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ON UNIFORM EXPONENTIAL N-DICHOTOMY

M. MEGAN and D.R. LATCU

The problem of uniform exponential N-dichotomy of evolutionary processes in Banach spaces is discussed. Generalizations of the some well-known results of R. Datko, Z. Zabczyk, S. Rollewicz and A. Ichikawa are obtained. The results are applicable for a large class of nonlinear differential equations.

I. - INTRODUCTION.

Let $X$ be a real or complex Banach space with the norm $\| \cdot \|$. Let $T$ be the set defined by

$$T = \{(t, t_0) : 0 \leq t_0 \leq t < \infty\}$$

Let $\Phi(t, t_0)$ with $(t, t_0) \in T$ be a family of operators with domain $X_{t_0} \subset X$.

Definition 1.1

The family $\Phi(t, t_0)$ with $(t, t_0) \in T$ is called an evolutionary process if:

$e_1$) $\Phi(t, t_0)x_0 \in X_t$ for all $(t, t_0)$ and $x_0 \in X_{t_0}$ ;

$e_2$) $\Phi(t, t_1)\Phi(t_1, t_0)x_0 = \Phi(t, t_0)x_0$ for $(t, t_1), (t_0, t_0) \in T$ and $x_0 \in X_{t_0}$ ;

$e_3$) $\Phi(t, t)x = x$ for all $t \geq 0$ and $x \in X_t$ ;

$e_4$) for each $t_0 \geq 0$ and $x_0 \in X_{t_0}$ the function $t \mapsto \Phi(t, t_0)x_0$ is continuous on $[t_0, \infty]$ ;

$e_5$) there is a positive nondecreasing function $\varphi : (0, \infty) \to (0, \infty)$ such that $\| \Phi(t, t_0)x_0 \| \leq \varphi(t - t_0)\| x_0 \|$ for all $(t, t_0) \in T$ and $x_0 \in X_{t_0}$.

Throughout in this paper for each $t_0 \geq 0$ we denote by

$$X^1_{t_0} = \{x_0 \in X_{t_0} : \Phi(., t_0)x_0 \in L^\infty_{t_0}(X)\} \quad \text{and} \quad X^2_{t_0} = X^2_{t_0} = X_{t_0} \setminus X^1_{t_0}$$

where $L^\infty_{t_0}(X)$ is the Banach space of $X$-valued function $f$ defined a.e. on $[t_0, \infty)$, such that $f$ is strongly measurable and essentially bounded.
Remark 1.1. If \( x_0 \in X_{t_0}^1 \) and \( t \geq t_0 \) then \( \Phi(t, t_0)x_0 \in X_t^1 \).

Indeed, if \( x_0 \in X_{t_0}^1 \) then
\[
\Phi(\cdot, t)\Phi(t, t_0)x_0 = \Phi(\cdot, t_0)x_0 \in L_\infty^0(X) \subset L_\infty^t(X)
\]
and hence \( \phi(t, t_0)x_0 \in X_t^1 \).

Let \( N \) be the set of strictly increasing real functions \( N \) defined on \([0, \infty)\) which satisfies:

\[
\lim_{t \to 0} N(t) = 0 \quad \text{and} \quad N(t, t_0) \leq N(t)N(t_0)
\]
for all \( t, t_0 \geq 0 \).

Remark 1.2. It is easy to see that if \( N \in N \) then

i) \( N(t) > 0 \) for every \( t > 0 \);

ii) \( N(0) = 0 \) and \( N(1) \geq 1 \);

iii) \( \lim_{t \to \infty} N(t) = \infty \).

Definition 1.2. Let \( N \in N \). The evolutionary process \( \Phi(\cdot, \cdot) \) is said to be \textit{uniformly exponentially \( N \)-dichotomic} (and we write u.e.-\( N \)-d.) if there are \( M_1, M_2, \nu_1, \nu_2 > 0 \) such that for all \( t \geq s \geq t_0 \geq 0 \) and \( x_1 \in X_{t_0}^1, x_2 \in X_{t_0}^2 \) we have:

\[
Nd_1) N(\| \Phi(t, t_0)x_1 \|) \leq M_1 e^{-\nu_1(t-s)}N(\| \Phi(s, t_0)x_1 \|), \quad \text{and}
\]

\[
Nd_2) N(\| (t, t_0)x_2 \|) \leq M_2 e^{\nu_2(t-s)}N(\| \Phi(s, t_0)x_2 \|).
\]

Particularly, for \( N(t) = t \), if \( \Phi(\cdot, \cdot) \) is u.e.-\( N \)-d. then \( \Phi(\cdot, \cdot) \) is called an \textit{uniform exponential dichotomic} (and we write u.e.d.) process. If \( \Phi(\cdot, \cdot) \) is u.e.-\( N \)-d. (respectively u.e.d.) and \( X_{t_0}^1 = X_{t_0}^2 \) for every \( t_0 \leq 0 \) then \( \Phi(\cdot, \cdot) \) is called an \textit{uniform exponential -\( N \)-stable} (respectively uniform exponential stable) \textit{process}.

