M. Megan
D.R. Latcu

On uniform exponential $N$-dichotomy

Annales mathématiques Blaise Pascal, tome 1, n° 2 (1994), p. 33-41

<http://www.numdam.org/item?id=AMBP_1994__1_2_33_0>
The problem of uniform exponential $N$-dichotomy of evolutionary processes in Banach spaces is discussed. Generalizations of the some well-known results of R. Datko, Z. Zabczyk, S. Rollewicz and A. Ichikawa are obtained. The results are applicable for a large class of nonlinear differential equations.

I. INTRODUCTION.

Let $X$ be a real or complex Banach space with the norm $\|\cdot\|$. Let $T$ be the set defined by

$$T = \{(t, t_0) : 0 \leq t_0 \leq t < \infty\}$$

Let $\Phi(t, t_0)$ with $(t, t_0) \in T$ be a family of operators with domain $X_{t_0} \subset X$.

**Definition 1.1**

The family $\Phi(t, t_0)$ with $(t, t_0) \in T$ is called an evolutionary process if:

1. $\Phi(t, t_0)x_0 \in X_t$ for all $(t, t_0)$ and $x_0 \in X_{t_0}$;
2. $\Phi(t, t_1)\Phi(t_1, t_0)x_0 = \Phi(t, t_0)x_0$ for $(t, t_1), (t_0, t_0) \in T$ and $x_0 \in X_{t_0}$;
3. $\Phi(t, t)x = x$ for all $t \geq 0$ and $x \in x_t$;
4. for each $t_0 \geq 0$ and $x_0 \in X_{t_0}$ the function $t \mapsto \Phi(t, t_0)x_0$ is continuous on $[t_0, \infty]$;
5. there is a positive nondecreasing function $\varphi : (0, \infty) \to (0, \infty)$ such that $\|\Phi(t, t_0)x_0\| \leq \varphi(t - t_0)\|x_0\|$ for all $(t, t_0) \in T$ and $x_0 \in X_{t_0}$.

Throughout in this paper for each $t_0 \geq 0$ we denote by

$$X_{t_0}^1 = \{x_0 \in X_{t_0} : \Phi(., t_0)x_0 \in L_{t_0}^\infty(X)\} \quad \text{and} \quad X_{t_0}^2 = X_{t_0}^2 = X_{t_0} \setminus X_{t_0}^1$$

where $L_{t_0}^\infty(X)$ is the Banach space of $X$-valued function $f$ defined a.e. on $[t_0, \infty]$, such that $f$ is strongly measurable and essentially bounded.
Remark 1.1. If \( x_0 \in X_{t_0}^1 \) and \( t \geq t_0 \) then \( \Phi(t, t_0)x_0 \in X_{t}^1 \).

Indeed, if \( x_0 \in X_{t_0}^1 \) then
\[
\Phi(\cdot, t)\Phi(t, t_0)x_0 = \Phi(\cdot, t_0)x_0 \in L_t^\infty(X) \subseteq L_{t_0}^\infty(X)
\]
and hence \( \phi(t, t_0)x_0 \in X_{t}^1 \).

Let \( \mathcal{N} \) be the set of strictly increasing real functions \( N \) defined on \( [0, \infty) \) which satisfies:
\[
\lim_{t \to 0} N(t) = 0 \quad \text{and} \quad N(t_0) \leq N(t)N(t_0)
\]
for all \( t, t_0 \geq 0 \).

Remark 1.2. It is easy to see that if \( N \in \mathcal{N} \) then

i) \( N(t) > 0 \) for every \( t > 0 \);

ii) \( N(0) = 0 \) and \( N(1) \geq 1 \);

iii) \( \lim_{t \to \infty} N(t) = \infty \).

