Representative subalgebra of a complete ultrametric Hopf algebra

REPRESENTATIVE SUBALGEBRA OF A COMPLETE ULTRAMETRIC HOPF ALGEBRA

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ABSTRACT. Let \((H, m, c, \eta, \sigma)\) be a complete ultrametric Hopf algebra over a complete ultrametric valued field \(K\), \(e\) be the unit of \(H\) and \(k\) the canonical map of \(K\) in \(H\). In order words, \(H\) is a Banach algebra with multiplication \(m : H \otimes H \to H\), coproduct \(c : H \to H \otimes H\) a continuous algebra homomorphism, inversion or antipode \(\eta : H \to H\) a continuous linear map and counit \(\sigma : H \to K\) a continuous algebra homomorphism. The coassociativity and counitary axioms hold, and

\[ m \circ (\eta \otimes 1_H) \circ c = k \circ \sigma \circ m \circ (1_H \otimes \eta) \circ c. \]

We define the representative subalgebra \(\mathcal{R}(H)\) of \(H\), i.e. the subalgebra of \(H\) generated by the coefficient "functions" associated with the finite dimensional left \(H\)-comodules. Under some conditions on \(H, \mathcal{R}(H)\) is a direct sum of finite dimensional subcoalgebras and is dense in \(H\). But in general, \(\mathcal{R}(H)\) is not dense in \(H\). The algebra \(\mathcal{R}(H)\) is a generalization of the algebra of representative functions on a group. Notice that when the valuation of \(K\) and the norm of \(H\) are trivial, one obtains the well known fact that \(H\) is equal to its representative subalgebra.

INTRODUCTION.

Let \((H, m, c, \eta, \sigma)\) be a complete ultrametric Hopf algebra over the complete ultrametric valued field \(K\). An ultrametric Banach space \(E\) over \(K\) is said to be a left Banach \(H\)-comodule if there exists a continuous linear map \(\Delta_E : E \to H \otimes E\), called coproduct, such that

(i) \((c \otimes 1_E) \circ \Delta_E = (1_H \otimes \Delta_E) \circ \Delta_E\)

(ii) \((\sigma \otimes 1_E) \circ \Delta_E = 1_E\)

A closed linear subspace \(E\) of \(E\) is a (left) Banach submodule of \(E\) if \(\Delta_E(M) \subseteq H \otimes M\).
Let \((E, \Delta_E)\) and \((F, \Delta_F)\) be two left Banach comodules. A continuous linear map \(u : E \to F\) is a Banach comodule morphism if \(\Delta_F \circ u = (1_H \otimes u) \circ \Delta_E\).

It is associated with any left Banach \(H\)-comodule \((E, \Delta_E)\) the closed linear subspace \(R(\Delta_E)\) of \(H\) spanned by the coefficient "functions" \((1_H \otimes x') \circ \Delta(x), x' \in E', x \in E\), where \(E'\) if the Banach space dual of \(E\). Furthermore, let \(\mathcal{R}(H)\) be the linear subspace of \(H\) spanned by all the \(R(\Delta_E)\) where \((E, \Delta_E)\) is a finite dimensional left \(H\)-comodule. Then \(\mathcal{R}(H)\) is a (non necessary closed) sub-Hopf-algebra of \(H\); \(\mathcal{R}(H)\) is called the representative subalgebra of \(H\). In general, \(\mathcal{R}(H)\) is not dense in \(H\) (cf. [1] or [5], [6]). However, with additional conditions on \(H\) it will be shown that \(\mathcal{R}(H)\) is dense in \(H\).

If \(E\) and \(F\) are ultrametric Banach spaces over \(K\), we denote by \(E \hat{\otimes} F\) the complete tensor product, that is the completion of \(E \otimes F\) with respect to the norm \(\|z\| = \inf_{z = x_j \otimes y_j} (\max_j \|x_j\|, \|y_j\|)\). In the sequel all Banach spaces are ultrametric.

### I - LEFT BANACH COMODULES

#### I - 1 Tensor products of left Banach comodules

Let \((E, \Delta_E)\) and \((F, \Delta_F)\) be two left Banach comodules. One has the continuous linear map \(\Delta_{E \hat{\otimes} F} : E \hat{\otimes} F \to H \hat{\otimes} E \hat{\otimes} F \to H \hat{\otimes} H \hat{\otimes} E \hat{\otimes} F \to H \hat{\otimes} E \hat{\otimes} F\) where

\[
\Delta_{E \hat{\otimes} F} = (m \otimes 1_E \otimes 1_F) \circ (1_H \otimes \tau_{E \hat{\otimes} F} \otimes 1_F) \circ (\Delta_E \otimes \Delta_F) \text{ and } \tau_{E \hat{\otimes} F}(x \otimes a) = a \otimes x.
\]

**Proposition 1 :** \(\Delta_{E \hat{\otimes} F} : E \hat{\otimes} F \to H \hat{\otimes} E \hat{\otimes} F\) is the coproduct of a left Banach \(H\)-comodule structure on \(E \hat{\otimes} F\).

**Proof:** Put, for \(x \in E\) and \(y \in F\), \(\Delta_E(x) = \sum_{j \geq 1} a_j x_j \in H \hat{\otimes} E\) and \(\Delta_F(y) = \sum_{t \geq 1} b_t y_t \in H \hat{\otimes} F\). Therefore, one has \(\Delta_{E \hat{\otimes} F}(x \otimes y) = \sum_{j \geq 1} a_j b_t x_j \otimes y_t\).

(i) It follows immediately that \((\sigma \otimes 1_{E \hat{\otimes} F}) \circ \Delta_{E \hat{\otimes} F}(x \otimes y) = \sum_{j \geq 1} \sum_{t \geq 1} \sigma(a_j b_t) x_j \otimes y_t = \sum_{j \geq 1} \sigma(a_j) x_j \otimes \sum_{t \geq 1} \sigma(b_t) y_t = x \otimes y = 1_{E \hat{\otimes} F}(x \otimes y)\). From what, one deduces \((\sigma \otimes 1_{E \hat{\otimes} F}) \circ \Delta_{E \hat{\otimes} F} = 1_{E \hat{\otimes} F}\)

(ii) Also, one has for \(x \in E, y \in F\)

\[
\sigma (c \otimes 1_E) \circ \Delta_E(x) = \sum_{j \geq 1} c(a_j) x_j = \sum_{j \geq 1, s \geq 1} a^1_{s,j} \otimes a^2_{s,j} x_j = (1_H \otimes \Delta_E) \circ \Delta_E(x) = \sum_{j \geq 1} a_j \otimes \Delta_E(x_j) = \sum_{j \geq 1, k \geq 1} a_j \otimes \gamma_{k,j} \otimes x_{k,j}
\]
and 
\[ \Delta_F(y) = \sum_{\ell \geq 1} c(b_{\ell}) \otimes y_{\ell} = \sum_{\ell \geq 1} \sum_{i \geq 1} \beta_{1,\ell} \otimes \beta_{2,\ell} \otimes y_{\ell} = (1_H \otimes \Delta_F) \circ \Delta_F(y) = \sum_{\ell \geq 1} b_{\ell} \otimes \Delta_F(y_{\ell}) = \sum_{\ell \geq 1} \sum_{m \geq 1} b_{\ell} \otimes \rho_{m,\ell} \otimes y_{m,\ell} \]

