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*p-adic Clifford algebras*


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P-ADIC CLIFFORD ALGEBRAS

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In a previous paper [2], we gave the index of the standard quadratic form of rank $n$ over the field of $p$-adic numbers. Here, we recover, as a consequence, the structure of the associated Clifford algebra.

The classification of all (equivalence classes of) quadratic forms over a $p$-adic field is well known (cf. [5]), with this classification, one is able to classify all $p$-adic Clifford algebras.

I - INTRODUCTION

Let $K$ be a field of characteristic $\neq 2$ and $E$ a vector space over $K$ of finite dimension $n$. A mapping $q : E \to K$ is a quadratic form over $E$ if there exists a bilinear symmetric form $f : E \times E \to K$ such that

$$q(x) = f(x, x) \quad \text{and} \quad f(x, y) = \frac{1}{2}[q(x + y) - q(x) - q(y)]$$

We assume that $q$ is regular, that is $f$ is non-degenerated.

An element $x \in E$ is isotropic if $q(x) = 0$. Let $V$ be a subspace of $E$; the orthogonal subspace of $V$ is the set $V^\perp = \{y \in E | f(x, y) = 0 \text{ for all } x \in V\}$. The subspace $V$ is called totally isotropic if $V \subset V^\perp$. It is well known (cf. for example [1]) that any totally isotropic subspace is contained in a maximal totally isotropic subspace. The maximal totally isotropic subspaces have the same dimension $\nu$, called the index of $q$ and $2\nu \leq n$. If $2\nu = n$, then $(E, q)$ is called a hyperbolic space and for the case $n = 2$, one says hyperbolic plane. The index $\nu = 0$ iff $q(x) \neq 0$ for $x \neq 0$ i.e. $(E, q)$ is anisotropic.
Let $E = K^n$ and $B = (e_1, \ldots, e_n)$ be the canonical basis of $E$; the standard quadratic form $q_0$ is the quadratic form associated to the bilinear form

$$< x, y > = \sum_{j=1}^{n} x_j y_j ; \text{ where } x = \sum_{j=1}^{n} x_j e_j \text{ and } y = \sum_{j=1}^{n} y_j e_j ;$$

hence $q_0(x) = <x, x> = \sum_{j=1}^{n} x_j^2$.

Let $(E, q)$ be a quadratic space, possibly non regular; an algebra $C = C(E, q)$ over $K$, with unit 1, is said to be a Clifford algebra for $(E, q)$ if

(i) There exists a one-to-one linear mapping $\rho : E \to C$ such that $\rho(x)^2 = q(x) \cdot 1$.

(ii) For every algebra $A$ with unit 1 and linear mapping $\phi : E \to A$ satisfying $\phi(x)^2 = q(x) \cdot 1$, there exists an algebra homomorphism $\phi : C \to A$ such that $\phi \circ \rho = \phi$.

Clifford algebra exists and is unique up algebra isomorphism (cf. for instance [1] or [3]). For example, let $K < X_1, \ldots, X_n >$ be the free algebra with free system of generators $X_1, \ldots, X_n$ and $I$ be the two-sided ideal of $K < X_1, \ldots, X_n >$ generated by $X_iX_j + X_jX_i - 2f(e_i, e_j) \cdot 1, 1 \leq i, j \leq n$, where $(e_1, \ldots, e_n)$ is an orthogonal basis of $(E, q)$; then $C(E, q) = K < X_1, \ldots, X_n > / I$.

II - THE P-ADIC STANDARD QUADRATIC FORM $q_0$

II - 1. The index of $q_0$

Let $p$ be a prime number and $\mathbb{Q}_p$ be the p-adic field i.e. the completion of the field of rational numbers $\mathbb{Q}$ for the p-adic absolute value.

We denote by $[\alpha]$ the integral part of the real number $\alpha$.

Proposition 1 [2]

The standard quadratic form $q_0(x) = \sum_{j=1}^{n} x_j^2$ over $E = \mathbb{Q}_p^n$ has index

(i) $\nu = \left[ \frac{n}{2} \right]$ if $p \equiv 1 \pmod{4}$

(ii) $\nu = \left[ \frac{n}{2} \right]$ if $p \equiv 3 \pmod{4}$ and $n \not\equiv 2 \pmod{4}$

(iii) $\nu = \left[ \frac{n}{2} \right] - 1$ if $p \equiv 3 \pmod{4}$ and $n \equiv 2 \pmod{4}$
Proof:

1°) If $p \equiv 1 \pmod{4}$, it is well known that $i = \sqrt{-1} \in \mathbb{Q}_p$. Let $\nu = \left[ \frac{n}{2} \right]$ and $e_j = i e_{2j-1} + e_{2j}$, $1 \leq j \leq \nu$, then $V = \bigoplus_{j=1}^{\nu} \mathbb{Q}_p e_j$ is a maximal totally isotropic subspace of $E = \mathbb{Q}_p^n$.

2°) $p \equiv 3 \pmod{4}$

Therefore $i \notin \mathbb{Q}_p$ and if $n = 2$ the index of $q_0$ is 0.

If $n = 3$, applying Chevalley's theorem and Newton's method to $q_0(x) = x_1^2 + x_2^2 + x_3^2$ we find $a, b \in \mathbb{Q}_p$, $a \neq 0$, $b \neq 0$, such that $a^2 + b^2 + 1 = 0$. Therefore $e_1 = a e_1 + b e_2 + e_3$ is isotropic in $\mathbb{Q}_p^3$ and $\nu = \left[ \frac{3}{2} \right] = 1$.

(a) For $n = 4m$, put $e_{2j-1} = a e_{4j-3} + b e_{4j-2} + e_{4j-1}$ and $e_{2j} = -b e_{4j-3} + a e_{4j-2} + e_{4j}$, $1 \leq j \leq m$. It is clear that $q_0(e_{2j-1}) = q_0(e_{2j}) = a^2 + b^2 + 1 = 0$ and $< e_{2j-1}, e_{2j} > = -ab + ab = 0$. Therefore $V = \bigoplus_{j=1}^{m} (\mathbb{Q}_p e_{2j-1} \oplus \mathbb{Q}_p e_{2j})$ is a totally isotropic subspace of $\mathbb{Q}_p^n$ and $\nu = 2m = \left[ \frac{3}{2} \right]$.

If $n = 4m + 1$, with the same notations as above the subspace $V$ is totally isotropic in $\mathbb{Q}_p^n$ and $\nu = 2m = \left[ \frac{3}{2} \right]$.

On the other hand if $n = 4m + 3$ the subspaces $V = \bigoplus_{j=1}^{m} (\mathbb{Q}_p e_{2j-1} \oplus \mathbb{Q}_p e_{2j})$ and $\mathbb{Q}_p e_{2m+1}$ where $e_{2m+1} = a e_{4m+1} + b e_{4m+2} + e_{4m+3}$, are totally isotropic and orthogonal. Therefore $V_0 = V \oplus \mathbb{Q}_p e_{2m+1}$ is totally isotropic and $\nu = 2m + 1 = \left[ \frac{n}{2} \right]$.

(b) If $n = 4m + 2$, let $V = \bigoplus_{j=1}^{m} (\mathbb{Q}_p e_{2j-1} \oplus \mathbb{Q}_p e_{2j})$ be as above. It is easy to verify that if $x \in \mathbb{Q}_p^n$ is isotropic and $x$ is orthogonal to $V$ then $x \in V$. Therefore $V$ is a maximal totally isotropic subspace of $\mathbb{Q}_p^n$ and $\nu = 2m = \left[ \frac{n}{2} \right] - 1$. 
Proposition 2: Let $p = 2$.

Let $n = 8m + s$, $0 \leq s \leq 7$.

