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The Mackey-Arens and Hahn-Banach theorems for spaces over valued fields


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The MacKee-Arens and Hahn-Banach Theorems

For Spaces Over Valued Fields

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Abstract. Characterizations of the spherical completeness of a non-archimedean complete non-trivially valued field in terms of classical theorems of Functional Analysis are obtained.

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Spherical completeness

Throughout this paper $K = (K, |.|)$ will denote a non-archimedean complete valued field with a non-trivial valuation $|.|$. It is well-known that the absolute value function $|.|$ of the field of the real numbers $\mathbb{R}$ or the complex numbers $\mathbb{C}$ satisfies the following properties:

(i) $0 \leq |x|, |x| = 0$ iff $x = 0$,
(ii) $|x + y| \leq |x| + |y|$, 
(iii) $|xy| = |x||y|$, $x, y \in \mathbb{R}$ or $x, y \in \mathbb{C}$.

If $K$ is a field, then by a valuation on $K$ we will mean a map $|.|$ of $K$ into $\mathbb{R}$ satisfying the above properties; in this case $(K, |.|)$ will be called a valued field. We will assume that $K$ is complete with respect to the natural metric of $K$.

It turns out that if $K$ is not isomorphic to $\mathbb{R}$ or $\mathbb{C}$, then its valuation satisfies the following strong triangle inequality, cf. e.g. [12],

(ii') $|x + y| \leq \max \{|x|, |y|\}$, $x, y \in K$.

A valued field $K$ whose valuation satisfies (ii') will be called non-archimedean and its valuation non-archimedean.

Let us first recall the following well-known result of Cantor

Theorem 0 Let $(X, \rho)$ be a metric space. Then it is complete iff every shrinking sequence of closed balls whose radii tend to zero has non-empty intersection.
Consider the set \( \mathbb{N} \) of the natural numbers endowed with the following metric \( \rho \) defined by 
\[
\rho(m, n) = 0 \text{ if } m = n \text{ and } 1 + \max\left(\frac{1}{m}, \frac{1}{n}\right) \text{ if } m \neq n.
\]

Then the metric \( \rho \) is non-archimedean, i.e. \( \rho(m, n) = 0 \) iff either \( m = n \), or 
\[
\rho(m, n) \leq \max\{\rho(m, k), \rho(k, n)\}, \text{ for all } m, n, k \in \mathbb{N}.
\]

It is easy to see that every shrinking sequence of balls in \( \mathbb{N} \) whose radii tend to zero has non-empty intersection; note that every ball whose radius is smaller than 1 contains exactly one point. On the other hand, the balls \( B_{1+\frac{1}{n}}(1), B_{1+\frac{1}{n}}(2), \ldots \), form a decreasing sequence and their intersection is empty. This suggests the following, see Ingleton \([3]\):

A non-archimedean metric space \((X, \rho)\) will be said to be \textit{spherically complete} if the intersection of every shrinking sequence of its balls is non-empty.

Clearly spherical completeness implies completeness; the converse fails: The space \((\mathbb{N}, \rho)\) is complete but not spherically complete. We refer to \([11]\) and \([12]\) for more information concerning this property.

**Theorem 1** Let \((X, \rho)\) be a non-archimedean metric space. Then \((X, \rho)\) is spherically complete iff given an arbitrary family \(B\) of balls in \(X\), no two of which are disjoint, then the intersection of the elements of \(B\) is non-empty.

The aim of this note is to collect a few characterizations of the spherical completeness of \(K\) in terms of the Mackey-Arens, Hahn-Banach and weak Schauder basis theorems, respectively, see \([5]\), \([6]\), \([7]\), \([12]\).

