The construction of normal bases for the space of continuous functions on $V_q$, with the aid of operators


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THE CONSTRUCTION OF NORMAL BASES FOR THE SPACE OF CONTINUOUS FUNCTIONS ON $V_q$, WITH THE AID OF OPERATORS

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Abstract. Let $a$ and $q$ be two units of $\mathbb{Z}_p$, $q$ not a root of unity, and let $V_q$ be the closure of the set $\{aq^n \mid n = 0, 1, 2, \ldots\}$. $K$ is a non-archimedean valued field, $K$ contains $\mathbb{Q}_p$, and $K$ is complete for the valuation $|\cdot|$, which extends the $p$-adic valuation. $C(V_q \to K)$ is the Banach space of continuous functions from $V_q$ to $K$, equipped with the supremum norm. Let $\mathcal{E}$ and $D_q$ be the operators on $C(V_q \to K)$ defined by $(\mathcal{E}f)(x) = f(qx)$ and $(D_qf)(x) = (f(qx) - f(x))/(x(q-1))$. We will find all linear and continuous operators that commute with $\mathcal{E}$ (resp. with $D_q$), and we use these operators to find normal bases $(r_n(x))$ for $C(V_q \to K)$. If $f$ is an element of $C(V_q \to K)$, then there exist elements $\alpha_n$ of $K$ such that $f(x) = \sum_{n=0}^{\infty} \alpha_n r_n(x)$ where the series on the right-hand-side is uniformly convergent. In some cases it is possible to give an expression for the coefficients $\alpha_n$.

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1. Introduction

Let $p$ be a prime, $\mathbb{Z}_p$ the ring of the $p$-adic integers, $\mathbb{Q}_p$ the field of the $p$-adic numbers. $K$ is a non-archimedean valued field, $K \supset \mathbb{Q}_p$, and we suppose that $K$ is complete for the valuation $|\cdot|$, which extends the $p$-adic valuation. Let $a$ and $q$ be two units of $\mathbb{Z}_p$ (i.e. $|a| = |q| = 1$), $q$ not a root of unity. Let $V_q$ be the closure of the set $\{aq^n \mid n = 0, 1, 2, \ldots\}$. We denote by $C(V_q \to K)$ (resp. $C(\mathbb{Z}_p \to K)$) the set of all continuous functions $f : V_q \to K$ (resp. $f : \mathbb{Z}_p \to K$) equipped with the supremum norm. If $f$ is an element of $C(V_q \to K)$ then we define the operators $\mathcal{E}$ and $D_q$ as follows:

$(\mathcal{E}f)(x) = f(qx)$
(D_q f)(x) = \frac{f(qx) - f(x)}{x(q-1)}

We remark that the operator E does not commute with D_q. Furthermore, the operator
D_q lowers the degree of a polynomial with one, whereas the operator E does not.

If \mathcal{L} is a non-archimedean Banach space over a non-archimedean valued field \mathcal{L}, and
\epsilon_1, \epsilon_2, \ldots is a finite or infinite sequence of elements of \mathcal{L}, then we say that this sequence
is orthogonal if ||\epsilon_1 + \cdots + \epsilon_k|| = \max\{||\epsilon_i|| : i = 1, \ldots, k\} for all k \in \mathbb{N} (or for all k
that do not exceed the length of the sequence) and for all \epsilon_1, \ldots, \epsilon_k in \mathcal{L}. An orthogonal
sequence \epsilon_1, \epsilon_2, \ldots is called orthonormal if ||\epsilon_i|| = 1 for all i. A family (\epsilon_i) of elements of
\mathcal{L} forms a(n) (ortho)normal basis of \mathcal{L} if the family (\epsilon_i) is orthonormal and also a basis.
We will call a sequence of polynomials (p_n(x)) a polynomial sequence if p_n is exactly of
degree n for all natural numbers n.

The aim here is to find normal bases for C(V_q \to K), which consist of polynomial
sequences. Therefore we will use linear, continuous operators which commute with D_q or
with E. If (r_n(x)) is such a polynomial sequence, and if f is an element of C(V_q \to K),
there exist coefficients \alpha_n in K such that \[ f(x) = \sum_{n=0}^{\infty} \alpha_n r_n(x) \] where the series on the right-
hand-side is uniformly convergent. In some cases it is possible to give an expression for
the coefficients \alpha_n.