The standard quadratic form $q_0(x) = \sum_{j=1}^{n} x_j^2$ over $E = \mathbb{Q}_2^n$ has index

(i) $\nu = 4m$ if $0 \leq s \leq 4$

(ii) $\nu = 4m + t$ if $s = 4 + t$, $1 \leq t \leq 3$

Proof:

1°) If $1 \leq n \leq 4$, then the index of $q_0$ is 0.

Indeed, this is clear when $n = 1$.

If $n = 2$, let $x = x_1e_1 + x_2e_2 \in \mathbb{Q}_2^2$ be isotropic and different from 0 i.e. $q_0(x) = x_1^2 + x_2^2 = 0$ and say $x_2 \neq 0$. Therefore $1 + a^2 = 0$ with $a = x_1x_2^{-1}$ and $v_2(a) = 0$ i.e. $a = 1 + 2^\mu a_0$, $\mu \geq 1$, $v_2(a_0) = 0$.

Then $1 + a^2 = 2 + 2^{\mu+1}a_0 + 2^{2\mu}a_0^2 = 0$ or $1 + 2^\mu a_0 + 2^{2\mu-1}a_0 = 0$; in other words $1 \equiv 0 \pmod{2}$; a contradiction.

In the same way, one shows that if $n = 3$ or 4, the index of $q_0$ is 0.

2°) $n = 5$

Let $x_0 = 2e_1 + e_2 + e_3 + e_4 + e_5 \in \mathbb{Q}_2^5$, then $q_0(x_0) = 8$ and $\frac{\partial q_0}{\partial x_j}(x_0) = 2 \neq 0 \pmod{4}$, $2 \leq j \leq 5$.

By Newton’s method there exists $x = \sum_{j=1}^{5} x_je_j \in \mathbb{Q}_2^5$ such that $q_0(x) = 0$ with $x_1 \equiv 2 \pmod{8}, x_j \equiv 1 \pmod{8}, 2 \leq j \leq 5$.

Put $a = x_1x_5^{-1}$, $b = x_2x_5^{-1}$, $c = x_3x_5^{-1}$, $d = x_4x_5^{-1}$, then $a^2 + b^2 + c^2 + d^2 + 1 = 0$.

The two following elements of $\mathbb{Q}_2^5$

\[ e_1 = a \ e_1 + b \ e_2 + c \ e_3 + d \ e_4 + e_5 \]
\[ e'_1 = -a \ e_1 - b \ e_2 - c \ e_3 - d \ e_4 + e_5 \]

are isotropic with $<e_1, e'_1> = 2$. Hence $H = \mathbb{Q}_2e_1 \oplus \mathbb{Q}_2e'_1$ is a hyperbolic plane in $\mathbb{Q}_2^5$. Let $U = H^\perp$ be the orthogonal subspace of $H$ in $\mathbb{Q}_2^5$. The following three elements of $\mathbb{Q}_2^5$:

\[ u_1 = b \ e_1 - a \ e_2 + d \ e_3 - c \ e_4 \]
\[ u_2 = e_1 - \frac{ac + bd}{c^2 + d^2} e_3 + \frac{bc - ad}{c^2 + d^2} e_4 \]
$u_3 = e_2 + \frac{ad - bc}{c^2 + d^2} e_3 - \frac{ac + bd}{c^2 + d^2} e_4$

are elements of $U$, with

$q_0(u_1) = -1$, $q_0(u_2) = -\frac{1}{c^2 + d^2} = q_0(u_3)$

Furthermore $\langle u_i, u_j \rangle = 0$ if $1 \leq i \neq j \leq 3$, and $(u_1, u_2, u_3)$ is a basis of $U$.

For every $u = y_1 u_1 + y_2 u_2 + y_3 u_3 \in U$ we have

$q_0(u) = y_1^2 q_0(u_1) + y_2^2 q_0(u_2) + y_3^2 q_0(u_3) = -\frac{c^2 y_1^2 + d^2 y_1^2 + y_2^2 + y_3^2}{c^2 + d^2}$

and $q_0(u) = 0$ iff $u = 0$ because the standard quadratic form of rank 4 is anisotropic. In other words $(U, q_0)$ is anisotropic and $Q_2^5 = H \perp U$ is a Witt decomposition of $(Q_2^5, q_0)$. Hence the index of $q_0$ is 1.

3°) $n = 8m + s$, $0 \leq s \leq 4$.

Put, for $0 \leq j \leq m - 1$

$$
\begin{align*}
\begin{cases}
\epsilon_{j,1} & = a e_{8j+1} + b e_{8j+2} + c e_{8j+3} + d e_{8j+4} + e_{8j+5} \\
\epsilon_{j,2} & = -b e_{8j+1} + a e_{8j+2} + d e_{8j+3} - c e_{8j+4} + e_{8j+6} \\
\epsilon_{j,3} & = -d e_{8j+1} + c e_{8j+2} - b e_{8j+3} + a e_{8j+4} + e_{8j+7} \\
\epsilon_{j,4} & = c e_{8j+1} + d e_{8j+2} - a e_{8j+3} - b e_{8j+4} + e_{8j+8}
\end{cases}
\end{align*}
$$

(1)

and

$$
\begin{align*}
\begin{cases}
\epsilon'_{j,1} & = -a e_{8j+1} - b e_{8j+2} - c e_{8j+3} - d e_{8j+4} + e_{8j+5} \\
\epsilon'_{j,2} & = b e_{8j+1} - a e_{8j+2} - d e_{8j+3} + c e_{8j+4} + e_{8j+6} \\
\epsilon'_{j,3} & = d e_{8j+1} - c e_{8j+2} + b e_{8j+3} - a e_{8j+4} + e_{8j+7} \\
\epsilon'_{j,4} & = -c e_{8j+1} - d e_{8j+2} + a e_{8j+3} + b e_{8j+4} + e_{8j+8}
\end{cases}
\end{align*}
$$

(2)

A straightforward computation shows that $\langle \epsilon_{i,k}, \epsilon_{j,l} \rangle = 0 \Rightarrow \langle \epsilon'_{i,k}, \epsilon'_{j,l} \rangle$, $0 \leq i, j \leq m - 1$ ; $1 \leq k, l \leq 4$ and $\langle \epsilon_{j,l}, \epsilon'_{j,l} \rangle = 2$ ; $0 \leq j \leq m - 1$ ; $1 \leq l \leq 4$. Furthermore $\langle \epsilon_{i,k}, \epsilon'_{j,l} \rangle = 0$ if $(i, k) \neq (j, l)$.

Hence the subspaces $V = \bigoplus_{j=0}^{m-1} Q_2 \epsilon_{j,l}$ and $W = \bigoplus_{j=0}^{m-1} Q_2 \epsilon'_{j,l}$ are isotropic with

$V \cap W = (0)$

Therefore $H = V \oplus W$ is a hyperbolic subspace of $E = Q_2^{8m+s}$, with $\dim V = \dim W = 4m$.  


But $E = E_m \perp E_s$ (orthogonal sum) where $E_m = \bigoplus_{j=1}^{8m} Q_2 e_j$ and $E_s = \bigoplus_{k=1}^{s} Q_2 e_{8m+k} \simeq Q_2^s$.

If $s = 0$, we have $E = E_m = V \oplus W = H$ and $(E, q_0)$ is a hyperbolic space with index $4m$.

If $1 \leq s \leq 4$; $E = E_m \perp E_s$ with $E_m = V \oplus W = H$. Since $1 \leq \dim E_s = s \leq 4$, the standard quadratic space $(E_s, q_0)$ is anisotropic. Consequently $E = (V \oplus W) \perp E_s$ is a Witt decomposition of $E$ and the index of $q_0$ is $4m$.