Definition 1.2. Let \( N \in \mathcal{N} \). The evolutionary process \( \Phi(\cdot, \cdot) \) is said to be uniformly exponentially \( N \)-dichotomic (and we write u.e.-N-d.) if there are \( M_1, M_2, \nu_1, \nu_2 > 0 \) such that for all \( t \leq s \leq t_0 \geq 0 \) and \( x_1 \in X_{t_0}^1 \), \( x_2 \in X_{t_0}^2 \) we have:
\[
Nd_1) \quad N(\|\Phi(t, t_0)x_1\|) \leq M_1 e^{-\nu_1(t-s)}N(\|\Phi(s, t_0)x_1\|), \quad \text{and}
\]
\[
Nd_2) \quad N(\|\Phi(t, t_0)x_2\|) \leq M_2 e^{\nu_2(t-s)}N(\|\Phi(s, t_0)x_2\|).
\]

Particularly, for \( N(t) = t \), if \( \Phi(\cdot, \cdot) \) is u.e.-N-d. then \( \Phi(\cdot, \cdot) \) is called an uniform exponential dichotomic (and we write u.e.d.) process. If \( \Phi(\cdot, \cdot) \) is u.e-N-d. (respectively u.e.d.) and \( X_{t_0}^1 = X_{t_0}^2 \) for every \( t_0 \leq 0 \) then \( \Phi(\cdot, \cdot) \) is called an uniform exponential -N-stable (respectively uniform exponential stable) process.

Remark 1.3. \( \Phi(\cdot, \cdot) \) is u.e-N-d. if and only if the inequalities \( (d_1) \) and \( (d_2) \) from Definition 1.2. hold for all \( t \leq s + 1 > s \leq t_0 \leq 0 \).

Indeed, if \( t_0 \geq s \geq t \geq s + 1 \), \( x_1 \in X_{t_0}^1 \) and \( T_2 \in X_{t_0}^2 \) then
\[
N(\|\Phi(t, t_0)x_1\|) \leq N(\varphi(t-s)).N(\|\Phi(s, t_0)x_1\|) \geq N(\varphi(1)).N(\|\Phi(s, t_0)x_1\|) \geq \leq N(\varphi(1)).e^{\nu_1(t-s)}N(\|\Phi(s, t_0)x_1\|)
\]
and
\[
M_2. e^{\nu_2}N(\|\Phi(s, t_0)x_2\|) \leq N(\|\Phi(s+1, t_0)x_2\|) \leq \leq N(\varphi(s+1-t)).N(\|\Phi(t, t_0)x_2\|) \leq \leq N(\varphi(1)).e^{\nu_2(t-s)}N(\|\Phi(t, t_0)x_2\|).
\]
A necessary and sufficient condition for the uniform exponential stability of a linear evolutionary process in a Banach space has been proved by Datko in [1]. The extension of Datko's theorem for uniform exponential dichotomy has been obtained by Preda and Megan in [3].

The case of uniform exponential-$N$-stable processes has been considered by Ichikawa in [2]. The particular case when the process is a strongly continuous semigroup of bounded linear operators has been studied by Zabczyk in [5] and Rolewicz in [4].

In this paper we shall extend these results in two directions. First, we shall give a characterization of u.e.-$N$-dichotomy, which can be considered as a generalization of Datko's theorem. Second, we shall not assume the linearity and boundedness of the process $\Phi(.,.)$. The obtained results are applicable for a large class of nonlinear differential equations described in [2].

II - PRELIMINARY RESULTS

An useful characterization of the uniform exponential-$N$-dichotomy property is given by

**Proposition 2.1**

The evolutionary process $\Phi(.,.)$ is u.e.-$N$-d. if and only if there are two continuous functions $\varphi_1, \varphi_2 : [0, \infty) \to (0, \infty)$ with the properties:

1. $N(\|\Phi(t,t_0)x_1\|) \leq \varphi_1(t-s)N(\|\Phi(s,t_0)x_1\|)$
2. $N(\|\Phi(t,t_0)x_2\|) \leq \varphi_2(t-s)N(\|\Phi(s,t_0)x_2\|)$
3. $\lim_{t \to \infty} \varphi_1(t) = 0$ and $\lim_{t \to \infty} \varphi_2(t) = \infty$ for all $t \geq s \geq t_0 \geq 0$, $x_1 \in X_{t_0}^1$ and $x_2 \in X_{t_0}^2$.

