

LUCIEN VAN HAMME

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THE P-ADIC Z-TRANSFORM

Lucien van Hamme

Abstract. Let $a + p^n \mathbb{Z}_p$ be a ball in $\mathbb{Z}_p$ and assume that a is the smallest natural number contained in the ball. We define a measure μ_z on $\mathbb{Z}_p$ by putting $\mu_z(a + p^n \mathbb{Z}_p) = \frac{z^a}{1-z^{p^n}}$ where $z \in \mathbb{C}_p, |z-1|_p \geq 1$. Let f be a continuous function defined on $\mathbb{Z}_p$. The mapping $f \rightarrow \int_{\mathbb{Z}_p} f(x) \mu_z(x)$ is similar to the classical Z-transform. We use this transform to give new proofs of several known results : the Mahler expansion with remainder for a continuous function, the Van der Put expansion, the expansion of a function in a series of Sheffer polynomials. We also prove some new results.

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1. Introduction

Let $\mathbb{Z}_p$ be the ring of p-adic integers, where p is a prime.

$\mathbb{Q}_p$ and $\mathbb{C}_p$ denote, as usual, the field of the p-adic numbers and the completion of the algebraic closure of $\mathbb{Q}_p$. $|\cdot|$ denotes the normalized p-adic valuation on $\mathbb{C}_p$.

We start by defining a measure on $\mathbb{Z}_p$.

Let $a + p^n \mathbb{Z}_p$ be a ball in $\mathbb{Z}_p$. We may assume that a is the smallest natural number contained in the ball. Our measure will depend on a parameter $z \in \mathbb{C}_p$.

Put $\mu_z(a + p^n \mathbb{Z}_p) = \frac{z^a}{1-z^{p^n}}$.

It is well-known that this defines a distribution on $\mathbb{Z}_p$.

Let D denote the set $\{z \in \mathbb{C}_p \mid |z-1|_p \geq 1\}$.

An easy calculation shows that if $z \in D$ then $\left| \frac{z^a}{1-z^{p^n}} \right| \leq 1$.

Throughout this paper we will assume that $z \in D$. Hence μ_z is a measure.

Now let $f : \mathbb{Z}_p \rightarrow \mathbb{C}_p$ be a continuous function.

If we associate with f the integral $F(z) = \int_{\mathbf{Z}_p} f(x) \mu_z(x)$ we get a transformation that we call the p -adic Z -transform since it is similar to the classical Z -transform used by engineers. The aim of this paper is to show how this transform can be used to obtain a number of results in p -adic analysis. In section 2 we start by studying the integral $F(z)$. In sections 3 and 4 we use the p -adic Z -transform to give new proofs of several known results : the Mahler expansion with remainder for a continuous function, the Van der Put expansion, the expansion of a function in a series of Sheffer polynomials. In section 5 we use the results of section 2 to find approximations to the p -adic logarithm of 2. We prove e.g. that the following congruence is valid in $\mathbf{Z}_p$

$$2 \left(1 - \frac{1}{p} \right) \lg 2 \equiv \sum_{\substack{k=1 \\ (k,p)=1}}^{p^n} \frac{(-1)^{k+1}}{k} \equiv 4(-1)^{n \cdot \frac{p-1}{2}} \sum_{\substack{k=0 \\ (2k+1,p)=1}}^{\frac{p^n-3}{2}} \frac{(-1)^k}{2k+1} \pmod{p^{2n} \mathbf{Z}_p}$$

2. The integral $\int_{\mathbf{Z}_p} f(x) \mu_z(x)$

This integral has already been studied and used by Y. Amice and others in [1] and [4].

A fundamental property of this integral is

Proposition : $F(z)$ is an analytic element in D (in the sense of Krasner).

This means that $F(z)$ is the uniform limit of a sequence of rational functions with poles outside D . But, by definition

$$F(z) = \int_{\mathbf{Z}_p} f(x) \mu_z(x) = \lim_{n \rightarrow \infty} \frac{\sum_{k=0}^{p^n-1} f(k) z^k}{1 - z^{p^n}} \quad (1)$$

It is not difficult to show that the sequence in (1) is uniformly convergent. Since the zeroes of $1 - z^{p^n}$ are outside D , $F(z)$ is an analytic element in D .

Corollary : F satisfies the "principle of analytic continuation" i.e. if $F(z)$ is zero on a ball in D it is zero in the whole of D .