Let \( E_x = E[(x_j, j \geq 1) \cup (x_{k, j}, k \geq 1, j \geq 1)] \) be the closed linear subspace of \( E \) spanned by \( (x_j, j \geq 1) \cup (x_{k, j}, k \geq 1, j \geq 1) \), and \( F_y = E[(y_{\ell, \ell} \geq 1) \cup (y_{m, \ell}, m \geq 1, \ell \geq 1)] \) be the closed linear subspace of \( F \) spanned by \( (y_{\ell, \ell} \geq 1) \cup (y_{m, \ell}, m \geq 1, \ell \geq 1) \). It is clear that the Banach spaces \( E_x \) and \( F_y \) are of countable type. Furthermore, if \( x' \in E_x' \) and \( y' \in F_y' \) one has

\[ (1_H \otimes 1_H \otimes x') \circ (c \otimes 1_E) \otimes \Delta_E(x) = \sum_{j \geq 1} \sum_{k \geq 1} <x', x_j > \alpha_{s,j} \otimes \alpha_{s,j} = \]

\[ (1_H \otimes 1_H \otimes x') \circ (1_H \otimes \Delta_E) \circ \Delta_E(y) = \sum_{j \geq 1} \sum_{k \geq 1} <y', y_{\ell} > \beta_{1,\ell} \otimes \beta_{2,\ell} = \]

\[ (1_H \otimes 1_H \otimes y') \circ (c \otimes 1_F) \circ \Delta_F(x) = \sum_{j \geq 1} \sum_{k \geq 1} <y', y_{\ell} > \beta_{1,\ell} \otimes \beta_{2,\ell} \]

On one hand, one has, \( (c \otimes 1_{E \otimes F}) \circ \Delta_{E \otimes F}(x \otimes y) = \sum_{j \geq 1} \sum_{k \geq 1} c(a_j b_k) \otimes x_j \otimes y_{\ell} = \)

\[ \sum_{j \geq 1} \sum_{k \geq 1} \sum_{l \geq 1} \sum_{m \geq 1} a_j b_k \otimes \gamma_{k,j} \otimes x_{k,j} \otimes y_{m,l} \]

\[ \sum_{j \geq 1} \sum_{k \geq 1} \sum_{l \geq 1} \sum_{m \geq 1} a_j b_k \otimes \gamma_{k,j} \otimes x_{k,j} \otimes y_{m,l} \]

Hence, if \( x' \in E_x' \) and \( y' \in F_y' \); first, one has

\[ (1_H \otimes 1_H \otimes x' \otimes y') \circ (c \otimes 1_{E \otimes F}) \circ \Delta_{E \otimes F}(x \otimes y) = \sum_{j \geq 1} \sum_{k \geq 1} <x', x_j > \alpha_{s,j} \otimes \alpha_{s,j} \sum_{l \geq 1} \sum_{m \geq 1} <y', y_{\ell}> \]

\[ \beta_{1,\ell} \otimes \beta_{2,\ell} = (1_H \otimes 1_H \otimes x') \circ (c \otimes 1_E) \circ \Delta_E(x) \cdot (1_H \otimes 1_H \otimes y') \circ (c \otimes 1_F) \circ \Delta_F(y) = \]

\[ (1_H \otimes 1_H \otimes x') \circ (1_H \otimes \Delta_E) \circ \Delta_E(x) \cdot (1_H \otimes 1_H \otimes y') \circ (1_H \otimes \Delta_F) \circ \Delta_F(y). \]

And, second, one has

\[ (1_H \otimes 1_H \otimes x' \otimes y') \circ (1_H \otimes \Delta_{E \otimes F}) \circ \Delta_{E \otimes F}(x \otimes y) = \sum_{j \geq 1} \sum_{k \geq 1} <x', x_{k,j} > a_j \otimes \gamma_{k,j} \sum_{l \geq 1} \sum_{m \geq 1} <y', y_{m,l}> \]

\[ b_{\ell} \otimes \rho_{m,\ell} = (1_H \otimes 1_H \otimes x') \circ (1_H \otimes \Delta_E) \circ \Delta_E(x) \cdot (1_H \otimes 1_H \otimes y') \circ \Delta_E \circ \Delta_F(y). \]

Therefore, for any \( x' \in E_x' \) and any \( y' \in F_y' \), we have

\[ (a) : (1_H \otimes 1_H \otimes x' \otimes y') \circ (c \otimes 1_{E \otimes F}) \circ \Delta_{E \otimes F}(x \otimes y) = (1_H \otimes \Delta_{E \otimes F}) \circ \Delta_{E \otimes F}(x \otimes y) = 0 \]

\( \gamma \) Since \( E_x \) [resp. \( F_y \)] is of countable type, there exist \( \alpha_0 > 0, \alpha_1 > 0 \) and
(e_j)_{j \geq 1} \subset E_z \ [\text{resp. } (f_t)_{t \geq 1} \subset F_y] \text{ such that for } z \in E_z \ [\text{resp. } \zeta \in F_y] \text{ one has }
z = \sum_{j \geq 1} \lambda_j e_j \ [\text{resp. } \zeta = \sum_{t \geq 1} \mu_t f_t] \text{ with } \alpha_0 \sup_{j \geq 1} |\lambda_j| \leq \|z\| \leq \alpha_1 \sup_{j \geq 1} |\lambda_j| \ [\text{resp. } \alpha_0 \sup_{t \geq 1} |\mu_t| \leq \|\zeta\| \leq \alpha_1 \sup_{t \geq 1} |\mu_t|] \ (cf. [4]).

Moreover, one has \(E_x \hat{\otimes} F_y \simeq c_0(\mathbb{N}^* \times \mathbb{N}^*, K)\) and \((H \hat{\otimes} H) \hat{\otimes} (E_x \hat{\otimes} E_y) \simeq c_0(\mathbb{N}^* \times \mathbb{N}^*, H \otimes H)\) \(\text{cf. [7]}\); any \(Z \in (H \hat{\otimes} H) \hat{\otimes} (E_x \hat{\otimes} E_y)\) can be written in the unique form \(Z = \sum_{j \geq 1} A_{jt} \otimes e_j \otimes f_t \) with \(A_{jt} \in H \hat{\otimes} H\) and \(\alpha_0^j \sup_{j,t} \|A_{jt}\| \leq \|Z\| \leq \alpha_1^j \sup_{j,t} |A_{jt}|\).

Let \(e'_j \in E'_x \ [\text{resp. } f'_t \in F'_y]\) be the continuous linear form defined by \(<e'_j, e_{j_1} >= \delta_{j,j_1} \ [\text{resp. } <f'_t, f_{t_1} >= \delta_{t,t_1}].\) Setting \((c \otimes 1_{E \hat{\otimes} F}) \circ \Delta_{E \hat{\otimes} F}(x \otimes y) - (1_H \otimes \Delta_{E \hat{\otimes} F} \circ \Delta_{E \hat{\otimes} F}(x \otimes y) = Z_0 = \sum_{j,t} A_{jt} \delta_{j_1,j} \delta_{t_1,t} \in H \hat{\otimes} H \hat{\otimes} E_x \hat{\otimes} E_y, \) for any \(j_1 \geq 1\) and any \(t_1 \geq 1,\) by (a), one has

\[ (1_H \otimes 1_H \otimes e'_j \otimes f'_t)(Z_0) = \sum_{j,t} A_{jt} \delta_{j_1,j} \delta_{t_1,t} = A^0_{j_1,t_1} = 0 \].

It follows that \(Z_0 = 0,\) i.e.

\[ (c \otimes 1_{E \hat{\otimes} F}) \circ \Delta_{E \hat{\otimes} F}(x \otimes y) = (1_H \otimes \Delta_{E \hat{\otimes} F} \circ \Delta_{E \hat{\otimes} F}(x \otimes y). \] From what, one deduces that

\[ (c \otimes 1_{E \hat{\otimes} F}) \circ \Delta_{E \hat{\otimes} F} = (1_H \otimes \Delta_{E \hat{\otimes} F} \circ \Delta_{E \hat{\otimes} F}. \]

Corollary: Let \(M \ [\text{resp. } N] \) be a left Banach \(H\)-subcomodule of \(E \ [\text{resp. } F].\) Then \(M \hat{\otimes} N\) is a left Banach subcomodule of \(E \hat{\otimes} F.\)

I - 2 Banach comodule morphisms

I - 2 - 1 Range and kernel

Proposition 2: Let \(u : E \to F\) be a Banach comodule morphism.