4°) $n = 8m + 4 + t$, $1 \leq t \leq 3$.

a) $n = 8m + 5$

With the same notations as above, we have $E = E_m \perp E_5$ where $E_5 = \bigoplus_{k=1}^{5} Q_2 e_{8m+k} \simeq Q_2^5$.

Let us write, as for $n = 5$,

\begin{align*}
\begin{cases}
    e_{4m+1} = a e_{8m+1} + b e_{8m+2} + c e_{8m+3} + d e_{8m+4} + e e_{8m+5} \\
    e'_{4m+1} = -a e_{8m+1} - b e_{8m+2} - c e_{8m+3} - d e_{8m+4} + e e_{8m+5}
\end{cases}
\end{align*}

and

\begin{align*}
\begin{cases}
    u_{m+1} = b e_{8m+1} - a e_{8m+2} + d e_{8m+3} - c e_{8m+4} \\
    u_{m+2} = e_{8m+1} - \frac{ad-bc}{a^2+b^2} e_{8m+3} + \frac{bc-ad}{a^2+b^2} e_{8m+4} \\
    u_{m+3} = e_{8m+2} + \frac{ad-bc}{a^2+b^2} e_{8m+3} - \frac{bc-ad}{a^2+b^2} e_{8m+4}
\end{cases}
\end{align*}

The subspace $U_5 = \bigoplus_{h=1}^{3} Q_2 u_{m+h}$ of $E_5$ is anisotropic. On the other hand, $q_0(e_{4m+1}) = 0 = q_0(e'_{4m+1})$; $<e_{4m+1}, e'_{4m+1}> = 2$ and $e_{4m+1}, e'_{4m+1}$ are orthogonal to $U_5$. Therefore $V_0 = V \oplus Q_2 e_{4m+1}$ and $W_0 = W \oplus Q_2 e'_{4m+1}$ are isotropic subspaces of $E$ and $E = (V_0 \oplus W_0) \perp U_5$ is a Witt decomposition of $E$. Hence the index of $q_0$ is $\dim V_0 = \dim W_0 = 4m+1$.

(b) $n = 8m + 6$.

As before, we have $E = E_m \perp E_6$ where $E_6 = \bigoplus_{k=1}^{6} Q_2 e_{8m+k} \supset E_5$; hence $e_{4m+1}$ and $e'_{4m+1} \in E_6$. 
Let us put
\begin{align}
\begin{cases}
\epsilon_{4m+2} &= -b \epsilon_{8m+1} + a \epsilon_{8m+2} + d \epsilon_{8m+3} - c \epsilon_{8m+4} + \epsilon_{8m+6} \\
\epsilon_{4m+2}' &= b \epsilon_{8m+1} - a \epsilon_{8m+2} - d \epsilon_{8m+3} + c \epsilon_{8m+4} + \epsilon_{8m+6}
\end{cases}
\end{align}

and
\begin{align}
\begin{cases}
\omega_{m+1} &= \epsilon_{8m+1} + \frac{bd-ac}{c^2+d^2} \epsilon_{8m+3} - \frac{ad-bc}{c^2+d^2} \epsilon_{8m+4} \\
\omega_{m+2} &= \epsilon_{8m+2} - \frac{bc+ad}{c^2+d^2} \epsilon_{8m+3} + \frac{ac-bd}{c^2+d^2} \epsilon_{8m+4}
\end{cases}
\end{align}

The subspace $U_6 = \mathbb{Q}_2 \omega_{m+1} \oplus \mathbb{Q}_2 \omega_{m+2}$ of $E_6$ is anisotropic. Moreover, $q_0(\epsilon_{4m+2}) = 0 = q_0(\epsilon_{4m+2}')$; $\langle \epsilon_{4m+2}, \epsilon_{4m+2}' \rangle = 2$ and $\epsilon_{4m+2}, \epsilon_{4m+2}'$ are orthogonal to $U_6$. Therefore $V_1 = V_0 \oplus \mathbb{Q}_2 \epsilon_{4m+2}$ and $W_1 = W_0 \oplus \mathbb{Q}_2 \epsilon_{4m+2}'$ are isotropic subspaces of $E$ and $E = (V_1 \oplus W_1) \perp U_6$ is a Witt decomposition of $E$. Hence the index of $q_0$ is $\dim V_1 = \dim W_1 = 4m + 2$.

(c) $n = 8m + 7$.

We have $E = E_7 \perp E_7$, where $E_7 = \bigoplus_{k=1}^{7} \mathbb{Q}_2 \epsilon_{8m+k} \supset E_6$.

Let us write
\begin{align}
\begin{cases}
\epsilon_{4m+3} &= -d \epsilon_{8m+1} + c \epsilon_{8m+2} - b \epsilon_{8m+3} + a \epsilon_{8m+4} + \epsilon_{8m+7} \\
\epsilon_{4m+3}' &= d \epsilon_{8m+1} - c \epsilon_{8m+2} + b \epsilon_{8m+3} - a \epsilon_{8m+4} + \epsilon_{8m+7}
\end{cases}
\end{align}

and
\begin{align}
\mathbf{u}_m &= c \epsilon_{8m+1} + d \epsilon_{8m+2} - a \epsilon_{8m+3} - b \epsilon_{8m+4}
\end{align}

The subspace $U_7 = \mathbb{Q}_2 \mathbf{u}_m$ of $E_7$ is anisotropic. Furthermore $q_0(\epsilon_{4m+3}) = 0 = q_0(\epsilon_{4m+3}')$; $\langle \epsilon_{4m+3}, \epsilon_{4m+3}' \rangle = 2$ and $\epsilon_{4m+3}, \epsilon_{4m+3}'$ are orthogonal to $U_7$. Therefore $V_2 = V_1 \oplus \mathbb{Q}_2 \epsilon_{4m+2}$ and $W_2 = W_1 \oplus \mathbb{Q}_2 \epsilon_{4m+2}'$ are isotropic subspaces of $E$ and $E = (V_2 \oplus W_2) \perp U_7$ is a Witt decomposition of $E$. Hence the index of $q_0$ is $\dim V_2 = \dim W_2 = 4m + 3$.

Remark

Let $K$ be a non formally real field. The level of $K$ is the least integer $s$ such that $-1 = \sum_{j=1}^{s} a_j^2$ where $a_j \in K, a_j \neq 0$. It is well known that $s = 2^r, r \geq 0$ (c.f. [3] or [4]).
The level of a $p$-adic field is $1$ if $p \equiv 1 \pmod{4}$; $2$ if $p \equiv 3 \pmod{4}$ and $4$ if $p = 2$.

If the level of a field $K$ is $1$ (resp. $2$, resp. $4$) then the index of the standard quadratic form over $K^n$ is given by Proposition 1 - (i) [resp. Prop.1 - (ii) - (iii), resp. Prop.2].

More generally let $K$ be a field of level $s = 2^r, r \geq 0$. If we write for any integer $n, n = m2^{r+1} + a$ where $0 \leq a \leq 2^{r+1} - 1$; then the index of the standard quadratic form over $K^n$ is

(i) $\nu = m2^r$ if $0 \leq a \leq 2^r$
(ii) $\nu = m2^r + t$ if $a = 2^r + t, 1 \leq t \leq 2^r - 1$.

II - 2 The Clifford algebra $C(Q_p^n, q_0)$

The following results can be deduced from a general setting (cf. [3] p. 128-129). Here we establish them by using the computation of the index of $q_0$ made in II-1.

Let us recall that if $E$ is a vector space over a field $K$ then the exterior algebra $\Lambda(E)$ is the Clifford algebra associated to the null quadratic form over $E$.