The Mackey-Arens and Hahn-Banach theorems

The terms "\(K\)-space", "topology", "seminorm or norm" will mean a Hausdorff locally convex space (lcs) over \(K\), a locally convex topology (in the sense of Monna) and a non-archimedean seminorm (norm), respectively. A seminorm on a vector space \(E\) over \(K\) is \textit{non-archimedean} if it satisfies condition \((ii')\). Clearly the topology \(\tau\) generated by a norm is \textit{locally convex}. Recall that a \textit{topological vector space} (tvs) \((E, \tau)\) over \(K\) is \textit{locally convex} \([10]\) if \(\tau\) has a basis of absolutely convex neighbourhoods of zero. A subset \(U\) of \(E\) is \textit{absolutely convex} (in the sense of Monna \([10]\)) if \(\alpha x + \beta y \in U\), whenever \(x, y \in U\), \(\alpha, \beta \in K\); \(|\alpha| \leq 1, |\beta| \leq 1\). For the basic notions and properties concerning tvs and lcs over \(K\) we refer to \([10]\), \([11]\), \([13]\).

A locally convex (lc) topology \(\gamma\) on \((E, \tau)\) is called \textit{compatible} with \(\tau\), if \(\tau\) and \(\gamma\) have the same continuous linear functionals; \((E, \tau)^* = (E, \gamma)^*\). \((E, \tau)\) is \textit{dual-separating} if \((E, \tau)^*\) separates points of \(E\). If \(G\) is a vector subspace of \(E\), \(\tau|G\) and \(\tau/G\) denote the topology \(\tau\) restricted to \(G\) and the quotient topology of the quotient space \(E/G\), respectively. If \(\alpha\) is a finer lc. topology on \(E/G\), we denote by \(\gamma := \tau \vee \alpha\) the weakest lc. topology on \(E\) such that \(\tau \leq \gamma\), \(\gamma/G = \alpha\), \(\gamma/G = \tau/G\), cf. e.g. \([1]\). The sets \(U \cap g^{-1}(V)\) compose a basis of neighbourhoods of zero for \(\gamma\), where \(U, V\) run over bases of neighbourhoods of zero for \(\tau\) and \(\alpha\), respectively, \(g := EE/G\) is the quotient map. By \(\text{sup}\{\tau, \alpha\}\) we denote the weakest lc. topology on \(E\) which is finer than \(\tau\) and \(\alpha\).
By the *Mackey topology* $\mu(E, E^*)$ associated with a lcs $E = (E, \tau)$ we mean the finest locally convex topology on $E$ compatible with $\tau$. In [14] Van Tiel showed that every lcs over spherically complete $K$ admits the Mackey topology.

In [3] Ingleton obtained a non-archimedean variant of the Hahn-Banach theorem for normed spaces, where $K$ is spherically complete.

**Theorem 2** If $E = (E, \| \cdot \|)$ is a normed space over $K$ and $K$ is spherically complete and $D$ is a subspace of $E$, then for every continuous linear functional $g \in D^*$ there exists a continuous linear extension $f \in E^*$ of $g$ such that $\|g\| = \|f\|$.

This suggests the following: A lcs $E$ will be said to have the *Hahn-Banach Extension Property* (HBEP) [9] if for every subspace $D$ every $g \in D^*$ can be extended to $f \in E^*$. It is known that every lcs over spherically complete $K$ has the HBEP, cf. e.g. [11].

The following theorem characterizes the spherical completeness of $K$ in terms of classical theorems of Functional Analysis; cf. also [5], [6] and [12], Theorem 4.15. The proof of our Theorem 3 uses some ideas of [4] extended to the non-archimedean case.

$I^\infty$ (resp. $c_0$) denotes the space of the bounded sequences (resp. the sequences of limit 0) with coefficients in $K$.

**Theorem 3** The following conditions on $K$ are equivalent:

(i) $K$ is spherically complete.

(ii) There exists $g \in (I^\infty)^*$ such that $g(x) = \sum_n x_n$ for every $x \in c_0$.

(iii) $(I^\infty/c_0)^* \neq 0$.

(iv) Every lcs over $K$ admits the Mackey topology.

(v) Every lcs over $K$ (resp. $K$-normed space) has the HBEP.

(vi) The completion of a dual-separating lcs over $K$ (resp. $K$-normed space) is dual-separating.

