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ON THE DEFINITION OF A COMPACTOID

S. Oortwijn

Abstract: This is a paper about locally convex modules over the valuation ring of a non-archimedean valued field. We will discuss the definition of a compactoid module and we will give some relations between compactoidity and other compact-like properties.

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0. Introduction

In this paper K is a non-archimedean complete valued field with a non-trivial valuation $|\cdot|$. B_K is the valuation ring of K , i.e. $B_K = \{\lambda \in K \mid |\lambda| \leq 1\}$.

One problem in defining convex-compact sets in a locally convex space over K is that K is in general not locally compact, which means that ‘convex-compact’ becomes a trivial notion. To overcome this problem, Springer proposed in 1965 in [11] the notion of *c-compactness*, which is based on a convexification of the intersection property of compactness. However, this notion is only suitable for spherically complete K .

Later, in 1974, Gruson and van der Put [4] introduced a new notion: *compactoidity*, based upon a convexification of precompactness. This notion is meaningful for general (non-archimedean) fields K . Gruson proved in [3] that in locally convex spaces over a spherically complete K , ‘c-compact & bounded’ is the same as ‘complete compactoid’.

In this paper we will discuss how to extend the definition of compactoidity to arbitrary locally convex modules over the valuation ring. This is the content of section 2. Section 1 is introductory while in section 3 modules of finite rank will be treated. These modules play an important role in the definition of locally compactoid modules and modules of finite type in section 4. In this last section we also compare the different types of compact-like properties.

No proofs are given here. Proofs of the theorems stated in this note will be published elsewhere.

1. Locally convex modules

Theorem 1.1 Let A be a B_K -module and $\lambda \in B_K$. Then $\lambda A := \{\lambda x \mid x \in A\}$ is a submodule of A .

Theorem 1.2 Let A be a B_K -module and B a submodule of A . Then A/B is a B_K -module. The quotient map $\pi : A \rightarrow A/B$ defined by $\pi(x) = x + B$ ($x \in A$) is a homomorphism.

Definition 1.3 Let A be a B_K -module. A submodule B of A is called *absorbing* if for every $x \in A$ there exists a $\lambda \in B_K$, $\lambda \neq 0$, such that $\lambda x \in B$.

Definition 1.4 Let A be a B_K -module and X a subset of A . Then the B_K -module generated by X is denoted $\text{co } X$.

A is called *finitely generated* if there exists a finite subset X of A such that $A = \text{co } X$. A is called *countably generated* if there exists a countable subset X of A such that $A = \text{co } X$.

Definition 1.5 Let A, B be B_K -modules. We call A *embeddable in* B if there exists an injective homomorphism $A \rightarrow B$. If there exists a bijective homomorphism $A \rightarrow B$, then A and B are called *isomorphic*. We will denote this by $A \sim B$.

Definition 1.6 A *seminorm* on a B_K -module is a map $p : A \rightarrow [0, \infty)$ with the properties:

- (i) $p(0) = 0$
- (ii) $p(x + y) \leq \max(p(x), p(y))$ ($x, y \in A$)
- (iii) $p(\lambda x) \leq p(x)$ ($\lambda \in B_K, x \in A$)
- (iv) If $x \in A$ and $(\lambda_n)_{n \in \mathbb{N}} \in B_K$ such that $\lambda_n \rightarrow 0$, then $p(\lambda_n x) \rightarrow 0$ ($n \rightarrow \infty$).

A seminorm p is called *faithful* if $p(\lambda x) = |\lambda|p(x)$ ($\lambda \in B_K, x \in A$).

Definition 1.7 A *norm* on a B_K -module is a seminorm p with the property

$$p(x) = 0 \Leftrightarrow x = 0 \quad (x \in A). \text{ Norms are usually denoted } \| \cdot \|.$$

A *normed B_K -module* is a pair $(A, \| \cdot \|)$, where A is a B_K -module and $\| \cdot \|$ is a norm on A .

Theorem 1.8 Let A be a B_K -module and p a seminorm on A . Then $\text{Ker } p$ is a submodule of A .