On the other hand, let $(E, q)$ be a regular quadratic space over $K$. If $E = V \oplus W$ is a hyperbolic space ($V$ and $W$ being maximal totally isotropic subspaces), it is well known that the Clifford algebra $C(E, q)$ is isomorphic to $\text{End}(\Lambda(V))$, the space of linear endomorphisms of the vector space $\Lambda(V)$. Furthermore the subalgebra of the even elements of $C(E, q)$, say $C_+(E, q)$ is isomorphic to $\text{End}(\Lambda_+(V)) \times \text{End}(\Lambda_-(V))$ where $\Lambda_+(V)$ (resp. $\Lambda_-(V)$) is the subspace of the even (resp. odd) elements of $\Lambda(V)$.

Generally, if $E = (V \oplus W) \perp U$ is a Witt decomposition of $E$, then $C(E, q) \simeq \text{End}(\Lambda(V)) \otimes_2 C(U, q)$, the tensor product of $\mathbb{Z}/2\mathbb{Z}$-graded algebras (cf. for example [1]).

If $\dim E = n$, then $\dim C(E, q) = 2^n = \dim \Lambda(E)$.

If $a, b \in K^*$, we denote by $\left(\frac{a, b}{K}\right)$ the associated quaternion algebra: i.e. the algebra generated by $i, j$ with $i^2 = a$; $j^2 = b$; $ij = -ji$. Also $\left(\frac{a, b}{K}\right)$ is the Clifford algebra of the rank $2$ quadratic form $q(x) = ax_1^2 + bx_2^2$.

Let us write $M(n, K)$ the algebra of the $n \times n$ matrices with coefficients in $K$. 


Theorem 1: \( p \equiv 1 \pmod{4} \)

(i) \( n = 2m \), then \( C(Q^n_p, q_0) \simeq M(2^m, Q_p) \)

(ii) \( n = 2m + 1 \), then \( C(Q^n_p, q_0) \simeq M(2^m, Q_p) \oplus M(2^m, Q_p) \)

Proof

Indeed, if \( n = 2m \), then \( (Q^n, q_0) \) is a hyperbolic space. It follows that \( C(Q^n_p, q_0) \simeq \text{End}(\wedge(Q^n_p)) \).

And, if \( n = 2m + 1 \), we have a Witt decomposition \( Q^n_p = (V \oplus W) \perp U \) where \( U = Q_p e_n \).

It follows that \( C(U, q_0) \simeq Q_p \oplus Q_p \) which gives (ii)

Theorem 2: \( p \equiv 3 \pmod{4} \)

(i) \( n = 4m \), then \( C(Q^n_p, q_0) \simeq M(2^m, Q_p) \)

(ii) \( n = 4m + 1 \), then \( C(Q^n_p, q_0) \simeq M(2^m, Q_p) \oplus M(2^m, Q_p) \)

(iii) \( n = 4m + 2 \), then \( C(Q^n_p, q_0) \simeq M(2^{m+1}, Q_p) \)

(iv) \( n = 4m + 3 \), then \( C(Q^n_p, q_0) \simeq M(2^{m+1}, Q_p[i]) \)

with \( i = \sqrt{-1} \).

Proof:

The case (i) is evident, since \( Q_p^{2m} \) is a hyperbolic space.

If \( n = 4m + 1 \), we have a Witt decomposition \( Q^n_p = (V \oplus W) \perp U \) where \( U = Q_p u \)
with \( u = a e_{4m-3} + b e_{4m-2} + e_{4m-1} - e_{4m+1} \) and \( q_0(u) = a^2 + b^2 + 1 + 1 = 1 \). It follows that \( C(U, q_0) \simeq Q_p \oplus Q_p \), which gives (ii).

If \( n = 4m + 2 \), we have a Witt decomposition \( Q^n_p = (V \oplus W) \perp U \) where
\[
U = Q_p u_1 \oplus Q_p u_2 \quad \text{and} \quad u_1 = a e_{4m-3} + b e_{4m-2} + e_{4m-1} + a e_{4m+1} + b e_{4m+2}
\]
\[
u_2 = -b e_{4m-3} + a e_{4m-2} + e_{4m} - b e_{4m+1} + a e_{4m+2}
\]
Furthermore \( <u_1, u_2> = 0 \), \( q_0(u_1) = -1 = q_0(u_2) \) and \( C(U, q_0) \simeq \left( \frac{-1, -1}{Q_p} \right) \). This quaternion algebra contains an element \( z \) with \( N(z) = a^2 + b^2 + 1 = 0 \). Hence \( \left( \frac{-1, -1}{Q_p} \right) \simeq M(2, Q_p) \) and finally we have \( C(Q^n_p, q_0) \simeq M(2^m, Q_p) \otimes_2 M(2, Q_p) \simeq M(2^{m+1}, Q_p) \).

If \( n = 4m + 3 \), we have a Witt decomposition \( Q^n_p = (V \oplus W) \perp U \) where \( U = Q_p u \), with \( u = -b e_{4m+1} + a e_{4m+2} \) and \( q_0(u) = b^2 + a^2 = -1 \). Hence \( C(U, q_0) \simeq Q_p[i] \), because \( u^2 = q_0(u) = -1 \).
We conclude that \( C(\mathbb{Q}_p^n, q_0) \cong M(2^{2m+1}, \mathbb{Q}_p[i]) \).

In the proof of the forthcoming theorem, one needs the following lemma:

**Lemma:**

Let \( K \) be a field (char. \( \neq 2 \)), \( c, d \in K^* \) such that \( c^2 + d^2 \neq 0 \).

If \( \sigma = \frac{1}{c^2 + d^2} \), then \( \left( \frac{-\sigma, -\sigma}{K} \right) \cong \left( \frac{-1, -1}{K} \right) \).

If the two-rank quadratic forms \( q_1(x) = -\sigma x_1^2 - \sigma x_2^2 \) and \( q_2(x) = -x_1^2 - x_2^2 \) are equivalent, then their Clifford algebras are isomorphic. But, putting \( x_1 = cx_1' + dx_2' \) and \( x_2 = dx_1' - cx_2' \), we have \( q_1(u(x')) = -\sigma(cx_1' + dx_2')^2 - \sigma(dx_1' - cx_2')^2 = -\sigma(c^2 + d^2)(x_1'^2 + x_2'^2) = q_2(x') \). Hence \( q_1 \) and \( q_2 \) are equivalent and the lemma is proved.

**Remark**

The quaternion algebra \( \left( \frac{-1, -1}{\mathbb{Q}_2} \right) = \mathbb{H}_2 \) is a skew field.

Indeed, for any \( z \in \mathbb{H}_2 = \left( \frac{-1, -1}{\mathbb{Q}_2} \right), z \neq 0 \), the norm of \( z \) is \( N(z) = x_0^2 + x_1^2 + x_2^2 + x_3^2 \neq 0 \) (the standard quadratic form of rank 4 over \( \mathbb{Q}_2 \) is anisotropic).

