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Weighted means in non-archimedean fields


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WEIGHTED MEANS IN NON-ARCHIMEDEAN FIELDS

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§1. INTRODUCTION.
In developing summability methods in non-archimedean fields, Srinivasan [6] defined the analogue of the classical weighted means \( (N, p_n) \) under the assumption that the sequence \( \{p_n\} \) of weights satisfies the conditions:

\[
|p_0| < |p_1| < \cdots < |p_n| < \cdots ;
\]
and

\[
\lim_{n \to \infty} |p_n| = \infty.
\]

However, it turned out that these weighted means were equivalent to convergence. In the present paper, an attempt is made to remedy the situation by assuming that the sequence \( \{p_n\} \) of weights satisfies the conditions:

\[
p_n \neq 0, \quad n = 0,1,2,\ldots;
\]
and

\[
|p_i| \leq |p_j|, \quad i = 0,1,2,\ldots, j = 0,1,2,\ldots,
\]

where \( P_j = \sum_{k=0}^{j} p_k, \ j = 0,1,2,\ldots \). Note that (3) and (4) imply \( P_n \neq 0, \ n = 0,1,2,\ldots \).

(4) is equivalent to

\[
\max_{0 \leq i \leq j} |p_i| \leq |P_j|, \ j = 0,1,2,\ldots.
\]

Since the valuation is non-archimedean,

\[
|P_j| \leq \max_{0 \leq i \leq j} |p_j|
\]
so that (4) is equivalent to

\[
|P_j| = \max_{0 \leq i \leq j} |p_j| = |p_j|.
\]
The assumptions (3) and (4) make the method of summability arising out of the weighted means non-trivial in certain cases (Remark 4) and further make it possible to compare two regular weighted means (Theorem 3) or compare a regular weighted mean with a regular matrix method (Theorem 4 and Theorem 5). This helps us to obtain (§4) a strictly increasing scale of regular summability methods in $\mathbb{Q}_p$, the p-adic field for a prime p; analogous to the scale of Cesàro means in $\mathbb{R}$ (the field of real numbers). These arise out of taking the weights

\[
p_n = p^{nk}, \quad \text{if } n \text{ is odd; } \]
\[
= \frac{1}{p^{nk}}, \quad \text{if } n \text{ is even,}
\]

\(n = 0, 1, 2, \ldots, k = 0, 1, 2, \ldots\).

For a knowledge of $(\mathcal{N}, p_n)$ methods in the classical case, the reader may refer [2],[5] and for analysis in non-archimedean fields [1].

§2. PRELIMINARIES.

Throughout this paper, $K$ denotes a complete, non-trivially valued, non-archimedean field and infinite matrices and sequences have their entries in $K$. Given an infinite matrix $A = (a_{nk}), n, k = 0, 1, 2, \ldots$ and a sequence $\{x_k\}$, $k = 0, 1, 2, \ldots$, by the $A$-transform of $\{x_k\}$, we mean the sequence $\{(Ax)_n\}$ where

\[(Ax)_n = \sum_{k=0}^{\infty} a_{nk}x_k, \quad n = 0, 1, 2, \ldots,\]

it being assumed that the series on the right converge. If $\lim_{n \to \infty} (Ax)_n = s$, we say that $\{x_k\}$ is $A$-summable (or summable by the infinite matrix method $A$) to $s$. If $\lim_{n \to \infty} (Ax)_n = s$ whenever $\lim_{k \to \infty} x_k = s$, the matrix method $A$ is said to be regular. It is well-known (see [3], [4]) that $A$ is regular if and only if

(a) $\sup_{n,k} |a_{nk}| < \infty$ ;

(b) $\lim_{n \to \infty} a_{nk} = 0, \quad k = 0, 1, 2, \ldots$ ;

and

(c) $\lim_{n \to \infty} \left(\sum_{k=0}^{\infty} a_{nk}\right) = 1$ .