(i) If \(V\) is a Banach submodule of \(F,\) then \(u^{-1}(V)\) is a Banach submodule of \(E.\)

(ii) The closure \(\overline{u(E)}\) of \(u(E)\) is a Banach submodule of \(F\)

Corollary: Let \(V\) and \(W\) be Banach submodule of the left Banach \(H\)-comodule \(E;\) then \(V \cap W\) is a Banach submodule of \(E.\)

Proofs: Rather easy, or see [3].

Note: One can also see [3] for the spaces of comodule morphisms.

Remark 1: If \(M\) is a Banach submodule of the left Banach \(H\)-comodule \(E,\) it is induced on the quotient Banach space \(E/M\) a structure of Banach left \(H\)-comodule such that the canonical map \(E \to E/M\) is a comodule morphism.
Then, if $u : E \to F$ is a Banach comodule morphism and if $u$ is strict, the Banach comodule $E/\ker u$ and $u(E)$ are isomorphic. Also, one can define the cokernel of $u$ as being $F/\text{coker}(u)$.

**I - 2- 2 Comodule morphisms of $E$ into $H$ associated with $\Delta ; R(\Delta)$**

Put $\Delta = \Delta_E$ the coproduct of the left Banach $H$-comodule $E$. Obviously, $H$ is a left Banach $H$-comodule with respect to its coproduct $c$.

**Proposition 3:** For any $x' \in E'$, the linear map $A_{x'} = (1_H \otimes x') \circ \Delta : E \to H$ is a Banach comodule morphism.

**Proof:** It is easy to see that $\text{co}(1_H \otimes x') = c \otimes x' = (1_H \otimes 1_H \otimes x') \circ (c \otimes 1_E)$. Therefore $\text{co}A_{x'} = \text{co}(1_H \otimes x') \circ \Delta = (1_H \otimes 1_H \otimes x') \circ (c \otimes 1_E) \circ \Delta = (1_H \otimes \Delta) \circ \Delta = (1_H \otimes A_{x'}) \circ \Delta$.

**Corollary 1:**
(i) $\ker A_{x'}$ is a closed subcomodule of $E$.
(ii) $A_{x'}(E)$ is a left Banach subcomodule (= closed left coideal) of $H$.

**Corollary 2:** If $E$ is a space of countable type, one has $\ker A_{x'} \neq E$ for any $x' \in E, x' \neq 0$.

**Proof:** Indeed, if $x' \in E', x' \neq 0$ and $0 < \alpha < 1$, there exists a $\alpha$-orthogonal base $(e_j)_{j \geq 1} \subset E$ such that $< x', e_1 > = 1$ and $< x', e_j > = 0, j \geq 2$. Moreover for any $j \geq 1$, $\Delta(e_j) = \sum_{t \geq 1} a_{tj} \otimes e_t$ and $e_j = \sum_{t \geq 1} \sigma(a_{tj})e_t$; therefore $\sigma(a_{tj}) = \delta_{tj}$ and $A_{x'}(e_1) = (1_H \otimes x') \circ \Delta(e_1) = a_{11} \neq 0$ since $\sigma(a_{11}) = 1$.

**Corollary 3:** Assume that $H$ is a pseudo-reflexive Banach space; i.e. $H \to H''$ is isometric.

Let $E$ be a simple Banach left $H$-comodule, i.e. $E$ contains no proper closed submodule. Then $E$ is a Banach space of countable type and $A_{x'}$ is injective for each $x' \in E'$, $x' \neq 0$.

**Proof:** If $H$ is pseudo-reflexive, it is shown in [3] that any simple Banach left $H$-comodule is a space of countable type. Applying Corollary 2, one sees that $A_{x'}$ is injective for $x' \in E', x' \neq 0$. □

Let $\beta : E \otimes E' \to K$ be the continuous linear form defined upon $\beta(x \otimes x') = < x', x >$. Put $\rho_\Delta = (1_H \otimes \beta) \circ (\Delta \otimes 1_{E'}) \circ \tau : E' \otimes E \to H$, where $\tau(x' \otimes x) = x \otimes x'$. Then $\rho_\Delta$ is linear and continuous with $\|\rho_\Delta\| \leq \|\Delta\|$. Moreover for $x' \in E', x \in E$, one has $\rho_\Delta(x' \otimes x) = (1_H \otimes x') \circ \Delta(x)$.
Put \( R(\Delta) = \rho_\Delta(E' \hat{\otimes} E) \) the closure of \( \rho_\Delta(E' \hat{\otimes} E) \) in \( H \). Obviously, \( R(\Delta) \) is the closed linear subspace of \( H \) spaned by the elements \((1_H \otimes x') \circ \Delta(x)\), \( x' \in E'\), \( x \in E \), called the coefficients of the comodule \((E, \Delta)\).

**Proposition 4:** \( R(\Delta) = \rho_\Delta(E' \hat{\otimes} E) \) is a left Banach submodule (= closed left coideal) of \( H \).

**Proof:** Since \( c : H \to H \hat{\otimes} H \) is linear and is a homeomorphism of \( H \) onto \( c(H) \), one has \( c(R(\Delta)) = (c(\rho_\Delta(E' \hat{\otimes} E))) \), a closed linear subspace of \( H \).

It remains to show that if \( a = \rho_\Delta(x' \otimes x) = (1_H \otimes x') \circ \Delta(x) = A_{x'}(x)\), \( x' \in E'\), \( x \in E \); then \( c(a) \in H \hat{\otimes} R(\Delta) \). Writing \( \Delta(x) = \sum a_j \otimes x_j \); one has \( c(a) = c \circ A_{x'}(x) = (1_H \otimes A_{x'}) \circ \Delta(x) = \sum a_j \otimes A_{x'}(x_j) \in H \hat{\otimes} R(\Delta) \). \( \square \)

**Proposition 5:** If the left Banach comodules \( E \) and \( E_1 \) with coproduct respectively \( \Delta \) and \( \Delta_1 \) are isomorphic, then \( R(\Delta) = R(\Delta_1) \).

**Proof:** Let \( u : E \to E_1 \) be a comodule isomorphism, in other words, \( u \) is linear, continuous and bijective with \( \Delta_1 \circ u = (1_H \otimes u) \circ \Delta \). Moreover, the reciprocal map \( u^{-1} \) of \( u \) satisfies \( (1_H \otimes u^{-1}) \circ \Delta_1 = \Delta \circ u^{-1} \) and the transpose of \( u \), \( u^t : E_1' \to E' \) is linear, continuous and bijective with \( (u^t)^{-1} = u^{-1} \).

Set \( a = \rho_{\Delta_1}(z_1) \in \rho_{\Delta_1}(E'_1 \hat{\otimes} E_1) \) and \( z_1 = \sum y'_j \otimes y_j\), \( y'_j \in E'_1\), \( y_j \in E_1\), \( \lim_j \|y'_j\| \|y_j\| = 0 \).