**Proof.**

The necessity is obvious from Definition 1.2 for $\varphi_1(t) = M_1 e^{-\nu_1 t}$ and $\varphi_2(t) = M_2 e^{\nu_2 t}$.

The sufficiency. From ($Nd'_3$) it follows that there are $s_1, s_2 > 0$ such that $\varphi_1(s_1) < 1$ and $\varphi_2(s_2) > 1$. Then for all $t \geq s \geq t_0$ there are two natural numbers $n_1$ and $n_2$ such that $t - s = n_1 s_1 + r_1 = n_2 s_2 + r_2$, where $r_1 \in [0, s_2]$.

From ($e_3$) and ($Nd'_1$) it results that if $t \geq s \geq t_0 \geq 0$ and $x_1 \in X_{t_0}^1$ then $N(\|\Phi(t,t_0)x_1\|) \leq N(\varphi(r_1))N(\|\Phi(s + n_1 x_1, t_0)x_1\|) \leq N(\varphi(s_1))\varphi_1(s_1)$.
\[ N(\| \Phi(s, t_0) x_1 \|) \leq M_1 e^{-\nu_1 (t-s)} N(\| \Phi(s, t_0) x_1 \|) \]

where \( M_1 = N(\varphi(s_1)) e^{\nu_1 s_1} = \frac{N(\varphi(s_1))}{\varphi_1(x_1)} \) and \( \nu_1 = -\frac{\ln \varphi_1(s_1)}{x_1} \).

Similarly, if \( t \geq s \geq t_0 \geq 0 \) and \( x_2 \in X^2 \) then

\[ N(\| \Phi(t, t_0) x_2 \|) \geq \varphi_2(r_2) N(\| (s + n_2 s_2, t_0) x_2 \|) \geq \varphi_2(r_2) \varphi_2(s_2)^n. \]

\[ N(\| \Phi(s, t_0) x_2 \|) \geq m_2 e^{\nu_2 n_2 s_2} M(\| \Phi(s, t_0) x_2 \|) \]

\[ M_2 e^{\nu_2 (t-s)} N(\| \Phi(s, t_0) x_2 \|), \]

where \( m_2 = \inf_{0 \leq t \leq s_2} \varphi_2(t), \) \( M_2 = \frac{m_2}{\varphi_2(s_2)} \) and \( \nu_2 = \frac{\ln \varphi_2(s_2)}{s_2}. \)

In virtue of Definition 1.2 it follows that \( \Phi(., .) \) is u.e-N-d.

**Corollary 2.1.**

*The evolutionary process \( \Phi(., .) \) is u.e.d. if and only if there are two continuous functions \( \varphi_1, \varphi_2 : (0, \infty) \rightarrow (0, \infty) \) with the properties:

\[ d'_1 \| \Phi(t, t_0) x_1 \| \leq \varphi_1(t-s) \| \Phi(s, t_0) x_1 \|, \]

\[ d'_2 \| \Phi(t, t_0) x_2 \| \geq \varphi_2(t-s) \| \Phi(s, t_0) x_2 \|, \]

\[ (Nd_1) \lim_{t \rightarrow \infty} \varphi_1(t) = 0 \quad \text{and} \quad (Nd_2) \lim_{t \rightarrow \infty} \varphi_2(t) = \infty. \]

for all \( t \geq s \geq t_0 \geq 0, x_1 \in X^1_{t_0} \) and \( x_2 \in S^2_{t_0}. \)

**Proof.** Is obvious from Proposition 2.1 for \( N(t) = t. \)

The relation between u. e-N-d. and u.e.d. properties is given by

**Proposition 2.2 :**

*The evolutionary process \( \Phi(., .) \) is u.e.d. if and only if there is \( N \in \mathcal{N} \) such that \( \Phi(., .) \) is u.e-N-d.*

**Proof :**

The necessity is obvious from Definition 1.2.

