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Weighted means in non-archimedean fields


<http://www.numdam.org/item?id=AMBP_1995__2_1_191_0>
WEIGHTED MEANS IN NON-ARCHIMEDEAN FIELDS

P.N. Natarajan

§1. INTRODUCTION.

In developing summability methods in non-archimedean fields, Srinivasan [6] defined the analogue of the classical weighted means $(N, p_n)$ under the assumption that the sequence $\{p_n\}$ of weights satisfies the conditions:

\[ |p_0| < |p_1| < |p_2| < \ldots < |p_n| < \ldots ; \quad (1) \]

and

\[ \lim_{n \to \infty} |p_n| = \infty . \quad (2) \]

However, it turned out that these weighted means were equivalent to convergence. In the present paper, an attempt is made to remedy the situation by assuming that the sequence $\{p_n\}$ of weights satisfies the conditions:

\[ p_n \neq 0, \quad n = 0, 1, 2, \ldots ; \quad (3) \]

and

\[ |p_i| \leq |P_j|, \quad i = 0, 1, 2, \ldots, j, \quad j = 0, 1, 2, \ldots , \quad (4) \]

where $P_j = \sum_{k=0}^{j} p_k, \quad j = 0, 1, 2, \ldots$. Note that (3) and (4) imply $P_n \neq 0, \quad n = 0, 1, 2, \ldots . \quad (4')$

(4) is equivalent to

\[ \max_{0 \leq i \leq j} |p_i| \leq |P_j|, \quad j = 0, 1, 2, \ldots . \]

Since the valuation is non-archimedean,

\[ |P_j| \leq \max_{0 \leq i \leq j} |p_j| \]

so that (4) is equivalent to

\[ |P_j| = \max_{0 \leq i \leq j} |p_j| = |p_j|. \quad (4') \]
The assumptions (3) and (4) make the method of summability arising out of the weighted means non-trivial in certain cases (Remark 4) and further make it possible to compare two regular weighted means (Theorem 3) or compare a regular weighted mean with a regular matrix method (Theorem 4 and Theorem 5). This helps us to obtain (§4) a strictly increasing scale of regular summability methods in $\mathbb{Q}_p$, the $p$-adic field for a prime $p$; analogous to the scale of Cesàro means in $\mathbb{R}$ (the field of real numbers). These arise out of taking the weights

$$ p_n = \begin{cases} p^{nk}, & \text{if } n \text{ is odd;} \\ \frac{1}{p^{nk}}, & \text{if } n \text{ is even,} \end{cases} \quad n = 0, 1, 2, \ldots, \quad k = 0, 1, 2, \ldots. $$

For a knowledge of $(N, p_n)$ methods in the classical case, the reader may refer [2],[5] and for analysis in non-archimedean fields [1].

§2. PRELIMINARIES.

Throughout this paper, $K$ denotes a complete, non-trivially valued, non-archimedean field and infinite matrices and sequences have their entries in $K$. Given an infinite matrix $A = (a_{nk}), n, k = 0, 1, 2, \ldots$ and a sequence $\{x_k\}, k = 0, 1, 2, \ldots$, by the $A$-transform of $\{x_k\}$, we mean the sequence $\{(Ax)_n\}$ where

$$ (Ax)_n = \sum_{k=0}^{\infty} a_{nk}x_k, \quad n = 0, 1, 2, \ldots, $$

it being assumed that the series on the right converge. If $\lim_{n \to \infty} (Ax)_n = s$, we say that $\{x_k\}$ is $A$-summable (or summable by the infinite matrix method $A$) to $s$. If $\lim_{n \to \infty} (Ax)_n = s$ whenever $\lim_{k \to \infty} x_k = s$, the matrix method $A$ is said to be regular. It is well-known (see [3], [4]) that $A$ is regular if and only if

$$ \begin{cases} \sup_{n,k} |a_{nk}| < \infty \quad ; \\
\lim_{n \to \infty} a_{nk} = 0, \quad k = 0, 1, 2, \ldots \quad ; \\
\lim_{n \to \infty} \left( \sum_{k=0}^{\infty} a_{nk} \right) = 1 \quad . \end{cases} \quad (5) $$