Remark 1.3. \( \Phi(\cdot, \cdot) \) is u.e.-\( N \)-d. if and only if the inequalities \( (d_1) \) and \( (d_2) \) from Definition 1.2. hold for all \( t \leq s + 1 > s \leq t_0 \leq 0 \).

Indeed, if \( t_0 \geq s \geq t \geq s + 1, x_1 \in X_{t_0}^1 \) and \( T_2 \in X_{t_0}^2 \) then

\[
N(\| \Phi(t, t_0)x_1 \|) \leq N(\varphi(t-s)).N(\| \Phi(s, t_0)x_1 \|) \geq N(\varphi(1))N(\| (s, t_0)x_1 \|) \geq \leq N(\varphi(1)).e^{\nu_1-\nu_1(t-s)}.N(\| \Phi(s, t_0)x_1 \|)
\]
and

\[
M_2.e^{\nu_2}.N(\| \Phi(s, t_0)x_2 \|) \leq N(\| \Phi(s + 1, t_0)x_2 \|) \leq \leq N(\varphi(s + 1 - t)).N(\| \Phi(t, t_0)x_2 \|) \leq \leq N(\varphi(1)).e^{\nu_2-\nu_2(t-s)}.N(\| \Phi(t, t_0)x_2 \|).
\]
A necessary and sufficient condition for the uniform exponential stability of a linear evolutionary process in a Banach space has been proved by Datko in [1]. The extension of Datko's theorem for uniform exponential dichotomy has been obtained by Preda and Megan in [3].

The case of uniform exponential-N-stable processes has been considered by Ichikawa in [2]. The particular case when the process is a strongly continuous semigroup of bounded linear operators has been studied by Zabczyk in [5] and Rolewicz in [4].

In this paper we shall extend these results in two directions. First, we shall give a characterization of u.e.-N-dichotomy, which can be considered as a generalization of Datko's theorem. Second, we shall not assume the linearity and boundedness of the process \( \Phi(\cdot, \cdot) \). The obtained results are applicable for a large class of nonlinear differential equations described in [2].

II - PRELIMINARY RESULTS

An useful characterization of the uniform exponential-N-dichotomy property is given by

**Proposition 2.1**

The evolutionary process \( \Phi(\cdot, \cdot) \) is u.e.-N-d. if and only if there are two continuous functions \( \varphi_1, \varphi_2 : [0, \infty) \to (0, \infty) \) with the properties:

1. \( \varphi_1(t) \leq \varphi_1(t-s) \leq M_1 e^{\nu_1 t} \) for all \( t > s > 0 \),
2. \( \varphi_2(t) \geq M_2 e^{\nu_2 t} \) for all \( t > s > 0 \),
3. \( \lim_{t \to \infty} \varphi_1(t) = 0 \) and \( \lim_{t \to \infty} \varphi_2(t) = \infty \) for all \( t \geq s \geq t_0 \geq 0 \),
4. \( x_1 \in X_{t_0}^1 \) and \( x_2 \in X_{t_0}^2 \).

**Proof.**

The necessity is obvious from Definition 1.2 for \( \varphi_1(t) = M_1 e^{-\nu_1 t} \) and \( \varphi_2(t) = M_2 e^{\nu_2 t} \).

The sufficiency. From \( (Nd'_3) \) it follows that there are \( s_1, s_2 > 0 \) such that \( \varphi_1(s_1) < 1 \) and \( \varphi_2(s_2) > 1 \). Then for all \( t \geq s \geq t_0 \) there are two natural numbers \( n_1 \) and \( n_2 \) such that

\[
-t - s = n_1 s_1 + r_1 = n_2 s_2 + r_2,
\]

where \( r_1 \in [0, s_2] \). From \( (e_5) \) and \( (Nd'_1) \) it results that if \( t \geq s \geq t_0 \geq 0 \) and \( x_1 \in X_{t_0}^1 \) then

\[
N(\|\Phi(t, t_0)x_1\|) \leq N(\varphi(r_1))N(\|\Phi(s + n_1 x_1, t_0)x_1\|) \leq N(\varphi(s_1))\varphi_1(s_1).
\]
where 

\[ M_1 = N(\varphi(s_1))e^{\nu_1 s_1} = \frac{N(\varphi(s_1))}{\varphi_1(x_1)} \quad \text{and} \quad \nu_1 = -\frac{\ln\varphi_1(s_1)}{x_1}. \]