The fact that $F(z)$ is an analytic element in D is very useful in proving properties of the integral (1). As an example we prove that

$$\int_{\mathbf{Z}_p} f(x) \mu_z(x) = f(0) + z \int_{\mathbf{Z}_p} f(x+1) \mu_z(x) \quad \text{in } D \quad (2)$$

Proof : For $|z| < 1$ formula (1) reduces to

$$\int_{\mathbf{Z}_p} f(x) \mu_z(x) = \sum_{k=0}^{\infty} f(k) z^k \quad (3)$$

The trivial identity

$$\sum_{k=0}^{\infty} f(k)z^k = f(0) + z \sum_{k=0}^{\infty} f(k+1)z^k \quad (|z| < 1)$$

can be written as

$$\int_{\mathbf{Z}_p} f(x)\mu_z(x) = f(0) + z \int_{\mathbf{Z}_p} f(x+1)\mu_z(x)$$

This is a priori valid for $|z| < 1$. By analytic continuation it is valid in D .

We now list some properties of the integral $\int_{\mathbf{Z}_p} f(x)\mu_z(x)$. We only give a few indications about the proofs.

$$\mathbf{P1} \quad \int_{\mathbf{Z}_p} f(x)\mu_z(x) = \sum_{k=0}^{n-1} f(k)z^k + z^n \int_{\mathbf{Z}_p} f(x+n)\mu_z(x) \quad \text{in } D \quad (4)$$

Proof : This follows by iterating (2)

$$\begin{aligned} \mathbf{P2} \quad \int_{\mathbf{Z}_p} f(x)\mu_z(x) &= - \sum_{k=1}^n \frac{f(-k)}{z^k} + \frac{1}{z^n} \int_{\mathbf{Z}_p} f(x-n)\mu_z(x) \quad \text{in } D \\ &= - \sum_{k=1}^{\infty} \frac{f(-k)}{z^k} \quad \text{if } |z| > 1 \end{aligned} \quad (5)$$

Proof : Replace $f(x)$ by $f(x-1)$ in (2) to get

$$\int_{\mathbf{Z}_p} f(x)\mu_z(x) = -\frac{f(-1)}{z} + \frac{1}{z} \int_{\mathbf{Z}_p} f(x-1)\mu_z(x)$$

Iteration of this formula yields (5).

$$\begin{aligned} \mathbf{P3} \quad \int_{\mathbf{Z}_p} f(x)\mu_z(x) &= \sum_{k=0}^{n-1} (\Delta^k f)(0) \frac{z^k}{(1-z)^{k+1}} + \frac{z^n}{(1-z)^n} \int_{\mathbf{Z}_p} (\Delta^n f)(x)\mu_z(x) \\ &= \sum_{k=0}^{\infty} (\Delta^k f)(0) \frac{z^k}{(1-z)^{k+1}} \quad \text{in } D \end{aligned} \quad (6)$$

Here Δ is the difference operator defined by $(\Delta f)(x) = f(x+1) - f(x)$.

Proof : Write (2) in the form

$$\int_{\mathbf{Z}_p} f(x) \mu_z(x) = \frac{f(0)}{1-z} + \frac{z}{1-z} \int_{\mathbf{Z}_p} (\Delta f)(x) \mu_z(x)$$

then iterate.

Let E be the translation operator defined by $(Ef)(x) = f(x+1)$ and put $Q = \Delta E^{-1}$ then

$$\mathbf{P4} \quad \int_{\mathbf{Z}_p} f(x) \mu_z(x) = \sum_{k=0}^{n-1} \frac{(Q^k f)(-1)}{(1-z)^{k+1}} + \frac{1}{(1-z)^n} \int_{\mathbf{Z}_p} (Q^n f)(x) \mu_z(x) \quad (7)$$

Proof : This follows from the obvious

$$\int_{\mathbf{Z}_p} f(x) \mu_z(x) = \frac{f(-1)}{1-z} + \frac{1}{1-z} \int_{\mathbf{Z}_p} (Qf)(x) \mu_z(x)$$

$$\mathbf{P5} \quad \int_{\mathbf{Z}_p} f(x) \mu_z(x) + \int_{\mathbf{Z}_p} f(-x) \mu_{1/z}(x) = f(0) \quad \text{in } D \quad (8)$$

Proof : Suppose first that $|z| > 1$ and use (5) for the first integral and (3) for the second integral. The formula then reduces to the obvious identity.

$$-\sum_{k=1}^{\infty} \frac{f(-k)}{z^k} + \sum_{k=0}^{\infty} \frac{f(-k)}{z^k} = f(0)$$

The formula is valid in D by analytic continuation.

$$\mathbf{P6} \quad \text{If } f \text{ is an even function then } \int_{\mathbf{Z}_p} f(x) \mu_{-1}(x) = \frac{f(0)}{2} \quad (9)$$

Proof : Put $z = -1$ in (8).

$$\begin{aligned} \mathbf{P7} \quad & \text{If } F(z) = \int_{\mathbf{Z}_p} f(x) \mu_z(x), G(z) = \int_{\mathbf{Z}_p} g(x) \mu_z(x) \\ & \text{then } F(z)G(z) = \int_{\mathbf{Z}_p} (f * g)(x) \mu_z(x) \quad \text{in } D \end{aligned} \quad (10)$$

where $f * g$ the convolution of f and g .

