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The Weierstrass-Stone approximation theorem for $p$-adic $C^n$-functions  

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Abstract. Let $K$ be a non-Archimedean valued field. Then, on compact subsets of $K$, every $K$-valued $C^n$-function can be approximated in the $C^n$-topology by polynomial functions (Theorem 1.4). This result is extended to a Weierstrass-Stone type theorem (Theorem 2.10).

INTRODUCTION

The non-archimedean version of the classical Weierstrass Approximation Theorem - the case $n = 0$ of the Abstract - is well known and named after Kaplansky ([1], 5.28). To investigate the case $n = 1$ first let us return to the Archimedean case and consider a real-valued $C^1$-function $f$ on the unit interval. To find a polynomial function $P$ such that both $|f - P|$ and $|f' - P'|$ are smaller or equal than a prescribed $\varepsilon > 0$ one simply can apply the standard Weierstrass Theorem to $f'$ obtaining a polynomial function $Q$ for which $|f' - Q| \leq \varepsilon$. Then $x \mapsto P(x) := f(0) + \int_0^x Q(t)dt$ solves the problem.

Now let $f : X \to K$ be a $C^1$-function where $K$ is a non-archimedean valued field and $X \subseteq K$ is compact.

Lacking an indefinite integral the above method no longer works. There do exist continuous linear antiderivations ([3], §64) but they do not map polynomials into polynomials ([3], Ex. 30.C). A further complicating factor is that the natural norm for $C^1$-functions on $X$ is given by

$$f \mapsto \max\{|f(x)| : x \in X\} \vee \max\{|f(x) - f(y)| : x, y \in X, x \neq y\}$$

rather than the more classical formula

$$f \mapsto \max\{|f(x)| : x \in X\} \vee \max\{|f'(x)| : x \in X\}.$$

(Observe that in the real case both formulas lead to the same norm thanks to the Mean Value Theorem, see [3], §§26,27 for further discussions.)
Thus, to obtain non-archimedean \( C^n \)-Weierstrass-Stone Theorems for \( n \in \{1, 2, \ldots \} \) our methods will necessarily deviate from the 'classical' ones.

0. PRELIMINARIES

1. Throughout \( K \) is a non-archimedean complete valued field whose valuation \( | \cdot | \) is not trivial. For \( a \in K, r > 0 \) we write \( B(a, r) := \{ x \in K : |x-a| \leq r \} \), the 'closed' ball about \( a \) with radius \( r \). 'Clopen' is an abbreviation for 'closed and open'. The function \( x \mapsto x (x \in K) \) is denoted \( \mathcal{X} \). The \( K \)-valued characteristic function of a subset \( Y \) of \( K \) is written \( \xi_Y \). For a set \( Z \), a function \( f : Z \to K \) and a set \( W \subset Z \) we define \( \| f \|_W := \sup \{ |f(x)| : x \in W \} \) (allowing the value \( \infty \)). The cardinality of a set \( \Gamma \) is \( \#\Gamma \).

\[ N_0 := \{0, 1, 2, \ldots\}, N := \{1, 2, 3, \ldots\}. \]

We now recall some facts from [2], [3] on \( C^n \)-theory.

2. For a set \( Y' \subset K, n \in N \) we set \( \nabla^n Y := \{(y_1, y_2, \ldots, y_n) \in Y^n : i \neq j \Rightarrow y_i \neq y_j\} \).

For \( f : Y \to K, n \in N_0 \) we define its \( n \)th difference quotient \( \Phi_n f : \nabla^{n+1} Y \to K \) inductively by \( \Phi_0 f := f \) and the formula

\[
\Phi_n f(y_1, \ldots, y_{n+1}) = \frac{\Phi_{n-1} f(y_1, y_3, \ldots, y_{n+1}) - \Phi_{n-1} f(y_2, y_3, \ldots, y_{n+1})}{y_1 - y_2}
\]

\( f \) is called a \( C^n \)-function if \( \Phi_n f \) can be extended to a continuous function on \( Y^{n+1} \).

The set of all \( C^n \)-functions \( Y \to K \) is denoted \( C^n(Y \to K) \). The function \( f : Y \to K \) is a \( C^\infty \)-function if it is in \( C^\infty(Y \to K) := \bigcap_{n=0}^{\infty} C^n(Y \to K) \). The space \( C^0(Y \to K) \), consisting of all continuous functions \( Y \to K \) is sometimes written as \( C(Y \to K) \).

FROM NOW ON IN THIS PAPER \( X \) IS A NONEMPTY COMPACT SUBSET OF \( K \) WITHOUT ISOLATED POINTS.

