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EXPONENTIAL-BESSEL PARTIAL DIFFERENTIAL EQUATION AND FOX'S H-FUNCTION

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ABSTRACT: In this paper, we present and solve a two dimensional Exponential-Bessel partial differential equation, and obtain a particular solution of it involving Fox's H-function.

1. - INTRODUCTION. The object of this paper is to formulate a two dimensional Exponential-Bessel partial differential equation and obtain its double series solution. We further present a particular solution of our Exponential-Bessel equation involving Fox's H-function. It is interesting to note that the particular solution also yields a new two dimensional series expansion for Fox's H-function involving exponential functions and Bessel functions.

The H-function introduced by Fox [5, p. 408], will be represented as follows:

\[
H_{m,n}^{p,q} \left[ z \left\{ \left( a_1, e_1, \ldots, a_p, e_p \right) \right\} \right] \equiv H_{p,q}^{n,m} \left[ z \left\{ \left( a_1, e_1 \right) \right\} \right].
\]

In what follows for sake of brevity:

\[
\sum_{j=1}^{p} e_j - \sum_{j=1}^{q} f_j \equiv A, \quad \sum_{j=1}^{n} e_j - \sum_{j=n+1}^{p} e_j + \sum_{j=1}^{m} f_j - \sum_{j=m+1}^{q} f_j \equiv B.
\]

The following formulae are required in the proof:
The integral [2, p. 704, (2.2)] :

\[(1.2) \int_0^\Pi \cos 2ux (\sin \frac{x}{2})^{-2w_1} H_{p,q}^{m,n} \left[ z (\sin \frac{x}{2})^{-2h} (a_p, e_p) \right] dx = \sqrt{(\Pi)} H_{p+2,q+2}^{m+1,n+1 \to m+2,q+2} \left[ z \left( 1-w_1-2u, h \right), (a_p, e_p), (1-w_1+2u, h) \right], \]

where \( h > 0, \sum_{j=1}^p e_j - \sum_{j=1}^q f_j \equiv A \leq 0, \sum_{j=n+1}^n e_j - \sum_{j=1}^m e_j - \sum_{j=m+1}^q f_j \equiv B > 0, \)

\(|\text{arg}z| < 1/2B\Pi, \Re(1-2w_1) - 2h \max_{1\leq j \leq n} [\Re(a_j - 1)/e_j] > 0. \)

The integral [7, p. 94, (2.2)] :

\[(1.3) \int_0^\infty y^{w_2-1} \sin y J_\nu(y) H_{p,q}^{m,n} \left[ z y^{2k} (a_p, e_p) \right] dy = 2^{w_2-1} \sqrt{\Pi} H_{p+4,q+1}^{m+1,n+1 \to m+1,q+1} \left[ 2^{2k_z} \left( \frac{1-w_2-v}{2}, k \right), (a_p, e_p), (1 + \frac{v-w_2}{2}, k) \right], \]

where \( k > 0, A \leq 0, B > 0, |\text{arg}z| < 1/2B\Pi, \Re(w_2 + v) + 2k \min_{1\leq j \leq m} [\Re b_j / f_j] > 0. \)

The orthogonality property of the Bessel functions [6, p. 291, (6)] :

\[(1.4) \int_0^\infty x^{-1} J_{a+2n+1}(x) J_{a+2m+1}(x) dx = \begin{cases} 0, & \text{if } m \neq n; \\ (4n + 2a + 2)^{-1}, & \text{if } m = n, \Re a + m + n > -1. \end{cases} \]

The following orthogonality property :

\[(1.5) \int_0^\Pi e^{2imx} \cos 2nx dx = \begin{cases} 0, & m \neq n; \\ \Pi/2, & m = n \neq 0; \\ \Pi, & m = n = 0. \end{cases} \]
2. TWO DIMENSIONAL EXPONENTIAL-BESSEL PARTIAL DIFFERENTIAL EQUATION

Let us consider

\[
\frac{\partial u}{\partial t} = c \frac{\partial^2 u}{\partial x^2} + y^2 \frac{\partial^2 u}{\partial y^2} + y \frac{\partial u}{\partial y} + y^2 u,
\]

where \( u \equiv u(x, y, t) \) and \( u(x, y, 0) = f(x, y) \).

