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On the thickness of topological spaces


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ON THE THICKNESS OF TOPOLOGICAL SPACES

by Bernard BRUNET

We recall there are three classical definitions of the topological dimension: the small inductive dimension, denoted by $ind$, the large inductive dimension, denoted by $Ind$ and the covering dimension, denoted by $dim$. (For the definitions, one can see (2).)

In this paper, coming back on an idea of J.P. REVEILLES (7), we give a nonstandard definition of the topological dimension - the thickness, denoted by $ep$ (for épaisseur), - and we prove this definition coincides with the classical definitions in the class of separable metric spaces.

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1 : Preliminary.
In the sequel, we consider a topological space $X$ and an enlargment $\mathcal{E}$ (see, for example, (4)) containing $X$.

1) Definition 1.1 :
Let us consider a base $B$ of $X$, a point $a$ of $X$ and put $B_a = \{ B \in B : a \in B \}$. (In the special case where $a = *x$, $B_a = \{ B \in B : x \in B \}$.)
Then, we call halo in base $B$ of $a$, the set $h_B(a) = \bigcap_{B \in B_a} ^* B$.

Remark :
If $B'$ is the base consisting of finite intersections of elements of $B$, we have, for every point $a$ of $X$, $h_B(a) = h_{B'}(a)$, whence the convention : we will call base of $X$ only these bases of $X$ saturated by finite intersections.

Proposition 1.2 :
For every base $B$ of $X$ and for every point $a$ of $X$, there exists an element $\Omega$ of $B_a$ such that $\Omega \subseteq h_B(a)$.
Indeed, the relation $R \subseteq B_a \times B_a$ defined by « $ARB \iff A \subset B$ » is concurrent on $B_a$.

Corollary 1.3 :
For every base $B$ of $X$, every subset $A$ of $X$ and every $a \in X$, if $a \in \overline{A}$ (with $\overline{A}$ the closure of $A$ in the space $X$), then $h_B(a) \cap \overline{A} \neq \emptyset$.
Note that, in the special case where $a = *x$, $x \in \overline{A}$ if and only if $h_B(*x) \cap \overline{A} \neq \emptyset$.

2) Definition 1.4 :
Let us considerer a base $B$ of $X$ and $a$ and $b$ two elements of $X$.
Since $a$ belongs to $h_B(b)$ if and only if $h_B(a)$ is contained in $h_B(b)$, the relation $\leq$ defined by « $a \leq b \iff a \in h_B(b)$ » is a preorder on $X$, called the preorder associated to $B$.
Note this relation is not necessarily symmetric.
If we have $a \leq b$ and $b \leq a$, we will say that $a$ and $b$ are equivalent modulo $B$ and we will write $a \equiv b$.
Moreover, we will write $a < b$ if and only if $a \leq b$ but not $b \leq a$. 
Proposition 1.5:
For every base $B$ of $X$ and for every element $a$ of $X$, there exists an element $b$ of $X$ such that $b \leq a$ and $b$ be minimal for the preorder associated to $B$.

Indeed, the set $I = \{ b \in X : b \leq a \}$ is inductive.

Proposition 1.6:
Let $B$ be a base of $X$ and $a$ an element of $X$. If there exists $B \in B$ such that $a \in FrB$ (with FrB the boundary of $B$ in the space $X$), then $a$ is not minimal for the preorder associated to $B$.

Since $a \in FrB$, it follows from 1.3 that $h_B(a) \cap X \neq \emptyset$ and $h_B(a) \cap (X \setminus B) \neq \emptyset$. There exists then an element $b$ of $B$ such that $b \leq a$. If $a \equiv b$, we would have $h_B(a) = h_B(b) \subset B$ and consequently, $B \cap (X \setminus B) \neq \emptyset$ which is impossible. It follows that we have $b < a$, so that $a$ is not minimal.

2 : Thickness of a topological space $X$.