(5)

(cf. For criterion for the regularity of a matrix method in the classical case see [2], p.43, Theorem 2). If a regular matrix $A$ is such that $\lim_{n \to \infty} (Ax)_n = s$ implies $\lim_{k \to \infty} x_k = s$, the matrix method $A$ is said to be trivial. Given two infinite matrix methods $A, B$, we say that $A$ is included in $B$, written as $A \subset B$, if any sequence $\{x_k\}$ that is $A$-summable to $s$ is also $B$-summable to $s$. An infinite matrix $A = (a_{nk})$ is said to be triangular (or, more precisely, lower triangular) if $a_{nk} = 0, \quad k > n, \quad n = 0, 1, 2, \ldots$. 
Definition 1. The \((N, p_n)\) method is defined by the infinite matrix \((a_{nk})\) where

\[
\begin{align*}
  a_{nk} &= \frac{p_k}{p_n} , \quad k \leq n ; \\
  &= 0 , \quad k > n .
\end{align*}
\]

(6)

Remark 1. If \(|\frac{P_{n+1}}{P_n}| > 1, n = 0, 1, 2 \ldots\) and \(\lim_{n \to \infty} |P_n| = \infty\) i.e. \(|P_n|\) strictly increases to infinity, then the method \((N, p_n)\) is trivial. For \(|p_n| = |P_n - P_{n-1}| = |P_n|\), since \(|P_n| > |P_{n-1}|\). So (1) is satisfied. Since \(\lim_{n \to \infty} |P_n| = \infty\), \(\lim_{n \to \infty} |p_n| = \infty\) so that (2) is satisfied too. Hence \((N, p_n)\) is trivial because of Theorem 4.2 of [6].

In the sequel we shall suppose that the sequence \(\{p_n\}\) of weights satisfies conditions (3) and (4).

An example of such an \((N, p_n)\) method corresponds to defined by

\[
p_n = \begin{cases} 
p^n , & \text{if } n \text{ is odd} ; \\
\frac{1}{p^n} , & \text{if } n \text{ is even ,}
\end{cases}
\]

where \(K = \mathbb{Q}_p\).

Remark 2. We note that (4) is equivalent to

\[
|P_{n+1}| \geq |P_n| , \quad n = 0, 1, 2, \ldots .
\]

(7)

Proof. Let (4) hold. Now

\[
|P_{n+1}| = \max_{0 \leq i \leq n+1} |p_i| = \max \left[ \max_{0 \leq i \leq n} |p_i| , \quad |p_{n+1}| \right]
\]

\[
= \max \left[ |P_n| , \quad |p_{n+1}| \right]
\]

\[
\geq |P_n| , \quad n = 0, 1, 2, \ldots .
\]

Conversely, let (7) hold. For a fixed integer \(j \geq 0\) let \(0 \leq i \leq j\). Then

\[
|p_i| = |P_i - P_{i-1}|
\]

\[
\leq \max \left[ |P_i| , \quad |P_{i-1}| \right]
\]

\[
\leq |P_j| ,
\]

by (7).
§3. MAIN RESULTS.

Theorem 1. \((N, p_n)\) is regular if and only if

\[
\lim_{n \to \infty} |P_n| = \infty
\]  

(8)

Proof. Let the \((N, p_n)\) method be regular. Using (6) and (5)(b), we note that (8) holds. Conversely, let (8) hold. In view of (6) and (8) it follows that \(\lim_{n \to \infty} a_{nk} = 0, \ k = 0, 1, 2, \ldots\).

Now, \(|a_{nk}| = 0, \ k > n\). If \(k \leq n\), \(|a_{nk}| = \frac{|p_k|}{|P_n|} \leq 1\), in view of (4).

Also \(\sum_{k=0}^{\infty} a_{nk} = 1, \ n = 0, 1, 2, \ldots\) so that \(\lim_{n \to \infty} \left(\sum_{k=0}^{\infty} a_{nk}\right) = 1\). Thus, by (5) the method \((N, p_n)\) is regular.