$f * g$ is by definition the continuous function with value equal to $(f * g)(n) = \sum_{k=0}^n f(k)g(n-k)$

if n is a natural number.

Proof : For $|z| \leq 1$ the equality $F(z)G(z) = \int_{\mathbf{Z}_p} (f * g)(x) \mu_z(x)$ is simply

$$\left(\sum_{k=0}^{\infty} f(k) z^k \right) \left(\sum_{k=0}^{\infty} g(k) z^k \right) = \sum_{k=0}^{\infty} (f * g)(k) z^k$$

which is obvious. The formula is valid in D by analytic continuation.

$$\text{P8} \quad \left| \int_{\mathbf{Z}_p} f(x) \mu_z(x) \right| \leq \|f\| \quad (11)$$

where $\|f\|$ denotes the sup-norm.

Remark : It follows from (5) that $\lim_{z \rightarrow \infty} zF(z)G(z) = -(f * g)(-1)$.

But $\lim_{z \rightarrow \infty} zF(z)G(z) = -f(-1) \lim_{z \rightarrow \infty} G(z) = 0$.

Hence we deduce the (known) fact that $(f * g)(-1) = 0$, i.e. the convolution of the two continuous functions is 0 at the point -1 .

3. The p-adic Z-transform

Let $C(\mathbf{Z}_p)$ denote the Banach space of the all continuous functions from $\mathbf{Z}_p$ to $\mathbb{C}_p$, equipped with the sup-norm.

Let (a_n) be a sequence in $\mathbb{C}_p$. A series of the form

$$\sum_{k=0}^{\infty} a_k \frac{z^k}{(1-z)^{k+1}} \quad \text{with} \quad \lim_{k \rightarrow \infty} a_k = 0 \quad (12)$$

is convergent in D .

Let B be the set of all functions $F : D \rightarrow \mathbb{C}_p$ that are the sum of a series of the form (12) with $\lim_{k \rightarrow \infty} a_k = 0$.

If we define $\|F\| = \sup_{z \in D} |F(z)|$ then B is a Banach space.

Formula (6) shows that $F(z) = \int_{\mathbf{Z}_p} f(x) \mu_z(x)$ belongs to B if $f \in C(\mathbf{Z}_p)$.

Hence it makes sense to consider the mapping

$$T : C(\mathbf{Z}_p) \rightarrow B : f \rightarrow F(z) = \int_{\mathbf{Z}_p} f(x) \mu_z(x)$$

We will call $F(z)$ the p-adic z-transform of f for the following reason. If $|z| < 1$ then

$F(z) = \sum_{k=0}^{\infty} f(k)z^k$. In applied mathematics it is customary to call the "generating function" $F(z)$ the z -transform of f .

We now examine the properties of the z -transform.
It is easily verified that T is linear and continuous.

If $F(z)$ is identical 0 then $\sum_{k=0}^{\infty} f(k)z^k = 0$ for $|z| < 1$. Hence $f(x) \equiv 0$.

This proves that T is injective.

T is also surjective. To see this we start from a given $F(z) = \sum_{k=0}^{\infty} a_k \frac{z^k}{(1-z)^{k+1}}$ with

$\lim_{k \rightarrow \infty} a_k = 0$. It follows from (6) that the z -transform of the function $f(x) = \sum_{k=0}^{\infty} a_k \binom{x}{k}$

is equal to the given $F(z)$ since $(\Delta^k f)(0) = a_k$.

Although we do not need it in the sequel we will also prove that T is an isometry. For this we need a lemma.

Lemma 1

If $a = (a_k)$ is a sequence in $\mathbb{C}_p$, with $\lim_{k \rightarrow \infty} a_k = 0$, then

$$\sup |a_k| = \sup\{|a_0|, |a_0 + a_1|, |a_1 + a_2|, \dots, |a_k + a_{k+1}|, \dots\}.$$

Proof : Put $\|a\| = \sup |a_k|$, $|||a||| = \sup\{|a_0|, \dots, |a_k + a_{k+1}|, \dots\}$.

