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M. MEGAN
D.R. LATCU

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ON UNIFORM EXPONENTIAL N-DICHOTOMY

M. MEGAN and D.R. LATCU

The problem of uniform exponential N-dichotomy of evolutionary processes in Banach spaces is discussed. Generalizations of the some well-known results of R. Datko, Z. Zabczyk, S. Rollewicz and A. Ichikawa are obtained . The results are applicable for a large class of nonlinear differential equations.

I - INTRODUCTION.

Let X be a real or complex Banach space with the norm $\|.\|$. Let T be the set defined by

$$T = \{(t, t_0) : 0 \leq t_0 \leq t < \infty\}$$

Let $\Phi(t, t_0)$ with $(t, t_0) \in T$ be a family of operators with domain $X_{t_0} \subset X$.

Definition 1.1

The famitily $\Phi(t, t_0)$ with $(t, t_0) \in T$ is called an *evolutionary process* if :

e_1) $\Phi(t, t_0)x_0 \in X_t$ for all (t, t_0) and $x_0 \in X_{t_0}$;

e_2) $\Phi(t, t_1)\Phi(t_1, t_0)x_0 = \Phi(t, t_0)x_0$ for $(t, t_1), (t_1, t_0 \in T$ and $x_0 \in X_{t_0}$;

e_3) $\Phi(t, t)x = x$ for all $t \geq 0$ and $x \in x_t$;

e_4) for each $t_0 \geq 0$ and $x_0 \in X_{t_0}$ the function $t \mapsto \Phi(t, t_0)x_0$ is continuous on $[t_0, \infty]$;

e_5) there is a positive nondecreasing function $\varphi : (0, \infty) \to (0, \infty)$ such that $\|\Phi(t, t_0)x_0\| \leq \varphi(t - t_0)\|x_0\|$ for all $(t, t_0) \in T$ and $x_0 \in X_{t_0}$.

Throughout in this paper for each $t_0 \geq 0$ we denote by

$X^1_{t_0} = \{x_0 \in X_{t_0} : \Phi(., t_0)x_0 \in L^\infty_{t_0}(X)\}$ and $X^2_{t_0} = X^2_{t_0} = X_{t_0} \setminus X^1_{t_0}$

where $L^\infty_{t_0}(X)$ is the Banach space of X-valued function f defined a.e. on $[t_0, \infty]$, such that f is strongly measurable and essentially bounded.

Remark 1.1 If $x_0 \in X^1_{t_0}$ and $t \geq t_0$ then $\Phi(t,t_0)x_0 \in X^1_t$.

Indeed, if $x_0 \in X^1_{t_0}$ then

$\Phi(.,t)\Phi(t,t_0)x_0 = \Phi(.,t_0)x_0 \in L^\infty_{t_0}(X) \subset L^\infty_t(X)$

and hence $\phi(t,t_0)x_0 \in X^1_t$.

Let $\mathcal{N}$ be the set of strictly increasing real functions N defined on $[0,\infty]$ wich satisfies :

$$\lim_{t\to 0} N(t) = 0 \quad \text{and} \quad N(t.t_0) \leq N(t)N(t_0)$$

for all $t, t_0 \geq 0$.

Remark 1.2. It is easy to see that if $N \in \mathcal{N}$ then

i) $N(t) > 0$ for every $t > 0$;

ii) $N(0) = 0$ and $N(1) \geq 1$;

iii) $\lim_{t\to\infty} N(t) = \infty$.

Definition 1.2. Let $N \in \mathcal{N}$. The evolutionary process $\Phi(.,.)$ is said to be *uniformaly exponentially N-dichotomic* (and we write u.e - N-d.) if there are $M_1, M_2, \nu_1, \nu_2 > 0$ such that for all $t \geq s \geq t_0 \geq 0$ and $x_1 \in X^1_{t_0}, x_2 \in X^2_{t_0}$ we have :

$Nd_1)$ $N(\|\Phi(t,t_0)x_1\|) \geq M_1 e^{-\nu_1(t-s)}.N(\|\Phi(s,t_0)x_1\|)$, and

$Nd_2)$ $N(\|(t,t_0)x_2\|) \leq M_2 e^{\nu_2(t-s)}.N(\|\Phi(s,t_0)x_2\|)$.