We remark that all the results (with proofs) in this paper can be found in [5], except
for theorem 5.

2. Notations.

Let V_q, K and C(V_q \to K) be as in the introduction. The supremum norm on
C(V_q \to K) will be denoted by ||·||. We introduce the following:
\[ A_0(x) = 1, A_n(x) = (x - q^{n-1})A_{n-1}(x) (n \geq 1), \]
\[ B_n(x) = A_n(x)/A_n(q^n), C_n(x) = q^{
(n-1)/2(q-1)n}B_n(x) \]

It is clear that (A_n(x)), (B_n(x)) and (C_n(x)) are polynomial sequences. The sequence
(C_n(x)) forms a basis for C(V_q \to K) and the sequence (B_n(x)) forms a normal basis for
C(V_q \to K). From this it follows that ||B_n|| = 1 and ||C_n|| = ||(q-1)n||. Let E and D_q be
as in the introduction. Then we introduce the following:

**Definition.** Let f be a function from V_q to K. We define the following operators:
\[ (D_q^n f)(x) = (D_q D_q^{n-1} f)(x) \]
\[ (E^n f)(x) = f(q^n x) \]
\[ D f(x) = D^{(1)}(x) = f(qx) - f(x) = ((E - 1)f)(x) \]
\[ D^{(n)} f(x) = ((E - 1)D^{(n-1)}(x) = ((E - q^{n-1}) f)(x), D^{(0)} f(x) = f(x) \]

The operator D_q does not commute with D. The following properties are easily
verified:
\[ D_q^j C_k(x) = C_{k-j}(x) \text{ if } j \geq k, \quad D_q^j C_k(x) = 0 \text{ if } j < k. \] So D_q^j lowers the degree of a
polynomial with j
\[ D(j)B_k(x) = (x/a)^j q^{j-k} B_{k-j}(x) \text{ if } j \leq k, \quad D(j)B_k(x) = 0 \text{ if } j > k \]

If \( p(x) \) is a polynomial of degree \( n \), then \( (D(j)p)(x) \) is a polynomial of degree \( n \) if \( n \) is at least \( j \), and \( (D(j)p)(x) \) is the zero-polynomial if \( n \) is strictly smaller than \( j \).

If \( f \) is an element of \( C(V_q \to K) \), then we also have

i) \( (D(n)f)(x) = x^n q^{n(n-1)/2} (q-1)^n (D_q^n f)(x) \)

ii) \( (q-1)^n D_q^n f(x) \to 0 \) uniformly

iii) \( D(n)f(x) \to 0 \) uniformly

( i) can be found in [1], p. 60, ii) can be found in [3], p. 124-125, iii) follows from i) and ii).

3. Linear Continuous Operators which Commute with \( E \) or with \( D_q \)

Let us start this section with the following known result:

If \( f \) is an element of \( C(\mathbb{Z}_p \to K) \), then the translation operator \( E \) on \( C(\mathbb{Z}_p \to K) \) is the operator defined by \( Ef(x) = f(x+1) \).

If we put \( G_n(x) = \binom{x}{n} \) (the binomial polynomials), then L. Van Hamme ([4]) proved the following theorem:

A linear, continuous operator \( Q \) on \( C(\mathbb{Z}_p \to K) \) commutes with the translation operator \( E \) if and only if the sequence \( (g_n) \) is bounded, where \( g_n = QG_n(0) \).

Such an operator \( Q \) can be written in the following way: \( Q = \sum_{i=0}^{\infty} g_i \Delta^i \), where \( \Delta \) is the operator defined as follows: \( (\Delta f)(x) = f(x+1) - f(x) \).

We can prove analogous theorems for the operators \( E \) and \( D_q \) on \( C(V_q \to K) \):

**Theorem 1** An operator \( Q \) on \( C(V_q \to K) \) is continuous, linear and commutes with \( E \) if and only if the sequence \( (b_n) \) is bounded, where \( b_n = (Q B_n)(a) \).

From the proof of the theorem it follows that \( Q \) can be written in the form \( Q = \sum_{i=0}^{\infty} b_i D(i) \).