Since $|a_k + a_{k+1}| \leq \max\{|a_k|, |a_{k+1}|\} \leq \|a\|$ we see that $|||a||| \leq \|a\|$.

Put $b_0 = a_0, b_1 = a_0 + a_1, \dots, b_k = a_{k-1} + a_k, \dots$.

Then $a_k = b_k - b_{k-1} + b_{k-2} - \dots \pm b_0$.

Hence $|a_k| \leq \max\{|b_0|, |b_1|, \dots, |b_k|\} \leq |||a|||$

thus $\|a\| \leq |||a|||$ and the lemma is proved.

Proposition : T is an isometry.

Proof : Let $F(z) = \sum_{k=0}^{\infty} a_k \frac{z^k}{(1-z)^{k+1}}$ be the z -transform of $f(x) = \sum_{k=0}^{\infty} (\Delta^k f)(0) \binom{x}{k}$.

$\|f\| = \sup_k |(\Delta^k f)(0)|$ since the polynomials $\binom{x}{k}$ form an orthogonal base for $C(\mathbb{Z}_p)$

$$= \sup |a_k|$$

$$= \sup\{|a_0|, |a_0 + a_1|, \dots, |a_k + a_{k+1}|, \dots\} \text{ by lemma 1}$$

Writing $u = \frac{z}{1-z}$ we observe that $z \in D$ if and only if $|u + 1| \leq 1$.

Now

$$\begin{aligned}
||f|| &= \sup\{|a_0|, |a_0 + a_1|, \dots, |a_k + a_{k+1}|, \dots\} \\
&= \sup_{|u| \leq 1} \{a_0 + (a_0 + a_1)u + \dots + (a_{k-1} + a_k)u^k + \dots\} \\
&= \sup_{|u+1| \leq 1} \{a_0 + (a_0 + a_1)u + \dots + (a_{k-1} + a_k)u^k + \dots\} \\
&= \sup_{z \in D} |F(z)| = ||F||
\end{aligned}$$

We now show how the z-transform can be used in p-adic analysis.

Application 1 Mahler's expansion with remainder

We start from formula (6)

$$F(z) = \sum_{k=0}^{n-1} (\Delta^k f)(0) \frac{z^k}{(1-z)^{k+1}} + \frac{z^n}{(1-z)^n} \int_{\mathbf{Z}_p} (\Delta^n f)(x) \mu_z(x) \quad (6)$$

If $f(x) = \binom{x}{n-1}$ all terms on the R.H.S. vanish except the term $\frac{z^{n-1}}{(1-z)^n}$. This means that the z-transform of $\binom{x}{n-1}$ is $\frac{z^{n-1}}{(1-z)^n}$.

Hence every term of (3) is the transform of a function in $C(\mathbf{Z}_p)$. Taking the inverse transform we get something of the form

$$f(x) = \sum_{k=0}^{n-1} (\Delta^k f)(0) \binom{x}{k} + r_n(x)$$

where $r_n(x)$ is the inverse transform of

$$z \cdot \frac{z^{n-1}}{(1-z)^n} \cdot \int_{\mathbf{Z}_p} (\Delta^n f)(x) \mu_z(x) \quad (13)$$

Using (10) we see that $r_n(x) = \left\{ \binom{x}{n-1} * \Delta^n f \right\} (x-1)$.

The presence of the first factor z in the product (13) makes it necessary to evaluate the convolution of $\binom{x}{n-1}$ and $\Delta^n f$ at the point $x-1$ instead of x .

This gives Mahler's expansion with an expression for the remainder

$$f(x) = \sum_{k=0}^{n-1} (\Delta^k f)(0) \binom{x}{k} + \left\{ \binom{x}{n-1} * \Delta^n f \right\} (x-1)$$

This was obtained in [5] by a different method.

Remark : Until now we have assumed that the functions of $C(\mathbb{Z}_p)$ take their values in $\mathbb{C}_p$. If we replace $\mathbb{C}_p$ by a field that is complete for a non archimedean valuation containing $\mathbb{Q}_p$, the method still works. The only restriction is that we can no longer use any property whose proof uses analytic continuation.

Application 2 Van der Put's expansion

Notation : If $n = a_0 + a_1p + \dots + a_s p^s$ with $a_s \neq 0$ then we put $m(n) = s$ and $n_- = a_0 + a_1p + \dots + a_{s-1}p^{s-1}$.