There exist, for \( j \geq 1 \) unique \( x'_j \in E' \) and \( x_j \in E \) such that \( y'_j = u(x_j) \); moreover \( \lim_j \|x'_j\| \|x_j\| = 0 \). Therefore \( a = \rho_{\Delta_1}(z_1) = \sum_j \rho_{\Delta_1}(y'_j \otimes y_j) = \sum_{j \geq 1} (1_H \otimes y'_j) \circ \Delta_1(y_j) = \sum_{j \geq 1} (1_H \otimes x'_j \circ u^{-1}) \circ \Delta \circ u(x_j) = \sum_{j \geq 1} (1_H \otimes x'_j \circ u^{-1}) \circ \Delta_1 \circ u(x_j) = \sum_{j \geq 1} \rho_\Delta(x'_j \otimes x_j) = \rho_\Delta \left( \sum_{j \geq 1} x'_j \otimes x_j \right) \). Hence, \( a = \rho_\Delta(z) \in \rho_\Delta(E' \hat{\otimes} E) \) where \( z = \sum x'_j \otimes x_j \); that is \( \rho_{\Delta_1}(E'_1 \hat{\otimes} E_1) \subset \rho_\Delta(E' \hat{\otimes} E) \). Likewise, one has \( \rho_\Delta(E' \hat{\otimes} E) \subset \rho_{\Delta_1}(E'_1 \hat{\otimes} E_1) \).

Therefore \( \rho_\Delta(E' \hat{\otimes} E) = \rho_{\Delta_1}(E'_1 \hat{\otimes} E_1) \) and \( R(\Delta) = R(\Delta_1) \). \( \square \)

Assume that \( E \) is a free Banach space i.e. \( E \simeq c_0(I, K) = \{(\lambda_j)_{j \in I} \subset K / \lim_j \lambda_j = 0 \} \). In other words, there exist \((e_j)_{j \in I} \subset E\), \( \alpha_0, \alpha_1 \in \mathbb{R}^*_+ \) such that any \( x \in E \) can be written...
in the form \( x = \sum_{j \in I} \lambda_j e_j \), \( \lambda_j \in K \) and \( \alpha_0 \sup_{j \in I} |\lambda_j| \leq \|x\| \leq \alpha_1 \sup_{j \in I} |\lambda_j| \). For any continuous linear form \( x' \in E' \), one has \( \frac{1}{\alpha_0} \sup_{j \in I} |< x', e_j >| \leq \frac{1}{\alpha_0} \sup_{j \in I} |< x', e_j >| \). Let \( e'_j \) be the element of \( E' \) defined by \( < e'_j, e_\ell > = \delta_{j \ell} \). Put \( E'_0 = E[(e'_j)_{j \in I}] \), the closed linear subspace of \( E' \) spaned by \( (e'_j)_{j \in I} \). Hence each \( x' \in E'_0 \) can be written in the unique form \( x' = \sum_{j \in I} \mu_j e'_j \), \( \mu_j \in K \), \( \mu_j = 0 \). Moreover, if \( v \in E'_0 \hat{\otimes} E \subset E' \hat{\otimes} E \), one has

\[
v = \sum_{j \in I} \mu_{\ell j} e'_\ell \otimes e_j, \quad \mu_{\ell j} \in K, \quad \lim_{j \to \ell} |\mu_{\ell j}| = 0.
\]

On the other hand, one has \( H \hat{\otimes} E \simeq c_0(I, H) = \{(a_j)_{j \in I} \in H/ \lim_{j \to \ell} a_j = 0\} \). For any \( z \in H \hat{\otimes} E \) one has \( z = \sum_{j \in I} a_j \otimes e_j, a_j \in H \) with \( \lim_{j \to \ell} |a_j| = 0 \) and \( \alpha_0 \sup_{j \in I} |a_j| \leq \|z\| \leq \alpha_1 \sup_{j \in I} |a_j| \). Hence, if \( (E, \Delta) \) is a left Banach \( H \)-comodule, for \( x \in E \), one has \( \Delta(x) = \sum_{j \in I} A_j(x) \otimes e_j \). In particular \( \Delta(e_\ell) = \sum_{j \in I} A_j(e_\ell) \otimes e_j = \sum_{j \in I} a_{\ell j} \otimes e_j \) and \( (c \otimes 1_E) \circ \Delta(e_\ell) = \sum_{j \in I} c(a_{\ell j}) \otimes e_j \), thus one obtains

<table>
<thead>
<tr>
<th>Expression</th>
<th>Description</th>
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<tbody>
<tr>
<td>(1) ( \sigma(a_{\ell j}) = \sum_{k \in I} a_{\ell k} \otimes a_{k j} ); ( \ell, j \in I )</td>
<td></td>
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<tr>
<td>(2) ( \delta_{\ell j} ); ( \ell, j \in I )</td>
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<tr>
<td>(3) ( \sum_{k \in I} a_{\ell k} \eta(a_{k j}) = \delta_{\ell j} \cdot e = \sum_{k \in I} \eta(a_{\ell k}) \otimes a_{k j} ); ( \ell, j \in I )</td>
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**Proposition 6**: \( R_0(\Delta) = \rho_\Delta(E'_0 \hat{\otimes} E) \) is a closed subcoalgebra of \( H \). In other words \( c(R_0(\Delta)) \subset R_0(\Delta) \hat{\otimes} R_0(\Delta) \)

**Proof**: Since \( (e'_j \otimes e_\ell)_{(\ell, j) \in I \times I} \) is a total family of \( E'_0 \hat{\otimes} E \) and \( \rho_\Delta \) is linear and continuous, the family \( (\rho_\Delta(e'_j \otimes e_\ell))_{(\ell, j) \in I \times I} \) is total in \( \rho_\Delta(E'_0 \hat{\otimes} E) = R_0(\Delta) \) = the closed linear subspace of \( H \) spaned by \( (1_H \otimes x') \circ \Delta(x) \), \( x' \in E'_0 \), \( x \in E \).

To see that \( c(R_0(\Delta)) \subset R_0(\Delta) \hat{\otimes} R_0(\Delta) \), it suffices to show that for \( \ell, j \in I \) one has \( c(\rho_\Delta(e'_j \otimes e_\ell)) \in R_0(\Delta) \hat{\otimes} R_0(\Delta) \). However, by definition, \( \rho_\Delta(e'_j \otimes e_\ell) = (1_H \otimes e'_j) \circ \Delta(e_\ell) = a_{\ell j} \in R_0(\Delta) \). Then, one deduces from (1) that \( c(\rho_\Delta(e'_j \otimes e_\ell)) = c(a_{\ell j}) = \sum_{k \in I} a_{\ell k} \otimes a_{k j} \in R_0(\Delta) \).
Note: If \( v = \sum_{t,j} \mu_{tj} e_j^t \otimes e_\ell \in E'_0 \otimes E \), one has \( \rho_\Delta(v) = \sum_{t,j} \mu_{tj} a_{tj} \) and \( a \in R_0(\Delta) \) iff there exist \( v_n \in E' \otimes E \) such that \( a = \lim_{n \to +\infty} \rho_\Delta(v_n) \).

Remark 2: Let \((E, \Delta)\) and \((E_1, \Delta_1)\) be two isomorphic left Banach comodules that are free Banach spaces. If \( u : E \to E_1 \) is a comodule isomorphism, \( (e_j)_{j \in I} \) a base of \( E \) and \( (e_j)_{j \in I} \) the base of \( E_1 \) defined by \( e_j = u(e_j) \); then, with the above notations, one has \( R_0(\Delta) = R_0(\Delta_1) \).

Remark 3: If \( \dim E = n < +\infty \), one has \( R(\Delta) = R_0(\Delta) = \rho_\Delta(E' \otimes E) \) and \( \dim R(\Delta) \leq n^2 \).