If \( f \) is an element of \( C(V_q \to K) \), then \( (Qf)(x) = \sum_{i=0}^{\infty} b_i (D(i)f)(x) \) and the series on the right-hand-side is uniformly convergent (since \( D(n)f(x) \to 0 \) uniformly). Clearly we have \( b_n = (Q B_n)(a) \), since \( (Q B_n)(a) = \sum_{i=0}^{\infty} b_i D(i) B_n(a) = \sum_{i=0}^{n} b_i (x/a)^i q^{i(i-n)} B_{n-i}(a) = b_n \).

Furthermore, \( Qx^n \) is a \( K \)-multiple of \( x^n \).

If \( b_0 = \ldots = b_{N-1} = 0, b_N \neq 0 \), and if \( p(x) \) is a polynomial, then \( x^N \) divides \( (Qp)(x) \).

**Some examples**

1) For the operator \( E \) we have: \( (EB_n)(x) = B_n(qx) \), so \( (EB_0)(a) = 1 \), \( (EB_1)(a) = 1 \), and \( (EB_n)(a) = 0 \) if \( n \geq 2 \). This gives us \( E = D(0) + D(1) \).
2) The operator $\mathcal{E} \circ \mathcal{D} = \mathcal{E}\mathcal{D}$ clearly commutes with $\mathcal{E}$. We have $((\mathcal{E}\mathcal{D})\mathcal{B}_0)(a) = 0$, and since $(n \geq 1)$ $((\mathcal{E}\mathcal{D})\mathcal{B}_n)(x) = (\mathcal{E} (\frac{q}{a} q^{1-n} B_{n-1}(qx))) = \frac{q}{a} q^{1-n} B_{n-1}(qx)$, we find $((\mathcal{E}\mathcal{D})\mathcal{B}_1)(a) = q$, $((\mathcal{E}\mathcal{D})\mathcal{B}_2)(a) = 1$ and $((\mathcal{E}\mathcal{D})\mathcal{B}_n)(a) = 0$ if $n \geq 3$. We conclude that $\mathcal{E}\mathcal{D} = q^2 \mathcal{D}^{(1)} + \mathcal{D}^{(2)}$.

Analogous to theorem 1 we have:

**Theorem 2** An operator $Q$ on $C(V_q \to \mathbb{K})$ is continuous, linear and commutes with $D_q$ if and only if the sequence $(c_n/(q-1)^n)$ is bounded, where $c_n = (QC_n)(a)$.

Such an operator $Q$ can be written in the form $Q = \sum_{i=0}^{\infty} c_i D_q^i$, and if $f$ is an element of $C(V_q \to \mathbb{K})$ it follows that $(Qf)(x) = \sum_{i=0}^{\infty} c_i (D_q^i f)(x)$, where the series on the right-hand-side converges uniformly (since $(q-1)^n D_q^n f(x) \to 0$ uniformly). Furthermore, we have $c_n = (QC_n)(a)$ since

$$
(QC_n)(a) = \left(\sum_{i=0}^{\infty} c_i D_q^i C_n\right)(a) = \sum_{i=0}^{n} c_i C_{n-i}(a) = c_n.
$$

**Remarks**

1) Let $R$ and $Q$ be linear, continuous operators on $C(V_q \to \mathbb{K})$, with $R$ of the form $R = \sum_{i=1}^{\infty} b_i D_q^{(i)}$ (i.e. $R$ commutes with $\mathcal{E}$, $b_0 = 0$), and $Q$ of the form $Q = \sum_{i=1}^{\infty} c_i D_q^i$ (i.e. $Q$ commutes with $D_q$, $c_n - \mu$). The main difference between the operators $Q$ and $R$ is that $Q$ lowers the degree of each polynomial with at least one, where $R$ does not necessarily lower the degree of a polynomial.

2) If $Q_1$ and $Q_2$ both commute with $D_q$ and if $Q_1 = \sum_{i=0}^{\infty} c_{1;i} D_q^i$,

$$Q_2 = \sum_{i=0}^{\infty} c_{2;i} D_q^i,$$

then $(Q_1 o Q_2)(f) = (Q_2 o Q_1)(f) = \sum_{k=0}^{\infty} D_q^k f \left(\sum_{j=0}^{k} c_{1;j} c_{2;k-j}\right)$.