Let $\bar{p} : A/\text{Ker } p \rightarrow [0, \infty)$ be defined as $\bar{p}(x + \text{Ker } p) = p(x)$ ($x \in A$). Then $(A/\text{Ker } p, \bar{p})$ is a normed B_K -module.

Definition 1.9 A *topological B_K -module* is a pair (A, τ) , where A is a B_K -module and τ is a Hausdorff topology on A , such that the addition: $A \times A \longrightarrow A$ and the scalar multiplication: $B_K \times A \longrightarrow A$ are continuous maps.

A topological B_K -module (A, τ) is called *locally convex* if there exists a base of zero neighbourhoods consisting of submodules of A .

Theorem 1.10 A *normed B_K -module is locally convex.*

Definition 1.11 A locally convex B_K -module is called *bounded* if for every open submodule U of A there exists a $\lambda \in B_K$, $\lambda \neq 0$, such that $\lambda A \subset U$.

Theorem 1.12 A *product of locally convex B_K -modules, equipped with the product topology, is again a locally convex B_K -module.*

Definition 1.13 Let (A, τ) be a locally convex B_K -module and B a submodule of A . Then the restricted topology on B is denoted $\tau|B$.

Definition 1.14 Let (A, τ) and (B, ν) be locally convex B_K -modules. We say (A, τ) is *topologically embeddable in (B, ν)* if there exists a injective homomorphism ρ from A into B , such that the map $\rho|A : (A, \tau) \longrightarrow (\rho(A), \nu|_{\rho(A)})$ is a homeomorphism.

Theorem 1.15

- (i) *Each Hausdorff locally convex space over K is a locally convex B_K -module.*
- (ii) *If A is a locally convex B_K -module and B is a submodule of A , then $(B, \tau|B)$ is a locally convex B_K -module.*

Theorem 1.16 Let (A, τ) be a locally convex B_K -module. Then:

(A, τ) is *topologically embeddable in a locally convex space* $\iff$ *There exists a separating collection of faithful seminorms on A generating τ .*

Definition 1.17 A normed B_K -module $(A, \| \cdot \|)$ is called *of countable type* if there exists a countable subset X of A such that $\text{co } X$ is dense in A .

A locally convex B_K -module is called *of countable type* if for every continuous seminorm p on A the normed space $(A/\text{Ker } p, \bar{p})$ is of countable type.

In the rest of this paper we will call a B_K -module shortly a module.

2. Compactoidity

For an absolutely convex subset of a locally convex space we know the following definition of compactoidity [2].

Definition 2.1 Let (E, τ) be a locally convex space and A an absolutely convex subset of E . A is called *compactoid* if for every zero neighbourhood U of E there exist $n \in \mathbb{N}$ and $x_1, \dots, x_n \in E$ such that $A \subset U + \text{co}\{x_1, \dots, x_n\}$.

This definition of compactoid seems to depend on the space (E, τ) of which A is a subset, but that is only partially true. It only depends on the restricted topology $\tau|_A$ on A .

Katsaras has proved:

Theorem 2.2 (Katsaras' Theorem [5]) $A \subset E$ is compactoid $\iff$ For every $\lambda \in K$ with $|\lambda| > 1$ and every zero neighbourhood U of E there exist $n \in \mathbb{N}$ and $x_1, \dots, x_n \in \lambda A$ such that $A \subset U + \text{co}\{x_1, \dots, x_n\}$.

Combining Katsaras' Theorem and the proof of lemma 10.5 in [6], saying that for every $\lambda \in K$ with $|\lambda| > 1$, $\tau|_A$ can on only one way be extended to a topology ν on λA that is induced by a locally convex topology on a vector space, one can prove:

Theorem 2.3 A is compactoid in $(E, \tau) \iff A$ is compactoid in every locally convex space (F, ν) in which $(A, \tau|_A)$ can be topologically embedded.

A first attempt to generalize the notion of compactoidity to arbitrary locally convex modules is a translation of Definition 2.1.