II - REPRESENTATIVE SUBALGEBRA

II - 1 Conjugate comodule of a finite dimensional comodule

Let \((E, \Delta)\) be a \( (\text{Banach}) \) left \( H \)-comodule of finite dimension and \( (e_j)_{1 \leq j \leq n} \) a \( K \)-base of \( E \). As above, for any \( x \in E \), one has \( \Delta(x) = \sum_{j=1}^n A_j(x) \otimes e_j \) and \( A_j(x) = (1_H \otimes e_j^t) \circ \Delta(x) = \rho_\Delta(e_j^t \otimes x) \). In particular \( \Delta(e_j) = \sum_{j=1}^n a_{tj} e_j \) where \( a_{tj} = A_j(e_t) = \rho_\Delta(e_j^t \otimes e_t) \); and we have the relations (1), (2) and (3), with \( I = [1, n] \).

The relation (3) means here, that the matrix \( A = (a_{tj})_{1 \leq t, j \leq n} \in \text{Mat}_n(H) \) is invertible with inverse \( A^{-1} = (\eta(a_{tj}))_{1 \leq t, j \leq n} \).

Fix the base \( (e_j)_{1 \leq j \leq n} \) of \( E \) and define the linear map \( \Delta^\vee : E' \to H \otimes E' \) by setting

\[
\Delta^\vee(e_j^t) = \sum_{t=1}^n \eta(a_{tj}) \otimes e_j^t, \quad 1 \leq j \leq n.
\]

Hence for \( x' = \sum_{j=1}^n \mu_j e_j^t \in E' \), one has \( \Delta^\vee(x') = \sum_{t=1}^n \sum_{j=1}^n \mu_j \eta(a_{tj}) \otimes e_j^t = \sum_{t=1}^n A_t^\vee(x') \otimes e_t^t \).

Lemma 1: \((E', \Delta^\vee)\) is a left \( H \)-comodule.

Proof: One verifies that \( \sigma \circ \eta = \sigma \); indeed, if \( a \in H \), then \( c(a) = \sum_{t \geq 1} a_t^1 \otimes a_t^2 \). Hence, one has

\[
m \circ (\eta \otimes 1_H) \circ c(a) = \sum_{t \geq 1} \eta(a_t^1) a_t^2 = \sigma(a)e \quad \text{and} \quad a = (1_H \otimes \sigma) \circ c(a) = \sum_{t \geq 1} a_t^1 \sigma(a_t^2).\]
follows that $\eta(a) = \sum_{t \geq 1} \eta(a_t^1) \sigma(a_t^2)$ and $\sigma \circ \eta(a) = \sum_{t \geq 1} \sigma(\eta(a_t^1)) \sigma(a_t^2) = \sigma\left(\sum_{t \geq 1} \eta(a_t^1) a_t^2\right) = \sigma(\sigma(a)e) = \sigma(a)$.

Since $\sigma(a_{t_j}) = \delta_{t_j}$, one has $(\sigma \otimes 1_{E'}) \circ \Delta^\vee(e_j') = \sum_{t=1}^n \sigma \circ \eta(a_{t_j}) e'_t = \sum_{t=1}^n \sigma(a_{t_j}) e'_j = e_j', \ 1 \leq j \leq n$. It follows, by linearity, that $(\sigma \otimes 1_{E'}) \circ \Delta^\vee = 1_{E'}$.

Let us remember that $c \circ \eta = \tau \circ (\eta \otimes \eta) \circ c$ where $\tau(a \otimes b) = b \otimes a$. Hence, we have $c \circ \eta(a_{t_j}) = \sum_{k=1}^n \eta(a_{k_j}) \otimes \eta(a_{k_k})$. Therefore $(c \otimes 1_{E'}) \circ \Delta^\vee(e_j') = (c \otimes 1_{E'})\left(\sum_{t=1}^n \eta(a_{t_j}) \otimes e'_t\right) = \left(\sum_{k=1}^n \eta(a_{k_j}) \otimes e'_t\right) = (c \otimes 1_{E'}) \circ \Delta^\vee(e_j')$, and $(c \otimes 1_{E'}) \circ \Delta^\vee = (1_H \otimes \Delta^\vee) \circ \Delta^\vee$.

**Corollary**: $R(\Delta^\vee) = \eta(R(\Delta))$.

**Proof**: Identifying $E''$ with $E$, one has $R(\Delta^\vee) = \rho_{\Delta^\vee}(E \otimes E')$. Set $z = \sum_{1 \leq t,j \leq n} \lambda_{t_j} e_t \otimes e_j' \in E \otimes E'$; hence $\rho_{\Delta^\vee}(z) = \sum_{1 \leq t,j \leq n} \lambda_{t_j} \rho_{\Delta^\vee}(e_t \otimes e_j')$. However $\rho_{\Delta^\vee}(e_t \otimes e_j') = (1_H \otimes e_t) \circ \Delta^\vee(e_j') = \eta(a_{t_j}) = \eta(\rho_{\Delta}(e_j' \otimes e_t))$; therefore $\rho_{\Delta^\vee}(z) = \sum_{1 \leq t,j \leq n} \lambda_{t_j} \eta(\rho_{\Delta}(e_j' \otimes e_t)) = \eta(\rho_{\Delta}(z_1))$ where $z_1 = \sum_{1 \leq t,j \leq n} \lambda_{t_j} e_j' \otimes e_t \in E' \otimes E$. It follows that $R(\Delta^\vee) \subset \eta(R(\Delta))$. The same formulae show that if $a = \rho_{\Delta}(z_1) \in R(\Delta)$, where $z_1 = \sum_{1 \leq t,j \leq n} \lambda_{t_j} e_j' \otimes e_t \in E' \otimes E$, one has $\eta(a) = \rho_{\Delta^\vee}(z)$ where $z = \sum_{1 \leq t,j \leq n} \lambda_{t_j} e_t \otimes e_j' \in E \otimes E'$, hence $\eta(R(\Delta)) \subset R(\Delta^\vee)$.

**II- 2 Direct sum of Banach comodules**

Let $(E_s)_{1 \leq s \leq m}$ be a finite family of left Banach $H$-comodules with $\Delta_s$ the coproduct of $E_s$. The direct sum $E = \bigoplus_{s=1}^m E_s$ equipped with any norm equivalent to the norm $\left\|\sum_{s=1}^m x_s\right\| = \max_{1 \leq s \leq m} \|x_s\|$ is a Banach space. Put $\Delta = \bigoplus_{s=1}^m \Delta_s$, i.e. $\Delta\left(\sum_{s=1}^m x_s\right) = \bigoplus_{s=1}^m \Delta_s(x_s)$. It is readily seen that $(E, \Delta)$ is a left Banach comodule. Moreover, if $p_s : E \to E$ is the projection of $E$ onto $E_s$, then $1_H \otimes p_s$ is a projection of $H \otimes E$ onto $H \otimes E_s$ and one has
Proposition 7: With the above notations, one has
\[ \rho_{\Delta}(E' \hat{\otimes} E) = \sum_{s=1}^{m} \rho_{\Delta_s}(E'_s \hat{\otimes} E_s) \]
and \( R(\Delta) \) is the closure of \( \sum_{s=1}^{m} R(\Delta_s) \) in \( H \).