(vii) Every closed subspace of a dual-separating lcs over $K$ (resp. $K$-normed space) is weakly closed.

(viii) For every lcs over $K$ (resp. $K$-normed space) every weakly convergent sequence is convergent.

(ix) Every weak Schauder basis in a lcs over $K$ (resp. $K$-normed space) is a Schauder basis.

**Proof** By Theorem 4.15 of [12] conditions (i), (ii), (iii) are equivalent. (i) implies (iv) : [14], Theorem 4.17. (i) implies (v) : [3], [11]. The implications (v) implies (vi), (v) implies (vii) are obvious. (i) implies (viii) : see [7], Theorem 3, [2], Proposition 4.3. (viii) implies (ix) is obvious.

(iv) implies (i) : Assume that $K$ is not spherically complete and consider the space $I^\infty$ of $K$-valued bounded sequences endowed with the topology $\tau$ generated by the norm $\|x\| = \sup_n |x_n|$, $x = (x_n) \in I^\infty$. Let $f$ be a non-zero linear function on $I^\infty$ with $f|_{c_0} = 0$. Set $E := I^\infty$ and $F := c_0$. Define a linear functional $h$ on the quotient space $E/F$ by $h(q(x)) = f(x)$, where $q : E \to E/F$ is the quotient map. Let $\alpha$ be the quotient topology.
of $E/F$. Since $(E/F, \alpha)^* = 0$, see (iii) implies (i), $F$ is dense in the weak topology $\sigma(E, E^*)$ (recall that $E^* = F$, [12], Theorem 4.17). Observe that on $E/F$ there exists a $K$-normed topology $\beta$ such that $(E/F, \alpha)$ and $(E/F, \beta)$ are isomorphic and $h$ is continuous in the topology $\sup\{\alpha, \beta\}$. Indeed, choose $x_0 \in E/F$ such that $h(x_0) = 2$ and define a linear map $T : E/F \to E/F$ by $T(x) := x - h(x)x_0$, $x \in E/F$. Then $T^2 = \text{id}$. Define $\beta := T(\alpha)$ (the image topology). Then $h$ is continuous in the topology $\sup\{\alpha, \beta\}$.

Set $\gamma_\alpha := \sigma(E, E^*) \vee \alpha$, $\gamma_\beta := \sigma(E, E^*) \vee \beta$. Then $\gamma_\alpha$ and $\gamma_\beta$ are compatible with $\sigma(E, E^*)$, hence with $\tau$. Assume that $E$ admits the finest locally convex topology $\mu$ compatible with $\tau$. Then $\sigma(E, E^*) \leq \sup\{\gamma_\alpha, \gamma_\beta\} \leq \mu$.

On the other hand $\sup\{\gamma_\alpha, \gamma_\beta\} / F = \sup\{\alpha, \beta\}$. Therefore $f$ is continuous in $\sup\{\gamma_\alpha, \gamma_\beta\}$. Since $f$ is not continuous in $\sigma(E, E^*)$ we get a contradiction. The proof is complete.

(vi) implies (i) : Assume that $K$ is not spherically complete. By the Baire category theorem we find a dense subspace $G$ of $E$ with dim$(E/G) = \dim(E/F)$, where $E$ and $F$ are defined as above. Indeed, let $\{x_s\}_{s \in S}$ be a Hamel basis of $E$ and $(S_n)$ a partition of $S$ such that $S = \bigcup_{n \in \mathbb{N}} S_n$ and card $S_n = \text{card } S$, $n \in \mathbb{N}$.