Definition 2.4 Let A be a submodule of a locally convex module (B, τ) . A is called a *compactoid in B* if for every zero neighbourhood U in B there exist $n \in \mathbb{N}$ and $x_1, \dots, x_n \in B$ such that $A \subset U + \text{co}\{x_1, \dots, x_n\}$.

A disadvantage of this definition is that it depends on the module B . For instance: If the valuation on K is dense, then $B_K^- := \{\lambda \in K \mid |\lambda| < 1\}$ is a compactoid in K , but not in B_K^- .

To overcome this obstruction we could try the following definition.

Definition 2.5 A locally convex module is called a *compactoid1-module* if A is compactoid in every locally convex module in which it can be topologically embedded.

If (A, τ) is such a compactoid module, then, in particular, A is compactoid in itself. And it is not hard to see that the converse is also true. So our new definition of compactoidity is equivalent to the following.

Definition 2.6 A locally convex module (A, τ) is called a *compactoid1-module* if for every zero neighbourhood U in A there exist $n \in \mathbb{N}$ and $x_1, \dots, x_n \in A$ such that $A \subset U + \text{co}\{x_1, \dots, x_n\}$.

This definition of a compactoid module now depends only on the module A itself, but it is too restricted for being a generalization of the notion of a compactoid for absolutely convex subsets of locally convex spaces.

For example, if the valuation on K is dense, then B_K^- , provided with the valuation topology is a compactoid in the sense of Definition 2.1, but not a compactoid1-module.

Definition 2.4 gives rise to yet another definition of compactoidity that also avoids the dependence on the module B ; we simply replace in Definition 2.5 ‘every’ by ‘some’:

Definition 2.7 A locally convex module (A, τ) is called a *compactoid2-module* if A is compactoid in some locally convex module in which it can be topologically embedded.

This definition turns out to be equivalent with our final definition of compactoidity (2.12) at the end of this section.

A second attempt for a generalization of compactoidity is based on Katsaras’ Theorem. This theorem does not directly carry over to a definition that is suitable for arbitrary locally convex modules, since for a module A the set λA , where $\lambda \in K$ with $|\lambda| > 1$, is not defined.

A slight modification in the formulation of Katsaras’ Theorem gives us a workable definition of a compactoid module.

Definition 2.8 Let (A, τ) be a locally convex module. (A, τ) is called a *compactoid3-module* if for every $\lambda \in B_K$, with $|\lambda| < 1$ and every zero neighbourhood U in A there exist $n \in \mathbb{N}$ and $x_1, \dots, x_n \in A$ such that $\lambda A \subset U + \text{co}\{x_1, \dots, x_n\}$.

This definition is a good one in the sense that it only depends on the module A and the topology τ on A . Moreover, it is a generalization of the notion of compactoid for absolutely convex subsets of locally convex spaces, as we can see in the following theorem.

Theorem 2.9 *Let A be an absolutely convex subset of a locally convex space (E, τ) . Then: A is compactoid in $(E, \tau) \iff (A, \tau|_A)$ is a compactoid3-module.*

The definition of a compactoid3-module has been used by Wim Schikhof in [10]. Yet we propose a slight modification of this definition, because of the following reasons.

1) If A is an absolutely convex subset of a locally convex space (E, τ) , then the linear span of A is of countable type, hence the module $(A, \tau|_A)$ is also of countable type. However, a locally convex compactoid3-module need not be of countable type.

2) If A is an absolutely convex subset of a locally convex metrizable space (E, τ) , then $(A, \tau|_A)$ is topologically embeddable in a compactly generated module, that is a locally convex module (B, ν) , such that there exists a compact subset X of B with $B = \overline{\text{co}} X$. However, a locally convex metrizable compactoid3-module can not always be topologically embedded in a compactly generated module.

For example, let $(E, \| \cdot \|)$ be a Banach space with an orthonormal base, which is not of countable type. Let $A = \{x \in E : \|x\| \leq 1\} / \{x \in E : \|x\| < 1\}$. Let $\| \cdot \|'$ on A be defined as

$$\|x\|' = \begin{cases} 1, & \text{if } x \neq 0; \\ 0, & \text{if } x = 0. \end{cases}$$

Then $\| \cdot \|'$ is a norm on A . ($\lambda x = 0$, for every $x \in A$ and every $\lambda \in B_K$, with $|\lambda| < 1$. Hence, $\|\lambda x\| \rightarrow 0$, if $\lambda \rightarrow 0$.)