Particulary, for $N(t) = t$, if $\Phi(.,.)$ is u.e-N-d. then $\Phi(.,.)$ is called an *uniform exponential dichotomic* (and we write u.e.d.) process. If $\Phi(.,.)$ is u.e-N-d. (respectively u.e.d.) and $X^1_{t_0} = X_{t_0}$ for every $t_0 \leq 0$ then $\Phi(.,.)$ is called an *uniform exponential -N-stable* (respectively uniform exponential stable) *process.*

Remark 1.3. $\Phi(.,.)$ is u.e-N-d. if and only if the inequalities (d_1) and (d_2) from Definition 1.2. hold for all $t \leq s+1 > s \leq t_0 \leq 0$.

Indeed, if $t_0 \geq s \geq t \geq s+1, x_1 \in X^1_{t_0}$ and $T_2 \in X^2_{t_0}$ then

$$N(\|\Phi(t,t_0)x_1\|) \leq N(\varphi(t-s)).N(\|\Phi(s,t_0)x_1\|) \geq N(\varphi(1))N(\|(s,t_0)x_1\|) \geq$$
$$\leq N(\varphi(1)).e^{\nu_1-\nu_1(t-s)}.N(\|\Phi(s,t_0)x_1\|)$$

and

$$M_2.e^{\nu_2}.N(\|\Phi(s,t_0)x_2\|) \leq N(\|\Phi(s+1,t_0)x_2\|) \leq$$
$$\leq N(\varphi(s+1-t)).N(\|\Phi(t,t_0)x_2\|) \leq$$
$$\leq N(\varphi(1)).e^{\nu_2-\nu_2(t-s)}.N(\|\Phi(t,t_0)x_2\|).$$

A necessary and sufficient condition for the uniform exponential stability of a linear evolutionary process in a Banach space has been proved by Dakto in [1]. The extension of Datko's theorem for uniform exponential dichotomy has been obtained by Preda and Megan in [3].

The case of uniform exponential-N-stable processes has been considered by Ichikawa in [2]. The particular case when the process is a strongly continuous semigroup of bounded linear operators has been studied by Zabczyk in [5]and Rolewicz in [4].

In this paper we shall extend these results in two directions. First, we shall give a characterization of u.e.-N-dichotomy, which can be considered as a generalization of Datko's theorem. Second, we shall not assume the linearity and boundedness of the process $\Phi(.,.)$. The obtained results are applicable for a large class of nonlinear differential equations described in [2].

II - PRELIMINARY RESULTS

An useful characterization of the uniform exponential-N-dichotomy property is given by

Proposition 2.1

The evolutionary process $\Phi(.,.)$ is u.e-N-d. if and only if there are two continuous functions $\varphi_1, \varphi_2 : [0,\infty] \to (0,\infty)$ with the properties :

$Nd'_1)$ $N(\|\phi(t,t_0)x_1\|) \leq \varphi_1(t-s)N(\|\Phi(s,t_0)x_1\|)$

$Nd'_2)$ $N(\|\phi(t,t_0)x_2\|) \leq \varphi_2(t-s)N(\|\Phi(s,t_0)x_2\|)$

$Nd'_3)$ $\lim\limits_{t\to\infty} \varphi_1(t) = 0$ and $\lim\limits_{t\to\infty} \varphi_2(t) = \infty$ for all $t \geq s \geq t_0 \geq 0$,

$x_1 \in X^1_{t_0}$ *and* $x_2 \in X^2_{t_0}$.

Proof.

The necessity is obvious from Definition 1.2 for $\varphi_1(t) = M_1 e^{-\nu_1 t}$ and $\varphi_2(t) = M_2 e^{\nu_2 t}$.

The sufficiency. From (Nd'_3) it follows that there are $s_1, s_2 > 0$ such that $\varphi_1(s_1) < 1$ and $\varphi_2(s_2) > 1$. Then for all $t \geq s \geq t_0$ there are two natural numbers n_1 and n_2 such that

$$t - s = n_1 s_1 + r_1 = n_2 s_2 + r_2 \text{ , where } r_1 \in [0, s_2].$$

From (e_5) and (Nd'_1) it results that if $t \geq s \geq t_0 \geq 0$ and $x_1 \in X^1_{t_0}$ then

$N(\|\Phi(t,t_0)x_1\|) \leq N(\varphi(r_1))N(\|\Phi(s + n_1 x_1, t_0)x_1\|) \leq N(\varphi(s_1))\varphi_1(s_1)$

$$.N(\|\Phi(s,t_0)x_1\|- \leq M_1 e^{-\nu_1(t-s)}.N\|(\Phi(s,t_0)x_1\|$$

where $M_1 = N(\varphi(s_1))e^{\nu_1 s_1} = \dfrac{N(\varphi(s_1))}{\varphi_1(x_1)}$ and $v_1 = -\dfrac{\ell n\varphi_1(s_1)}{x_1}$.