(cf. For criterion for the regularity of a matrix method in the classical case see [2], p.43, Theorem 2). If a regular matrix $A$ is such that $\lim_{n \to \infty} (Ax)_n = s$ implies $\lim_{k \to \infty} x_k = s$, the matrix method $A$ is said to be trivial. Given two infinite matrix methods $A, B$, we say that $A$ is included in $B$, written as $A \subset B$, if any sequence $\{x_k\}$ that is $A$-summable to $s$ is also $B$-summable to $s$. An infinite matrix $A = (a_{nk})$ is said to be triangular (or, more precisely, lower triangular) if $a_{nk} = 0, \quad k > n, \quad n = 0, 1, 2, \ldots$. 
**Definition 1.** The \((\bar{N}, p_n)\) method is defined by the infinite matrix \((a_{nk})\) where

\[
a_{nk} = \begin{cases} 
\frac{p_k}{p_n}, & k \leq n \\
0, & k > n
\end{cases}
\]  

(6)

**Remark 1.** If \(\left|\frac{p_{n+1}}{p_n}\right| > 1\), \(n = 0, 1, 2, \ldots\) and \(\lim_{n \to \infty} |p_n| = \infty\) i.e. \(|p_n|\) strictly increases to infinity, then the method \((\bar{N}, p_n)\) is trivial. For \(|p_n| = |P_n - P_{n-1}| = |P_n|\), since \(|P_n| > |P_{n-1}|\). So (1) is satisfied. Since \(\lim_{n \to \infty} |P_n| = \infty\), \(\lim_{n \to \infty} |p_n| = \infty\) so that (2) is satisfied too. Hence \((\bar{N}, p_n)\) is trivial because of Theorem 4.2 of [6].

In the sequel we shall suppose that the sequence \(\{p_n\}\) of weights satisfies conditions (3) and (4).

An example of such an \((\bar{N}, p_n)\) method corresponds to \(\{p_n\}\) defined by

\[
p_n = p^n, \quad \text{if } n \text{ is odd};
\]

\[
= \frac{1}{p^n}, \quad \text{if } n \text{ is even},
\]

where \(K = \mathbb{Q}_p\).

**Remark 2.** We note that (4) is equivalent to

\[
|p_{n+1}| \geq |P_n|, \quad n = 0, 1, 2, \ldots.
\]

(7)

**Proof.** Let (4) hold. Now

\[
|P_{n+1}| = \max_{0 \leq i \leq n+1} |p_i|
\]

\[
= \max \left[ \max_{0 \leq i \leq n} |p_i|, \quad |p_{n+1}| \right]
\]

\[
= \max \left[ |P_n|, \quad |p_{n+1}| \right]
\]

\[
\geq |P_n|, \quad n = 0, 1, 2, \ldots
\]

Conversely, let (7) hold. For a fixed integer \(j \geq 0\) let \(0 \leq i \leq j\). Then

\[
|p_i| = |P_i - P_{i-1}|
\]

\[
\leq \max \left[ |P_i|, \quad |P_{i-1}| \right]
\]

\[
\leq |P_i|
\]

\[
\leq |P_j|
\]

by (7).
§3. MAIN RESULTS.

Theorem 1. \((\overline{N}, p_n)\) is regular if and only if

\[
\lim_{n \to \infty} |P_n| = \infty
\]  

Proof. Let the \((\overline{N}, p_n)\) method be regular. Using (6) and (5)(b), we note that \((8)\) holds. Consequently, let \((8)\) hold. In view of (6) and (8) it follows that \(\lim_{n \to \infty} a_{nk} = 0, \ k = 0, 1, 2, \ldots\).

Now, \(|a_{nk}| = 0, \ k > n\). If \(k \leq n\), \(|a_{nk}| = \frac{|p_k|}{|P_n|} \leq 1\), in view of (4).

Also \(\sum_{k=0}^{\infty} a_{nk} = 1, \ n = 0, 1, 2, \ldots\) so that \(\lim_{n \to \infty} \left( \sum_{k=0}^{\infty} a_{nk} \right) = 1\). Thus, by (5) the method \((\overline{N}, p_n)\) is regular.