Now $(A, \| \cdot \|')$ is a locally convex compactoid3-module, because $\lambda A = 0$ for every $\lambda \in B_K$ with $|\lambda| < 1$.

If the module $(A, \| \cdot \|')$ is of countable type, then also the space $(E, \| \cdot \|)$ must be of countable type. Hence, $(A, \| \cdot \|')$ is not of countable type. And one can prove that neither $(A, \| \cdot \|')$ is topologically embeddable in a compactly generated module.

For the modification of Definition 2.9 we shall look again closely to the definition of compactoidity in locally convex space. First we observe that for an absolutely convex subset of a locally convex space (E, τ) we have the following theorem.

Theorem 2.10 *A is compactoid $\iff$ For every open submodule (= an open absolutely convex subset) U of E there exist $n \in \mathbb{N}$ and $x_1, \dots, x_n \in E$ such that $A \subset U + \text{co}\{x_1, \dots, x_n\}$.*

For an open submodule U of E we obtain the following diagram.

$$\begin{array}{ccc}
 E & \xrightarrow{\pi} & E/U \\
 \text{injection} \uparrow & & \uparrow \text{injection} \\
 A & \xrightarrow{\pi|_A} & \pi(A)
 \end{array}$$

Here, π is the quotient map $E \longrightarrow E/U$, which is a homomorphism.

Then $\pi|_A : A \longrightarrow \pi(A)$ is a surjective homomorphism. Now $\text{Ker } \pi|_A = A \cap U$. This means that $A/(A \cap U) \sim \pi(A)$.

There exist $n \in \mathbb{N}$ and $x_1, \dots, x_n \in E$ such that $A \subset U + \text{co}\{x_1, \dots, x_n\}$. Then $\pi(A) \subset \text{co}\{\pi(x_1), \dots, \pi(x_n)\}$. We see that $A/(A \cap U)$ is embeddable in a finitely generated module. It is not hard to see that any submodule of A that is open in the restricted topology has the form $A \cap U$, where U is an open submodule of E .

Now one can prove the following.

Theorem 2.11 *Let A be an absolutely convex subset of a locally convex space. Then: A is a compactoid $\iff$ For every submodule V of A that is open in the restricted topology, A/V is embeddable in a finitely generated module.*

These observations lead to the following definition, meaningful for arbitrary locally convex modules.

Definition 2.12 (Final definition of compactoidity) Let (A, τ) be a locally convex module. (A, τ) is called a *compactoid module* if for every open submodule U of A , the quotient A/U is embeddable in a finitely generated module.

From Theorem 2.11 it is clear that this definition is a generalization of the notion of compactoidity for absolutely convex subsets of locally convex spaces. Moreover, this definition is beautiful, because it does not involve embeddings into other modules like in Definition 2.1, 2.5 and 2.7.

This definition of compactoidity was also proposed by Caenepeel in an unpublished note. The basic theory of compactoidity in locally convex space can be generalized to locally convex compactoid modules, and also the difficulties we had with the compactoid3-modules, mentioned after Theorem 2.9 vanish (see Theorem 4.4).

We will conclude with some theorems comparing the various definitions of compactoidity given in this section.

Theorem 2.13 *Let A be a submodule of a locally convex module (B, τ) , such that A is compactoid in B (in the sense of Definition 2.4). Then $(A, \tau|_A)$ is a compactoid module.*

Theorem 2.14 *Let (A, τ) be a locally convex module, then:*

(A, τ) is a compactoid module $\iff$ There exists a locally convex module (B, ν) , in which (A, τ) can be topologically embedded, such that A is compactoid in B (i.e. (A, τ) is a compactoid2-module).