1) Definition 2.1:
Let $x \in X$ and $B$ be a base of $X$. We will call chain of length $p$ ($p \in \mathbb{N}$) of $h_B(x)$ every finite subset $\{ a_p, \ldots, a_i \}$ of $h_B(x)$ such that $a_p \leq \ldots \leq a_i \leq x$ and we will say that:

i) the thickness in $x$ of $B$ is less than $n$ (and we will write $ep(x, B) \leq n$) if and only if, for every chain $\{ a_p, \ldots, a_i \}$ of $h_B(x)$, we have $p \leq n$.

ii) the thickness in $x$ of $B$ is equal to $n$ if and only if $ep(x, B) = n$ and $ep(x, B) > n - 1$.

Note our definition of thickness is the same as the « intended » definition in (7), provided the notion of « consecutive halos » is corrected therein p. 707.

2) Definition 2.2:
Let $B$ be a base of $X$. We will call thickness of $B$, the element of $D = \{ n \in \mathbb{Z} : n \geq -1 \} \cup \{ +\infty \}$, denoted by $ep B$, defined by $ep B = \sup \{ ep(x, B) : x \in X \}$.

Remark:
Note that one can give another definition of the thickness of a base $B$, using the thickness of $B$ in all the points of $X$, standard or not. This thickness, denoted $Ep B$ ($= \sup \{ ep(a, B) : a \in X \}$), is such of course that $ep B \leq Ep B$ and it might happen that $ep B < Ep B$. However, one can prove that for the « complemented » bases $B$, one has $ep B = Ep B$ and that, for every base $B$, there exists a « complemented » base $C$ such that $ep C \leq ep B$, so that, if necessary, one only
considers « complemented » bases of $X$. All these results will be proved in another paper of the author.

We now discuss some examples.

Proposition 2.3:

Let us suppose $X$ non empty and let $B$ be a base of $X$. Then $ep\, B = 0$ if and only if $B$ consists of open-closed subsets of $X$.

i) Suppose $ep\, B = 0$. Let us consider an element $B$ of $B$ and $x$ an element of $\overline{B}$. Then, we have $h_B(x) \cap \overline{B} \neq \emptyset$. Let $a \in \overline{B}$ such that $a \leq x$. Since $ep\, B = 0$, we have $a \equiv x$ and therefore $x \in B$, so that $B$ is closed.

ii) Suppose all the elements of $B$ are open-closed. Let $x \in X$ and $a \leq x$. Let us prove that we have $x \leq a$. Let $B \in B$ such that $a \in \overline{B}$. Then, we have $h_B(a) \cap \overline{B} \neq \emptyset$ and therefore $h_B(x) \cap \overline{B} \neq \emptyset$, so that $x \in \overline{B}$. Since $B$ is closed, we have $x \in B$ and therefore $x \leq a$.

Proposition 2.4:

i) For every totally ordered space $X$ (totally ordered set $X$ with its order topology), if we denote by $B_o$ the base of $\overline{X}$ consisting of all open intervals, we have $ep\, B_o \leq 1$.

ii) In the special case where $X = IR$, we have $ep\, B_o = 1$.

Proof:

i) For every $x \in X$ and every $a \in h_B(x)$, we have $h_{B_o}(a) = h_{B_o}(x)$ or $h_{B_o}(a) = h_{B_o}(x) \cap \overline{x} \rightarrow [ \text{ or } h_{B_o}(a) = h_{B_o}(x) \cap ] \leftarrow, \overline{x}$. 

ii) If $X = IR$, since $B_o$ is not a base consisting of open-closed subsets of $IR$, we have $ep\, B_o > 0$ and therefore $ep\, B_o = 1$.

Proposition 2.5:

Let $X$ a topological space, $B$ a base of $X$ and $A$ a subset of $X$. If we denote by $C$ the trace of $A$ on $X$, we have $ep\, C \leq ep\, B$.

Indeed, for every couple $(a, b) \in \overline{A} \times \overline{A}$, the relations « $a < b$ modulo $C$ » and « $a < b$ modulo $B$ » are equivalent.