Remark 3. If \((\overline{N}, p_n)\) is non-trivial, then \((1)\) cannot be satisfied. Suppose \((1)\) holds, then \(|p_n| = |P_n|\) so that \((2)\) also holds. Thus \((\overline{N}, p_n)\) is trivial by Theorem 4.2 of [6], a contradiction. This establishes the claim.

Remark 4. There are non-trivial \((\overline{N}, p_n)\) methods. Let \(\alpha \in K\) such that \(0 < c = |\alpha| < 1\), this being possible since \(K\) is non-trivially valued. Let

\[
\{p_n\} = \left\{ \alpha, \frac{1}{\alpha^2}, \frac{1}{\alpha^3}, \ldots \right\}
\]

and

\[
\{s_n\} = \left\{ \frac{1}{\alpha}, \frac{1}{\alpha^2}, \frac{1}{\alpha^3}, \alpha, \ldots \right\}
\]

It is clear that \(\{s_n\}\) does not converge. If \(\{t_n\}\) is the \((\overline{N}, p_n)\) transform of \(\{s_k\}\),

\[
|t_{2k}| = \left| \frac{2k}{\alpha + \frac{1}{\alpha^2} + \frac{1}{\alpha^3} + \ldots + \frac{1}{\alpha^{2k}}} \right|
\]

\[
= \frac{|2k|}{\left( \frac{1}{c^{2k}} \right)}
\]

\[
|t_{2k+1}| = \left| \frac{2k + 1}{\alpha + \frac{1}{\alpha^2} + \frac{1}{\alpha^3} + \ldots + \frac{1}{\alpha^{2k}} + \frac{1}{\alpha^{2k+1}}} \right|
\]

\[
= \frac{|2k + 1|}{\left( \frac{1}{c^{2k}} \right)}
\]

\[
\leq c^{2k}
\]
so that \( \lim_{n \to \infty} t_n = 0 \). Thus \( \{s_n\} \), though non convergent, is summable \((\overline{N}, p_n)\) (in fact, to 0). This establishes our claim.

**Theorem 2.** (Limitation theorem) If \( \{s_n\} \) is summable \((\overline{N}, p_n)\) to \( s \), then

\[
|s_n - s| = o\left( \frac{|P_n|}{p_n} \right), \quad n \to \infty.
\]

**Proof.** If \( \{t_n\} \) is the \((\overline{N}, p_n)\) transform of \( \{s_k\} \), then

\[
\left| \frac{p_n(s_n - s)}{p_n} \right| = \left| \frac{p_n s_n - p_n s}{p_n} \right| = \left| \frac{p_n t_n - p_{n-1} t_{n-1} - s(P_n - P_{n-1})}{p_n} \right| = \left| \frac{p_n(t_n - s) - P_{n-1}(t_{n-1} - s)}{p_n} \right| \leq \max \left[ |t_n - s|, \frac{|P_{n-1}|}{p_n} |t_{n-1} - s| \right] \leq \max \left[ |t_n - s|, |t_{n-1} - s| \right]
\]

since \( \left| \frac{P_{n-1}}{p_n} \right| \leq 1 \), by (7). Since \( \lim_{n \to \infty} t_n = s \), it follows that \( \lim_{n \to \infty} \left| \frac{p_n(s_n - s)}{p_n} \right| = 0 \). Thus

\[
|s_n - s| = o\left( \frac{|P_n|}{p_n} \right), \quad n \to \infty.
\]

**Theorem 3.** (Comparison theorem for two regular weighted means). If \((\overline{N}, p_n), (\overline{N}, q_n)\) are two regular methods and if

\[
\left| \frac{p_n}{q_n} \right| \leq H \left| \frac{Q_n}{q_n} \right|, \quad n = 0, 1, 2, \ldots , (9)
\]

where \( H > 0 \) is a constant and \( Q_n = \sum_{k=0}^{\infty} q_k \), then \((\overline{N}, p_n) \subset (\overline{N}, q_n)\).