Theorem 2.15 *Let (A, τ) be a locally convex module. Then:*

(A, τ) is a compactoid in itself (i.e. a compactoid1-module) $\iff$ For every open submodule U of A the module A/U is finitely generated.

Theorem 2.16 *Let (A, τ) be a locally convex module, then:*

(A, τ) is a compactoid3-module $\iff$ For every $\lambda \in B_K$ with $|\lambda| < 1$ the set λA is a compactoid module.

3. Modules of finite rank

To know more about the structure of compactoid modules we need to investigate those modules that are embeddable in a finitely generated module.

Examples of these modules are bounded absolutely convex subsets of finite-dimensional vector spaces. The module B_K/B_K^- is generated by one element and not absolutely convex (for it is a torsion module).

The class of all modules that are embeddable in a finitely generated module is denoted $\mathcal{B}_K$.

Theorem 3.1 *$\mathcal{B}_K$ is the smallest class $\mathcal{C}$ of modules for which:*

1. $B_K \in \mathcal{C}$
2. $\mathcal{C}$ is closed with respect to submodules
3. $\mathcal{C}$ is closed with respect to finite direct sums
4. $\mathcal{C}$ is closed with respect to quotients.

It turns out that $\mathcal{B}_K$ is the class consisting of all B_K -modules A for which there exist an $n \in \mathbb{N}$, a bounded absolutely convex subset B of K^n and a surjective homomorphism $\varphi :$

$B \longrightarrow A$. Hence, $\mathcal{B}_K$ is the class of all quotients of bounded finite-dimensional absolutely convex sets.

This last description of $\mathcal{B}_K$ is very helpful as it directly links $\mathcal{B}_K$ to the well-known class of finite-dimensional absolutely convex sets.

Another class of modules, related to $\mathcal{B}_K$, is the class $\mathcal{F}_K$ of all modules that are quotient of a (not necessarily bounded) finite-dimensional absolutely convex set.

The module K is a member of $\mathcal{F}_K$, but not of $\mathcal{B}_K$.

Theorem 3.2 *$\mathcal{F}_K$ is the smallest class $\mathcal{C}$ of modules for which:*

1. $K \in \mathcal{C}$
2. $\mathcal{C}$ is closed with respect to submodules
3. $\mathcal{C}$ is closed with respect to finite direct sums
4. $\mathcal{C}$ is closed with respect to quotients.

The class $\mathcal{F}_K$ plays an important role in the definition of a locally compactoid module in the following section.

Definition 3.3 A module $A \in \mathcal{F}_K$ is called *bounded* if for every absorbing submodule B of A there exists a $\lambda \in B_K$ with $\lambda \neq 0$ such that $\lambda A \subset B$. (i.e. It is bounded with respect to any locally convex topology on A in the sense of Definition 1.11.)

Theorem 3.4 *The class of all bounded modules in $\mathcal{F}_K$ equals $\mathcal{B}_K$.*

$\mathcal{F}_K$ is also known as the class of all B_K -modules of finite Fleischer rank defined in [1] as follows.

Definition 3.5 Let A be a torsion-free module. Then $\text{rank } A := \dim K \otimes_{B_K} A$.

Definition 3.6 Let A be a module. A is called *of finite Fleischer rank* if there exists a torsion-free module B , with $\text{rank } B < \infty$ and a surjective homomorphism $\varphi : B \rightarrow A$.

If A is a module of finite Fleischer rank, then the *Fleischer rank* of A is the minimal among all $n \in \mathbb{N}$ for which there exist a torsion-free module B , with $\text{rank } B = n$ and a surjective homomorphism $\varphi : B \rightarrow A$.

4. Compact-like modules

Besides the definition of a compactoid module we will discuss in this section the definitions of locally compactoid modules, modules of finite type, c-compact modules and some relations between these types of modules.

First we recall our final definition of compactoidity from section 2.

A locally convex module (A, τ) is called a *compactoid module* if for every open submodule U of A the quotient A/U is embeddable in a finitely generated module.

This definition can now be modified into:

Definition 4.1 A locally convex module (A, τ) is called a *compactoid module* if $A/U \in \mathcal{B}_K$ for every open submodule U of A .