If we take two formal power series $q_1(t) = \sum_{i=0}^{\infty} c_{1;i} t^i$, $q_2(t) = \sum_{i=0}^{\infty} c_{2;i} t^i$, then

$$q_1(t) \cdot q_2(t) = \sum_{k=0}^{\infty} t^k \left(\sum_{j=0}^{k} c_{1;j} c_{2;k-j}\right),$$

so the composition of two operators which commute with $D_q$, corresponds with multiplication of power series.
This is not the case if we take two operators which commute with $\mathcal{E}$: Take e.g. $\mathcal{E} = \mathcal{D}^{(0)} + \mathcal{D}^{(1)}$ and $\mathcal{D}^{(1)}$, then $\mathcal{E} \circ \mathcal{D}^{(1)} = \mathcal{E}\mathcal{D}^{(1)} = q\mathcal{D}^{(1)} + \mathcal{D}^{(2)}$, whereas for power series this gives $q_1(t) = 1 + t$, $q_2(t) = t$ and $q_1(t) \cdot q_2(t) = t + t^2$.

4. Normal bases for $C(V_q \to K)$

We use the operators of theorems 1 and 2 to make polynomials sequences $(p_n(x))$ which form normal bases for $C(V_q \to K)$. If $Q$ is an operator as found in theorem 1, with $b_0$ equal to zero, we associate a (unique) polynomial sequence $(p_n(x))$ with $Q$. We remark that the operator $R = \sum_{i=0} \delta_i \mathcal{D}^{(i)}$ does not necessarily lowers the degree of a polynomial.

**Proposition 1** Let $Q = \sum_{i=N}^\infty b_i \mathcal{D}^{(i)} \ (N \geq 1)$ with $|b_N| > |b_n|$ if $n > N$. There exists a unique polynomial sequence $(p_n(x))$ such that $(Q p_n)(x) = x^N p_{n-N}(x)$ if $n \geq N$, $p_n(a q^i) = 0$ if $n \geq N$, $0 \leq i < N$ and $p_n(x) = B_n(x)$ if $n < N$.

In the same way as in proposition 1 we have.

**Proposition 2** Let $Q = \sum_{i=N}^\infty c_i \mathcal{D}_q^{i} \ (N \geq 1)$, $c_N \neq 0$, $(c_n/(q - 1)^n)$ bounded. Then there exists a unique polynomial sequence $(p_n(x))$ such that $(Q p_n)(x) = p_{n-N}(x)$ if $n \geq N$, $p_n(a q^i) = 0$ if $n \geq N$, $0 \leq i < N$ and $p_n(x) = B_n(x)$ if $n < N$.

We use the operators of theorems 1 and 2 to make polynomials sequences $(p_n(x))$ which form normal bases for $C(V_q \to K)$. If $f$ is an element of $C(V_q \to K)$, there exist coefficients $\alpha_n$ such that $f(x) = \sum_{n=0}^\infty \alpha_n p_n(x)$ where the series on the right-hand-side is uniformly convergent. In some cases, it is also possible to give an expression for the coefficients $\alpha_n$.

**Theorem 3** Let $Q = \sum_{i=N}^\infty b_i \mathcal{D}^{(i)} \ (N \geq 1)$ with $|b_n| < |b_N| = 1$ if $n > N$

1) There exists a unique polynomial sequence $(p_n(x))$ such that $(Q p_n)(x) = x^N p_{n-N}(x)$ if $n \geq N$, $p_n(a q^i) = 0$ if $n \geq N$, $0 \leq i < N$ and $p_n(x) = B_n(x)$ if $n < N$. This sequence forms a normal basis for $C(V_q \to K)$ and the norm of $Q$ equals one.

2) If $f$ is an element of $C(V_q \to K)$, then $f$ can be written as a uniformly convergent series $f(x) = \sum_{n=0}^\infty \beta_n p_n(x)$, $\beta_n = ((D^{(i)}(x^{-N}Q^k)f)(a))$ if $n = i + kN$ ($0 \leq i < N$), with $||f|| = \max_{0 \leq k, 0 \leq i < N} |((D^{(i)}(x^{-N}Q^k)f)(a))|$, where $x^{-N}Q$ is a linear continuous operator with norm equal to one.
And analogous to theorem 3 we have

**Theorem 4** Let $Q = \sum_{i=N}^{\infty} c_i D_q^i$ ($N \geq 1$) with $|c_N| = |(q-1)^N|$, $|c_n| \leq |(q-1)^n|$ if $n > N$.