Take $f \in C(\mathbb{Z}_p)$ and let f_r denote the locally constant function defined by

$$\begin{aligned} f_r(k) &= f(k) \quad \text{for } k = 0, 1, \dots, p^r - 1 \\ f_r(x) &= f_r(x + p^r) \end{aligned}$$

By induction on r we can verify that

$$\sum_{0 \leq n < p^r} (f(n) - f(n_-)) \frac{z^n}{1 - z^{m(n)}} = \frac{\sum_{n=0}^{p^r-1} f(n) z^n}{1 - z^{p^r}} \quad (14)$$

Using the definition (1) we see that the R.H.S. of (14) is the z -transform of f_r . In the same way we can verify that $\frac{z^n}{1 - z^{m(n)}}$ is the z -transform of the function

$$\begin{aligned} e_n(x) &= 1 \quad \text{if } |x - n| < \frac{1}{n} \\ e_n(x) &= 0 \quad \text{if } |x - n| \geq \frac{1}{n} \end{aligned}$$

The inverse transform of (8) gives the identity

$$\sum_{0 \leq n < p^r} [f(n) - f(n_-)] e_n(x) = f_r(x)$$

If $r \rightarrow \infty$ we recover the Van der Put expansion of $f(x)$.

Application 3

If we put $f(x) = \binom{x+n}{n}$ in (7) we see that z -transform of $\binom{x+n}{n}$ is $\frac{1}{(1-z)^{n+1}}$. The inverse of (7) yields

$$f(x) = \sum_{k=0}^n (Q^k f)(-1) \binom{x+k}{k} + \left\{ \binom{x+n}{n} * Q^{n+1} f \right\}(x) \quad Q = \Delta E^{-1}$$

4. The expansion of a continuous function in a series of Sheffer polynomials

In this section we will use the p-adic z-transform to generalize the main theorem of [6]. We first recall a few elements of the p-adic umbral calculus developed in [6].

Let R be a linear continuous operator on $C(\mathbb{Z}_p, K)$, where K is a field containing $\mathbb{Q}_p$ that is complete for a non archimedean valuation. If R commutes with E it can be written in the form $R = \sum_{i=0}^{\infty} b_i \Delta^i$ where (b_i) is a bounded sequence in K . The result that we want to generalize is the following.

Proposition [6]

If $Q = \sum_{i=0}^{\infty} b_i \Delta^i$ is a linear continuous operator on $C(\mathbb{Z}_p, K)$ such that $b_0 = 0, |b_1| = 1, |b_i| \leq 1$ for $i \geq 2$ then

a) there exists a unique sequence of polynomials $p_n(x)$ such that

$$Qp_n = p_{n-1}, \deg p_n = n, p_n(0) = 0 \text{ for } n \geq 1 \text{ and } p_0 = 1$$

b) every continuous function $f : \mathbb{Z}_p \rightarrow K$ has a uniformly convergent expansion of the form

$$f(x) = \sum_{n=0}^{\infty} (Q^n f)(0) p_n(x) \quad (15)$$

With an operator $R = \sum_{i=0}^{\infty} b_i \Delta^i$ we can associate a measure on $\mathbb{Z}_p$ by means of the functional sending a $f \in C(\mathbb{Z}_p, K)$ to $(Rf)(0)$.

Example : Take $R = \frac{1}{1-Ez}$ with $z \in D$. Then

$$R = \frac{1}{1-z+\Delta z} = \sum_{k=0}^{\infty} \Delta^k \frac{z^k}{(1-z)^{k+1}}$$

Formula (6) shows that the measure obtained in this way is the measure introduced in section 1.

Now let $Q = \sum_{i=0}^{\infty} b_i \Delta^i$ and $S = \sum_{i=0}^{\infty} s_i \Delta^i$ be two operators commuting with E where S is invertible.

If $b_0 = 0$, any operator R , commuting with E , can be written in the form

$$R = \sum_{n=0}^{\infty} r_n Q^n, \quad r_n \in K$$

We can see this as an equality between operators or as an identity between formal power series in Δ . If we take $R = \frac{S}{1-Ez}$ the coefficients r_n will depend on z . Let us write it in the form

$$\frac{S}{1-Ez} = \sum_{n=0}^{\infty} \frac{T_n(z)}{(1-z)^{n+1}} Q^n \quad (16)$$

Writing out everything as a powerseries in Δ and comparing the coefficient of Δ^n we see that $T_n(z)$ is a polynomial of degree n in z . If, moreover, $|b_1| = 1$ the sequence $\frac{T_n(z)}{(1-z)^{n+1}}$ is bounded.