Remark 3. If \((N, p_n)\) is non-trivial, then (1) cannot be satisfied. Suppose (1) holds, then \(|p_n| = |P_n|\) so that (2) also holds. Thus \((N, p_n)\) is trivial by Theorem 4.2 of [6], a contradiction. This establishes the claim.

Remark 4. There are non-trivial \((N, p_n)\) methods. Let \(\alpha \in K\) such that \(0 < c = |\alpha| < 1\), this being possible since \(K\) is non-trivially valued. Let

\[
\{p_n\} = \left\{\alpha, \frac{1}{\alpha^2}, \frac{1}{\alpha^3}, \ldots\right\}
\]

and

\[
\{s_n\} = \left\{\frac{1}{\alpha}, \frac{1}{\alpha^2}, \frac{1}{\alpha^3}, \ldots\right\}
\]

It is clear that \(\{s_n\}\) does not converge. If \(\{t_n\}\) is the \((N, p_n)\) transform of \(\{s_k\}\),

\[
|t_{2k}| = \left|\frac{2k}{\alpha + \frac{1}{\alpha^2} + \frac{1}{\alpha^3} + \ldots + \frac{1}{\alpha^{2k}}}\right| \\
= \frac{|2k|}{\left(\frac{1}{c^{2k}}\right)} \\
\leq c^{2k}
\]

\[
|t_{2k+1}| = \left|\frac{2k + 1}{\alpha + \frac{1}{\alpha^2} + \frac{1}{\alpha^3} + \ldots + \frac{1}{\alpha^{2k}} + \alpha^{2k+1}}\right| \\
= \frac{|2k + 1|}{\left(\frac{1}{c^{2k}}\right)} \\
\leq c^{2k}
\]
so that \( \lim_{n \to \infty} t_n = 0 \). Thus \( \{s_n\} \), though non convergent, is summable \((N, p_n)\) (in fact, to 0). This establishes our claim.

**Theorem 2.** (Limitation theorem) If \( \{s_n\} \) is summable \((N, p_n)\) to \( s \), then

\[
|s_n - s| = o\left(\frac{P_n}{p_n}\right), \quad n \to \infty.
\]

**Proof.** If \( \{t_n\} \) is the \((N, p_n)\) transform of \( \{s_k\} \), then

\[
\left| \frac{P_n(s_n - s)}{p_n} \right| = \frac{|P_n s_n - p_n s|}{p_n} = \frac{|P_n t_n - P_n - P_{n-1} t_{n-1} - s (P_n - P_{n-1})|}{p_n} = \frac{|P_n(t_n - s) - P_{n-1}(t_{n-1} - s)|}{p_n}
\]

\[
\leq \max \left[ |t_n - s|, \frac{|P_{n-1}|}{p_n} |t_{n-1} - s| \right]
\]

since \( \left| \frac{P_{n-1}}{p_n} \right| \leq 1 \), by (7). Since \( \lim_{n \to \infty} t_n = s \), it follows that \( \lim_{n \to \infty} \left| \frac{P_n(s_n - s)}{p_n} \right| = 0 \). Thus

\[
|s_n - s| = o\left(\frac{P_n}{p_n}\right), \quad n \to \infty.
\]

**Theorem 3.** (Comparison theorem for two regular weighted means). If \((N, p_n), (N, q_n)\) are two regular methods and if

\[
\frac{P_n}{p_n} \leq H \frac{Q_n}{q_n}, \quad n = 0, 1, 2, \ldots ,
\]

where \( H > 0 \) is a constant and \( Q_n = \sum_{k=0}^{\infty} q_k \), then \((N, p_n) \subset (N, q_n)\).