Proposition 2.6:

Let $X$ and $Y$ be two topological spaces. For every base $B$ of $X$ and every base $C$ of $Y$, we have $ep\, (B \times C) \leq ep\, B + ep\, C$.

Indeed, for every $(a, b) \in \overline{X} \times \overline{Y}$, we have $h_{B \times C}(a, b) = h_B(a) \times h_C(b)$. 
3) Definition 2.7:
Let $X$ a topological space. We will call *thickness of* $X$ the element of $D$, denoted by $\text{ep } X$, defined by $\text{ep } X = \inf \{ \text{ep } B : B \in B(X) \}$, where $B(X)$ is the set of all bases of $X$.

It follows from this definition and the previous results that:

2.8: 1) If $X$ is non empty, $\text{ep } X = 0$ if and only if $X$ has a base consisting of open-closed subsets.

2) For every totally ordered space $X$, we have $\text{ep } X \leq 1$.

In particular, since $\mathbb{R}$ is connected, we have $\text{ep } \mathbb{R} = 1$.

3) For every topological space $X$ and every subset $A$ of $X$, we have $\text{ep } A \leq \text{ep } X$.

4) For every topological spaces $X$ and $Y$, we have $\text{ep } (X \times Y) \leq \text{ep } X + \text{ep } Y$.

2.9: Remarks.
1) It follows from 2.8.2) and 2.8.4) that, for every $n \geq 1$, $\text{ep } \mathbb{R}^n \leq n$. (In the sequel, we will prove that $\text{ep } \mathbb{R}^n = n$).

2) In contrast to the classical definitions, there is no need for any special hypothesis for 2.8.3) and 2.8.4) to be true: recall, for example, there exists (3) two compact spaces $X$ and $Y$ such that $\text{ind}(X \times Y) > \text{ind } X + \text{ind } Y$.

3: Comparison between thickness and classical dimensions.
1) Theorem 3.1:
For every topological space $X$, we have:

a) $\text{ep } X = 0$ if and only if $\text{ind } X = 0$,

b) $\text{ind } X \leq \text{ep } X$.

Proof:

a) is immediate since these two assertions are equivalents to « there exists a base of $X$ consisting of open-closed subsets ».

b) The theorem is obvious if $\text{ep } X = +\infty$, so that we can suppose $\text{ep } X < +\infty$.

Let us prove the theorem by induction on $n = \text{ep } X$.

It follows from a) that the statement holds for $n = 0$.

Suppose it holds for every space $Y$ such that $\text{ep } Y \leq n - 1$ and let us prove then that $\text{ind } X \leq n$, i.e., that, for every point $x$ of $X$ and every neighbourhood $V$ of $x$, there exists an open subset $0$ such that $x \in 0 \subseteq V$ and $\text{ind } (\text{Fr } 0) \leq n - 1$.

Since $\text{ep } X = n$, there exists a base $B$ of $X$ such that $\text{ep } B = n$. Let us prove then that, for every $B \in B$, we have $\text{ep } (\text{Fr } B) \leq n - 1$, which by the induction
hypothesis, implies \( \text{ind} X \leq n \).
Let \( B \in B \). Put \( F = \text{Fr} B \) and call \( C \) the trace of \( B \) on \( F \).
Let us prove that \( \text{ep} C \leq n - 1 \). Let \( x \in F \) and \( \{ a_p, \ldots, a_i \} \) be a chain of \( h_C(\ast x) \).
Since \( h_C(\ast x) = h_B(\ast x) \cap \ast F \), it follows from 1.6 that \( a_p \) is not minimal for the preorder associated to \( B \). Consequently, there exists an element \( a_{p+1} \) of \( \ast X \) such that \( \{ a_{p+1}, a_p, \ldots, a_i \} \) is a chain of \( h_B(\ast x) \). Since \( \text{ep} B = n \), we have necessarily \( p \leq n - 1 \), which implies \( \text{ep} (x, C) \leq n - 1 \) and therefore \( \text{ep} C \leq n - 1 \). Since \( \text{ep} F \leq \text{ep} C \), we conclude \( \text{ep} F \leq n - 1 \).