For every $n \in \mathbb{N}$ we denote by $G_n$ the vector space generated by the elements $x_s$ when $s$ runs in $\bigcup_{k=1}^n S_k$. Then we have $E = \bigcup_{n \in \mathbb{N}} G_n$ and $\dim G_n = \dim(E/G_n) = \dim E$, $n \in \mathbb{N}$. Then there exists $m \in \mathbb{N}$ such that $G_m$ is dense in $E$. Hence we obtain a subspace $G$ as required. Let $\alpha$ be a $K$-normed topology on $E/G$ such that the spaces $(E/G, \alpha)$ and $(E/F, \tau/F)$ are isomorphic. Then the topology $\gamma := \tau \vee \alpha$ is compatible with $\tau$ and strictly finer than $\tau$. Let $E_0$ be the completion of the dual-separating $K$-normed space $(E, \gamma)$. Choose $x \in E_0 \setminus E$. There exists a sequence $(x_n)$ in $E$ and $y \in E$ such that $x_n \to x$ in $E_0$ and $x_n \to y$ in $(E, \tau)$. Then $f(x - y) = 0$ for all $f \in E_0^*$ but $x - y \neq 0$. This completes the proof.

(vii) implies (i) : Assume that $K$ is not spherically complete. The space $G$ constructed in the previous case is closed in $(E, \gamma)$ and dense in $(E, \sigma(E, E^*))$, where $E^* := (E, \gamma)^*$. (v) implies (i) : Assume that $K$ is not spherically complete. Let $(e_n)$ be the sequence of the unit vectors in $E$, where $E$ is as above. Then $e_n \to 0$ in $\sigma(E, E^*)$, [13]. Clearly $(e_n)$ is a normalized Schauder basis in $F$. If $x = (x_n) \in F$, then $x = \sum_n x_ne_n$. Set $g(x) := \sum_n x_n$. Then $g$ is a well-defined continuous linear functional on $F$. Suppose that $g$ has a continuous linear extension $f$ to the whole space $E$. Then $f(e_n) \to 0$ but $g(e_n) = 1$ for all $n \in \mathbb{N}$, a contradiction.

(viii) implies (i) : See the proof of the previous implication.

(ix) implies (i) : Assume that $K$ is not spherically complete. The sequence $(e_n)$ is a Schauder basis in $(E, \sigma(E, E^*))$ but it is not a Schauder basis in the original topology of $E$. The second part of this sentence follows from the fact that $E$ is not of countable type, cf. e.g. [12]. On the other hand, by Theorem 4.17 of [12] (and its proof) the space $E$ is reflexive and for every $g \in E^*$ there exists $(a_n) \in F$ such that $g(x) = \sum_n x_na_n$ for every
\( x = (x_n) \in E. \) Since \((E, \sigma(E, E^*))\) is a sequentially complete lcs [12], Theorem 9.6, then \( \sum_{k=1}^{n} x_k e_k \) weakly converges to \( x = (x_n) \).

**Remark** In [9] Martinez-Maurica and Perez-Garcia proved that whenever \( K \) is spherically complete, then the local convexity is a *three space property*, i.e. if \( E \) is an A-Banach tvs over \( K \) and \( F \) its subspace such that \( F \) and \( E/F \) are locally convex, then \( E \) is locally convex. Is the converse also true?

By \( L(E, F) \) we denote the space of all continuous linear maps between lcs \( E \) and \( F \). A topology \( \alpha \) on \( E \) will be called *compatible* with the pair \((E, L(E, F))\) if \( L((E, \alpha), F) = L(E, F) \); if \( F = \), as usual we shall say that \( \alpha \) is compatible with the dual pair \((E, E^*)\), where \( E^* := L(E, K) \).

A lcs space \( F \) will be said to have the *Mackey-Arens property* (MA-property) if for every lcs space \( E \) the finest topology \( \mu(E, L(E, F)) \) compatible with \((E, L(E, F))\) exists, [7].