Multiplying (16) with S^{-1} and applying the operators on both sides to a function $f \in C(\mathbb{Z}_p, K)$ we get the series

$$F(z) = \sum_{n=0}^{\infty} (S^{-1}Q^n f)(0) \frac{T_n(z)}{(1-z)^{n+1}} \quad (17)$$

This series is uniformly convergent since $\lim_{n \rightarrow \infty} (S^{-1}Q^n f)(0) = 0$.

The idea is now to take the inverse z -transform of (17).

Now the z -transform of $\binom{x}{n}$ is $\frac{z^n}{(1-z)^{n+1}}$. Hence the z -transform of a polynomial of degree n is of the form $\frac{P_n(z)}{(1-z)^{n+1}}$ where $P_n(z)$ is also a polynomial of degree n .

Taking the inverse transform of (17) we get

$$f(x) = \sum_{n=0}^{\infty} (S^{-1}Q^n f)(0) t_n(x) \quad (18)$$

where $t_n(x)$ is a polynomial of degree n .

This is the expansion we wanted to obtain.

To see that (18) is a generalization of (15) take S equal to the identity operator and take f equal to the polynomial p_n in (15). (18) then reduces to $p_n(x) = t_n(x)$.

In the general case the polynomials $t_n(x)$ are called "Sheffer polynomials" in umbral calculus.

Remark

It is possible to work in an even more general situation. Let $Q_1, Q_2, \dots, Q_n, \dots$ be a sequence operators satisfying the same conditions as the operator Q above. There exists a sequence of polynomials $T_n(z)$, $\deg T_n = n$, such that

$$\frac{S}{1 - Ez} = \sum_{n=0}^{\infty} \frac{T_n(z)}{(1 - z)^{n+1}} Q_1 Q_2 \dots Q^n$$

5. A formula for $\lg 2$

The formula

$$2\left(1 - \frac{1}{p}\right) \lg 2 = \lim_{n \rightarrow \infty} \sum_{\substack{k=1 \\ (k, p)=1}}^{p^n} \frac{(-1)^{k+1}}{k}, \quad p \neq 2$$

is proved in [2] p. 180 and [3] p. 38. Here $\lg 2$ is the p-adic logarithm.

In this section we show that it is possible to refine this result using the properties of the integral studied in section 2.

$$\begin{aligned} \text{Let } f(x) &= 0 & \text{for } |x| < 1 \\ &= \frac{1}{x} & \text{for } |x| = 1 \end{aligned}$$

In [1] (lemma 6.4, chapter 12) it is proved that, for $z \in D$,

$$\int_{\mathbf{Z}_p} f(x) \mu_z(x) = \frac{1}{p} \lg \frac{1 - z^p}{(1 - z)^p} \quad (19)$$

If $U_p = \mathbf{Z}_p \setminus p\mathbf{Z}_p$ denotes the group of units of $\mathbf{Z}_p$ the integral can be written as

$$\int_{U_p} \frac{\mu_z(x)}{x} = \frac{1}{p} \lg \frac{1 - z^p}{(1 - z)^p}$$

Putting $z = -1$ we get

$$\int_{U_p} \frac{\mu_{-1}(x)}{x} = -\left(1 - \frac{1}{p}\right) \lg 2 \quad (20)$$

The idea is to construct approximations for the integral on the LHS of (20). This will yield the following theorem.

Theorem : If $p \neq 2$ then

$$\text{a) } 2\left(1 - \frac{1}{p}\right) \lg 2 \equiv \sum_{k=1, (k,p)=1}^{p^n} \frac{(-1)^{k+1}}{k} \pmod{p^{2n}}$$

$$\text{b) } 2\left(1 - \frac{1}{p}\right) \lg 2 \equiv 4\varepsilon_n \sum_{\substack{k=0 \\ (2k+1,p)=1}}^{\frac{p^n-3}{2}} \frac{(-1)^{k+1}}{2k+1} \pmod{p^{2n}}$$

$$\text{where } \varepsilon_n = (-1)^{n \cdot \frac{p-1}{2}}$$

$$\text{c) } -2\left(1 - \frac{1}{p}\right) \lg 2 \equiv \sum_{\substack{k=1 \\ (k,p)=1}}^{p^n} \frac{(-1)^{k+1}}{k} - 8\varepsilon_n \sum_{k=0}^{\frac{p^n-3}{2}} \frac{(-1)^{k+1}}{2k+1} \pmod{p^{4n}}$$

For the proof we need the value of a few integrals. We collect these results in the following lemma. i denotes a squareroot of -1 .