Proof: If \( x'_s \in E'_s, x_t \in E_t \) and \( s \neq t \), then \((1_H \otimes x'_s) \circ \Delta(x_t) = (1_H \otimes x'_s) \circ \Delta_t(x_t) = 0 \). Set \( z = \sum_{j \geq 1} x'_j \otimes x_j \in E' \hat{\otimes} E \) and \( \Delta_t(x_{t,j}) = \sum_{j \geq 1} x'_{s,j} \otimes x_{s,j} \). It follows that
\[ \rho_{\Delta}(z) = \sum_{j \geq 1} \sum_{s=1}^{m} \rho_{\Delta_s}(x'_{s,j} \otimes x_{t,j}) = \sum_{j \geq 1} \sum_{s=1}^{m} \rho_{\Delta_s}(x'_s) \circ \Delta(x_{t,j}) = \sum_{s=1}^{m} \rho_{\Delta_s}(z_s) \]. If \( z_s \in E'_s \hat{\otimes} E_s \subseteq E' \hat{\otimes} E, 1 \leq s \leq m \), one has \( \rho_{\Delta}(z_s) = \rho_{\Delta_s}(z_s) \). Therefore, on one hand, \( \rho_{\Delta}(E' \hat{\otimes} E) \subseteq \sum_{s=1}^{m} \rho_{\Delta_s}(E'_s \hat{\otimes} E_s) \), and on the other hand, \( \rho_{\Delta_s}(E'_s \hat{\otimes} E_s) \subseteq \rho_{\Delta}(E' \hat{\otimes} E) \). Hence, one has \( \rho_{\Delta}(E' \hat{\otimes} E) = \sum_{s=1}^{m} \rho_{\Delta_s}(E'_s \hat{\otimes} E_s) \). One verifies readily that \( R(\Delta) \) is equal to the closure of \( \sum_{s=1}^{m} R(\Delta_s) \) in \( H \).

Corollary: If \( \dim E_s < +\infty, 1 \leq s \leq m \), then one has \( R(\Delta) = \sum_{s=1}^{m} R(\Delta_s) \) where
\[ E = \bigoplus_{s=1}^{m} E_s, \Delta = \bigoplus_{s=1}^{m} \Delta_s. \]

Remark 3: If the comodules \((E_s, \Delta_s), 1 \leq s \leq m\), are pair wise isomorphic, then for the comodule \((E, \Delta)\) where \( E = \bigoplus_{s=1}^{m} E_s, \Delta = \bigoplus_{s=1}^{m} \Delta_s \), one has \( R(\Delta) = R(\Delta_s), 1 \leq s \leq m \).
II - 3  The representative subalgebra of $H$

Let $S(H)$ be the set of all elements of the form $a = (1_H \otimes x') \circ \Delta(x)$ of $H$ where $(E, \Delta)$ is a finite dimensional left $H$-comodule and $x' \in E'$, $x \in E$. Let us put $\dim E = \dim \Delta$

Lemma 2 :  $S(H)$ is a multiplicative, unitary submonoid of $H$.

Proof :  Set $a = (1_H \otimes x') \circ \Delta(x)$ and $b = (1_H \otimes y') \circ \Delta_1(y) \in S(H)$ where $(E, \Delta)$ and $(E, \Delta_1)$ are left $H$-comodules of finite dimension and $x' \in E'$, $x \in E$, $y' \in E_1$, $y \in E_1$.

One has $\Delta(x) = \sum_{j=1}^{p} a_j \otimes x_j$, $\Delta_1(y) = \sum_{t=1}^{q} b_t \otimes y_t$ and $\Delta_{E \otimes E_1}(x \otimes y) = \sum_{j=1}^{p} \sum_{t=1}^{q} a_j b_t \otimes x_j \otimes y_t$.

Hence, $ab = (1_H \otimes x') \circ \Delta(x) \cdot (1_H \otimes y') \circ \Delta(y) = \sum_{j=1}^{p} \sum_{t=1}^{q} a_j b_t \otimes x_j \otimes y_t \in S(H)$.

Since $c(e) = e \otimes e$, $E = K.e$ is a left subcomodule of $H$ of dimension 1, one has $e = (1_H \otimes \sigma) \circ c(e) \in S(H)$.

Let $R(H)$ be the linear subspace of $H$ spaned by $S(H)$. Then $R(H)$ is an unitary subalgebra of $H$. Indeed, if $a = \sum_{j=1}^{p} \lambda_j a_j$ and $b = \sum_{t=1}^{q} \mu_t b_t$ are two elements of $R(H)$, since $a_j b_t \in S(H)$, one has $ab = \sum_{j=1}^{p} \sum_{t=1}^{q} \lambda_j \mu_t a_j b_t \in R(H)$. One says that $R(H)$ is the representative subalgebra of $H$.

Note :  Put, for the left $H$-comodule $(E, \Delta)$ of finite dimension, $S(\Delta) = \{a = (1_H \otimes x') \circ \Delta(x) \in H; x' \in E, x \in E\}$. As in Proposition 5, $S(\Delta)$ depends only of the isomorphism class $\tilde{\Delta}$ of $(E, \Delta)$. Furthermore, one has $S(H) = \bigcup_{\dim \Delta < +\infty} S(\Delta)$.\[\square\]

Also, it is clear that the $K$-linear vector space $R(\Delta) = \rho(\Delta)(E' \otimes E)$ is spaned by $S(\Delta)$. Hence one has $R(H) = \bigcup_{\dim \Delta < +\infty} R(\Delta)$. Moreover, if $(E_1, \Delta_1)$ and $(E_2, \Delta_2)$ are two comodules, then $R(\Delta_1 \oplus \Delta_2) = R(\Delta_1) + R(\Delta_2)$ contains $R(\Delta_1)$ and $R(\Delta_2)$ i.e. the family $(R(\Delta))_\Delta$ ordered by inclusion is directed upward.

Theorem 1 :  The representative subalgebra $R(H)$ of $H$ is such that $c(R(H)) \subset R(H) \otimes R(H)$. Moreover $(R(H), m, c, \eta, \sigma)$ is a Hopf algebra.

Proof :  It follows from Proposition 6 and Remark 3 that if $\Delta$ is a coproduct of finite dimension, then $c(R(\Delta)) \subset R(\Delta) \otimes R(\Delta)$ : that is $R(\Delta)$ is a coalgebra. Since
\( \mathcal{R}(H) = \bigcup_{\dim \Delta < +\infty} R(\Delta) \) is the union of coalgebras, it is a coalgebra. On the other hand, one deduces from the Corollary of Lemma 1 that \( \eta(\mathcal{R}(H)) \subset \mathcal{R}(H) \). The Theorem 1 is proved.

**II - 4 Simple comodules of finite dimension**

Let \( (e_j)_{1 \leq j \leq n} \) be a base of the finite dimensional left \( H \)-comodule \( (E, \Delta) \). Let us remember that \( A_j = (1_H \otimes e_j') \circ \Delta \) is a comodule morphism. One sees that \( \bigcap_{1 \leq j \leq n} \ker A_j = \{0\} \) and \( (A_j)_{1 \leq j \leq n} \) is free in \( \mathcal{L}(E, H) \). Since \( A_j(e_j) = a_{jj} \), one deduces from (2) or from Corollary 2 of Proposition 3 that \( e_j \notin \ker A_j \) and \( \ker A_j \neq E \).

Put \( H_j = A_j(E) \); then \( H_j \) is a left subcomodule of \( H \) of dimension \( \leq n \). Furthermore, with previous notations, one has \( R(\Delta) = \rho_\Delta(E' \otimes E) = \bigoplus_{j=1}^{n} H_j \) and \( H_j = \bigoplus_{t=1}^{n} K \cdot a_{tj} \), also \( R(\Delta) \) is a subcoalgebra of dimension \( \leq n^2 \). One can have \( \dim R(\Delta) < n^2 \); for example, if \( E_q = \bigoplus_{t=1}^{q} E \) and \( \Delta_q = \bigoplus_{t=1}^{q} \Delta \), \( q \geq 2 \), one has \( R(\Delta_q) = R(\Delta) \) and \( \dim R(\Delta_q) = \dim R(\Delta) \leq n^2 < (qn)^2 = (\dim(E_q))^2 \).