As we have already mentioned Van Tiel [14] proved that if \( K \) is spherically complete, then \( K \) has the MA-property, i.e. every \( K \)-space \( E \) over spherically complete \( K \) admits the finest topology \( \mu(E, E^*) \) compatible with the dual pair \((E, E^*)\). We have already proved the converse: If \( K \) is not spherically complete, then \( \ell^\infty \) does not admit the Mackey topology \( \mu(\ell^\infty, (\ell^\infty)^*) \). Hence

**Corollary** \( K \) is spherically complete iff it has the MA-property.

On the other hand one has the following

**Theorem 4** Every spherically complete normed \( K \)-space \( F = (F, \| \cdot \|) \) has the MA-property.

We shall need the following

**Lemma 1** Let \( E, F \) be two vector spaces over \( K \), where \( F \) is endowed with a norm \(| | \) and \( p, q \) are seminorms on \( E \). Let \( T : E \to F \) be a linear map such that \(|T(x)|| \leq \max(p(x), q(x)) \). If \( F \) is spherically complete, then there exists two linear maps \( T_i : E \to F \), \( i = 1, 2 \), such that \( T = T_1 + T_2 \) and \(|T(x)||| \leq p(x), \|T(x)||| \leq q(x), x \in E \).

**Proof** Set \( P(x, x) = T(x), U(x, y) = \max\{p(x), q(y)\} \), \( x, y \in E \). Then \( U(x, y) \) is a seminorm on \( E \times E \) and \(|P(x, x)|| = |T(x)|| \leq \max\{p(x), q(x)\} = U(x, x) \). Since \( F \) is spherically complete, then by Ingleton theorem, cf. e.g. [6], Theorem 4.18, there exists a linear map \( P_0 : E \times E \to F \) extending \( P \) such that \(|(P_0(x, y))|| \leq U(x, y), x, y \in E \). To complete the proof it is enough to put \( T_1(x) = P_0(x, 0), T_2(x) = P_0(0, x) \).

We shall also need the following lemma. Its proof uses some ideas of [1] and [4].

**Lemma 2** Let \( E, F \) be two dual-separating \( K \)-spaces over non-spherically complete \( K \) and such that \( F \) is complete and \( E \) is an infinite dimensional metrizable and complete. Then \( E \) admits two topologies \( \tau_1 \) and \( \tau_2 \) strictly finer than the original one of \( E \) and compatible with the pair \((E, L(E, F))\) and such that the topology \( \sup\{\tau_1, \tau_2\} \) is not compatible with \((E, L(E, F))\).
Proof: Observe that $E$ contains a dense subspace $G$ with $\dim(E/G) = \dim(l^\infty/c_0)$. Let $h$ be a non-zero linear functional on $E$ vanishing on $G$. As above we construct on $E$ two topologies $\tau_1$ and $\tau_2$ strictly finer than the original one $\tau$ of $E$ such that $\tau_j|G = \tau|G$ and $(E/G, \tau_j/G)$ is isomorphic to the quotient space $l^\infty/c_0$, $j = 1, 2$, and $h$ is continuous in $\sup\{\tau_1, \tau_2\}$. We show that the topologies $\tau_j$, $j = 1, 2$, are compatible with the pair $(E, L(E,F))$. Fix $j \in \{1, 2\}$ and non-zero $T \in L((E, \tau_j), F)$. There exists $x_0 \in E$ and $f \in F^*$ such that $f(T(x_0)) \neq 0$. Suppose that $T|G = \{0\}$. Then the map $q(x) \to f(Tx)$ defines a non-zero continuous linear functional on $(E/G, \tau_j/G)$, $q : E \to E/G$ is the quotient map. Since $(l^\infty/c_0)^* = \{0\}$, [12], Corollary 4.3, we get a contradiction. Hence $T|G$ is non-zero. Since $G$ is dense in $E$ and $\tau$ and $\tau_j$ coincide on $G$, there exists a continuous linear extension $W$ of $T$ to $E$. It is easy to see that $T = W$. Hence $T \in L(E,F)$. Finally the map $x \to h(x)y$, for fixed $y \in F$, defines a $\tau$-discontinuous linear map $H$ of $E$ into $F$ such that $H \in L((E, \sup \tau_1, \tau_2), F)$.