Lemma 2

$$(1) \quad \int_{U_p} \frac{\mu_{-1}(x)}{x^2} = \int_{U_p} \frac{\mu_{-1}(x)}{x^4} = 0$$

$$(2) \quad \int_{U_p} \frac{\mu_i(x)}{x^2} + \int_{U_p} \frac{\mu_{-i}(x)}{x^2} = 0$$

$$\int_{U_p} \frac{\mu_i(x)}{x^4} + \int_{U_p} \frac{\mu_{-i}(x)}{x^4} = 0$$

$$(3) \quad \int_{U_p} \frac{\mu_i(x)}{x} = \int_{U_p} \frac{\mu_{-i}(x)}{x} = -\frac{1}{2}\left(1 - \frac{1}{p}\right) \lg 2 \quad \text{for } p \neq 2$$

$$(4) \quad \int_{U_p} \frac{\mu_i(x)}{x^3} = \int_{U_p} \frac{\mu_{-i}(x)}{x^3} = \frac{1}{8} \int_{U_p} \frac{\mu_{-1}(x)}{x^3}$$

Proof of the lemma

- (1) These are special cases of formula (9).
- (2) These are special cases of (8) with $z = i$.

(3) Suppose first that $p \equiv 1 \pmod{4}$. Then $i^p = i$, hence

$$\int_{U_p} \frac{\mu_i(x)}{x} = \frac{1}{p} \lg \frac{1-i}{(1-i)^p} = -(1 - \frac{1}{p}) \lg(1-i)$$

Since $(1-i)^2 = -2i$ and $\lg i = 0$ we see that $\lg(1-i) = \frac{1}{2} \lg 2$ and the assertion is proved. If $p \equiv 3 \pmod{4}$ we have $i^p = -i$ and we get

$$\int_{U_p} \frac{\mu_i(x)}{x} = \frac{1}{p} \lg \frac{1+i}{(1-i)^p}$$

Since $\frac{1+i}{1-i} = i$ and $\lg i = 0$ we conclude that

$$\frac{1}{p} \lg \frac{1+i}{(1-i)^p} = -(1 - \frac{1}{p}) \lg(1-i) = -\frac{1}{2}(1 - \frac{1}{p}) \lg 2$$

The integral $\int_{U_p} \frac{\mu_{-i}(x)}{x}$ is calculated in the same way.

(4) Let k be a natural number and let $\zeta(s)$ be the Riemann zeta function. It is well-known that the numbers $\zeta(-k)$ are rational and that the sequence $k \rightarrow (1-p^k)\zeta(-k)$ can be interpolated p-adically. This can be deduced from the following formula (see [1] p. 295).

$$(1-p^k)\zeta(-k) = \frac{1}{q^{k+1}-1} \sum \int_{U_p} x^k \mu_\theta(x) \quad (21)$$

The sum is extended over all primitive q -th roots of unity θ with $\theta \neq 1$. q is an integer prime to p .

In [1] the author supposes that q is a prime but this restriction is not necessary.

Clearly the LHS of (21) is independent of q . Taking respectively $q = 2$ and $q = 4$ we get

$$\frac{1}{2^{k+1}-1} \int_{U_p} x^k \mu_{-1}(x) = \frac{1}{4^{k+1}-1} \left\{ \int_{U_p} x^k \mu_{-1}(x) + \int_{U_p} x^k \mu_i(x) + \int_{U_p} x^k \mu_{-i}(x) \right\}$$

or

$$2^{k+1} \int_{U_p} x^k \mu_{-1}(x) = \int_{U_p} x^k \mu_i(x) + \int_{U_p} x^k \mu_{-i}(x) \quad (22)$$

If k remains in a fixed residue class mod $(p-1)$ the LHS of (21) is a continuous function of k . Hence (21) and (22) remain valid for negative integers (except possibly for $k = -1$). Taking $k = -3$ we get

$$4 \int_{U_p} \frac{\mu_{-1}(x)}{x^3} = \int_{U_p} \frac{\mu_i(x)}{x^3} + \int_{U_p} \frac{\mu_{-i}(x)}{x^3}$$

Since (8) implies that $\int_{U_p} \frac{\mu_i(x)}{x^3} = \int_{U_p} \frac{\mu_{-i}(x)}{x^3}$ the last assertion of lemma 2 is proved.