Similary, if $t \geq s \geq t_0 \geq 0$ and $x_2 \in X_t^2$ then

$N(\|\Phi(t,t_0)x_2\|) \geq \varphi_2(r_2)N(\|(s+n_2s_2,t_0)x_2\|) \geq \varphi_2(r_2)\varphi_2(s_2)^n$.

$N((\|\Phi(s,t_0)x_2\|) \geq m_2 e^{\nu_2 n_2 s_2}.M(\|\Phi(s,t_0)x_2\|)$

$M_2 e^{\nu_2(t-s)}N((\|\Phi(s,t_0)x_2\|)$,

where $m_2 = \inf\limits_{0\leq t\leq s_2} \varphi_2(t)$, $M_2 = \dfrac{m_2}{\varphi_2(s_2)}$ and $\nu_2 = \dfrac{\ell n\varphi_2(s_2)}{s_2}$.

In virtue of Definition 1.2 it follows that $\Phi(.,.)$ is u.e-N-d.

Corollary 2.1.

The evolutionary process $\Phi(.,.)$ is u.e.d. if and only if there are two continuous functions $\varphi_1, \varphi_2 : (0,\infty) \to (0,\infty)$ with the properties :

d_1') $\|\Phi(t,t_0)x_1\| \leq \varphi_1(t-s).\|\Phi(s,t_0)x_1\|$,

d_2') $\|(t,t_0)x_2\| \geq \varphi_2(t-s).\|\phi(s,t_0)x_2\|$,

Nd_3') $\lim\limits_{t\to\infty} \varphi_1(t) = 0$ and $\lim\limits_{t\to\infty} \varphi_2(t) = \infty$.

for all $t \geq s \geq t_0 \geq 0$, $x_1 \in X_{t_0}^1$ and $x_2 \in S_{t_0}^2$.

Proof. : Is obvious from Proposition 2.1 for $N(t) = t$.

The relation between u. e-N-d. and u.e.d. properties is given by

Proposition 2.2 :

The evolutionary process $\Phi(.,.)$ is u.e.d. if and only if there is $N \in \mathcal{N}$ such that $\Phi(.,.)$ is u.e-N-d.

Proof :

The necessity is obvious from Definition 1.2.

The sufficiency. Suppose that there is $N \in \mathcal{N}$ such that $\Phi(.,.)$ satisfies the condition (Nd_1) and (Nd_2) from Definition 1.2.

Let $s_1, s_2, s_3 > 0$ such that $M_1N(2) < e^{\nu_1 s_1}, N(2) < M_2 e^{\nu_2 s_2}$ and $N(s_3) < M_2$. If $t \geq s \geq t_0 0$ then there are two natural numbers n_1 and n_2 such that $t - s = n_1 s_1 + r_1 = n_2 s_2 + r_2$, where $r_1 \in (0, s_1)$ and $r_1 \in (0, s_2)$. Then for $s \geq t_0 \geq 0$ and $x_1 \in X_{t_0}^1$ we have

$$N(\|\Phi(s,t_0)x_1\|) \geq \frac{e^{\nu_1 s_1}}{M_1}N(\|\Phi(s_1+s,t_0x_1)$$
$$\geq N(2)N(\|\Phi(s+s_1,t_0)x_1\|) \geq N(2\|\Phi(s+s_1,t_0)x_1\|)$$

and hence (because N is nondecreasing)

$\|\Phi(s,t_0)x_1\| \geq 2.\|(s+s_1,t_0)x_1\|$ and (by induction)

$\|\Phi(s,t_0)x_1\| \geq 2^n\|\Phi(s+ns_1,t_0)x_1\|$ for every natural number n.

Therefore for $t \geq s \geq t_0 \geq 0$ and $x_1 \in X^1_{t_0}$ we obtain that

$$\|\phi(t,t_0)x_1\| \leq \varphi(r_1)\|\Phi(s+n_1s_1,t_0)x_1\| \leq \frac{\varphi(s_1)}{2^{n_1}}\|\Phi(s,t_0)x_1\|$$

and hence

(2.1) $\|\phi(t,t_0)x_1\| \leq \varphi_1(t-s)\|\Phi(s,t_0)x_1\|$ for $t \geq s \geq t_0$ and $x_1 \in X^1_{t_0}$, where $\varphi_1(u) = \dfrac{\varphi(s_1)}{2^{u/s_1}}$.