Proof of Theorem 4 Let $E = (E, \tau)$ be a lcs and $\mathcal{F}$ the family of all topologies on $E$ compatible with $(E, L(E,F))$. It is enough to show that the topology $\mu := \sup \mathcal{F}$ belongs to $\mathcal{F}$. Let $T : (E, \mu) \to F$ be a continuous linear map. There exist seminorms $p_j$ on $E$, $j = 1, \ldots, n$, continuous in topologies $\gamma_j$ ($\gamma_j \in \mathcal{F}$), respectively, and $M > 0$ such that $\|T(x)\| \leq M \max_{1 \leq j \leq n} p_j(x)$ for every $x \in E$. Using Lemma 1 one shows that $T$ is $\tau$-continuous.

Remarks (1) There exist complete normed $K$-spaces having the MA-property which are not spherically complete. In fact, assume that $K$ is spherically complete; then $l^\infty$ is spherically complete [12], p. 97; hence $l^\infty$ has the MA-property (by our Theorem 4). On the other hand there exists on the space $l^\infty$ another norm $\nu$ which is equivalent with the usual norm, such that $(l^\infty, \nu)$ is not spherically complete [12], p. 50 and p. 98. On the other hand the space $(l^\infty, \nu)$ has the MA-property.

(2) Let $E$ be an infinite dimensional normed and complete $K$-space. Since $F := \prod_n E_n / \bigoplus_n E_n$, where $E_n = E$ for every $n \in \mathbb{N}$, is spherically complete for any $K$ [12], Theorem 4.1, then by our Theorem 4 the space $F$ has the MA-property. For concrete spaces put $E = l^\infty$; then $F = l^\infty/c_0$. If $K$ is not spherically complete, then by Lemma 2 the space $l^\infty$ does not admit the Mackey topology $\mu(l^\infty, l^\infty)^*$ but $l^\infty/c_0$ has the MA-property. In particular there exists on $l^\infty$ the finest topology $\mu$ compatible with $(l^\infty, L(l^\infty, l^\infty/c_0))$.

(3) Let $E$ and $F$ be $K$-spaces and assume that $E$ admits the Mackey topology $\mu = \mu(E, E^*)$. Then the finest topology on $E$ compatible with $((E, \mu), L((E, \mu), F))$ exists and equals $\mu$.

(4) In [13], Corollary 7.9, Schikhof proved that for polarly barrelled or polarly bornological $K$-spaces $(E, \tau)$ where $K$ is not spherically complete, the finest polar topology $\mu(E, E^*)$ compatible with $(E, E^*)$ exists and equals $\tau$. 

Proof: Observe that $E$ contains a dense subspace $G$ with $\dim(E/G) = \dim(l^\infty/c_0)$. Let $h$ be a non-zero linear functional on $E$ vanishing on $G$. As above we construct on $E$ two topologies $\tau_1$ and $\tau_2$ strictly finer than the original one $\tau$ of $E$ such that $\tau_j|G = \tau|G$ and $(E/G, \tau_j/G)$ is isomorphic to the quotient space $l^\infty/c_0$, $j = 1, 2$, and $h$ is continuous in $\sup\{\tau_1, \tau_2\}$. We show that the topologies $\tau_j$, $j = 1, 2$, are compatible with the pair $(E, L(E,F))$. Fix $j \in \{1, 2\}$ and non-zero $T \in L((E, \tau_j), F)$. There exists $x_0 \in E$ and $f \in F^*$ such that $f(T(x_0)) \neq 0$. Suppose that $T|G = \{0\}$. Then the map $q(x) \to f(Tx)$ defines a non-zero continuous linear functional on $(E/G, \tau_j/G)$, $q : E \to E/G$ is the quotient map. Since $(l^\infty/c_0)^* = \{0\}$, [12], Corollary 4.3, we get a contradiction. Hence $T|G$ is non-zero. Since $G$ is dense in $E$ and $\tau$ and $\tau_j$ coincide on $G$, there exists a continuous linear extension $W$ of $T$ to $E$. It is easy to see that $T = W$. Hence $T \in L(E,F)$. Finally the map $x \to h(x)y$, for fixed $y \in F$, defines a $\tau$-discontinuous linear map $H$ of $E$ into $F$ such that $H \in L((E, \sup \tau_1, \tau_2), F)$.
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