**Definition:** A left Banach \( H \)-comodule \( E \) is called simple or topologically irreducible if \( E \) is not the null space and does not contain any closed subcomodule different from \( \{0\} \) and \( E \).

Let \( \text{Hom} \cdot \text{com}(E, E_1) \) be the Banach space of the left Banach comodule morphisms of \( (E, \Delta) \) into \( (E_1, \Delta_1) \) and \( \text{End} \cdot \text{com}(E) = \text{Hom} \cdot \text{com}(E, E) \), this later is a Banach algebra.

**Remark 4:** Schur's Lemma. Let \( (E, \Delta) \) and \( (E_1, \Delta_1) \) be two simple, finite dimensional left \( H \)-comodules.

(i) If \( E \) and \( E_1 \) are not isomorphic, one has \( \text{Hom} \cdot \text{com}(E, E_1) = \{0\} \)

(ii) In the alternative case, any non null comodule morphism of \( E \) into \( E_1 \) is an isomorphism. In particular, \( \text{End} \cdot \text{com}(E) \) is a (skew) field of finite dimension \( \leq (\dim E)^2 \). If \( K \) is algebraically closed, then \( \text{End} \cdot \text{com}(E) = K.1_E \).

**Proposition 8:** Let \( (E, \Delta) \) be a simple Banach left \( H \)-comodule of finite dimension \( n \). Let \( (e_j)_{1 \leq j \leq n} \) be a base of \( E \) and \( A_j = (1_H \otimes e_j') \circ \Delta \), \( 1 \leq j \leq n \).

Then \( H_j = A_j(E) \) is a simple left \( H \)-comodule of \( H \) of finite dimension \( n \). Furthermore, there exists \( J \subset [1, n] \) such that \( R(\Delta) = \bigoplus_{j \in J} H_j \) (a direct sum of comodules).

**Proof:** It is the same as in semi-simple module theory. Indeed, since \( \ker A_j \neq E \) and \( E \) is a simple comodule of finite dimension \( n \), the map \( A_j : E \to H_j = A_j(E) \) is a comodule
isomorphism. Hence $H_j$ is a simple comodule of dimension $n$ with base $(a_{lj})_{1 \leq l \leq n}$. If $1 \leq j, q \leq n$, one has $H_j \cap H_q = (0)$, or $H_j = H_q$. Changing the order if necessary, we may assume that $(H_1, \ldots, H_m)$ is the family of the distinct comodules $H_j$; $m \leq n$. Hence

$$R(\Delta) = \sum_{j=1}^{m} H_j, H_j \neq H_q \text{ for } j \neq q.$$ 

Since $H_1 \cap H_2 = (0)$, one has the direct sum of comodules $H_1 \oplus H_2$. Let $j_0$ be the least integer $\geq 3$ such that $(H_1 \oplus H_2) \cap H_{j_0} = (0)$. Hence, one has the direct sum $H_1 \oplus H_2 \oplus H_{j_0}$ and for $j < j_0$, one has $(H_1 \oplus H_2) \cap H_j \neq (0)$, therefore $H_j \subset H_1 \oplus H_2$. Hence, by induction, one obtains $J = \{1, 2, j_0, \ldots, j_k = m\} \subset [1, m]$ and the direct sum of comodules $\bigoplus_{j \in J} H_j$ such that for $\ell \notin J$, $H_\ell \subset \bigoplus_{j \in J} H_j$. It follows that $R(\Delta) = \bigoplus_{j \in J} H_j$.

**Corollary:** Let $(E, \Delta)$ and $(E_1, \Delta_1)$ be two simple left $H$-comodules of finite dimension that are not isomorphic; then $R(\Delta \oplus \Delta_1) = R(\Delta) \oplus R(\Delta_1)$, a direct sum of comodules.

**Proof:** With previous notations, put $R(\Delta) = \bigoplus_{j \in J} H_j$ and $R(\Delta_1) = \bigoplus_{\ell \in L} H_\ell$. Let $p_j$ [resp. $p_\ell$] be the projection of $R(\Delta)$ [resp. $R(\Delta_1)$] onto $H_j$ [resp. $H_\ell$]. Suppose that $R(\Delta) \cap R(\Delta_1) \neq (0)$; this finite dimensional comodule must contain at least one simple comodule $V$. There exists $j \in J$ [resp. $\ell \in L$] such that $p_j(V) \neq (0)$ [resp. $p_\ell(V) \neq (0)$]; therefore $p_j(V) = H_j$ [resp. $p_\ell(V) = H_\ell$]. Since $V$ is simple, $p_j|_V$ [resp. $p_\ell|_V$] is an isomorphism of $V$ onto $H_j$ [resp. $H_\ell$]. It follows that $H_j$ and $H_\ell$ are isomorphic. Hence $E$ and $E_1$ are isomorphic; a contradiction. Therefore $R(\Delta) \cap R(\Delta_1) = (0)$ and $R(\Delta \oplus \Delta_1) = R(\Delta) \oplus R(\Delta_1)$.

**Remark 5:** Notations and hypothesis as above. If $K$ is algebraically closed, then the $H_j$, $1 \leq j \leq n$, are pairwise distinct.

**Proof:** Indeed, if $H_j = H_q$ for $j \neq q$, then $u = A_j \circ A_q^{-1}$ is an automorphism of the finite dimensional simple comodule $H_j$. By Schur's lemma, one has $u = \lambda \cdot 1_{H_j}$, $\lambda \in k$, $\lambda \neq 0$. Hence $A_j = \lambda A_q$ and $a_{jj} = A_j(e_j) = \lambda A_q(e_j) = \lambda a_{jq}$. Therefore $\sigma(a_{jj}) = 1 = \lambda \sigma(a_{jq}) = \lambda \delta_{jq} = 0$; a contradiction. \hfill \Box

Let $H'$ be the Banach space dual of $H$; if we set for $a', b' \in H'$, $a' \ast b' = (a' \otimes b') \circ c$, then $H'$ becomes a complete normed algebra with unit $\sigma$. If $(E, \Delta)$ is a left Banach comodule, setting for $a' \in H'$, and $x \in E$, $a' \cdot x = (a' \otimes 1_E) \circ \Delta(x)$, one induces on $E$ a complete normed right $H'$-module structure. Moreover, if $H$ is a pseudo-reflexive Banach space, then any closed right $H'$-submodule of $E$ is a Banach left $H$-subcomodule of $E$ and reciprocally (cf. [3]).
Let \((E', \Delta^\vee)\) be the conjugate of the finite dimensional left \(H\)-comodule \((E, \Delta)\). One has for any \(a' \in H', x' \in E'\) and \(x \in E, <a' \cdot x', x> = <x', \eta(a') \cdot x>\). Therefore, if \(M\) is a \(H'\)-submodule of \(E\), then \(M^\perp = \{x' \in E'/ <x', x> = 0, x \in M\}\) is a \(H'\)-submodule of \(E'\). Reciprocally, if \(\eta\) is bijective and if \(M'\) is \(H'\)-subcomodule of \(E'\), then \(M'^\perp\) is a \(H'\)-submodule of \(E'\).

**Proposition 9:** Let \(H\) be a complete ultrametric Hopf algebra that is a pseudo-reflexive Banach space such that \(\eta\) is bijective.

Then, a finite dimensional left \(H\)-comodule \((E, \Delta)\) is simple if and only if \((E', \Delta^\vee)\) is simple.

**Proof:** Indeed, suppose that \((E, \Delta)\) is simple; if \(M'\) is a left \(H\)-subcomodule of \((E', \Delta^\vee)\) then \(M'^\perp\) is a left \(H\)-subcomodule of \(E\); therefore \(M'^\perp = (0)\) or \(M'^\perp\) and \(M' = E'\) or \(M' = (0)\). By the same way, one shows the reciprocal.