On the other hand, for $s \geq t_0 \geq 0$ and $x_2 \in X^2_{t_0}$ we have

$$\|\phi(t,t_0)x_2\| \geq M_2e^{\nu_2 s_2}N(\|\Phi(s,t_0)x_2\|) \geq N(2)N(\|\phi(s,t_0)x_2\|)$$
$$\geq N(2\|\Phi(s,t_0)x_2\|)$$

and hence

$\|\phi(s+s_2,t_0)x_2\| \geq 2\|\phi(s,t_0)x_2\|$ and (by induction)

$\|\phi(s+ns_2,t_0)x_2\| \geq 2^n\|\Phi(s,t_0)x_2\|$ for all $s \geq t_0 \geq 0$, $x_2 \in X^2_{t_0}$ and every natural number n.

hence, if $t \geq s \geq t_0 \geq 0$ and $x_2 \in X^2_{t_0}$ then

$$N(\|\Phi(t,t_0)x_2\|) = N(\|\Phi(s+n_2s_2+r_2,t_0)x_2\|)M_2e^{\nu_2 r_2}N(\|\Phi(s+n_2s_2,t_0)x_2\|)$$
$$\geq N(s_3\|\Phi(s+n_2s_2,t_0)x_2\|),$$

which implies

$\|\Phi(t,t_0)x_2\| \geq s_3\|\Phi(s+n_2s_2,t_0)x_2\| \geq 2^{n_2}.s_3\|\Phi(s,t_0)x_2\|$

and hence

(2.2) $\|\Phi(t,t_0)x_2\| \geq \varphi_2(t-s)\|\Phi(s,t_0)x_2\|$ for $t \geq s \geq 0$ and $x_2 \in X^2_{t_0}$, where $\varphi_2(u) = \dfrac{s_3}{2}2^{u/s_2}$.

From (2.1) , (2.2) and Corollary 2.1 it follows that $\Phi(.,.)$ is u.e.d.

3 - THE MAIN RESULTS.

The following theorem is an extension of Datko's theorem ([1]) to the general case of uniform exponential-N-dichtomy.

Theorem 3.1.

The evolutionary process $\Phi(.,.)$ is u.e-N-d. if and only if there are $M, m > 0$ such that

$$(Nd_1'') \quad \int_t^\infty N(\|\Phi(s,t_0)x_1\|)ds \ \leq \ M.N(\|\Phi(t,t_0)x_1\|),$$

$$(Nd_2'') \quad \int_{t_0}^t N(\|\Phi(s,t_0)x_2\|)dsM.N(\|\Phi(t,t_0)x_2\|),$$

$$(Nd_3'') \quad N(\|\Phi(t+1,t_0)x_2\|)m.N(\|\Phi(t,t_0)x_2\|)$$

for all $t \geq t_0 \geq 0, x_1 \in X_{t_0}^1$ and $x_2 \in X_{t_0}^2$.

Proof. The necessity is simply verified. Now we prove the sufficiency part.

Let $s \geq t_0 \geq 0, x_1 \in X_{t_0}^1$ and $\frac{1}{M_0} = \int_0^1 \frac{dt}{\psi(t)}$, where $\psi = N.\varphi$.

If $t \geq s+1$ then

$$\frac{N(\|\Phi(t,t_0)x_1\|)}{M_0} = \int_0^1 \frac{N(\|\Phi(t,t_0)x_1\|)}{\psi(r)}dr \leq \int_s^1 \frac{N(\|\Phi(t,t_0)x_1\|)}{\psi(t-v)}dr \leq$$

$$\leq \int_s^t N(\|\Phi(v,t_0)x_1\|)dv \leq \int_s^\infty N(\|\Phi(t,t_0)x_1\|)dv \leq M.N(\|\Phi(s,t_0)x_1\|)$$

and hence

$N(\|\Phi(t,t_0)x_1\|) \leq M.M_0 N(\|\Phi(s,t_0)x_1\|)$, for all $t \geq s+1 \geq t_0 \geq 0$ and $x_1 \in X_{t_0}^1$.