Similarly, if \( t \geq s \geq t_0 \geq 0 \) and \( x_2 \in X_2^1 \) then

\[ N(\||\Phi(t, t_0)x_2||) \geq \varphi_2(r_2)N(\||\Phi(s, t_0)x_2||) \geq \frac{\varphi_2(r_2)^{\varphi_2(s_2)}}{M_2}. \]

\[ M_2 e^{\nu_2(t-s)}N(\||\Phi(s, t_0)x_2||), \]

where \( m_2 = \inf_{0 \leq t \leq s_2} \varphi_2(t), \ M_2 = \frac{m_2}{\varphi_2(s_2)} \quad \text{and} \quad \nu_2 = \frac{\ln\varphi_2(s_2)}{s_2}. \]

In virtue of Definition 1.2 it follows that \( \Phi(., .) \) is u.e-N-d.

**Corollary 2.1.**

The evolutionary process \( \Phi(., .) \) is u.e.d. if and only if there are two continuous functions \( \varphi_1, \varphi_2 : (0, \infty) \to (0, \infty) \) with the properties:

\[ d_1') \quad \|\Phi(t, t_0)x_1\| \leq \varphi_1(t-s)\|\Phi(s, t_0)x_1\|, \]

\[ d_2') \quad \|\Phi(t, t_0)x_2\| \geq \varphi_2(t-s)\|\phi(s, t_0)x_2\|, \]

\[ N d_2') \lim_{t \to -\infty} \varphi_1(t) = 0 \quad \text{and} \quad \lim_{t \to -\infty} \varphi_2(t) = \infty. \]

for all \( t \geq s \geq t_0 \geq 0, \ x_1 \in X_1^{t_0} \) and \( x_2 \in S_2^{t_0} \).

**Proof.** : Is obvious from Proposition 2.1 for \( N(t) = t \).

The relation between u. e-N-d. and u.e.d. properties is given by

**Proposition 2.2 :**

The evolutionary process \( \Phi(., .) \) is u.e.d. if and only if there is \( N \in \mathcal{N} \) such that \( \Phi(., .) \) is u.e-N-d.

**Proof :**

The necessity is obvious from Definition 1.2.

The sufficiency. Suppose that there is \( N \in \mathcal{N} \) such that \( \Phi(., .) \) satisfies the condition (\( Nd_1) \) and (\( Nd_2) \) from Definition 1.2.

Let \( s_1, s_2, s_3 > 0 \) such that \( M_1 N(2) < e^{\nu_1 s_1}, N(2) < M_2 e^{\nu_2 s_2} \) and \( N(s_3) < M_2. \) If \( t \geq s \geq t_0 > 0 \) then there are two natural numbers \( n_1 \) and \( n_2 \) such that \( t-s = n_1 s_1 + r_1 = n_2 s_2 + r_2, \) where \( r_1 \in (0, s_1) \) and \( r_2 \in (0, s_2). \) Then for \( s \geq t_0 \geq 0 \) and \( x_1 \in X_1^{t_0} \) we have
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\[ N(||\Phi(s, t_0)x_1||) \geq \frac{e^{\nu_1 s_1}}{1} N(||\Phi(s + s_1, t_0)x_1||) \]
\[ \geq N(2)N(||\Phi(s + s_1, t_0)x_1||) \geq N(2||\Phi(s + s_1, t_0)x_1||) \]

and hence (because \( N \) is nondecreasing)

\[ ||\Phi(s, t_0)x_1|| \geq 2||\Phi(s + s_1, t_0)x_1|| \text{ and (by induction)} \]
\[ ||\Phi(s, t_0)x_1|| \geq 2^n||\Phi(s + ns_1, t_0)x_1|| \text{ for every natural number } n. \]

Therefore for \( t \geq s \geq t_0 \geq 0 \) and \( x_1 \in X_{t_0}^1 \) we obtain that

\[ ||\Phi(t, t_0)x_1|| \leq \varphi(r_1)||\Phi(s + n_1s_1, t_0)x_1|| \leq \frac{\varphi(s_1)}{2^{n_1}} ||\Phi(s, t_0)x_1|| \]

and hence

\[ (2.1) \ ||\Phi(t, t_0)x_1|| \leq \varphi_1(t - s)||\Phi(s, t_0)x_1|| \text{ for } t \geq s \geq t_0 \text{ and } x_1 \in X_{t_0}^1, \text{ where } \varphi_1(u) = \frac{\varphi(s_1)}{2^{u/s_1}}. \]