Corollary 3.2 :
For every \( n \geq 1 \), we have \( \text{ep} \mathbb{R}^n = n \).
Indeed, we know that \( \text{ind} \mathbb{R}^n = n \) (see for example (2)) and \( \text{ep} \mathbb{R}^n \leq n \).

Corollary 3.3 :
For every totally ordered space \( X \), we have \( \text{ind} X = \text{ep} X \leq 1 \)
This assertion follows from 3.1 and 2.8.2).

Remark :
In another paper (1), we have proved that, for every totally ordered space \( X \), \( \text{ind} X = \text{Ind} X = \text{dim} X \leq 1 \).

2) An example of a space \( X \) such that \( \text{ind} X = \text{Ind} X < \text{ep} X \).
In (3), V.V. FILIPPOV has proved there exists two compact (non metric) spaces \( X_1 \) and \( X_2 \) such that \( \text{ind} X_1 = \text{Ind} X_1 = 1, \text{ind} X_2 = \text{Ind} X_2 = 2 \) and \( \text{ind}(X_1 \times X_2) = \text{Ind}(X_1 \times X_2) \geq 4 \). It follows from this example that \( X_1 \) or \( X_2 \) is such that \( \text{ind} X_i = \text{Ind} X_i < \text{ep} X_i \). Indeed, if \( \text{ind} X_1 = \text{Ind} X_1 = \text{ep} X_1 \) and \( \text{ind} X_2 = \text{Ind} X_2 = \text{ep} X_2 \), we would have, from 2.8.4), \( \text{ep} (X_1 \times X_2) \leq 3 \), which is impossible since \( \text{ep} (X_1 \times X_2) \geq \text{ind}(X_1 \times X_2) \) and \( \text{ind}(X_1 \times X_2) \geq 4 \).
Note the space we are looking for is the space \( X_2 \). Indeed, it is not the space \( X_1 \) because \( X_1 \) is by definition the quotient of a product of a compact totally disconnected space \( Z^* \) by a long line \( L \). Since \( \text{ep} Z^* = \text{ind} Z^* = 0 \) and \( \text{ep} L = \text{ind} L = 1 \) (use 3.3), we have \( \text{ep} (Z^* \times L) = 1 \) and therefore \( \text{ep} X_1 \text{ind} X_1 = 1 \).
Note the description of the space \( X_2 \) is quite complicated so that it will not be reproduced here.
3) An example of space $X$ such that $\text{ep } X = \text{ind } X < \text{Ind } X = \text{dim } X$.

In (8), P. ROY has proved there exists a completely metric space $X$ such that $\text{ind } X = 0$ and $\text{Ind } X = \text{dim } X = 1$. It follows from 3.1 a) that, for this space, $\text{ep } X = \text{ind } X = 0$ and $\text{ep } X < \text{Ind } X = \text{dim } X$.

4) An example of space $X$ such that $\text{dim } X < \text{ep } X$.

In (5), O.V. LOKUCIEVSKII has proved there exists a compact (non metric) space such that $\text{dim } X = 1 < 2 = \text{ind } X = \text{Ind } X$. For this space, we have $\text{dim } X < \text{ind } X < \text{ep } X$.

4 : The case of metric spaces.

Theorem 4.1 : 

For every metric space $X$, we have $\text{ind } X < \text{ep } X < \text{dim } X = \text{Ind } X$.

Since, for every topological space $Y$, we have $\text{ind } Y < \text{ep } Y$ and, for every metric space $Z$, we have $\text{dim } Z = \text{Ind } Z$ (see for example (2)), it suffices to prove that, for every metric space $X$, we have $\text{ep } X \leq \text{dim } X$.