The sufficiency. Suppose that there is \( N \in \mathcal{N} \) such that \( \Phi(., .) \) satisfies the condition 

\( (Nd_1) \) and \( (Nd_2) \) from Definition 1.2.

Let \( s_1, s_2, s_3 > 0 \) such that \( M_1 N(2) < e^{\nu_1 s_1}, N(2) < M_2 e^{\nu_2 s_2} \) and \( N(s_3) < M_2. \) If \( t \geq s \geq t_0 \) then there are two natural numbers \( n_1 \) and \( n_2 \) such that \( t - s = n_1 s_1 + r_1 = n_2 s_2 + r_2, \) where \( r_1 \in (0, s_1) \) and \( r_1 \in (0, s_2). \) Then for \( s \geq t_0 \geq 0 \) and \( x_1 \in X^1_{t_0} \) we have
and hence (because $N$ is nondecreasing)

\[ \|\Phi(s, t_0)x_1\| \geq 2\cdot\|(s + s_1, t_0)x_1\| \text{ and (by induction)} \]

\[ \|\Phi(s, t_0)x_1\| \geq 2^n\|\Phi(s + ns_1, t_0)x_1\| \text{ for every natural number } n. \]

Therefore for $t \geq s \geq t_0 \geq 0$ and $x_1 \in X^1_{t_0}$ we obtain that

\[ \|\phi(t, t_0)x_1\| \leq \varphi(r_1)\|\Phi(s + n_1s_1, t_0)x_1\| \leq \frac{\varphi(s_1)}{2^{n_1}}\|\Phi(s, t_0)x_1\| \]

and hence

(2.1) \[ \|\phi(t, t_0)x_1\| \leq \varphi_1(t - s)\|\Phi(s, t_0)x_1\| \text{ for } t \geq s \geq t_0 \text{ and } x_1 \in X^1_{t_0}, \text{ where } \varphi_1(u) = \frac{\varphi(s_1)}{2^{u/s_1}}. \]

On the other hand, for $s \geq t_0 \geq 0$ and $x_2 \in X^2_{t_0}$ we have

\[ \|\phi(t, t_0)x_2\| \geq M_2e^{\nu_2r_2}N(\|\Phi(s, t_0)x_2\|) \geq N(2\|\Phi(s, t_0)x_2\|) \]

\[ \geq N(2\|\Phi(s, t_0)x_2\|) \]

and hence

\[ \|\phi(s + s_2, t_0)x_2\| \geq 2\|\phi(s, t_0)x_2\| \text{ and (by induction)} \]

\[ \|\phi(s + ns_2, t_0)x_2\| \geq 2^n\|\Phi(s, t_0)x_2\| \text{ for all } s \geq t_0 \geq 0, x_2 \in X^2_{t_0} \text{ and every natural number } n. \]

hence, if $t \geq s \geq t_0 \geq 0$ and $x_2 \in X^2_{t_0}$ then

\[ N(\|\Phi(t, t_0)x_2\|) = N(\|\Phi(s + n_2s_2 + r_2, t_0)x_2\|)M_2e^{\nu_2r_2}N(\|\Phi(s + n_2s_2, t_0)x_2\|) \]

\[ \geq N(s_3\|\Phi(s + n_2s_2, t_0)x_2\|), \]

which implies

\[ \|\Phi(t, t_0)x_2\| \geq s_3\|\Phi(s + n_2s_2, t_0)x_2\| \geq 2^n.s_3\|\Phi(s, t_0)x_2\| \]

and hence

(2.2) \[ \|\Phi(t, t_0)x_2\| \geq \varphi_2(t - s)\|\Phi(s, t_0)x_2\| \text{ for } t \geq s \geq 0 \text{ and } x_2 \in X^2_{t_0}, \text{ where } \varphi_2(u) = \frac{s_3}{2^{u/s_2}}. \]
From (2.1), (2.2) and Corollary 2.1 it follows that \( \Phi(\cdot, \cdot) \) is u.e.d.