To solve (2.1), we assume that (2.1) has a solution of the form:

\[
u(x, y, t) = e^{4t c r^2 t + (v + 2s + 1)^2 t} X(i x) Y(y).
\]

The substitution of (2.2) into (2.1) yields:

\[
-c \left[ X'' + 4r^2 X \right] Y + X \left[ y^2 Y'' + y Y' + \{ y^2 - (v + 2s + 1)^2 \} Y \right] = 0.
\]

We see that \( X'' + 4r^2 X = 0 \) has a solution \( X = e^{2riz} \) and \( y^2 Y'' + y Y' + \{ y^2 - (v + 2s + 1)^2 \} Y = 0 \) is Bessel equation \([1, p. 200, (6.25)]\), with solution \( Y = J_{v + 2s + 1}(y) \). Therefore the solution of (2.1) is of the form:

\[
u(x, y, t) = e^{4t c r^2 t + (v + 2s + 1)^2 t} e^{2riz} J_{v + 2s + 1}(y).
\]

In view of the principle of superposition, the general solution of (2.1) is given by

\[
u(x, y, t) = \sum_{r = -\infty}^{\infty} \sum_{s = 0}^{\infty} A_{r,s} e^{4t c r^2 t + (v + 2s + 1)^2 t + 2riz} J_{v + 2s + 1}(y).
\]

In (2.5), putting \( t = 0 \), we get

\[
f(x, y) = \sum_{r = -\infty}^{\infty} \sum_{s = 0}^{\infty} A_{r,s} e^{2riz} J_{v + 2s + 1}(y).
\]

Multiplying both sides of (2.6) by \( y^{-1} \cos 2ux J_{v + 2u + 1}(y) \), integrating with respect to \( y \) from 0 to \( \infty \) and with respect to \( x \) from 0 to \( \Pi \), then using (1.4) and (1.5), the Fourier Exponential-Bessel coefficients are given by

\[
A_{r,s} = \frac{4}{\Pi (v + 2s + 1)} \times \int_0^{\Pi} \int_0^{\infty} f(x, y) y^{-1} \cos 2ux J_{v + 2s + 1}(y) dy dx.
\]
In view of the theory of double and multiple Fourier series given by Carslaw and Jaeger [3, pp. 180-183], and many other references, such as Erdélyi [4, pp. 64-65] etc., the double series (2.6) is convergent, provided the function $f(x, y)$ is defined in the region $0 < x < \pi, 0 < y < \infty$. In brief, the double series (2.6) converges, if the double integral on the right hand side of (2.7) exists.

In the subsequent section, we take $f(x, y)$ as Fox's $H$-function and present another method to obtain Fourier exponential-Bessel coefficients $A_{r,s}$.

3. PARTICULAR SOLUTION INVOLVING FOX'S $H$-FUNCTION

The particular solution to be obtained is

$$(3.1) \quad u(x, y, t) = 2^{W_2 + 1} \sum_{r=\infty}^{\infty} \sum_{s=0}^{\infty} e^{4cr^2t + (v + 2s + 1)^2t + 2rix(v + 2 + s + 1)} j_{v+2s+1}(y)$$

$$\times H^{m+2,n+2}_{p+6,q+3} \left[ \begin{array}{c}
(1 - w_1 - 2r, h), \left( -\frac{w_1 + v + 2s}{2}, k \right), (a_p, c_p), \\
(1 - w_1 + 2r, h), \left( 1 + \frac{v + 2s + 1 - w_1}{2}, k \right), \\
\left( 1 + \frac{v + 2s + 1 + w_2}{2}, k \right), \left( 1 + \frac{v + 2s - w_2}{2}, k \right), \\
\left( \frac{1}{2} - w, h \right), \left( \frac{1}{2} - w_2, k \right), (b_q, f_q) \end{array} \right]$$

valid under the conditions of (1.2), (1.3) and (1.4).

Proof. Let

$$(3.2) \quad f(x, y) = \left( \sin \frac{x}{2} \right)^{-2w_1} y^{w_2} \sin y H^{m,n}_{p,q} \left[ z \left( \sin \frac{x}{2} \right)^{-2h} y^{2k} \right]$$

$$= \sum_{r=-\infty}^{\infty} \sum_{s=0}^{\infty} A_{r,s} e^{2irx} J_{v+2s+1}(y).$$

Equation (3.2) is valid, since $f(x, y)$ is defined in the region $0 < x < \Pi, 0 < y < \infty$.

Multiplying both sides of (3.2) by $y^{-1} J_{v+2w+1}(y)$ and integrating with respect to $y$ from $0$ to $\infty$, then using (1.3) and (1.4). Now multiplying both sides of the resulting
expression by $\cos 2ux$ and integrating with respect to $x$ from 0 to $\Pi$, then using (1.2) and (1.5), we obtain the value of $A_{r,s}$. Substituting this value of $A_{r,s}$ in (2.5), the expansion (3.1) is obtained.

**NOTE 1** : The value of $A_{0,s}$ is one-half the value of $A_{r,s}$.

**NOTE 2** : If we put $t = 0$ in (3.1), it reduces to a new two dimensional series expansion for Fox's $H$-function involving exponential functions and Bessel functions.

Since on specializing the parameters Fox's $H$-function yields almost all special functions appearing in applied mathematics and physical sciences. Therefore, the result (3.1) presented in this paper is of a general character and hence may encompass several cases of interest.
REFERENCES