Therefore

$$(t-s-1)N(\|\Phi(t,t_0)x_1\|) = \int_s^{t-1} N(\|\Phi(t,t_0)x_1\|)ds \ leqM.M_0 \int_s^\infty N(\|\Phi(t,t_0)x_1\|)dv \leq$$

$$\leq M^2.M_0.N(\|\Phi(s,t_0)x_1\|),$$

which implies

$$(3.1) \quad N(\|\Phi(t,t_0)x_1\|) \leq \varphi_1(t-s)N(\|\Phi(s,t_0)x_1\|),$$

for all $t \geq s+1 \geq s \geq t_0 \geq 0$ and $x_1 \in X^1_{t_0}$, where

$$\varphi_1(v) = \frac{M.M_0(1+M)}{1+v}$$

Let $t_0 \geq 0, x_2 \in X^2_{t_0}$ and $s \geq t_0 + 1$. Then

$$\frac{N(\|\Phi(s,t_0)x_2\|)}{M_0} \leq N(\|\Phi(s,t_0)x_2\|)\int_{t_0}^{s} \frac{dv}{\psi(s-v)} \leq \int_{t_0}^{s} N(\|\Phi(v,t_0)x_2\|)dv \leq$$

$$\leq \int_{t_0}^{t} N(\|\Phi(v,t_0)x_2\|)dv \leq M.N(\|\Phi(t,t_0)x_2\|)$$

and hence

$N(\|\Phi(t,t_0)x_2\|) \geq \dfrac{N(\|\Phi(s,t_0)x_2\|)}{M.M_0}$ for all $t \geq s \geq t_0+1$ and $x_2 \in X^2_{t_0}$.

If $t \geq s+1 \geq s \geq t_0$ then (by preceding inequality and Nd_3'')

$$N(\|\Phi(y,t_0)x_2\|) \geq \frac{N(\|\Phi(s+1,t_0)x_2\|)}{M.M_0} \geq \frac{mN(\|\Phi(s,t_0)x_2\|)}{M.M_0} \geq$$

$$\geq \frac{N(\|\Phi(s,t_0)x_2\|)}{M_2}$$

for all $x_2 \in X^2_{t_0}$, where $\frac{1}{M_2} = min\{\frac{1}{M.M_0}, \frac{m}{M.M_0}\}$.

Therefore

$$(t-s-1)N(\|\Phi(t,t_0)x_2\|) \leq M_2\int_{s+1}^{t} N(\|\Phi(v,t_0)x_2\|)dv \leq M_2\int_{t_0}^{t} N(\|\Phi(v,t_0)x_1\|)dv$$

$$M.M_2N(\|\Phi(t,t_0)x_2\|),$$

which implies

(3.2) $N(\|\Phi(t,t_0)x_2\|) \geq \varphi_2(t-s)N(\|\Phi(s,t_0)x_2\|)$

for all $t \geq s+1 \geq s \geq t_0 \geq 0$ and $x_2 \in X^2_{t_0}$, where $\varphi_2 = \frac{v+1}{M_2.(M+1)}$.

From (3.1) , (3.2) and Proposition 2.1 it follows that $\Phi(.,.)$ is u.e-N-d.

As a particular case we obtain

Corollary 3.1

The evolutionary process $\Phi(.,.)$ is u. e. d. if and only if there are two positive constants M and m such that

$$(d_1'') \quad \int_t^\infty \|\Phi(s,t_0)x_1\|ds \le M.\|\Phi(t,t_0)x_1\|,$$

$$(d_2'') \quad \int_{t_0}^t \|\Phi(s,t_0)x_2\|ds \le M.\|\Phi(t,t_0)x_2\|,$$

$$(d_3'') \quad \|\Phi(t+1,t_0)x_2\|ds \ge m\|\Phi(t,t_0)x_2\|,$$

for all $t \ge t_0 \ge 0, x_1 \in X_{t_0}^1$ and $x_2 \in X_{t_0}^2$.

Proof. Is obvious from Theorem 3.1 for $N(t) = t$.

Remark 3.1 Corollary 3.1 is a nonlinear version of Theorem 3.2 form [3]. It is an extension of Theorem 2.1 from [2]from the general case of uniform exponential dichotomy.

Remark 3.2. Corollary 3.1 remains valid if the power 1 from (d_1'') and (d_2'') is replaced by any $p \in (1,\infty)$, i.e. the inequalities (d_1'') and (d_2'') can by replaced, respectively, by

$$(d_1'') \quad \int_t^\infty \|\Phi(s,t_0)x_1\|^p ds \le M.\|\Phi(t,t_0)x_1\|^p$$

and

$$(d_2'') \quad \int_{t_0}^t \|\Phi(s,t_0)x_2\|^p ds \le M.\|\Phi(t,t_0)x_2\|^p.$$

The proof follows almost verbatim from those given in the case $p = 1$ for $N(t) = t$.

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M. MEGAN AND D. R. LATCU
Department of Mathematics, University of Timisoara
B-dul V. Parvan Nr. 4 1900 - Timisoara
ROMANIA

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