On the other hand, for \( s \geq t_0 \geq 0 \) and \( x_2 \in X_{t_0}^2 \) we have

\[ ||\Phi(t, t_0)x_2|| \geq M_2e^{r_2 s_2}N(||\Phi(s, t_0)x_2||) \geq N(2)||\Phi(s, t_0)x_2|| \]
\[ \geq N(2||\Phi(s, t_0)x_2||) \]

and hence

\[ ||\Phi(s + s_2, t_0)x_2|| \geq 2||\Phi(s, t_0)x_2|| \text{ and (by induction)} \]
\[ ||\Phi(s + ns_2, t_0)x_2|| \geq 2^n||\Phi(s, t_0)x_2|| \text{ for all } s \geq t_0 \geq 0, \ x_2 \in X_{t_0}^2 \text{ and every natural number } n. \]

hence, if \( t \geq s \geq t_0 \geq 0 \) and \( x_2 \in X_{t_0}^2 \) then

\[ N(||\Phi(t, t_0)x_2||) = N(||\Phi(s + n_2s_2 + r_2, t_0)x_2||)M_2e^{r_2 s_2}N(||\Phi(s + n_2s_2, t_0)x_2||) \]
\[ \geq N(s_3||\Phi(s + n_2s_2, t_0)x_2||), \]

which implies

\[ ||\Phi(t, t_0)x_2|| \geq s_3||\Phi(s + n_2s_2, t_0)x_2|| \geq 2^{n_2} s_3||\Phi(s, t_0)x_2|| \]

and hence

\[ (2.2) \ ||\Phi(t, t_0)x_2|| \geq \varphi_2(t - s)||\Phi(s, t_0)x_2|| \text{ for } t \geq s \geq 0 \text{ and } x_2 \in X_{t_0}^2, \text{ where } \varphi_2(u) = \frac{s_3}{2^{u/s_2}}. \]
From (2.1), (2.2) and Corollary 2.1 it follows that $\Phi(.,.)$ is u.e.d.

3 - THE MAIN RESULTS.

The following theorem is an extension of Datko's theorem ([1]) to the general case of uniform exponential-N-dichotomy.

**Theorem 3.1.**

The evolutionary process $\Phi(.,.)$ is u.e.-N-d. if and only if there are $M, m > 0$ such that

\[
(Nd_1') \quad \int_t^\infty N(\|\Phi(s, t_0)x_1\|)ds \leq M.N(\|\Phi(t, t_0)x_1\|),
\]

\[
(Nd_2') \quad \int_{t_0}^t N(\|\Phi(s, t_0)x_2\|)ds M.N(\|\Phi(t, t_0)x_2\|),
\]

\[
(Nd_3') \quad N(\|\Phi(t+1, t_0)x_2\|)m.N(\|\Phi(t, t_0)x_2\|)
\]

for all $t \geq t_0 \geq 0, x_1 \in X_{t_0}^1$ and $x_2 \in X_{t_0}^2$.

**Proof.** The necessity is simply verified. Now we prove the sufficiency part.

Let $s \geq t_0 \geq 0, x_1 \in X_{t_0}^1$ and $\frac{1}{M_0} = \int_0^1 \frac{dt}{\psi(t)}$, where $\psi = N.\psi$.

If $t \geq s + 1$ then

\[
\frac{N(\|\Phi(t, t_0)x_1\|)}{M_0} = \int_0^1 \frac{N(\|\Phi(t, t_0)x_1\|)}{\psi(r)}dr \leq \int_0^1 \frac{N(\|\Phi(t, t_0)x_1\|)}{\psi(t-v)}dr \leq \int_s^t N(\|\Phi(v, t_0)x_1\|)dv \leq \int_s^\infty N(\|\Phi(t, t_0)x_1\|)dv \leq M.N(\|\Phi(s, t_0)x_1\|)
\]

and hence

\[
N(\|\Phi(t, t_0)x_1\|) \leq M.M_0 N(\|\Phi(s, t_0)x_1\|), \text{ for all } t \geq s + 1 \geq t_0 \geq 0 \text{ and } x_1 \in X_{t_0}^1.
\]