3 - THE MAIN RESULTS.

The following theorem is an extension of Datko's theorem (\cite{1}) to the general case of uniform exponential-N-dichotomy.

**Theorem 3.1.**  
*The evolutionary process \( \Phi(\cdot, \cdot) \) is u.e-N-d. if and only if there are \( M, m > 0 \) such that*

\[
(Nd_1') \quad \int_t^\infty N(\|\Phi(s, t_0)x_1\|)ds \leq M.N(\|\Phi(t, t_0)x_1\|),
\]

\[
(Nd_2') \quad \int_{t_0}^t N(\|\Phi(s, t_0)x_2\|)ds M.N(\|\Phi(t, t_0)x_2\|),
\]

\[
(Nd_3') \quad N(\|\Phi(t + 1, t_0)x_2\|)m.N(\|\Phi(t, t_0)x_2\|)
\]

*for all \( t \geq t_0 \geq 0, x_1 \in X^1_{t_0} \) and \( x_2 \in X^2_{t_0} \).*

**Proof.** The necessity is simply verified. Now we prove the sufficiency part.

Let \( s \geq t_0 \geq 0, x_1 \in X^1_{t_0} \) and \( \frac{1}{M_0} = \int_0^1 \frac{dt}{\psi(t)} \), where \( \psi = N.\varphi \).

If \( t \geq s + 1 \) then

\[
\frac{N(\|\Phi(t, t_0)x_1\|)}{M_0} = \int_0^1 \frac{N(\|\Phi(t, t_0)x_1\|)}{\psi(r)} dr \leq \int_0^1 \frac{N(\|\Phi(t, t_0)x_1\|)}{\psi(t-v)} dr \leq \int_s^t N(\|\Phi(v, t_0)x_1\|)dv \leq \int_s^\infty N(\|\Phi(t_0)x_1\|)dv \leq M.N(\|\Phi(s, t_0)x_1\|)
\]

and hence

\[
N(\|\Phi(t, t_0)x_1\|) \leq M.M_0 N(\|\Phi(s, t_0)x_1\|), \text{ for all } t \geq s + 1 \geq t_0 \geq 0 \text{ and } x_1 \in X^1_{t_0}.
\]

Therefore

\[
(t - s - 1)N(\|\Phi(t, t_0)x_1\|) = \int_s^{t-1} N(\|\Phi(t, t_0)x_1\|)ds \leq M.M_0 \int_s^\infty N(\|\Phi(t, t_0)x_1\|)dv \leq M^2.M_0 N(\|\Phi(s, t_0)x_1\|),
\]

which implies

\[
(3.1) \quad N(\|\Phi(t, t_0)x_1\|) \leq \varphi_1(t - s)N(\|\Phi(s, t_0)x_1\|),
\]
for all $t \geq s + 1 \geq s \geq t_0 \geq 0$ and $x_1 \in X^1_{t_0}$, where

$$\varphi_1(v) = \frac{M.M_0(1 + M)}{1 + v}$$

Let $t_0 \geq 0, x_2 \in X^2_{t_0}$ and $s \geq t_0 + 1$. Then

$$\frac{N(||\Phi(s, t_0)x_2||)}{M_0} \leq N(||\Phi(s, t_0)x_2||) \int_{t_0}^{s} \frac{dv}{\psi(s - v)} \leq \int_{t_0}^{s} N(||\Phi(v, t_0)x_2||)dv \leq \int_{t_0}^{t} N(||\Phi(v, t_0)x_2||)dv \leq M.N(||\Phi(t, t_0)x_2||)$$

and hence

$$N(||\Phi(t, t_0)x_2||) \geq \frac{N(||\Phi(s, t_0)x_2||)}{M.M_0} \text{ for all } t \geq s \geq t_0 + 1 \text{ and } x_2 \in X^2_{t_0}.$$