By using properties of modules in $\mathcal{B}_K$ one can prove:

Theorem 4.2

- (i) *A submodule of a compactoid module is a compactoid module.*
- (ii) *A continuous homomorphic image of a compactoid module is a compactoid module.*
- (iii) *A product of compactoid modules is a compactoid module.*
- (iv) *Each compactoid module is topologically embeddable in a product of finitely generated discrete torsion modules.*

From (iv) of this theorem we obtain the following corollary.

Corollary 4.3 *The completion of a compactoid module is a compactoid module.*

Theorem 4.4 (See the remarks following Theorem 2.9.)

- (i) *A compactoid module is of countable type*
- (ii) *Every compactoid module is topologically embeddable in a compactly generated module.*

Definition 4.5 A compactoid module (A, τ) is called a *pure compactoid module* if for every open submodule U of A the module A/U is finitely generated.

The following theorem is another formulation of Theorem 2.15.

Theorem 4.6 *Let (A, τ) be a locally convex module. Then:*

(A, τ) is a pure compactoid module $\iff$ For every open submodule U of A there exist $n \in \mathbf{N}$ and $x_1, \dots, x_n \in A$ such that $A = U + \text{co}\{x_1, \dots, x_n\}$.

Using this theorem the following is not hard to see.

Theorem 4.7 *Let A be an absolutely convex subset of a locally convex space (E, τ) . Then:*

A is a pure compactoid in $(E, \tau) \iff (A, \tau|_A)$ is a pure compactoid module.

(For the definition of pure compactoid in vector space see [7]).

A submodule of a pure compactoid module need not be a pure compactoid module. However, the following can be proved.

Theorem 4.8

- (i) *A continuous homomorphic image of a pure compactoid module is a pure compactoid module.*
- (ii) *A product of pure compactoid modules is a pure compactoid module.*
- (iii) *The completion of a pure compactoid module is a pure compactoid module.*
- (iv) *If the completion of a locally convex module is a pure compactoid module, then the module itself is a pure compactoid module.*

The following equivalence theorem about pure compactoid modules can also be proved.

Theorem 4.9 *Let (A, τ) be a locally convex module. Then the following assertions are equivalent.*

- (α) *(A, τ) is a pure compactoid module.*
- (β) *Every continuous seminorm on A is bounded.*
- (γ) *Every continuous seminorm on A has a maximum.*
- (δ) *If $U_1 \subset U_2 \subset U_3 \subset \dots$ are open submodules on A such that $\bigcup_{n \in \mathbf{N}} U_n = A$, then there exists an $n \in \mathbf{N}$ such that $U_n = A$.*

Interesting is the equivalence between (β) and (γ). Consider the case that A is an absolutely convex subset of a locally convex space (K, τ) . Let us call a seminorm on A *restricted*, if it is the restriction of a faithful seminorm on E . In [8] it is proved that the following two assertions are equivalent.

- (α') *$(A, \tau|_A)$ is a pure compactoid module*

(β') *Every continuous restricted seminorm on A has a maximum.*

But (β') is not equivalent with

(γ') *Every continuous restricted seminorm on A is bounded.*

For example, B_K^- is an absolutely convex subset of the locally convex space $(E, |\cdot|)$, and every continuous restricted seminorm on B_K^- is bounded, for if p is a restricted seminorm on B_K^- , then there exists a seminorm q on K such that $p = q|_{B_K^-}$. Then $p \leq q(1)$ on B_K^- . But the valuation, which is of course a continuous restricted seminorm, has no maximum on B_K^- .

The final theorem about compactoid modules we will give is the following.

Theorem 4.10 *Every compactoid module is topologically embeddable in a pure compactoid module.*

Now we will discuss the notion of a locally compactoid module.

We can generalize the definition of a locally compactoid absolutely convex subset of a locally convex space, as is given in [9], to arbitrary locally convex modules in the same spirit as we did it for compactoids by replacing $\mathcal{B}_K$ by $\mathcal{F}_K$.