3. Since \( X \) has no isolated points we have for an \( f \in C^n(X \to K) \) that the continuous extension of \( \Phi_n f \) to \( X^{n+1} \) is unique; we denote this extension by \( \overline{\Phi}_n f \). Also we write

\[
D_n f(a) := \overline{\Phi}_n f(a, a, \ldots, a) \quad (a \in X)
\]

The following facts are proved in [2] and [3].

**Proposition 0.3.**

(i) For each \( n \in N_0 \) the space \( C^n(X \to K) \) is a \( K \)-algebra under pointwise operations.

(ii) \( C^0(X \to K) \supset C^1(X \to K) \supset \ldots \)
(iii) If $f \in C^n(X \to K)$ then $f$ is $n$ times differentiable and $j!D_jf = f^{(j)}$ for each $j \in \{0,1,\ldots,n\}$. More generally, if $i,j \in \{0,1,\ldots,n\}$, $i+j \leq n$ then $\binom{i+j}{i}D_iD_jf = D_{i+j}f$.

(iv) If $f \in C^n(X \to K)$ then for $x,y \in X$ we have Taylor's formula

$$f(x) = f(y) + (x-y)D_if(y) + \cdots + (x-y)^{n-1}D_{n-1}f(y) + (x-y)^n \rho_1f(x,y),$$

where $\rho_1f(x,y) = \overline{f}f(x,y,y,\ldots,y)$.

4. Since $X$ is compact the difference quotients $\Phi_i f$ ($0 \leq i \leq n$) are bounded if $f \in C^n(X \to K)$. We set

$$\|f\|_{n,X} := \max\{\|\Phi_i f\|_{\nu+i+1,X} : 0 \leq i \leq n\}.$$ 

Then $\|f\|_{0,X} = \|f\|_X$. We quote the following from [2] and [3]. Recall that a function $f : X \to K$ is a local polynomial if for every $a \in X$ there is a neighbourhood $U$ of $a$ such that $f \mid X \cap U$ is a polynomial function.

**Proposition 0.4.** Let $n \in \mathbb{N}_0$.

(i) The function $\|\cdot\|_{n,X}$ is a norm on $C^n(X \to K)$ making it into a $K$-Banach algebra.

(ii) The local polynomials form a dense subset of $C^n(X \to K)$.

(iii) The function

$$f \mapsto \|f\|_{n,X} := \max_{0 \leq i \leq n-1} \|D_i f\|_X \vee \|\rho_1 f\|_X$$

(see Proposition 0.3 (iv)) also is a norm on $C^n(X \to K)$. We have

$$\|f\|_{n,X} = \max\{\|D_i f\|_{n-i,X} : 0 \leq i \leq n\} \quad (f \in C^n(X \to K)).$$

**Remarks**

1. Proposition 0.4 (ii) will also follow from Proposition 2.8.

2. In general $\|\cdot\|_{n,X}$ is not equivalent to $\|\cdot\|_{n,X}$ for $n \geq 3$ (see [3], Example 83.2).
1 THE WEIERSTRASS THEOREM FOR $C^n$-FUNCTIONS

The following product rule for difference quotients is easily proved by induction with respect to $j$. Let $f, g : X \to K$, let $j \in \mathbb{N}_0$. Then for all $(x_1, \ldots, x_{j+1}) \in \nabla^{j+1} X$ we have

$$\Phi_j(fg)(x_1, \ldots, x_{j+1}) = \sum_{k=0}^{j} \Phi_k f(x_1, \ldots, x_{k+1}) \Phi_{j-k} g(x_{k+1}, \ldots, x_{j+1}).$$

Or, less precise,

$$\Phi_j(fg)(x_1, \ldots, x_{j+1}) = \sum_{k=0}^{j} \Phi_k f(x_k) \Phi_{j-k} g(u_{j-k})$$

for certain $z_k \in \nabla^{k+1} X$, $u_{j-k} \in \nabla^{j-k+1} X$.

In the sequel we need an extension of this formula to finite products of functions. The proof is straightforward by induction with respect to $N$.