Notations : Let $\mathcal{F} = (F_i)_{i \in I}$ be an indexed family of subsets of $X$. Let us put, for every element $x$ of $X$, $\text{ord } (x, \mathcal{F}) = \{i \in I : x \in F_i\} - 1$ (where $|A|$ denotes the cardinal of $A$) and $\text{ord } \mathcal{F} = \sup \{\text{ord } (x, \mathcal{F}) : x \in X\}$ (ord $\mathcal{F}$ is called the order of $\mathcal{F}$).

Lemma 4.1.1 : 

For every base $B$ of $X$, let $\mathcal{F} = (FrB)_{B \in B}$, then $\text{ep } B \leq \text{ord } \mathcal{F} + 1$.

Let $x$ be an element of $X$ and $\{a_1, \ldots, a_p\}$ be a chain of $h_B(*x)$. There exists then $p$ distinct elements of $B$, $B_1, \ldots, B_p$ such that, for every $i \in \{1, \ldots, p\}$, $a_j \in *B_i$ if and only if $j \geq i$ and such that $x \in FrB_i$. Consequently, by the definition of $\text{ord } (x, \mathcal{F})$, we have $p \leq \text{ord } (x, \mathcal{F}) + 1$, which implies $\text{ep } (x, B) \leq \text{ord } (x, \mathcal{F}) + 1$. It follows then, from the definitions of $\text{ep } B$ and $\text{ord } \mathcal{F}$, that we have $\text{ep } B \leq \text{ord } \mathcal{F} + 1$.

4.1.2. : Proof of 4.1 : 

This assertion is obvious if $\text{dim } X = +\infty$.

If $\text{dim } X = n$, there exists (see, for example, (2) and 4.2.2.) a $\sigma$-locally finite base $B$ of $X$ such that, if we put $\mathcal{F} = (FrB)_{B \in B}$, we have $\text{ord } \mathcal{F} \leq n - 1$. It follows then, from 4.1.1., that, for this base $B$, we have $\text{ep } B \leq n$, which implies that $\text{ep } X \leq n$.

4.2 : Let us note that ROY's space is a metric space such that $\text{ind } X = \text{ep } X = 0 < \text{dim } X = \text{Ind } X = 1$. 
4.3 : Coincidence theorem for separable metric spaces.

For every separable metric space $X$, we have $ep X = indX = IndX = dimX$.

This assertion is an immediate consequence of 4.1 and the well-known theorem :
« For every separable metric space $X$, we have $indX = IndX = dimX$ ».

4.4. : One can give a direct proof of 4.3. Indeed, let $X$ be a separable metric space such that $indX = n$. Let us denote by $N_n^{2n+1}$ NOBELING's space (6), viz, the subspace of $\mathbb{R}^{2n+1}$ consisting of all points which have at most $n$ rational coordinates, and, by $C_n^{2n+1}$ the trace on $N_n^{2n+1}$ of the base $B^{2n+1}$ of $\mathbb{R}^{2n+1}$ consisting of all parallelepipeds with rational coordinates. One can prove that $ep C_n^{2n+1} \leq n$ which implies, since $ind N_n^{2n+1} = n$ (see, for example, (2) 1.8.5), that $ep N_n^{2n+1} = n$. Since $indX = n$ and $N_n^{2n+1}$ is universal for the class of separable metric spaces whose dimension is not larger than $n$ (see also (2), 1.11.5), $X$ is homeomorphic to a subspace of $N_n^{2n+1}$, which implies, from 2.8.3), that $ep X \leq ep N_n^{2n+1}$ and therefore that $ep X = n$.

4.5 : An example of a non separable metric space $X$ such that $ep X = indX = IndX = dimX$.

In (9), E.K. VAN DOUWEN proved there exists a non separable metric space $X$ such that $indX = IndX = dimX = 1$.

This space is therefore such that $ep X = indX = IndX = dimX$.

4.6 : Question : Does there exist a metric space $X$ such that $indX < ep X$ ?

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