**Theorem 3:** \( p = 2 \)

The Clifford algebra \( C(\mathbb{Q}_2^n, q_0) \) is isomorphic to:

(0) \( \text{End}(\wedge(\mathbb{Q}_2^{4m})) \cong M(2^{4m}, \mathbb{Q}_2), \) if \( n = 8m \)

(1) \( M(2^{4m}, \mathbb{Q}_2) \oplus M(2^{4m}, \mathbb{Q}_2), \) if \( n = 8m + 1 \)

(2) \( M(2^{4m+1}, \mathbb{Q}_2), \) if \( n = 8m + 2 \)

(3) \( M(2^{4m+1}, \mathbb{Q}_2[i]), \) with \( i = \sqrt{-1}, \) if \( n = 8m + 3 \)

(4) \( M(2^{4m+1}, \mathbb{H}_2), \) if \( n = 8m + 4 \)

(5) \( M(2^{4m+1}, \mathbb{H}_2) \oplus M(2^{4m+1}, \mathbb{H}_2), \) if \( n = 8m + 5 \)

(6) \( M(2^{4m+2}, \mathbb{H}_2), \) if \( n = 8m + 6 \)

(7) \( M(2^{4m+3}, \mathbb{Q}_2[i]), \) if \( n = 8m + 7 \)

**Proof**

According to the proof of Proposition 2, if \( n = 8m + s, 0 \leq s \leq 7 \), then \( \mathbb{Q}_2^n = (V \oplus W) \perp E_s \) where \( V \) and \( W \) are totally isotropic subspaces of dimension \( 4m \), and \( (E_s, q_0) \cong \)
It follows that \( C(Q_2^n, q_0) \simeq \text{End}(\wedge(Q_p^{4m})) \otimes_2 C(Q_2^n, q_0) \). It is easy to see that
\[
C(Q_2^5, q_0) \simeq Q_2 \oplus Q_2 ; \quad C(Q_2^3, q_0) \simeq \left( \frac{1,1}{Q_2} \right) \simeq M(2, Q_2) \quad \text{and} \quad C(Q_2^3, q_0) \simeq M(2, Q_2[i]).
\]

If \( s = 4 \), the subalgebra, generated by \( e_1e_2, e_2e_4 \) and \( e_1e_4 \), is isomorphic to \( \left( -\frac{1}{Q_2}, -\frac{1}{Q_2} \right) = H_2 \). Hence \( C(Q_2^4, q_0) \simeq M(2, H_2) \).

If \( s = 5 \), then \( Q_2^5 = F \perp U \), where \( F \) is a hyperbolic plane and \( U \) a three-dimensional anisotropic subspace, with orthogonal basis \((u_1, u_2, u_3)\) satisfying \( q_0(u_1) = -1, q_0(u_2) = -\sigma = q_0(u_3). (\sigma = \frac{-1}{x^2+y^2} \) and \( a, b, c, d \in Q_2 \) such that \( a^2 + b^2 + c^2 + d^2 + 1 = 0 \). Therefore \( C_+(U, q_0) \simeq \left( \frac{-\sigma}{Q_2} \right) \simeq H_2 \); \( C_+ \) stands for the even subalgebra. But in
\[
C(U, q_0), (u_1u_2u_3)^2 = \sigma^2 \quad \text{is a square in} \quad Q_2 \; \text{;} \quad \text{therefore} \quad C(U, q_0) \simeq H_2 \oplus H_2.
\]
Furthermore
\[
C(Q_2^5, q_0) \simeq C(F, q_0) \otimes_2 C(U, q_0) \simeq M(2, H_2) \oplus M(2, H_2), \quad \text{because} \quad C(F, q_0) \simeq M(2, Q_2).
\]

If \( s = 6 \), then \( Q_2^6 = F \perp U \), where \( F \) is a hyperbolic space of dimension 4 and \( U \) a two-dimensional anisotropic subspace with an orthogonal basis \((u_1, u_2)\) satisfying \( q_0(u_1) = -\sigma = q_0(u_2). \) Therefore \( C(U, q_0) \simeq \left( \frac{-\sigma}{Q_2} \right) \simeq H_2 \). And consequently \( C(Q_2^6, q_0) \simeq C(F, q_0) \otimes_2 C(U, q_0) \simeq M(2^2, H_2) \).

If \( s = 7 \), then \( Q_2^7 = F \perp U \), where \( F \) is a hyperbolic space of dimension 6 and \( U = Q_2u \), with \( q_0(u) = -1 \). Hence \( C(U, q_0) \simeq Q_2[i] \) and \( C(Q_2^7, q_0) \simeq M(2^3, Q_2[i]). \)

One deduces the isomorphisms of the theorem from \( C(Q_2^n, q_0) \simeq M(2^{4m}, Q_2) \otimes_2 C(Q_2^4, q_0). \)

\textbf{N.B :} A classical way to prove the above theorems is based on the isomorphisms
\[
C(K^{n+2}, q_0) \simeq C(K^n, -q_0) \otimes C(K^2, q_0)
\]
and
\[
C(K^{n+2}, -q_0) \simeq C(K^n, q_0) \otimes C(K^2, -q_0)
\]
which give first 8-periodicity, etc ...

\((-q_0 \) is the opposite of the standard quadratic form \( q_0)\)
III - THE FAMILIES OF P-ADIC CLIFFORD ALGEBRAS

III-1. Equivalent classes of the p-adic quadratic forms

Let \( a, b \in \mathbb{Q}_p^* = \mathbb{Q}_p \setminus \{0\} \). The Hilbert symbol \((a, b)\) is defined by \((a, b) = 1\) if the quadratic form of rank 3, \(q'(x) = x_0^2 - ax_1^2 - bx_2^2\) is isotropic \((a, b) = -1\) otherwise.

N.B. \((a, b) = 1\) iff \(\left(\frac{a}{b}\right) \simeq M(2, \mathbb{Q}_p)\).

Let \(E\) be a vector space over \(\mathbb{Q}_p\) of dimension \(n\). Let us consider a regular quadratic form \(q\) over \(E\). If \(\{e_j\}_{1 \leq j \leq n}\) is an orthogonal basis of \(E\) and \(a_j = q(e_j)\); then the discriminant \(d(q)\) of \(q\) is equal to \(a_1 \ldots a_n\) in the group \(M_p = \mathbb{Q}_p^*/\mathbb{Q}_p^2\). Let \(\epsilon(q) = \prod_{1 \leq i < j \leq n} (a_i, a_j)\).

Theorem A

(i) The p-adic regular quadratic forms \(q\) and \(q'\) of rank \(n\) are equivalent iff \(d(q) = d(q')\) and \(\epsilon(q) = \epsilon(q')\).

(ii) Let \(d \in M_p\) and \(\epsilon = \pm 1\). There exists a p-adic regular quadratic form \(q\) such that \(d(q) = d\) and \(\epsilon(q) = \epsilon\) iff

\[(a) \quad n = 1 \quad and \quad \epsilon = 1\]
\[(b) \quad n = 2 \quad and \quad (d, \epsilon) \neq (-1, -1)\]
\[(c) \quad n \geq 3\]

Proof: cf. [5]

According to that proof of Theorem A, one can give, explicitly, representatives of the equivalence classes of p-adic regular quadratic forms.

Let us recall that \(M_2 = \{\pm 1, \pm 2, \pm 5, \pm 10\}\) and \(M_p = \{1, p, \omega, \omega p\}\) if \(p \neq 2\), where \(\omega\) is a unit such that \(\left(\omega\right)_p = -1\); \(\left(\frac{-}{p}\right)\) = the Legendre symbol. Furthermore \(-1 = 1\) in \(M_p\) if \(p \equiv 1 (\mod. 4)\) and \(M_p = \{1, p, -1, -p\}\) if \(p \equiv 3 (\mod. 4)\).

We are content ourself here, with the primes \(p\) different from 2. Then a complete set of representatives of the equivalent classes of regular p-adic quadratic forms is obtained as follows.
Then \( q^a(x) = ax^2, a \in M_p \); and the Clifford algebras \( C(Q_p, q^a) \) are isomorphic respectively to \( Q_p \oplus Q_p, Q_p[\sqrt{p}], Q_p[\sqrt{\omega}] \) and \( Q_p[\sqrt{p\omega}] \).