Definition 4.11 A locally convex module (A, τ) is called a *locally compactoid module* if $A/U \in \mathcal{F}_K$ for every open submodule U of A .

This definition is a generalization of the notion of locally compactoidity in locally convex space in the following sense.

Theorem 4.12 *Let A be an absolutely convex subset of a locally convex space (E, τ) . Then the following two assertions are equivalent.*

- (α) *$(A, \tau|_A)$ is a locally compactoid module.*
- (β) *There exists a locally convex space (F, ν) , in which $(A, \tau|_A)$ can be topologically embedded, such that A is a locally compactoid absolutely convex subset of F (in the sense of [9]).*

In general it is not true that if A is an absolutely convex subset of a locally convex space (E, τ) such that $(A, \tau|_A)$ is a locally compactoid module, then A is a locally compactoid subset of E . In [9] Schikhof gives a counterexample.

The following theorem is not very suprising.

Theorem 4.13 *Let (A, τ) be a locally convex module. Then:*

(A, τ) is a bounded locally compactoid module $\iff (A, \tau)$ is a compactoid module.

Again with the aid of properties of modules in $\mathcal{F}_K$ one can prove:

Theorem 4.14

- (i) *A submodule of a locally compactoid module is a locally compactoid module.*
- (ii) *A continuous homomorphic image of a locally compactoid module is a locally compactoid module.*
- (iii) *A product of locally compactoid modules is a locally compactoid module.*

From the theory of locally convex spaces we also know the notion of a space of finite type.

Definition 4.15 A faithful seminorm p on a vector space E is called *of finite type* if $E/\text{Ker } p$ is a finite-dimensional space.

A locally convex space (E, τ) is called *of finite type* if there exists a collection of faithful continuous seminorms of finite type on E generating τ .

One can prove that if (E, τ) is of finite type, then every faithful continuous seminorm is of finite type.

Next we move to arbitrary modules.

Definition 4.16 A seminorm p on a module A is called *of finite type* if $A/\text{Ker } p \in \mathcal{F}_K$.

A locally convex module (A, τ) is called a *module of finite type* if there exists a collection of continuous seminorms of finite type on A generating τ .

Contrary to the situation in locally convex space, not all continuous seminorms on a module of finite type need to be of finite type. We even have the following.

Theorem 4.17 *Let (A, τ) be a locally convex module such that every continuous seminorm on A is of finite type. Then $A \in \mathcal{F}_K$.*

We have the following representation theorem.

Theorem 4.18 *Let (A, τ) be a locally convex module. Then the following assertions are equivalent.*

- (α) *(A, τ) is of finite type.*

(β) (A, τ) is topologically embeddable in a product of normed modules in $\mathcal{F}_K$.

(γ) (A, τ) is topologically embeddable in a product of discrete torsion modules in $\mathcal{F}_K$.

Corollary 4.19 *The completion of a module of finite type is of finite type.*

The notion module of finite type gives us a new description of the class of locally compactoid modules, because one can prove the following.

Theorem 4.20 *Let (A, τ) be a locally convex module. Then:*

(A, τ) is of finite type $\iff (A, \tau)$ is a locally compactoid module.

We will conclude this section with a generalization of the notion of c -compactness. Let us recall the definition of c -compact sets in locally convex spaces.

Definition 4.21 An absolutely convex subset A of a locally convex space over a spherically complete K is called *c-compact* if for every family $\mathcal{C}$ of (relatively) closed convex subsets of A with the finite intersection property we have $\bigcap \mathcal{C} \neq \emptyset$.

This leads directly to:

Definition 4.22 A locally convex B_K -module (A, τ) , over a spherically complete K , is called *c-compact* if for every collection $\mathcal{C}$ of closed convex sets with the finite intersection property we have $\bigcap \mathcal{C} \neq \emptyset$.

Theorem 4.23 *Let K be spherically complete. Let A be a locally convex module. Then:*

(i) *(A, τ) is c-compact $\iff (A, \tau)$ is a complete locally compactoid module.*

(ii) *(A, τ) is c-compact and bounded $\iff (A, \tau)$ is a complete compactoid.*

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