**Proof.** Let, for a given sequence \( \{s_n\} \),

\[
t_n = p_0 s_0 + p_1 s_1 + \ldots + p_n s_n / P_n, \quad u_n = q_0 s_0 + q_1 s_1 + \ldots + q_n s_n / Q_n, \quad n = 0, 1, 2, \ldots .
\]
Then \( p_0 s_0 = P_0 t_0, \ p_n s_n = P_n t_n - P_{n-1} t_{n-1}, \) \( n = 1, 2, \ldots \). Now,

\[
 u_n = \frac{1}{Q_n} \left[ \frac{q_0}{p_0} P_0 t_0 + \frac{q_1}{p_1} (P_1 t_1 - P_0 t_0) + \ldots + \frac{q_n}{p_n} (P_n t_n - P_{n-1} t_{n-1}) \right]
\]

\[
 = \sum_{k=0}^{\infty} c_{nk} t_k,
\]

where

\[
c_{nk} = \left( \frac{q_k}{p_k} - \frac{q_{k+1}}{p_{k+1}} \right) P_k Q_n, \quad k < n;
\]

\[
 = \frac{q_k}{p_k} P_k Q_k, \quad k = n;
\]

\[
 = 0, \quad k > n.
\]

Since \( \lim_{n \to \infty} |Q_n| = \infty \), \( \lim_{n \to \infty} c_{nk} = 0, \ k = 0, 1, 2, \ldots \) if \( s_n = 1, \ n = 0, 1, 2, \ldots \).

\( t_n = u_n = 1, \ n = 0, 1, 2, \ldots \) so that \( \sum_{k=0}^{\infty} c_{nk} = 1, \ n = 0, 1, 2, \ldots \) and so \( \lim_{n \to \infty} \left( \sum_{k=0}^{\infty} c_{nk} \right) = 1. \)

Let \( k < n. \)

\[
|c_{nk}| = \left| \frac{q_k}{p_k} - \frac{q_{k+1}}{p_{k+1}} \right| \left| \frac{P_k}{Q_n} \right|
\]

\[
\leq \max \left[ \left| \frac{q_k}{p_k} \right| \left| \frac{P_k}{Q_n} \right|, \left| \frac{q_{k+1}}{p_{k+1}} \right| \left| \frac{P_k}{Q_n} \right| \right]
\]

\[
\leq \max \left[ \left| \frac{q_k}{p_k} \right| \left| \frac{P_k}{Q_n} \right|, \left| \frac{q_{k+1}}{p_{k+1}} \right| \left| \frac{P_{k+1}}{Q_n} \right| \right]
\]

\[
\leq H,
\]

by (9), since \( k < n \) implies \( |Q_k|, |Q_{k+1}| \leq |Q_n| \) and so \( \frac{1}{Q_n} \leq \frac{1}{Q_k}, \ \frac{1}{Q_{k+1}} \) and \( |P_k| \leq |P_{k+1}| \).

If \( k = n, \ |c_{nn}| = \left| \frac{q_n}{p_n} \frac{P_n}{Q_n} \right| \leq H \) and \( |c_{nk}| = 0 \leq H, \ k > n. \) Consequently \( \sup_{n,k} |a_{nk}| \leq H. \)

The method \((c_{nk})\) is thus regular, using (5) and so \((N, p_n) \subset (\bar{N}, q_n)\). The proof of the theorem is now complete.

Remark 5. Note that the classical counterpart of Theorem 3 (see [2], p.58, Theorem 14) has an additional hypothesis.

Theorem 4. (Comparison theorem for a regular \((\bar{N}, p_n)\) method and a regular matrix). Let \((\bar{N}, p_n)\) be a regular method and \(A\) be a regular matrix. If

\[
\lim_{k \to \infty} \frac{a_{nk} P_k}{p_k} = 0, \quad n = 0, 1, 2, \ldots ; \quad (10)
\]

and

\[
\sup_{n,k} \left| \frac{\frac{a_{nk}}{P_k} - \frac{a_{n,k+1}}{p_{k+1}}}{p_k} \right| < \infty, \quad (11)
\]
then $(\overline{N}, p_n) \subset A$.

