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P-ADIC ALMOST PERIODICITY

AND REPRESENTATIONS

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Abstract– In the first international conference on p-adic functional analysis, the question whether it is possible to get the structure of the Banach Algebra $A_c(G)$ of p-adic valued continuous almost periodic functions on a totally disconnected topological IB-group G through the structure of its non-archimedean Bohr compactification $\hat{G}$ was raised. We affirmatively answer this question here. This structure of $A_c(G)$ helps one to study the p-adic regular representation of G using the known theory of representations for compact groups.

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1 Introduction

Let G be a group and K a complete ultra-metric valued field. When G carries a topology under which G is a topological group, we have studied in earlier papers Rangan [5], [6], [7] and [8] continuous almost periodic functions on G with values in K. In Rangan [8] we conjectured that a structure theory for the Banach algebra $A = A_c(G)$ of continuous almost periodic functions on G can be obtained using the known structure theory of the group algebra of a compact group by going to the Bohr compactification $\hat{G}$ of G. In this paper we give an affirmative answer to the conjecture. The observation that G is an IB-group if and only if the Bohr compactification $\hat{G}$ is an IB-group or equivalently a p-free group, where p is the characteristic of the residue class field of K, which is implicitly contained in the results proved in Rangan [7], helps us to establish the conjecture.

When G is an arbitrary group and K is a locally compact field we consider the subgroup topology on G defined by the normal subgroups of finite index in G under which

G becomes a O-dimensional group. The space of continuous almost periodic functions on G described above coincides with the space of almost periodic functions $AP(G \to K)$ defined by Schikof [10] using compactoid. This enables us to prove that there exists an invariant mean on $AP(G \to K)$ or equivalently the pair (G, K) is a.p.i.m. in the sense of Diarra [2](p.23, N.B.(i)) if and only if G is a IB-group or equivalently a p-free group (see Rangan [7]). Thus in the case when the base field is locally-compact, the problem of characterising (G, K) pairs which are a.p.i.m posed by Diarra is solved. The problem still remains open for non-locally compact fields. This also gives rise to the structure theory for $AP(G \to K)$ which is got by going to its Bohr compactification.

The structure theory so arrived at for the algebra of almost periodic functions gives rise to a study of representations of G taking the base space for representation to be the space of almost periodic functions on G. This may give rise to an alternative approach to representation theory developed by Diarra [1] using Hopf algebras. We intend discussing the details in another paper. Using the structure theory of $AP(G \to K)$, we prove that the regular representation decomposes as a direct sum of finite-dimensional representations.

2 Notations and Definitions

G is a group and K is a complete ultra metric rank one valued field, p denotes the characteristic of the residue class field. For $f : G \to K$, $x, s \in G$ we put $f_s(x) := f(s^{-1}x)$, $f^s(x) := f(xs), f^\vee(x) := f(x^{-1}), f_G = \{f_s : s \in G\}$ and $f^G := \{f^s : s \in G\}$. A function f defined on G is called almost periodic if f_G is precompact or equivalently if for every $\epsilon > 0$ there exists a covering of G by a finite collection of subsets $A_1, A_2, \cdots, A_n$ such that for $x, y \in A_i$ for $i = 1, 2, \cdots, n$ $|f(cxd) - f(cyd)| < \epsilon$ for all $c, d \in G$ (See Maak [4]). Interestingly it turns out that for a given $\epsilon > 0$ and an almost periodic function f on G, the covering consisting of minimum number of subsets $A_1, A_2 \cdots, A_n$ such that for $x, y \in A_i$, $|f(cxd) - f(cyd)| < \epsilon$ for $i = 1, 2, \cdots, n$ is the covering by cosets of a suitable normal subgroup $H(f, \epsilon)$ called the ϵ -kernel of finite index n in G. If f is a continuous almost periodic function on a topological group $G, H(f, \epsilon)$ is also an open and closed subgroup of finite index in G. A (topological) group is called an IB-group (Index Bounded group) if $\inf |n| > 0$, as n varies over all the indices of (closed) subgroups of finite index of G. We take $c = \inf |n|$. G is p-free if only if $c = 1$ or equivalently $|n| = 1$ for each index n. There exists a Mean M with $\| M \| = 1$ (sup norm) on $A_c(G)$ if and only if G is a p-free group.