**II - 5** When \(H\) admits a left integral

**II - 5 - 1** Again some general facts

**Lemma 3:** Let \((E, \Delta)\) be a finite dimensional left \(H\)-comodule and let \(\Delta_c\) be the restriction of \(c\) to \(R(\Delta) = \rho_\Delta(E' \otimes E)\); then \(R(\Delta_c) = R(\Delta)\).

**Proof:** Let \((e_j)_{1 \leq j \leq n}\) be a base of \(E\). One has \(\Delta(e_\ell) = \sum_{j=1}^{n} a_{\ell j} \otimes e_j, 1 \leq \ell \leq n\) and \((a_{\ell j})_{1 \leq \ell, j \leq n}\) spans \(R(\Delta)\). Since \(\sigma|_{R(\Delta)} \in R(\Delta)'\), one has, according to (1), \(a_{\ell j} = (1_H \otimes \sigma) \circ c(a_{\ell j}) = \rho_{\Delta_c}(\sigma \otimes a_{\ell j}) \in R(\Delta_c)\) and \(R(\Delta) \subset R(\Delta_c)\). Reciprocally, if \(a' \in R(\Delta)'\) and \(a = \sum_{1 \leq \ell, j \leq n} \lambda_{\ell j} a_{\ell j} \in R(\Delta)\), one has \((1_H \otimes a') \circ \Delta_c(a) = \sum_{1 \leq \ell, j \leq n} \lambda_{\ell j} < a', a_{k j} > a_{k \ell} \in R(\Delta)\) and \(R(\Delta_c) \subset R(\Delta)\).

**Lemma 4:** Any finite dimensional left \(H\)-subcomodule \(E\) of \(H\) is contained in the representative subalgebra \(R(H)\) of \(H\).

**Proof:** If \((e_j)_{1 \leq j \leq n}\) is a base of \(E \subset H\), one has \(c(e_j) = \sum_{\ell=1}^{n} a_{\ell j} \otimes e_\ell\). Let \(c_E\) be the restriction of \(c\) to \(E\), then \(R(c_E)\) is spaned by \((a_{j \ell})_{1 \leq j, \ell \leq n}\). Since \(e_j = (1_H \otimes \sigma) \circ c(e_j) = \sum_{\ell=1}^{n} \sigma(e_\ell)a_{j \ell} \in R(c_E)\), one has \(E \subset R(c_E) \subset R(H)\).
Note: If $K$ and $H$ are discrete, one deduces from the above result and from Theorem 1 - (ii) - of [3] that $\mathcal{R}(H) = H$.

II - 5 - 2 Under the hypothesis: $H$ admits a left integral

Let $\Omega$ be the family of the isomorphic classes of the simple, finite dimensional left $H$-comodules; $\Omega$ is not empty: its contains the class of the left subcomodule $K.e$ of $H$. If $\omega \in \Omega$ is the class of $(E, \Delta)$, we set $R(\omega) = R(\Delta)$ that is independant of $(E, \Delta)$. It is readily seen that $\mathcal{R}_s(H) = \sum_{\omega \in \Omega} R(\omega)$ is a subcoalgebra of $\mathcal{R}(H)$. Moreover $\mathcal{R}_s(H) = \bigoplus R(\omega)$, a direct sum of coalgebras. Indeed for any finite subset $(\omega_1, \ldots, \omega_m)$ of $\Omega$, one has $\sum_{t=1}^{n} R(\omega_t) = \bigoplus_{t=1}^{m} R(\omega_t)$ : see Corollary of Propositions 8 and its proof. Furthermore, if $\eta$ is bijective, then $\mathcal{R}_s(H)$ is a sub-Hopf-algebra of $\mathcal{R}(H)$. 

By definition, a left integral for the complete Hopf algebra $H$ is an element $\nu$ of $H'$ such that $\mu \ast \nu = \langle \mu, e \rangle \nu$ for all $\mu \in H'$. The complete Hopf algebra $H$ is called supple if $H$ is a pseudo-reflexive Banach space and $\eta \circ \eta = 1_H$. For $H$, a supple complete Hopf algebra that admits a left integral $\nu$ such that $\langle \nu, e \rangle = 1$, we know that any simple left Banach $H$-comodule is finite dimensional (Theorem 3 - [3])

Theorem 2: Let $H$ a supple complete Hopf algebra that admits a left integral $\nu$ such that $\langle \nu, e \rangle = 1$. Then

(i) $R(H) = \bigoplus_{\omega \in \Omega} R(\omega)$ where $\Omega$ is the family of the isomorphic classes of simple Banach left $H$-comodules.

(ii) The Hopf algebra $\mathcal{R}(H)$ is dense in $H$, that is $H = \overline{\mathcal{R}(H)} = \bigoplus_{\omega \in \Omega} R(\omega)$.

Proof:

(i) One deduces from [2] - Theorem 3 that any finite dimensional $H$-comodule $(E, \Delta)$ is semi-simple i.e. $(E, \Delta) = \bigoplus (V_{t,r}, \Delta_{t,r})$ with $V_{t,r} \in \omega_r$ and $\omega_r \in \Omega$. Hence $R(\Delta) = \sum_{t} \sum_{r} R(\Delta_{t,r}) = \sum_{r} R(\omega_r) = \bigoplus R(\omega_r) \subset \mathcal{R}_s(H)$. It follows that $\mathcal{R}(H) = \mathcal{R}_s(H) = \bigoplus_{\omega \in \Omega} R(\omega)$.

(ii) The Hopf algebra $H$ is naturally a Banach left $H$-comodule with coproduct $c$. Let $a \in H$, $a \neq 0$; since $H$ is pseudo-reflexive, the Banach left subcomodule $E(a) = \overline{H' \cdot a}$ of $H$ contains $a$ and is a non null Banach space of countable type (cf. [3]).
With the hypothesis, we know that $E(a)$ contains simple left $H$-subcomodules (finite dimensional) (cf. [3]).

Let $(V_r)_{r \in T}$ be the family of all simple subcomodules of $E(a)$. Put $W = \sum_{r \in T} V_r$, there exists $S \subset T$ such that $W = \bigoplus_{r \in S} V_r$; one has $c(W) \subset H \otimes W$. Since $c$ is a homeomorphism of $H$ onto $c(H)$, setting $E_0 = \overline{W}$, one has $c(E_0) \subset H \otimes E_0$, i.e. $E_0$ is a Banach left subcomodule of $E(a)$. In fact $E_0 = E(a)$. Otherwise, one has a direct sum of Banach comodules $E(a) = E_0 \oplus E_1$ with $E_1 \neq (0)$ (cf. [2]). However $E_1$ must contain at least one simple comodule $V$ and by definition of $W$, one has $V \subset W$. Hence $E_0 \cap E_1 \neq (0)$; a contradiction.

Let $\omega_r$ be the isomorphic class of the simple comodule $V_r, r \in T$. By Lemma 4, $V_r \subset R(\omega_r), r \in T$. Hence, we have $W = \sum_{r \in T} V_r \subset \sum_{r \in T} R(\omega_r) \subset \bigoplus_{\omega \in \Omega} R(\omega) = R(H)$. It follows that $a \in E(a) = E_0 = \overline{W} \subset \overline{R(H)}$. We have proved that $H = \overline{R(H)} = \bigoplus_{\omega \in \Omega} R(\omega)$.

Note: The above results are abstract version of some results of representation theory of groups. In particular Theorem 2 is Peter-Weyl Theorem (cf. [3]).

REFERENCES


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