**Proof.** Let $\{s_n\}$ be any sequence, $\{t_n\}, \{\tau_n\}$ be its $(\overline{N}, p_n)$, $A$ transforms respectively so that

$$t_n = \frac{p_0s_0 + p_1s_1 + \ldots + p_ns_n}{p_n},$$

$$\tau_n = \sum_{k=0}^{\infty} a_{nk}s_k, \; n = 0, 1, 2, \ldots.$$ 

Now,

$$s_n = \frac{P_nt_n - P_{n-1}s_1t_{n-1}}{p_n}, \; P_{-1} = 0$$

Let $\lim_{n \to \infty} t_n = s$. $\tau_n = \sum_{k=0}^{\infty} a_{nk}s_k$ exists, $n = 0, 1, 2, \ldots$ and in fact

$$\tau_n = \sum_{k=0}^{\infty} a_{nk}s_k = \sum_{k=0}^{\infty} a_{nk}\left\{\frac{P_k t_k - P_{k-1} t_{k-1}}{p_k}\right\}$$

$$= \sum_{k=0}^{\infty} \left(\frac{a_{nk}}{p_k} - \frac{a_{n,k+1}}{p_{k+1}}\right)P_k t_k,$$

since $\lim_{k \to \infty} \frac{a_{n,k+1}}{p_{k+1}}P_k t_k = 0$ by (10) and using the fact that $\{t_k\}$ is convergent and so bounded and $\left|\frac{P_k}{P_{k+1}}\right| \leq 1$. We can now write

$$\tau_n = \sum_{k=0}^{\infty} b_{nk}t_k,$$

where

$$b_{nk} = \left(\frac{a_{nk}}{p_k} - \frac{a_{n,k+1}}{p_{k+1}}\right)P_k.$$ 

By (11), $\sup_{n,k} |b_{nk}| < \infty$. Since $A$ is regular, $\lim_{n \to \infty} a_{nk} = 0, \; k = 0, 1, 2, \ldots$ so that

$$\lim_{n \to \infty} b_{nk} = 0, \; k = 0, 1, 2, \ldots.$$ 

Let $s_n = 1, \; n = 0, 1, 2, \ldots$. Then $t_n = 1, \; n = 0, 1, 2, \ldots$.

It now follows that $\sum_{k=0}^{\infty} b_{nk} = \sum_{k=0}^{\infty} a_{nk}, \; n = 0, 1, 2, \ldots$. Consequently

$$\lim_{n \to \infty} \left(\sum_{k=0}^{\infty} b_{nk}\right) = \lim_{n \to \infty} \left(\sum_{k=0}^{\infty} a_{nk}\right) = 1.$$ 

The method $(b_{nk})$ is thus regular and so $\lim_{n \to \infty} t_n = s$ implies $\lim_{n \to \infty} \tau_n = s$, i.e. $(\overline{N}, p_n) \subset A$.

**Theorem 5.** $\overline{N}, p_n)$ is a regular method and $A = (a_{nk})$ is a regular triangular matrix. Then $(\overline{N}, p_n) \subset A$ if and only if (11) holds.
Proof. Let (11) hold. Since $A$ is a triangular matrix, (10) clearly holds. In view of Theorem 4, we have $(\mathcal{N}, p_n) \subset A$. Conversely, let $(\mathcal{N}, p_n) \subset A$. Following the notation of Theorem 4, let $\lim_{n \to \infty} t_n = s$. As in the proof of Theorem 4, let $\lim_{n \to \infty} a_{nk} s_k = \sum_{k=0}^{\infty} b_{nk} t_k$.

$$\tau_n = \sum_{k=0}^{\infty} a_{nk} s_k = \sum_{k=0}^{\infty} b_{nk} t_k,$$

where

$$b_{nk} = \left( \frac{a_{nk}}{p_k} - \frac{a_{n,k+1}}{p_{k+1}} \right) p_k.$$

Since $(\mathcal{N}, p_n) \subset A$, for every sequence $\{t_k\}$ with $\lim_{k \to \infty} t_k = s$, $\lim_{n \to \infty} \tau_n = s$. This means that $(b_{nk})$ is a regular matrix and so (11) holds. This complices the proof.

§ 4. A SCALE OF STRICTLY INCREASING WEIGHTED MEANS.