Schikhof [10] calls a function $f : G \to K$ almost periodic if f_G is a compactoid in $B(G, K)$, the space of bounded functions on G with the supremum norm. The set of all almost periodic functions from G to K is denoted by $AP(G \to K)$. The almost periodic functions which are analogous of the classical case discussed earlier are called strictly almost periodic and the space of such functions is denoted by $SAP(G \to K)$. When G is a topological group the space of continuous strictly almost periodic functions is the space $A_c(G)$ of the earlier papers of the author. In general $SAP(G \to K) \subset AP(G \to K)$; however when the base field is locally compact $SAP(G \to K) = AP(G \to K)$. Diarra [1] has shown that χ_N the characteristic function of a normal subgroup N belongs to $AP(G \to K)$ if and only if N is of finite index in G.

3 Existence of Mean

Theorem 3.1 *If G is a topological O-dimensional group then G is an IB-group if and only if its Bohr compactification $\hat{G}$ is an IB-group or equivalently a p-free group.*

Proof: Let G be an IB-group. Then Theorem 3.3. [5] implies that there exists a Mean M on $A_c(G)$. Again by Theorem 3.8. [7] M defines an invariant integral for continuous functions on $\hat{G}$ and so $\hat{G}$ is a p-free group or equivalently an IB-group.

Conversely if $\hat{G}$ is an IB-group or equivalently a p-free group, the integral on $\hat{G}$ induces an invariant mean on$A_c(G)$. and so G is a p-free group or an IB-group with $c = 1$. ■

Remark 1: When G is compact the collection of open and closed subgroups coincides with the collection of closed subgroups of finite index in G and so the p-free condition in the usual sense coincides with the IB-condition on G.

Remark 2: When the base field K is locally-compact Diarra has given (corollary 2, p.13, [1]) several equivalent criteria for the existence of mean on $AP(G \to K)$ in terms of almost periodic representations, existence of Haar measure on the Bohr compactification etc. The above theorem which gives a criterion for the existence of mean in $AP(G \to K)$ enables one to conclude that Diarra's equivalent formulations holds when and only when the group is p-free.

If G is an arbitrary group. Let τ_B be the subgroup topology on G for which the collection of all normal subgroups of finite index is a fundamental system of neighbourhoods at the identity of G. With this topology, G is a topological group.

Proposition 3.2 *When K is locally compact and G is an arbitrary group, $AP(G \to K) = SAP(G \to K) = A_c(G)$, where $A_c(G)$ is the space of all continuous (in the subgroup topology defined above) of almost periodic functions in the sense of Maak.*

Proof: When K is locally compact every closed bounded subset of K is compact and so $SAP(G \to K) = AP(G \to K)$ (See Schikhof [10], p.3); clearly $A_c(G) \subset AP(G \to K)$. If $f \in AP(G \to K), f \in SAP(G \to K)$. Hence for $\epsilon > 0$, there exists a normal subgroup of finite index $H = H(f, \epsilon)$ such that

(i) $G = \cup_{i=1}^{n} Hx_i, x_i \in G$

(ii) for $x, y \in Hx_i, \ i = 1, 2, \cdots, n$

$$|f(cxd) - f(cyd)| < \epsilon \text{ for all } c, d \in G.$$

In particular for $x, y \in H, |f(x) - f(y)| < \epsilon$, i.e. f is uniformly continuous with respect to the subgroup topology τ_B on G and so $f \in A_c(G)$. This proves the proposition. ∎

The next theorem gives a necessary and sufficient condition for the existence of Mean on $AP(G \to K)$ in tune with the earlier conditions for the existence of Haar measure etc. (see van Rooij [8]) where G is an arbitrary group which solves the problem posed by Schikhof [10] in the case of the locally compact base field K. See also Diarra [1] theorem 4 and Schikhof [10], Theorem 8.2.

Theorem 3.3 *Let K be a locally compact field. An invariant Mean M on $AP(G \to K)$ exists if and only if G is p-free.*

Proof: We consider the subgroup topology τ_B on G given by the normal subgroups of finite index as a neighbourhood base at the identity. By the earlier proposition 3.2, $AP(G \to K) = SAP(G \to K) = A_c(G)$. Now the Theorem follows from Theorem 3.3 of Rangan [5]. ∎

Example: Let G be any free-group. Then for every $x \in G, x$ different from the identity of G, there exists a normal subgroup of finite index $N, x \notin N$. (See Hewitt and Ross [3]). Hence the subgroup topology on G given by the family of normal subgroups of finite index as a neighbourhood base is a Hausdorff topology on G. Hence $AP(G \to K) = SAP(G \to K) = A_c(G)$. G is a maximally almost periodic group. An invariant Mean exists on $AP(G \to K)$ if and only if G is p-free.