(b) \( n = 2 \)

Then we have over \( Q_p^2 \) (with \( \omega = -1 \) if \( p \equiv 3 \mod{4} \))

\[
q_0(x) = x_1^2 + x_2^2 \quad q_4(x) = p x_1^2 + \omega p x_2^2 \quad \text{if } p \equiv 1 \mod{4}.
\]

\[
q_1(x) = x_1^2 + p x_2^2 \quad (\text{resp.}) \quad q_4(x) = p x_1^2 + p x_2^2 \quad \text{if } p \equiv 3 \mod{4}.
\]

\[
q_2(x) = \omega x_1^2 + \omega p x_2^2 \quad q_5(x) = x_1^2 + \omega p x_2^2
\]

\[
q_3(x) = x_1^2 + \omega x_2^2 \quad q_6(x) = p x_1^2 + \omega x_2^2
\]

Furthermore \( \epsilon(q_\ell) = 1 \) if \( \ell = 0, 1, 3, 5 \) and \( \epsilon(q_\ell) = -1 \) if \( \ell = 2, 4, 6 \).

N.B: If \( p = 2 \), then for \( n = 2 \), one has

8 regular quadratic forms \( q \) such that \( \epsilon(q) = 1 \)

and 7 regular quadratic forms \( q \) such that \( \epsilon(q) = -1 \).

(c) \( n = 3 \)

If \( (e_1, e_2, e_3) \) is the canonical basis of \( Q_p^3 \), then

- \( q_\ell(x) = q_\ell(x_1 e_1 + x_2 e_2 + x_3^2), 0 \leq \ell \leq 6 \)
  and

- \( q_\ell(x) = p x_1^2 + \omega x_2^2 + \omega p x_3^2 = q_6(x_1 e_1 + x_2 e_2) + \omega p x_3^2 \quad \text{if } p \equiv 1 \mod{4} \)
  resp.

- \( q_\ell(x) = p x_1^2 - x_2^2 + p x_3^2 = q_6(x_1 e_1 + x_2 e_2) + p x_3^2 \quad \text{if } p \equiv 3 \mod{4} \)

Furthermore \( d(q_\ell') = d(q_\ell), \epsilon(q_\ell') = \epsilon(q_\ell), 0 \leq \ell \leq 6 \) and \( d(q_6') = -1, \epsilon(q_6') = -1 \).

(d) \( n \geq 4 \)

Let \( (e_j)_{1 \leq j \leq n} \) be the canonical basis of \( Q_p^n \), then

- \( q_\ell''(x) = q_\ell(x_1 e_1 + x_2 e_2) + \sum_{j=3}^{n} x_j^2, 0 \leq \ell \leq 6 \).

In other words \( q_\ell''(x) = q_\ell(x_1 e_1 + x_2 e_2) + q_0 \left( \sum_{j=3}^{n} x_j e_j \right) \)

i.e. \( (Q_p^n, q_\ell'') \simeq (Q_p^2, q_\ell) \perp (Q_p^{n-2}, q_0), 0 \leq \ell \leq 6 \)

and
If \( n = 3 \), then the classes of regular quadratic forms have 15 representative forms \( q' \) with \( \epsilon(q') = 1 \), resp. \( \epsilon(q') = -1 \) and \( d(q') \neq -1 \), obtained from corresponding representative quadratic forms of ranks 2 by adding the rank 1 form \( x_3^2 \). The other representative form is \( q'_{15}(x) = -x_1^2 - x_2^2 - x_3^2 \) with \( \epsilon(q'_{15}) = -1 \) and \( d(q'_{15}) = -1 \).

And if \( n \geq 4 \), one proceeds as above.

### III - 2 The p-adic Clifford algebras

With the above notations, we have the following concrete propositions

**Proposition 3:** \( p \neq 2 \)

(i) \( C(\mathbb{Q}_p^2, q_{\ell}) \cong M(2, \mathbb{Q}_p) \) if \( \ell = 0, 1, 3, 5 \).

(ii) \( C(\mathbb{Q}_p^2, q_{\ell}) \cong \left( \frac{p, \omega}{\mathbb{Q}_p} \right) = H_p = the \ p\text{-adic quaternion field} \), if \( \ell = 2, 4, 6 \).

**Proof**

(i) Indeed, if \( \ell = 0, 1, 3, 5 \); then \( \epsilon(q_{\ell}) = 1 \). Therefore \( C(\mathbb{Q}_p^2, q_{\ell}) \cong M(2, \mathbb{Q}_p) \).

(ii) If \( \ell = 2, 4, 6 \) then the Clifford algebras \( C(\mathbb{Q}_p^2, q_{\ell}) \) are isomorphic to the quaternion algebras with norm respectively,

\[
N_2(z) = x_0^2 - \omega x_1^2 - \omega p x_2^2 + \omega^2 p x_3^2 ;
\]

\[
N_4(z) = x_0^2 - p x_1^2 - \omega p x_2^2 + \omega p^2 x_3^2 \quad \text{if} \quad p \equiv 1 \pmod{4} ;
\]

(resp. \( N_4(x) = x_0^2 - p x_1^2 - p x_2^2 + p^2 x_3^2 \quad \text{if} \quad p \equiv 3 \pmod{4} \))

and \( N_6(z) = x_0^2 - p x_1^2 - \omega x_2^2 + \omega p x_3^2 \).

It is easily seen that these quadratic forms are anisotropic and equivalent. Therefore

\( C(\mathbb{Q}_p^2, q_2) \cong C(\mathbb{Q}_p^2, q_4) \cong C(\mathbb{Q}_p^2, q_6) \cong \left( \frac{p, \omega}{\mathbb{Q}_p} \right) = H_p \) is a skew field. Hence \( H_p \) is the unique quaternion field over \( \mathbb{Q}_p \) (according isomorphism). This result obtained directly here is a general result for local fields (cf. [3]).
Proposition 4 : $p \equiv 1 \pmod{4}$

The Clifford algebra $C(\mathbb{Q}_p^3, q'_p)$ is isomorphic to

(i) $M(2, \mathbb{Q}_p) \oplus M(2, \mathbb{Q}_p)$ if $\ell = 0$

(ii) $M(2, \mathbb{Q}_p[\sqrt{p}])$ if $\ell = 1, 2$

(iii) $M(2, \mathbb{Q}_p[\sqrt{p}])$ if $\ell = 3, 4$

(iv) $M(2, \mathbb{Q}_p[\sqrt{p}])$ if $\ell = 5, 6$

(v) $\mathbb{H}_p \oplus \mathbb{H}_p$ if $\ell = 7$

Similarly we have

Proposition 4' : $p \equiv 3 \pmod{4}$

The Clifford algebra $C(\mathbb{Q}_p^3, q'_p)$ is isomorphic to

(i) $M(2, \mathbb{Q}_p[1])$ if $\ell = 0, 4$

(ii) $M(2, \mathbb{Q}_p[\sqrt{-p}])$ if $\ell = 1, 2$

(iii) $M(2, \mathbb{Q}_p) \oplus M(2, \mathbb{Q}_p)$ if $\ell = 3$

(iv) $M(2, \mathbb{Q}_p[\sqrt{p}])$ if $\ell = 5, 6$

(v) $\mathbb{H}_p \oplus \mathbb{H}_p$ if $\ell = 7$

Proof of Propositions 4 and 4'

Let us recall that if $(E, q)$ is a regular quadratic space over a field $K$ with $n = \dim E$ odd, then $C(E, q) \simeq Z \oplus C_+(E, q)$, where $Z$ is the centre of $C(E, q)$ and $C_+(E, q)$ the subalgebra of even elements. Furthermore, if $(e_1, \ldots, e_n)$ is an orthogonal basis of $(E, q)$ then $u = e_1 \ldots e_n$ is such that $u^2 = (-1)^{[q]}d(q)$ and $Z = K[u]$.