Proof of the theorem

Starting from (1) we have

$$\int_{U_p} \frac{\mu_z(x)}{x} = \sum_{\substack{k=1 \\ (k,p)=1}}^{p^n} \frac{z^k}{k} + z^{p^n} \int_{U_p} \frac{\mu_z(x)}{x + p^n}$$

Now
$$\frac{1}{x + p^n} = \frac{1}{x} - \frac{p^n}{x^2} + \frac{p^{2n}}{x^3} - \frac{p^{3n}}{x^4} + \frac{p^{4n}}{x^4(x + p)}$$

Integrating this over U_p and observing that (11) implies

$$\left| \int_{U_p} \frac{\mu_z(x)}{x^4(x + p^n)} \right| \leq 1$$

we see that the (p-adic) value of

$$(1 - z^{p^n}) \int_{U_p} \frac{\mu_z(x)}{x} - \sum_{\substack{k=1 \\ (k,p)=1}}^{p^n} k + z^{p^n} \left[p^n \int_{U_p} \frac{\mu_z(x)}{x^2} - p^{2n} \int_{U_p} \frac{\mu_z(x)}{x^3} + p^{3n} \int_{U_p} \frac{\mu_z(x)}{x^4} \right] \quad (23)$$

is $\leq \frac{1}{p^4}$.

For $z = -1$ the first assertion of lemma 2 implies that two of these integrals are zero. Since the other integrals clearly lie in $\mathbb{Z}_p$ we obtain the following congruence in $\mathbb{Z}_p$

$$2 \int_{U_p} \frac{\mu_{-1}(x)}{x} \equiv \sum_{\substack{k=1 \\ (k,p)=1}}^{p^n} \frac{(-1)^k}{k} - p^{2n} \int_{U_p} \frac{\mu_{-1}(x)}{x^3} \pmod{p^{4n}} \quad (24)$$

If we compare this with (20) we see that point (a) of the theorem is proved.

In order to prove (b) note that $i^p = (-1)^{\frac{p-1}{2}}$ and hence $i^{p^n} = \varepsilon_n i$.

Now put $z = i$ in (23). This gives

$$\left| (1 - \varepsilon_n i) \int_{U_p} \frac{\mu_i(x)}{x} - \sum_{\substack{k=1 \\ (k,p)=1}}^{p^n} \frac{i^k}{k} + p^n \varepsilon_n i \int_{U_p} \frac{\mu_i(x)}{x^2} - p^{2n} \varepsilon_n i \int_{U_p} \frac{\mu_z(x)}{x^3} + p^{3n} \varepsilon_n i \int_{U_p} \frac{\mu_z(x)}{x^4} \right| \leq \frac{1}{p^4}$$

Replace i by $-i$ and subtract. When the integrals are replaced by their values given in lemma 2 we obtain the congruence

$$\varepsilon_n i \left(1 - \frac{1}{p}\right) \lg 2 \equiv 2i \sum_{\substack{k=0 \\ (2k+1,p)=1}}^{\frac{p^n-1}{2}} \frac{(-1)^k}{2k+1} + \frac{\varepsilon_n i p^{2n}}{4} \int_{U_p} \frac{\mu_{-1}(x)}{x^3} \pmod{p^{4n}} \quad (25)$$

Neglecting the last term we see that (b) is proved.

To obtain (c) it is sufficient to take a linear combination of (24) and (25) such that the integral $\int_{U_p} \frac{\mu_{-1}(x)}{x^3}$ disappears.

We can deduce the following purely arithmetical result from the theorem.

Corollary

For $p \neq 2$

$$\begin{aligned} 2 \cdot \frac{2^{(p-1)} - 1}{p^2} &\equiv 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots - \frac{1}{p-1} \pmod{p^2} \\ &\equiv 4(-1)^{\frac{p-1}{2}} \left(1 - \frac{1}{3} + \frac{1}{5} - \dots \pm \frac{1}{p-2}\right) \pmod{p^2} \end{aligned}$$

Proof: Since $2^{(p-1)p} \equiv 1 \pmod{p^2}$ we have

$$p(p-1) \lg 2 = \lg(2^{(p-1)p} - 1 + 1) \equiv 2^{(p-1)p} - 1 \pmod{p^4}$$

and hence

$$\left(1 - \frac{1}{p}\right) \lg 2 \equiv \frac{2^{(p-1)p} - 1}{p^2} \pmod{p^4}$$

Combining this with the congruences (a) and (b) of the theorem (for $n = 1$) we see that the required congruences are established.

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Vrije Universiteit Brussel

Faculty of Applied Sciences

Pleinlaan 2

B - 1050 Brussels

Belgium