If $t \geq s + 1 \geq s \geq t_0$ then (by preceding inequality and $N d''_3$)

$$N(||\Phi(y, t_0)x_2||) \geq \frac{N(||\Phi(s + 1, t_0)x_2||)}{M.M_0} \geq \frac{mN(||\Phi(s, t_0)x_2||)}{M.M_0} \geq \frac{N(||\Phi(s, t_0)x_2||)}{M_2}$$

for all $x_2 \in X^2_{t_0}$, where $\frac{1}{M_2} = \min\{\frac{1}{M.M_0}, \frac{m}{M.M_0}\}$.

Therefore

$$(t - s - 1)N(||\Phi(t, t_0)x_2||) \leq M_2 \int_{s+1}^{t} N(||\Phi(v, t_0)x_2||)dv \leq M_2 \int_{t_0}^{t} N(||\Phi(v, t_0)x_1||)dv$$

$$M.M_2N(||\Phi(t, t_0)x_2||),$$

which implies

$$(3.2) \quad N(||\Phi(t, t_0)x_2||) \geq \varphi_2(t - s)N(||\Phi(s, t_0)x_2||)$$

for all $t \geq s + 1 \geq s \geq t_0 \geq 0$ and $x_2 \in X^2_{t_0}$, where $\varphi_2 = \frac{m^{s+1}}{M_2(M+1)}$.

From (3.1), (3.2) and Proposition 2.1 it follows that $\Phi(., .)$ is u.e-N-d.

As a particular case we obtain
Corollary 3.1

The evolutionary process $\Phi(\cdot, \cdot)$ is u.e.d. if and only if there are two positive constants $M$ and $m$ such that

\begin{align*}
(d'') & \quad \int_t^\infty \|\Phi(s, t_0) x_1\| ds \leq M \|\Phi(t, t_0) x_1\|, \\
(d'') & \quad \int_{t_0}^t \|\Phi(s, t_0) x_2\| ds \leq M \|\Phi(t, t_0) x_2\|,

\end{align*}

\begin{align*}
(d') & \quad \|\Phi(t+1, t_0) x_2\| ds \geq m \|\Phi(t, t_0) x_2\|,

\end{align*}

for all $t \geq t_0 \geq 0$, $x_1 \in X_{t_0}^1$ and $x_2 \in X_{t_0}^2$.

**Proof.** Is obvious from Theorem 3.1 for $N(t) = t$.

**Remark 3.1** Corollary 3.1 is a nonlinear version of Theorem 3.2 form [3]. It is an extension of Theorem 2.1 from [2] from the general case of uniform exponential dichotomy.

**Remark 3.2.** Corollary 3.1 remains valid if the power 1 from $(d''')$ and $(d''')$ is replaced by any $p \in (1, \infty)$, i.e. the inequalities $(d''')$ and $(d''')$ can be replaced, respectively, by

\begin{align*}
(d''') & \quad \int_t^\infty \|\Phi(s, t_0) x_1\|^p ds \leq M \|\Phi(t, t_0) x_1\|^p

\end{align*}

and

\begin{align*}
(d''') & \quad \int_{t_0}^t \|\Phi(s, t_0) x_2\|^p ds \leq M \|\Phi(t, t_0) x_2\|^p.

\end{align*}

The proof follows almost verbatim from those given in the case $p = 1$ for $N(t) = t$. 
REFERENCES


M. MEGAN AND D. R. LATCU
Department of Mathematics, University of Timisoara
B-dul V. Parvan Nr. 4 1900 - Timisoara
ROMANIA

Manuscrit reçu le 15 juillet 1994