In particular for $n = 3$ and $q(x) = \alpha x_1^2 + \beta x_2^2 + \gamma x_3^2$, we have $e_1^2 = \alpha, e_2^2 = \beta, e_3^2 = \gamma; u^2 = -\alpha\beta\gamma = \delta \neq 0$ and $C_+(E, q) = \langle 1, e_1e_2, e_1e_3, e_2e_3 \rangle =$ subspace generated by $1, \ldots, e_2e_3$. Put $E_1 = e_1e_2, E_2 = e_1e_3, E_3 = -\alpha e_2e_3$, hence $C_+(E, q) = \langle 1, E_1, E_2, E_3 \rangle$ with $E_1^2 = -\alpha\beta, E_2^2 = -\alpha\gamma, E_1E_2 = E_3 = -E_2E_1$. Therefore $C_+(E, q) \simeq \left(\frac{-\alpha\beta, -\alpha\gamma}{K}\right)$.

Consequently (1) if $\delta \in K^{*2}$, then $Z \simeq K \oplus K$ and $C(E, q) \simeq \left(\frac{-\alpha\beta, -\alpha\gamma}{K}\right) \oplus \left(\frac{-\alpha\beta, -\alpha\gamma}{K}\right)$.

(2) if $\delta \notin K^{*2}$, then $Z = K[u]$ is a field and $C(E, q) \simeq \left(\frac{-\alpha\beta, -\alpha\gamma}{K[u]}\right)$.

Applying these remarks to Propositions 4 and 4', one finds the desired isomorphisms. For example if $p \equiv 1 \pmod{4}$ and $\ell = 2$, then $\delta = -\omega^2 p = (i\omega)^2 p$ and $Z = \mathbb{Q}_p[\sqrt{p}]$, hence
C(Q_p^2, g_2) \simeq \left( -\omega^2 p, -\omega \right) = \left( \frac{p\omega}{Q_p[\sqrt{p}]} \right) \simeq M(2, Q_p[\sqrt{p}]) : \tilde{g}(v) = px^2 + \omega y^2 \text{ represents 1 over } Q_p[\sqrt{p}]. \text{ Also if } \ell = 7, \text{ then } \delta = -p^2\omega^2 = (i\omega p)^2, \text{ hence } Z \simeq Q_p \oplus Q_p \text{ and since } \\
\left( \frac{-p\omega, -\omega p^2}{Q_p} \right) \simeq \left( \frac{p\omega, \omega}{Q_p} \right) \simeq H_p \text{ we have } C(Q_p^3, q_7^2) \simeq H_p \oplus H_p.

In the case } p \equiv 3(\text{mod. 4}), \text{ for example if } \ell = 0(\text{resp. } \ell = 3) \text{ we have } \delta = -1 \text{ (resp. } = 1) \text{ and } Z \simeq Q_p[i], (\text{resp. } Z \simeq Q_p \oplus Q_p). \text{ Hence } C(Q_p^3, q_0^3) \simeq \left( \frac{-1, -1}{Q_p} \right) \simeq M(2, Q_p[i]), \left( \text{resp. } C(Q_p^3, q_3^3) \simeq \left( \frac{-1, -1}{Q_p} \right) \oplus \left( \frac{-1, -1}{Q_p} \right) \simeq M(2, Q_p) \oplus M(2, Q_p) \right).

The other verifications are left to the reader.

**Lemma 2:** \( p \neq 2 \)

\[ C(Q_p^4, q_7^4) \simeq M(2, H_p). \quad \square \]

Indeed, since \( q_7^4 = p x_1 + \omega x_2 + \omega' p x_3 + \omega x_4 \text{ where } \omega' = \omega \text{ if } p \equiv 1 \text{ (mod.4)} \text{ and } \omega = -1, \omega' = 1 \text{ if } p \equiv 3 \text{ (mod. 4)} \); we have \( C(Q_p^4, q_7^4) \simeq \left( \frac{p, \omega}{Q_p} \right) \otimes \left( \frac{\omega', 1}{Q_p} \right) \simeq H_p \otimes M(2, Q_p) \simeq M(2, H_p). \)

**Theorem 4:** \( p \equiv 1 \text{ (mod. 4)} \); \( n \geq 4 \)

1°) If \( n = 2m \), then the Clifford algebra \( C(Q_p^n, q_\ell^m) \) is isomorphic to

(i) \[ M(2^m, Q_p) \text{ if } \ell \equiv 0, 1, 3, 5 \]

(ii) \[ M(2^{m-1}, H_p) \text{ if } \ell \equiv 2, 4, 6, 7 \]

2°) If \( n = 2m + 1 \), then the Clifford algebra \( C(Q_p^n, q_\ell^m) \) is isomorphic to

(i) \[ M(2^m, Q_p) \oplus M(2^m, Q_p) \text{ if } \ell \equiv 0 \]

(ii) \[ M(2^m, Q_p[\sqrt{\tau}]) \text{ if } \ell \equiv 1, 2, 3, 4, 5, 6 \]

with \( \tau = p \text{ (resp. } \omega, \text{ resp. } \omega p) \text{ for } \ell \equiv 1, 2 \text{ (resp. } \ell \equiv 3, 4 \text{ ; resp. } 5, 6 \).

(iii) \[ M(2^{m-1}, H_p) \oplus M(2^{m-1}, H_p) \text{ if } \ell \equiv 7 \]

**Proof:**

1°) \( n = 2m \)

Notice that \( C(Q_p^n, q_\ell^m) \simeq C(Q_p^{n-2}, q_0^0) \otimes M(2, Q_p) \) if \( \ell \equiv 0, 1, 3, 5 \) and \( C(Q_p^n, q_\ell^m) \simeq H_p \text{ if } \ell \equiv 0, 1, 3, 5 \).
2, 4, 6. Since $C(Q_p^{n-2}, q_0) \simeq M(2^{m-1}, Q_p)$ by Theorem 1 - (i) - , we have $C(Q_p^n, q''_\ell) \simeq M(2^{m-1}, Q_p) \otimes_2 M(2, Q_p) \simeq M(2^m, Q_p)$ if $\ell = 0, 1, 3, 5$ and $C(Q_p^n, q''_\ell) \simeq M(2^{m-1}, Q_p) \otimes_2 H_p \simeq M(2^{m-1}, H_p)$ if $\ell = 2, 4, 6$.

For $\ell = 7$ , applying Lemma 2 and Theorem 1 - (i) - we obtain $C(Q_p^n, q''_\ell) \simeq C(Q_p^{n-4}, q_0) \otimes_2 C(Q_p^4, q''_\ell) \simeq M(2^{m-2}, Q_p) \otimes_2 M(2, H_p) \simeq M(2^{m-1}, H_p)$.

2°) $n = 2m+1$

If $1 \leq \ell \leq 6$, then we have $C(Q_p^n, q''_\ell) \simeq C(Q_p^{n-2}, q_0) \otimes_2 C(Q_p^3, q''_\ell) \simeq M(2^{m-1}, Q_p) \otimes_2 C(Q_p^3, q''_\ell)$.

Applying Proposition 4, we obtain the isomorphism $C(Q_p^n, q''_\ell) \simeq M(2^m, Q_p[\sqrt{\tau}])$ as claimed.

The case $\ell = 0$ is Theorem 1 - (ii) -

If $\ell = 7$, then $C(Q_p^3, q''_\ell) \simeq H_p \oplus H_p$ and $C(Q_p^n, q''_\ell) \simeq M(2^{m-1}, Q_p) \otimes_2 (H_p \oplus H_p) \simeq M(2^{m-1}, H_p) \oplus M(2^{m-1}, H_p)$.