**Proof.** Let, for a given sequence \( \{s_n\} \),

\[
t_n = \frac{p_0 s_0 + p_1 s_1 + \ldots + p_n s_n}{P_n},
\]

\[
u_n = \frac{q_0 s_0 + q_1 s_1 + \ldots + q_n s_n}{Q_n}, \quad n = 0, 1, 2, \ldots .
\]
Then \( p_0s_0 = P_0t_0, \ p_ns_n = P_nt_n - P_{n-1}t_{n-1}, \ n = 1, 2, \ldots \). Now,

\[
\begin{align*}
  u_n &= \frac{1}{Q_n} p_0t_0 + \frac{q_1}{p_1} (P_1t_1 - P_0t_0) + \ldots + \frac{q_{n}}{p_{n}} (P_{nt_n} - P_{n-1}t_{n-1}) \\
  &= \sum_{k=0}^{\infty} c_{nk} t_k,
\end{align*}
\]

where

\[
  c_{nk} = \begin{cases} 
    \left( \frac{q_k}{p_k} - \frac{q_{k+1}}{p_{k+1}} \right) \frac{P_k}{Q_n}, & k < n; \\
    \frac{q_k}{p_k} \frac{P_k}{Q_k}, & k = n; \\
    0, & k > n.
  \end{cases}
\]

Since \( \lim_{n \to \infty} |Q_n| = \infty, \ lim_{n \to \infty} c_{nk} = 0, \ k = 0, 1, 2, \ldots \). If \( s_n = 1, \ n = 0, 1, 2, \ldots \),

\[
t_n = u_n = 1, \ n = 0, 1, 2, \ldots \ 
\]

so that \( \sum_{k=0}^{\infty} c_{nk} = 1, \ n = 0, 1, 2, \ldots \) and so \( \lim_{n \to \infty} \left( \sum_{k=0}^{\infty} c_{nk} \right) = 1. \)

Let \( k < n. \)

\[
|c_{nk}| = \left| \frac{q_k}{p_k} - \frac{q_{k+1}}{p_{k+1}} \right| \frac{|P_k|}{|Q_n|} 
\leq \max \left[ \left| \frac{q_k}{p_k} \right| \frac{|P_k|}{|Q_n|}, \left| \frac{q_{k+1}}{p_{k+1}} \right| \frac{|P_k|}{|Q_k|} \right] 
\leq \max \left[ \left| \frac{q_k}{p_k} \right| \frac{|P_k|}{|Q_n|}, \left| \frac{q_{k+1}}{p_{k+1}} \right| \frac{|P_{k+1}|}{|Q_{k+1}|} \right] 
\leq H, 
\]

by (9), since \( k < n \) implies \( |Q_k|, |Q_{k+1}| \leq |Q_n| \) and so \( \frac{1}{Q_n} \leq \frac{1}{Q_k}, \ \frac{1}{Q_{k+1}} \) and \( |P_k| \leq |P_{k+1}|. \)

If \( k = n, \ |c_{nn}| = \frac{|q_n|}{p_n} \frac{P_n}{|Q_n|} \leq H \) and \( |c_{nk}| = 0 \leq H, \ k > n. \) Consequently \( \sup_{n,k} |a_{nk}| \leq H. \)

The method \((c_{nk})\) is thus regular, using (5) and so \((N, p_n) \subset (N, q_n)\). The proof of the theorem is now complete.

**Remark 5.** Note that the classical counterpart of Theorem 3 (see [2], p.58, Theorem 14) has an additional hypothesis.

**Theorem 4.** (Comparison theorem for a regular \((N, p_n)\) method and a regular matrix). Let \((N, p_n)\) be a regular method and \(A\) be a regular matrix. If

\[
\lim_{k \to \infty} \frac{a_{nk} P_k}{p_k} = 0, \ n = 0, 1, 2, \ldots ;
\]

and

\[
\sup_{n,k} \left| \frac{a_{nk} P_k}{p_k} - \frac{a_{n,k+1} P_{k+1}}{p_{k+1}} \right| < \infty,
\]

then \( p_{ns_n} = P_{nt_n} - P_{n-1}t_{n-1}, \ n = 1, 2, \ldots \). Now,

\[
\begin{align*}
  u_n &= \frac{1}{Q_n} p_0t_0 + \frac{q_1}{p_1} (P_1t_1 - P_0t_0) + \ldots + \frac{q_{n}}{p_{n}} (P_{nt_n} - P_{n-1}t_{n-1}) \\
  &= \sum_{k=0}^{\infty} c_{nk} t_k,
\end{align*}
\]

