from 3.1 and 2.8.2).

Remark :

In another paper (1), we have proved that, for every totally ordered space X, $indX = IndX = dimX \leq 1$.

2) An example of a space X such that $indX = IndX < ep\ X$.

In (3), V.V. FILIPPOV has proved there exists two compact (non metric) spaces X_1 and X_2 such that $indX_1 = IndX_1 = 1, indX_2 = IndX_2 = 2$ and $ind(X_1 \times X_2) = Ind(X_1 \times X_2) \geq 4$. It follows from this example that X_1 or X_2 is such that $indX_i = IndX_i < ep\ X_i$. Indeed, if $indX_1 = IndX_1 = ep\ X_1$ and $indX_2 = IndX_2 = ep\ X_2$, we would have, from 2.8.4), $ep\ (X_1 \times X_2) \leq 3$, which is impossible since $ep\ (X_1 \times X_2) \geq ind(X_1 \times X_2)$ and $ind(X_1 \times X_2) \geq 4$.

Note the space we are looking for is the space X_2. Indeed, it is not the space X_1 because X_1 is by definition the quotient of a product of a compact totally disconnected space Z^* by a long line L. Since $ep\ Z^* = indZ^* = 0$ and $ep\ L = indL = 1$ (use 3.3), we have $ep\ (Z^* \times L) = 1$ and therefore $ep\ X_1 = indX_1 = 1$.

Note the description of the space X_2 is quite complicated so that it will not be reproduced here.

3) An example of space X such that $ep\ X = indX < IndX = dimX$.

In (8), P. ROY has proved there exists a completely metric space X such that $indX = 0$ and $IndX = dimX = 1$. It follows from 3.1 a) that, for this space, $ep\ X = indX = 0$ and $ep\ X < IndX = dimX$.

4) An example of space X such that $dimX < ep\ X$.

In (5), O.V. LOKUCIEVSKII has proved there exists a compact (non metric) space such that $dimX = 1 < 2 = indX = IndX$. For this space, we have $dimX < indX \leq ep\ X$.

4 : The case of metric spaces.

Theorem 4.1 :

For every metric space X, we have $indX \leq ep\ X \leq dimX = IndX$.

Since, for every topological space Y, we have $indY \leq ep\ Y$ and, for every metric space Z, we have $dimZ = IndZ$ (see for example (2)), it suffices to prove that, for every metric space X, we have $ep\ X \leq dimX$.

Notations : Let $\mathcal{F} = (F_i)_{i \in I}$ be an indexed family of subsets of X. Let us put, for every element x of X, $ord\ (x, \mathcal{F}) = |\{i \in I : x \in F_i\}| - 1$ (where $|\mathcal{A}|$ denotes the cardinal of $\mathcal{A}$) and $ord\ \mathcal{F} = sup\{\ ord\ (x, \mathcal{F}) : x \in X\}$ ($ord\ \mathcal{F}$ is called the order of $\mathcal{F}$).

Lemma 4.1.1 :

For every base $\mathcal{B}$ of X, let $\mathcal{F} = (FrB)_{B \in \mathcal{B}}$, then $ep\ \mathcal{B} \leq ord\ \mathcal{F} + 1$.

Let x be an element of X and $\{a_p, ..., a_1\}$ be a chain of $h_{\mathcal{B}}(^*x)$. There exists then p distinct elements of $\mathcal{B}$, $B_1, ..., B_p$ such that, for every $i \in \{1, ..., p\}, a_j \in {}^*B_i$ if and only if $j \geq i$ and such that $x \in FrB_i$. Consequently, by the definition of $ord\ (x, \mathcal{F})$, we have $p \leq ord\ (x, \mathcal{F}) + 1$, which implies $ep\ (x, \mathcal{B}) \leq ord\ (x, \mathcal{F}) + 1$. It follows then, from the definitions of $ep\ \mathcal{B}$ and $ord\ \mathcal{F}$, that we have $ep\ \mathcal{B} \leq ord\ \mathcal{F} + 1$.

4.1.2. : Proof of 4.1 :

This assertion is obvious if $dimX = +\infty$.

If $dimX = n$, there exists (see, for example, (2) , 4.2.2.)) a σ-locally finite base $\mathcal{B}$ of X such that, if we put $\mathcal{F} = (FrB)_{B \in \mathcal{B}}$, we have $ord\ \mathcal{F} \leq n - 1$. It follows then, from 4.1.1., that, for this base $\mathcal{B}$, we have $ep\ \mathcal{B} \leq n$, which implies that $ep\ X \leq n$.

4.2 : Let us note that ROY's space is a metric space such that
$indX = ep\ X = 0 < dimX = IndX = 1$.

4.3 : Coïncidence theorem for separable metric spaces.

For every separable metric space X, we have $ep\ X = indX = IndX = dimX$.

This assertion is an immediate consequence of 4.1 and the well-knowed theorem : « For every separable metric space X, we have $indX = IndX = dimX$ ».

4.4. : One can give a direct proof of 4.3. Indeed, let X be a separable metric space such

that $indX = n$. Let us denote by N_n^{2n+1} NOBELING's space (6), viz, the subspace of $\mathbb{R}^{2n+1}$ consisting of all points which have at most n rational coordinates, and, by $\mathcal{C}_n^{2n+1}$ the trace on N_n^{2n+1} of the base $\mathcal{B}^{2n+1}$ of $\mathbb{R}^{2n+1}$ consisting of all parallelepipeds with rational coordinates. One can prove that $ep\ \mathcal{C}_n^{2n+1} \leq n$ which implies, since $indN_n^{2n+1} = n$ (see, for example, (2) 1.8.5), that $ep\ N_n^{2n+1} = n$. Since $indX = n$ and N_n^{2n+1} is universal for the class of separable metric spaces whose dimension is not larger than n (see also (2), 1.11.5), X is homeomorphic to a subspace of N_n^{2n+1}, which implies, from 2.8.3), that $ep\ X \leq ep\ N_n^{2n+1}$ and therefore that $ep\ X = n$.

4.5 : An example of a non separable metric space X such that $ep\ X = indX = IndX = dimX$.

In (9), E.K. VAN DOUWEN proved there exists a non separable metric space X such that $indX = IndX = dimX = 1$.

This space is therefore such that $ep\ X = indX = IndX = dimX$.

4.6 : Question : Does there exist a metric space X such that $indX < ep\ X$?

Aknowledgments : I express my deep gratitude to Professor Labib HADDAD for many valuable suggestions.

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