1) There exists a unique polynomial sequence $(p_n(x))$ such that $(Qp_n)(x) = p_{n-N}(x)$ if $n \geq N$, $p_n(aq^i) = 0$ if $n \geq N$, $0 \leq i < N$ and $p_n(x) = B_n(x)$ if $n < N$. This sequence forms a normal basis for $C(V_q \rightarrow K)$ and the norm of $Q$ equals one.

2) If $f$ is an element of $C(V_q \rightarrow K)$, there exists a unique, uniformly convergent expansion of the form $f(x) = \sum_{n=0}^{\infty} \gamma_n p_n(x)$, where $\gamma_n = a^i(q-1)^i q^{i(i-1)/2} (D_q^i Q f)(a)$ if $n = i + kN$ ($0 \leq i < N$), with $\|f\| = \max_{0 \leq k, 0 \leq i < N} \{ |(q-1)^i(D_q^i Q f)(a)|\}$.

**Remark.** Here we have $|c_n| \leq |c_N|$, in contrast with theorem 3, where we need $|b_n| < |b_N|$ ($n > N$).

An example

Let us consider the following operator $Q = (q-1)D_q$. Then $c_1 = (q-1)$ and $c_k = 0$ if $k \neq 1$. The polynomials $p_k(x)$ are given by $p_k(x) = C_k(x)/(q-1)^k$, and they form a normal basis for $C(V_q \rightarrow K)$. The expansion $f(x) = \sum_{k=0}^{\infty} ((q-1)^k D_q^k f)(a) p_k(x) = \sum_{k=0}^{\infty} (D_q^k f)(a) C_k(x)$ is known as Jackson's interpolation formula ([2],[3]).

If $Q$ is an operator as found in theorem 4, with $N$ equal to one, then we can prove a theorem analogous to theorem 2:

**Theorem 5** Let $Q$ be an operator such that $Q = \sum_{i=1}^{\infty} c_i D_q^i$, with $|c_1| = |(q-1)|$,

$|c_n| \leq |(q-1)^n|$ if $n > 1$, and let $p_n(x)$ be the polynomial sequence as found in theorem 4. An operator $T$ on $C(V_q \rightarrow K)$ is continuous, linear and commutes with $D_q$ if and only if $T$ is of the form $T = \sum_{i=0}^{\infty} d_i Q^i$, where the sequence $(d_n)$ is bounded, where $d_n = (Tp_n)(a)$.

**Remark.** In theorem 2 the sequence $(c_n/(q-1)^n)$ must be bounded, whereas here the sequence $(d_n)$ must be bounded. This follows from the fact that the norm of the operator $D_q$ equals $|q-1|^{-1}$, whereas the norm of the operator $Q$ equals 1.

5. More Normal Bases

We want to make more normal bases, using the ones we found in theorems 3 and 4. For operators which commute with $E$ we can prove the following theorem:
Theorem 6 Let \( (p_n(x)) \) be a polynomial sequence which forms a normal basis for \( C(V_q \to K) \), and let \( Q = \sum_{i=N}^{\infty} b_i D^{(i)}(N \geq 0) \) with \( 1 = |b_N| > |b_k| \) if \( k > N \). If \( Qp_n(x) = x^N r_{n-N}(x) \) \( (n \geq N) \), then the polynomial sequence \( (r_k(x)) \) forms a normal basis for \( C(V_q \to K) \).

And analogous for operators which commute with the operator \( D_q \) we have:

Theorem 7 Let \( (p_n(x)) \) be a polynomial sequence which forms a normal basis for \( C(V_q \to K) \), and let \( Q = \sum_{i=N}^{\infty} c_i D_q^i (N \geq 0) \) with \( |c_N| = |(q-1)^N| \), \( |c_n| \leq |(q-1)^n| \) if \( n > N \).

If \( (Qp_n)(x) = r_{n-N}(x) \) \( (n \geq N) \), then the polynomial sequence \( (r_k(x)) \) forms a normal basis for \( C(V_q \to K) \).

We remark that analogous results can be found on the space \( C(\mathbb{Z}_p \to K) \) for linear continuous operators which commute with the translation operator \( E \). The result analogous to theorems 3 and 4 for the case \( N \) equal to one, was found by L. Van Hamme (see [4]), and the extensive version of theorems 3 and 4, and the analogons of theorems 5, 6 and 7 can be found with proofs similar to the proofs of the theorems in this paper.

REFERENCES


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