where

\[
  c_{nk} = \begin{cases} 
    \left( \frac{q_k}{p_k} - \frac{q_{k+1}}{p_{k+1}} \right) \frac{P_k}{Q_n}, & k < n; \\
    \frac{q_k}{p_k} \frac{P_k}{Q_k}, & k = n; \\
    0, & k > n.
  \end{cases}
\]

Since \( \lim_{n \to \infty} |Q_n| = \infty, \ lim_{n \to \infty} c_{nk} = 0, \ k = 0, 1, 2, \ldots \). If \( s_n = 1, \ n = 0, 1, 2, \ldots \),

\[
t_n = u_n = 1, \ n = 0, 1, 2, \ldots \ 
\]

so that \( \sum_{k=0}^{\infty} c_{nk} = 1, \ n = 0, 1, 2, \ldots \) and so \( \lim_{n \to \infty} \left( \sum_{k=0}^{\infty} c_{nk} \right) = 1. \)

Let \( k < n. \)

\[
|c_{nk}| = \left| \frac{q_k}{p_k} - \frac{q_{k+1}}{p_{k+1}} \right| \frac{|P_k|}{|Q_n|} 
\leq \max \left[ \left| \frac{q_k}{p_k} \right| \frac{|P_k|}{|Q_n|}, \left| \frac{q_{k+1}}{p_{k+1}} \right| \frac{|P_k|}{|Q_k|} \right] 
\leq \max \left[ \left| \frac{q_k}{p_k} \right| \frac{|P_k|}{|Q_n|}, \left| \frac{q_{k+1}}{p_{k+1}} \right| \frac{|P_{k+1}|}{|Q_{k+1}|} \right] 
\leq H, 
\]

by (9), since \( k < n \) implies \( |Q_k|, |Q_{k+1}| \leq |Q_n| \) and so \( \frac{1}{Q_n} \leq \frac{1}{Q_k}, \ \frac{1}{Q_{k+1}} \) and \( |P_k| \leq |P_{k+1}|. \)

If \( k = n, \ |c_{nn}| = \frac{|q_n|}{p_n} \frac{P_n}{|Q_n|} \leq H \) and \( |c_{nk}| = 0 \leq H, \ k > n. \) Consequently \( \sup_{n,k} |a_{nk}| \leq H. \)

The method \((c_{nk})\) is thus regular, using (5) and so \((N, p_n) \subset (N, q_n)\). The proof of the theorem is now complete.

**Remark 5.** Note that the classical counterpart of Theorem 3 (see [2], p.58, Theorem 14) has an additional hypothesis.

**Theorem 4.** (Comparison theorem for a regular \((N, p_n)\) method and a regular matrix). Let \((N, p_n)\) be a regular method and \(A\) be a regular matrix. If

\[
\lim_{k \to \infty} \frac{a_{nk} P_k}{p_k} = 0, \ n = 0, 1, 2, \ldots ;
\]

and

\[
\sup_{n,k} \left| \frac{a_{nk} P_k}{p_k} - \frac{a_{n,k+1} P_{k+1}}{p_{k+1}} \right| < \infty,
\]
then \((N, p_n) \subset A\).