Remark: When K is locally compact for the study of continuous almost periodic functions on a totally disconnected topological group, only the topology τ_B on G matters. For if (G, τ) be a totally-disconnected topological group. G is a totally disconnected topological group also with respect to the topology τ_B defined by closed (in τ) normal subgroups of finite index in (G, τ). The topology τ_B is weaker than τ. By Theorem 4.1 Rangan [6], and proposition 3.2 above it follows that $A_c(G, \tau) = A_c(G, \tau_B) = AP(G \to K)$.

4 Structure of $A = A_c(G)$

Throughout this section we assume that K is locally compact and G is either a totally disconnected topological group or an arbitrary group G considered as a topological group with respect to the subgroup topology τ_B defined by the normal subgroups of finite index in G. So $A_c(G,\tau) = A_c(G,\tau_B) = AP(G \to K)$. We assume G to be a p-free group.

Theorem 4.1 *The algebra $A = A_c(G)$ is isometrically isomorphic to the group algebra $L(\hat{G})$ of the Bohr compactification $\hat{G}$ of G.*

Proof: The map $\theta : A \to L(\hat{G})$ given by $f \to \hat{f}$ where $\hat{f}$ is the associated continuous function on the compact group $\hat{G}$ to f (see Rangan [6], Theorem 4.4). If ρ is the homomorphism which imbeds G in $\hat{G}$, for $x \in G, f(x) = f(\rho(x))$. θ is one-to-one: For $\theta(f) = \theta(g) \Rightarrow \hat{f} = \hat{g} \Rightarrow f(x) = g(x)$ for all $x \in G \Rightarrow f = g$. θ is onto: if $h \in L(\hat{G}), h$ is a continuous function on $\hat{G}$. Define $f(x) = h(\rho(x))$ for $x \in G$ then $\hat{f} = h$. θ is an algebra homomorphism: For

$$\begin{aligned} f * g(x) &= M_y(f(y)g(y^{-1}x)) \\ &= \int_G f(y)g(y^{-1}x)dy = \hat{f} * \hat{g}(x) \end{aligned}$$

where the integral is the Haar integral and it exists since $\hat{G}$ is p-free, G being so.

θ is an isometry: When G is p-free $|n| = 1$ for every normal subgroup of finite index and so $c = 1$. Hence for $f \in A$,

$$\| f \| = \sup_{x\in G} |f(x)| = \sup_{x\in G} |\hat{f}(\rho(x))| = \sup_{x\in \hat{G}} |f(t)|$$

since $\rho(G)$ is dense in $\hat{G}$. ∎

Proposition 4.2 *A is the closure of the K-linear span of the idempotents of A.*

Proof: Since $A = A_c(G) = A_c(G,\tau_B) = AP(G \to K) = SAP(G \to K)$ the proposition follows from Lemma 4.4, Schikhof [10], which is now easily seen to be a restatement of the approximation Theorem 7.4 of Rangan [5]. ∎

Theorem 4.3 *For a p-free group $G, A = \oplus A_e$ where $A_e = e * A$ is a finite-dimensional two sided ideal of A and for every $f \in A$,*

$$f = \sum_{e\in E} e * f \text{and } \| f \| = \sup_{e\in E} \| e * f \|$$

and every non-zero minimal two sided ideal in A is an A_e for a suitable $e \in E$. If I is a closed two sided ideal in A then

$$I = cl \sum_{e\in I} A_e$$

where E is the set of all minimal non-zero central idempotents of A.

Proof: Follows from 8.14 Theorem van Rooij [9] since by the earlier theorem A and $L(\hat{G})$ are isometrically isomorphic. ■

It is not difficult to prove, using the existence of the approximate identity (U_H), (H varying over the collection Γ'_G of normal subgroups of finite index in G) that the closed ideals in A are same as closed invariant subspaces. For $f \in A$, defining $(L_a f)(x) = f(a^{-1}x)$ for $x \in G$, we get the (left) regular representation $a \to L_a$ on G. A_e being invariant subspaces in view of Theorem 4.3, L_a decomposes as a direct sum of finite-dimensional representations. Thus we get the following result.

Theorem 4.4 *The regular Representation decomposes as a direct sum of finite-dimensional representations.*

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