Theorem 5 : $p \equiv 3(\text{mod.} 4) ; n \geq 4$

The Clifford algebra $C(Q_p^n, q''_\ell)$ is isomorphic to the following matrix algebra or direct sum of two matrix algebras.

1°)

\begin{align*}
(i) & \quad M(2^{2m}, Q_p) \quad \text{if} \quad \ell = 0, 1, 3, 5 \\
(ii) & \quad M(2^{2m-1}, H_p) \quad \text{if} \quad \ell = 2, 4, 6, 7
\end{align*}

2°)

\begin{align*}
(i) & \quad M(2^{2m}, Q_p) \oplus M(2^{2m}, Q_p) \quad \text{if} \quad \ell = 0, 4 \\
(ii) & \quad M(2^{2m}, Q_p[\sqrt{\tau}]) \quad \text{if} \quad \ell = 1, 2, 3, 5, 6, 7
\end{align*}

with $\tau = p$ (resp. $-1$, resp. $-p$) for $\ell = 1, 2$ (resp.$\ell = 3, 7$, resp.$\ell = 5, 6$).

3°)

\begin{align*}
(i) & \quad M(2^{2m+1}, Q_p) \quad \text{if} \quad \ell = 0, 1, 3, 5 \\
(ii) & \quad M(2^{2m}, H_p) \quad \text{if} \quad \ell = 2, 4, 6, 7
\end{align*}
4°) \( n = 4m + 3 \)

(i) \( M(2^{2m+1}, Q_p) \oplus M(2^{2m+1}, Q_p) \) if \( \ell = 3 \)

(ii) \( M(2^{2m+1}, Q_p[\sqrt{\tau}]) \) if \( \ell = 0, 1, 2, 4, 5, 6 \),

with \( \tau = -1 \) (resp. \(-p, res.p\)) for \( \ell = 0, 4 \) (resp. \( \ell = 1, 2 \), resp. \( \ell = 5, 6 \)).

(iii) \( M(2^{2m}, H_p) \oplus M(2^{2m}, H_p) \) if \( \ell = 7 \).

Proof:

1°) \( n = 4m \)

As in Lemma 2, it is readily seen that \( C(Q_p^n, q_i^{\ell}) \simeq M(2^2, Q_p) \) if \( \ell = 0, 1, 3, 5 \) and \( C(Q_p^n, q_i^{\ell}) \simeq M(2, H_p) \) if \( \ell = 2, 4, 6, 7 \).

If \( n = 4m, \ m \geq 2 \), we have \( C(Q_p^n, q_i^{\ell}) \simeq C(Q_p^{n-4}, q_0) \otimes_2 C(Q_p^n, q_i^{\ell}) \). But Theorem 3 - (i) - gives \( C(Q_p^{n-4}, q_0) \simeq M(2^{m-2}, Q_p) \). Therefore \( C(Q_p^n, q_i^{\ell}) \simeq M(2^{2m-2}, Q_p) \otimes_2 M(2^2, Q_p) \simeq M(2^{2m}, Q_p) \) if \( \ell = 0, 1, 3, 5 \) and \( C(Q_p^n, q_i^{\ell}) \simeq M(2^{2m-2}, Q_p) \otimes_2 M(2^2, H_p) \simeq M(2^{2m-1}, H_p) \) if \( \ell = 2, 4, 6, 7 \).

2°) \( n = 4m + 1 \)

With notations used in the proof of Propositions 4 and 4' we have \( C(Q_p^n, q_i^{\ell}) \simeq Z \otimes C_+(Q_p^n, q_i^{\ell}) \) and \( Z = Q_p[u] \) where \( u^2 = d(q_i^{\ell}) \). Hence \( Z \) is isomorphic to \( Q_p \oplus Q_p \) if \( \ell = 0, 4 \); resp. \( Q_p[\sqrt{\tau}] \) if \( \ell = 1, 2 \); resp. \( Q_p[\sqrt{-1}] \) if \( \ell = 3, 7 \); resp. \( Q_p[\sqrt{-1}] \) if \( \ell = 5, 6 \). On the other hand \( C_+(Q_p^n, q_i^{\ell}) \simeq C_+(Q_p^n, x_n, x_n^2) \otimes C(Q_p^{n-1}, -q_i^{\ell}) \simeq C(Q_p^{n-1}, -q_i^{\ell}) \simeq M(2^{2m}, Q_p) \). Hence \( C(Q_p^n, q_i^{\ell}) \simeq Z \otimes M(2^{2m}, Q_p) \) which proves the isomorphisms.

3°) \( n = 4m + 2 \)

Since \( n - 2 = 4m \), we obtain \( C(Q_p^n, q_i^{\ell}) \simeq C(Q_p^{4m}, q_0) \otimes_2 C(Q_p^n, q_i^{\ell}) \).

By Theorem 2 - (i) - one has \( C(Q_p^{4m}, q_0) \simeq M(2^{2m}, Q_p) \) and by Proposition 3, \( C(Q_p^n, q_\ell) \simeq M(2, Q_p) \) if \( \ell = 0, 1, 3, 5 \) and \( C(Q_p^n, q_\ell) \simeq H_p \) if \( \ell = 2, 4, 6 \). It follows that \( C(Q_p^n, q_i^{\ell}) \simeq M(2^{m+1}, Q_p) \) if \( \ell = 0, 1, 3, 5 \) and \( C(Q_p^n, q_i^{\ell}) \simeq M(2^{2m}, H_p) \) if \( \ell = 2, 4, 6 \).

For the case \( \ell = 7 \), since \( n - 4 = 4(m - 1) + 2 \) we have \( C(Q_p^n, q_i^{\ell}) \simeq C(Q_p^{n-4}, q_0) \otimes_2 C(Q_p^n, q_i^{\ell}) \). By theorem 2 - (iii) -, \( C(Q_p^{n-4}, q_0) \simeq M(2^{m+1}, Q_p) \) and by Lemma 2, \( C(Q_p^n, q_i^{\ell}) \simeq M(2, H_p) \). Hence \( C(Q_p^n, q_i^{\ell}) \simeq M(2^{2m}, H_p) \).

Notice that in 1°) and 3°) the exponent of 2 is \( \frac{3}{2} \).
4°) \( n = 4m + 3 \)

Here, \( n - 3 = 4m \) and \( C(\mathbb{Q}_p^n, q''_\ell) \simeq C(\mathbb{Q}_p^{4m}, q_0) \otimes_2 C(\mathbb{Q}_p^3, q'_\ell) \). But \( C(\mathbb{Q}_p^{4m}, q_0) \simeq M(2^{2m}, \mathbb{Q}_p) \) and by Proposition 4', \( C(\mathbb{Q}_p^3, q'_\ell) \) is isomorphic to \( M(2, \mathbb{Q}_p) \oplus M(2, \mathbb{Q}_p) \) if \( \ell = 3 \), resp. \( H_p \oplus H_p \) if \( \ell = 7 \), resp. \( M(2, \mathbb{Q}_p[\sqrt{\tau}]) \) if \( \ell = 0, 1, 2, 4, 5, 6 \) with \( \tau = -1 \) for \( \ell = 0, 4 \); \( \tau = -p \) for 1, 2 and \( \tau = p \) for 5, 6.

Taking tensor product we obtain the desired isomorphisms.

Remark:

As for \( C(\mathbb{Q}_p^n, q_0) \), for the other Clifford algebras \( C(\mathbb{Q}_p^n, q''_\ell) \) we have 2-periodicity when \( p \equiv 1 \pmod{4} \) and 4-periodicity when \( p \equiv 3 \pmod{4} \).

N.B. When \( p = 2 \), in the same way one can give as above the table of the 2-adic Clifford algebras.

REFERENCES


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