**Proof.** Let \(\{s_n\}\) be any sequence, \(\{t_n\}\), \(\{\tau_n\}\) be its \((N, p_n)\), \(A\) transforms respectively so that

\[ t_n = \frac{p_0 s_0 + p_1 s_1 + \ldots + p_n s_n}{P_n}, \]

\[ \tau_n = \sum_{k=0}^{\infty} a_{nk} s_k, \quad n = 0, 1, 2, \ldots \]

Now,

\[ s_n = \frac{P_n t_n - P_{n-1} s_1 t_{n-1}}{p_n}, \quad P_{-1} = 0 \]

Let \(\lim_{n \to \infty} t_n = s\). \(\tau_n = \sum_{k=0}^{\infty} a_{nk} s_k\) exists, \(n = 0, 1, 2, \ldots\) and in fact

\[ \tau_n = \sum_{k=0}^{\infty} a_{nk} s_k = \sum_{k=0}^{\infty} a_{nk} \left( \frac{P_k t_k - P_{k-1} t_{k-1}}{p_k} \right) \]

\[ = \sum_{k=0}^{\infty} \left( \frac{a_{nk}}{p_k} - \frac{a_{n,k+1}}{p_{k+1}} \right) P_k t_k, \]

since \(\lim_{k \to \infty} \frac{a_{nk+1}}{p_{k+1}} P_k t_k = 0\) by (10) and using the fact that \(\{t_k\}\) is convergent and so bounded and \(|\frac{P_k}{P_{k+1}}| \leq 1\). We can now write

\[ \tau_n = \sum_{k=0}^{\infty} b_{nk} t_k, \]

where

\[ b_{nk} = \left( \frac{a_{nk}}{p_k} - \frac{a_{n,k+1}}{p_{k+1}} \right) P_k. \]

By (11), \(\sup_{n,k} |b_{nk}| < \infty\). Since \(A\) is regular, \(\lim_{n \to \infty} a_{nk} = 0, \quad k = 0, 1, 2, \ldots\) so that

\(\lim_{n \to \infty} b_{nk} = 0, \quad k = 0, 1, 2, \ldots\). Let \(s_n = 1, \quad n = 0, 1, 2, \ldots\) Then \(t_n = 1, \quad n = 0, 1, 2, \ldots\)

It now follows that \(\sum_{k=0}^{\infty} b_{nk} = \sum_{k=0}^{\infty} a_{nk}, \quad n = 0, 1, 2, \ldots\). Consequently \(\lim_{n \to \infty} \left( \sum_{k=0}^{\infty} b_{nk} \right) = \lim_{n \to \infty} \left( \sum_{k=0}^{\infty} a_{nk} \right) = 1\). The method \((b_{nk})\) is thus regular and so \(\lim_{n \to \infty} t_n = s\) implies \(\lim_{n \to \infty} \tau_n = s\), i.e. \((N, p_n) \subset A\).

**Theorem 5.** \((N, p_n)\) is a regular method and \(A = (a_{nk})\) is a regular triangular matrix. Then \((N, p_n) \subset A\) if and only if (11) holds.
Proof. Let (11) hold. Since $A$ is a triangular matrix, (10) clearly holds. In view of Theorem 4, we have $(\overline{N}, p_n) \subset A$. Conversely, let $(\overline{N}, p_n) \subset A$. Following the notation of Theorem 4, let $\lim_{n \to \infty} t_n = s$. As in the proof of Theorem 4,

$$
\tau_n = \sum_{k=0}^{\infty} a_n s_k = \sum_{k=0}^{\infty} b_n t_k,
$$

where

$$
b_{nk} = \left( \frac{a_{nk}}{p_k} - \frac{a_{n,k+1}}{p_{k+1}} \right) P_k.
$$

Since $(\overline{N}, p_n) \subset A$, for every sequence $\{t_k\}$ with $\lim_{k \to \infty} t_k = s$, $\lim_{n \to \infty} \tau_n = s$. This means that $(b_{nk})$ is a regular matrix and so (11) holds. This complices the proof.

§4. A SCALE OF STRICTLY INCREASING WEIGHTED MEANS.