We conclude the present paper by obtaining a strictly increasing scale of regular summability methods in $\mathbb{Q}_p$. We define, for $k = 0, 1, 2, \ldots$, the method $(\mathcal{N}, p_n^{(k)})$ by

$$p_n^{(k)} = p_n^k, \quad \text{if } n \text{ is odd; }$$

$$p_n^{(k)} = \frac{1}{p_{nk}}, \quad \text{if } n \text{ is even}.$$

We now establish that

$$(\mathcal{N}, p_n^{(k)}) \subsetneq (\mathcal{N}, p_n^{(k+1)}). \quad (12)$$

We apply Theorem 3 to prove this assertion. For convenience, let $p_n = p_n^{(k)}$ and $q_n = p_n^{(k+1)}$, $n = 0, 1, 2, \ldots$. If $n$ is odd,

$$\left| \frac{P_n}{p_n} \right| = \frac{1}{c^{n-1)(k+1)}} \cdot \frac{1}{c_{nk}} = \frac{1}{c^{(2n-1)(k+1)}},$$

so that

$$\left| \frac{P_n}{p_n} \right| \leq \left| \frac{Q_n}{q_n} \right|.$$

If $n$ is even,

$$\left| \frac{P_n}{p_n} \right| = \frac{1}{c_{nk}} \cdot c_{nk} = 1$$

$$\left| \frac{Q_n}{q_n} \right| = \frac{1}{c_{n(k+1)}} \cdot c_{n(k+1)} = 1.$$

Thus

$$\left| \frac{P_n}{p_n} \right| \leq \left| \frac{Q_n}{q_n} \right|.$$
in this case too. Consequently, by Theorem 3, \((\overline{N}, p_n^{(k)}) \subset (\overline{N}, p_n^{(k+1)})\). Let now

\[
s_n = \begin{cases} 0, & \text{if } n \text{ is even;} \\ \frac{1}{p^{n(k+1)+k(n-1)}}, & \text{if } n \text{ is odd.} \end{cases}
\]

Let \(\{\tau_n\}\) be the \((\overline{N}, q_n)\) transform of \(\{s_n\}\).

If \(n\) is odd,

\[
|\tau_n| = \left| \frac{0 + p^{k+1} \cdot \frac{1}{p^{k+1}} + 0 + p^{3(k+1)} \cdot \frac{1}{p^{3(k+1)+2k}} + \ldots + 0 + p^{n(k+1)} \cdot \frac{1}{p^{n(k+1)+k(n-1)}}}{1 + p^{k+1} + \frac{1}{p^{2(k+1)}} + \ldots + \frac{1}{p^{(n-1)(k+1)}} + p^{n(k+1)}} \right|
\]

\[
= \frac{1}{c^{k(n-1)}} \frac{1}{c^{(k+1)(n-1)}}
\]

\[
= c^{n-1}
\]

If \(n\) is even,

\[
|\tau_n| = \left| \frac{0 + p^{k+1} \cdot \frac{1}{p^{k+1}} + 0 + p^{3(k+1)} \cdot \frac{1}{p^{3(k+1)+2k}} + \ldots + 0 + p^{(n-1)(k+1)} \cdot \frac{1}{p^{(n-1)(k+1)+k(n-2)}} + 0}{1 + p^{k+1} + \frac{1}{p^{2(k+1)}} + \ldots + p^{(n-2)-(k+1)} + \frac{1}{p^{n(k+1)}}} \right|
\]

\[
= \frac{1}{c^{k(n-2)}} \frac{1}{c^{n(k+1)}}
\]

\[
= c^{n+2k}
\]

In both the cases, \(\lim_{n \to \infty} \tau_n = 0\). Thus \(\{s_n\}\) is summable \((\overline{N}, q_n)\) to 0. Let, now, \(\{t_n\}\) be the \((\overline{N}, p_n)\) transform of \(\{s_n\}\).
If $n$ is odd

$$|\tau_n| = \left| 0 + p^k \cdot \frac{1}{p^{k+1}} + 0 + p^{3k} \cdot \frac{1}{p^{3(k+1)+2k}} + \ldots + 0 + p^{nk} \cdot \frac{1}{p^{n(k+1)+k(n-1)} + 1} \right|$$

$$= \frac{1}{c^{n+k(n-1)}}$$

$$= \frac{1}{c^{n-1}k}$$

Since $\frac{1}{c} > 1$, $\lim_{n \to \infty} |t_n| = \infty$ that $\{t_n\}$ cannot converge. Thus $\{s_n\}$ is not $(N, p_n)$ summable and consequently (12) holds.

REFERENCES