Therefore

\[
(t - s - 1)N(\|\Phi(t, t_0)x_1\|) = \int_s^{t-1} N(\|\Phi(t, t_0)x_1\|)ds \leq q.M.M_0 \int_s^\infty N(\|\Phi(t, t_0)x_1\|)dv \leq M^2.M_0 N(\|\Phi(s, t_0)x_1\|),
\]

which implies

\[
(3.1) \quad N(\|\Phi(t, t_0)x_1\|) \leq \varphi(t - s)N(\|\Phi(s, t_0)x_1\|),
\]
for all $t \geq s + 1 \geq s \geq t_0 \geq 0$ and $x_1 \in X^1_{t_0}$, where

$$\varphi_1(v) = \frac{M M_0 (1 + M)}{1 + v}.$$

Let $t_0 \geq 0, x_2 \in X^2_{t_0}$ and $s \geq t_0 + 1$. Then

$$\frac{N(||\Phi(s, t_0)x_2||)}{M_0} \leq N(||\Phi(s, t_0)x_2||) \int_{t_0}^{s} \frac{dv}{\psi(s - v)} \leq \int_{t_0}^{s} N(||\Phi(v, t_0)x_2||)dv \leq \int_{t_0}^{t} N(||\Phi(v, t_0)x_2||)dv \leq M N(||\Phi(t, t_0)x_2||)$$

and hence

$$N(||\Phi(t, t_0)x_2||) \geq \frac{N(||\Phi(s, t_0)x_2||)}{M_0}$$

for all $t \geq s \geq t_0 + 1$ and $x_2 \in X^2_{t_0}$.

If $t \geq s + 1 \geq s \geq t_0$ then (by preceding inequality and $Nd_3$)

$$N(||\Phi(y, t_0)x_2||) \geq \frac{N(||\Phi(s + 1, t_0)x_2||)}{M M_0} \geq \frac{M N(||\Phi(s, t_0)x_2||)}{M M_0} \geq \frac{N(||\Phi(s, t_0)x_2||)}{M_2}$$

for all $x_2 \in X^2_{t_0}$, where $\frac{1}{M_2} = \min\{\frac{1}{M M_0}, \frac{m}{M M_0}\}$.

Therefore

$$(t - s - 1)N(||\Phi(t, t_0)x_2||) \leq M_2 \int_{s+1}^{t} N(||\Phi(v, t_0)x_2||)dv \leq M_2 \int_{t_0}^{t} N(||\Phi(v, t_0)x_1||)dv = M M_2 N(||\Phi(t, t_0)x_2||),$$

which implies

$$N(||\Phi(t, t_0)x_2||) \geq \varphi_2(t - s)N(||\Phi(s, t_0)x_2||)$$

for all $t \geq s + 1 \geq s \geq t_0 \geq 0$ and $x_2 \in X^2_{t_0}$, where $\varphi_2 = \frac{\psi(s+1)}{M_2 (M+1)}$.

From (3.1), (3.2) and Proposition 2.1 it follows that $\Phi(., .)$ is u.e-N-d.

As a particular case we obtain
Corollary 3.1

The evolutionary process $\Phi(.,.)$ is u.e.d. if and only if there are two positive constants $M$ and $m$ such that

\[(d_1') \int_t^\infty \|\Phi(s, t_0)x_1\| ds \leq M \|\Phi(t, t_0)x_1\|,\]

\[(d_2') \int_{t_0}^t \|\Phi(s, t_0)x_2\| ds \leq M \|\Phi(t, t_0)x_2\|,\]

\[(d_3') \|\Phi(t + 1, t_0)x_2\| ds \geq m \|\Phi(t, t_0)x_2\|,\]

for all $t \geq t_0 \geq 0$, $x_1 \in X_{t_0}^1$ and $x_2 \in X_{t_0}^2$.

Proof. Is obvious from Theorem 3.1 for $N(t) = t$.

Remark 3.1: Corollary 3.1 is a nonlinear version of Theorem 3.2 from [3]. It is an extension of Theorem 2.1 from [2] from the general case of uniform exponential dichotomy.

Remark 3.2. Corollary 3.1 remains valid if the power 1 from $(d_1')$ and $(d_2')$ is replaced by any $p \in (1, \infty)$, i.e. the inequalities $(d_1')$ and $(d_2')$ can be replaced, respectively, by

\[(d_1'') \int_t^\infty \|\Phi(s, t_0)x_1\|^p ds \leq M \|\Phi(t, t_0)x_1\|^p,\]

and

\[(d_2'') \int_{t_0}^t \|\Phi(s, t_0)x_2\|^p ds \leq M \|\Phi(t, t_0)x_2\|^p.\]

The proof follows almost verbatim from those given in the case $p = 1$ for $N(t) = t$. 
REFERENCES


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