We conclude the present paper by obtaining a strictly increasing scale of regular summability methods in $Q_p$. We define, for $k = 0, 1, 2, \ldots$, the method $(\overline{N}, p_n^{(k)})$ by

$$
p_n^{(k)} =
\begin{cases}
p_n, & \text{if } n \text{ is odd;} \\
\frac{1}{p_n}, & \text{if } n \text{ is even;}
\end{cases}
$$

We now establish that

$$
(\overline{N}, p_n^{(k)}) \subsetneq (\overline{N}, p_n^{(k+1)}).
$$

We apply Theorem 3 to prove this assertion. For convenience, let $p_n = p_n^{(k)}$ and $q_n = p_n^{(k+1)}$, $n = 0, 1, 2, \ldots$. If $n$ is odd,

$$
\left| \frac{P_n}{p_n} \right| = \frac{1}{c(n-1)k} \cdot \frac{1}{c^{nk}} = \frac{1}{c(2n-1)k}
$$

$$
\left| \frac{Q_n}{q_n} \right| = \frac{1}{c(n-1)(k+1)} \cdot \frac{1}{c^{n(k+1)}} = \frac{1}{c(2n-1)(k+1)}, \quad c = |p| < 1,
$$

so that

$$
\left| \frac{P_n}{p_n} \right| \leq \left| \frac{Q_n}{q_n} \right|
$$

If $n$ is even,

$$
\left| \frac{P_n}{p_n} \right| = \frac{1}{c^{nk}} \cdot c^{nk} = 1
$$

$$
\left| \frac{Q_n}{q_n} \right| = \frac{1}{c^{n(k+1)}} \cdot c^{n(k+1)} = 1.
$$

Thus

$$
\left| \frac{P_n}{p_n} \right| \leq \left| \frac{Q_n}{q_n} \right|
$$
in this case too. Consequently, by Theorem 3, $(\bar{N}, p_n^{(k)}) \subset (\bar{N}, p_n^{(k+1)})$. Let now

$$s_n = \begin{cases} 0, & \text{if } n \text{ is even;} \\ 1/p^{n(k+1)+k(n-1)}, & \text{if } n \text{ is odd.} \end{cases}$$

Let $\{\tau_n\}$ be the $(\bar{N}, q_n)$ transform of $\{s_n\}$.

If $n$ is odd,

$$|\tau_n| = \left| \frac{1 + p^{k+1} + 0 + p^{3(k+1)} + \ldots + 0 + p^{n(k+1)} + 1}{1 + p^{k+1} + 1/p^{2(k+1)} + \ldots + 1/p^{(n-1)(k+1)} + p^{n(k+1)}} \right|$$

$$= \frac{1}{c^{k(n-1)}} \frac{1}{c^{(k+1)(n-1)}}$$

$$= c^{n-1}$$

If $n$ is even,

$$|\tau_n| = \left| \frac{1 + p^{k+1} + 0 + p^{3(k+1)} + \ldots + 0 + p^{(n-1)(k+1)} + 1/p^{(n-1)(k+1)+k(n-2)} + 0}{1 + p^{k+1} + 1/p^{2(k+1)} + \ldots + p^{(n-1)-(k+1)} + 1/p^{n(k+1)}} \right|$$

$$= \frac{1}{c^{k(n-2)}} \frac{1}{c^{n(k+1)}}$$

$$= c^{n+2k}$$

In both the cases, $\lim_{n \to \infty} \tau_n = 0$. Thus $\{s_n\}$ is summable $(\bar{N}, q_n)$ to 0. Let, now, $\{t_n\}$ be the $(\bar{N}, p_n)$ transform of $\{s_n\}$. 
If $n$ is odd

$$ |\tau_n| = \left| 0 + \frac{1}{p^{k+1}} + 0 + \frac{1}{p^{3(k+1)+2k}} + \ldots + 0 + \frac{1}{p^{n(k+1)+k(n-1)}} \right|$$

$$ = \frac{1}{c^{n+k(n-1)}} \cdot \frac{1}{c^{(n-1)k}}$$

$$ = \frac{1}{c^n}$$

Since $\frac{1}{c} > 1$, $\lim_{n \to \infty} |t_n| = \infty$ that $\{t_n\}$ cannot converge. Thus $\{s_n\}$ is not $(N, p_n)$ summable and consequently (12) holds.

REFERENCES


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