Lemma 1.1. (Product Rule) Let $h_1, \ldots, h_N : X \to K$, let $j \in \mathbb{N}_0$. Then for all $(x_1, \ldots, x_{j+1}) \in \nabla^{j+1} X$ we have

$$\Phi_j(\prod_{s=1}^{N} h_s)(x_1, \ldots, x_{j+1}) = \sum_{\sigma} \prod_{s=1}^{N} \Phi_{j_s} h_s(z_{\sigma,s})$$

where the sum is taken over all $\sigma := (j_1, \ldots, j_N) \in \mathbb{N}_0^N$ for which $j_1 + \cdots + j_N = j$ and where $z_{\sigma,s} \in \nabla^{j_s+1} X$ for each $s \in \{1, \ldots, N\}$. (In fact, $z_{\sigma,1} = (x_1, \ldots, x_{j_1+1})$, $z_{\sigma,2} = (x_{j_1+1}, \ldots, x_{j_1+j_2+1}), \ldots, z_{\sigma,N} = (x_{j_1+\cdots+j_{N-1}+1}, \ldots, x_{j+1})$.)

The following key lemma grew out of [1], 5.28.

Lemma 1.2. Let $0 < \delta < 1$, $0 < \varepsilon < 1$, let $B = B_0 \cup B_1 \cup \cdots \cup B_m$ where $B_0, \ldots, B_m$ are pairwise disjoint 'closed' balls in $K$ of radius $\delta$. Then, for each $n \in \{0, 1, \ldots\}$ there exists a polynomial function $P : K \to K$ such that $\|P - \xi_{B_0}\|_{n,B} \leq \varepsilon$.

Proof. We may assume $0 \in B_0$. Choose $c_1 \in B_1, \ldots, c_m \in B_m$; we may assume that $|c_1| \leq |c_2| \leq \cdots \leq |c_m|$. Then $\delta < |c_1|$. We shall prove the following statement by induction with respect to $n$.

Let $k \in \mathbb{N}$ be such that $(\delta/|c_1|)^k \leq \varepsilon \delta^n$, $k > n$. Let $t_1, t_2, \ldots, t_m \in \mathbb{N}$ be such that for all $\ell \in \{1, \ldots, m\}$

$$\left| \frac{c_\ell}{c_1} \right|^{kt_1} \left| \frac{c_\ell}{c_2} \right|^{kt_2} \cdots \left| \frac{c_\ell}{c_{\ell-1}} \right|^{kt_{\ell-1}} \left( \frac{\delta}{|c_1|} \right)^{t_\ell} \leq \varepsilon \delta^n$$

(1)
It is easily seen that such \( k, t_1, \ldots, t_m \) exist since \( \delta / |c_1| < 1 \). Then the formula
\[
P(x) = \prod_{i=1}^{m} \left(1 - \left(\frac{x}{c_i}\right)^{k_i}\right)^{t_i}
\]
defines a polynomial function \( P : K \to K \) for which
\[
\|P - \xi_{B_0}\|_{n, B} \leq \varepsilon.
\]
The case \( n = 0 \) is proved in [1], 5.28. To prove the step \( n - 1 \to n \) we first observe that from the induction hypothesis (with \( \varepsilon \) replaced by \( \varepsilon \delta \)) it follows that
\[
\|P - \xi_{B_0}\|_{n-1, B} \leq \varepsilon \delta
\]
So it remains to be shown that
\[
|\Phi_n(P - \xi_{B_0})(x_1, \ldots, x_{n+1})| \leq \varepsilon
\]
for all \((x_1, \ldots, x_{n+1}) \in \mathcal{V}^{n+1} B_t \). If \(|x_i - x_j| > \delta \) for some \( i, j \in \{1, \ldots, n+1\} \) we have, using (2),
\[
|\Phi_n(P - \xi_{B_0})(x_1, \ldots, x_{n+1})| = |x_i - x_j|^{-1} |\Phi_n-1(P - \xi_{B_0})(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{n+1}) - \Phi_n-1(P - \xi_{B_0})(x_1, \ldots, x_i, x_{i+1}, \ldots, x_{n+1})| \leq \delta^{-1} \varepsilon \delta = \varepsilon.
\]
So this reduces the proof of (3) to the case where \(|x_i - x_j| \leq \delta \) for all \( i, j \in \{1, \ldots, n+1\} \); in other words we may assume that \( x_1, \ldots, x_{n+1} \) are all in the same \( B_\ell \) for some \( \ell \in \{0, 1, \ldots, m\} \). But then, after observing that \( n \geq 1 \), we have \( \Phi_n \xi_{B_0}(x_1, \ldots, x_{n+1}) = 0 \) so it suffices to prove the following.

If \( \ell \in \{0, 1, \ldots, m\} \) and \( x_1, \ldots, x_{n+1} \in B_\ell \) are pairwise distinct then
\[
|\Phi_n P(x_1, \ldots, x_{n+1})| \leq \varepsilon
\]
To prove it we introduce, with \( \ell \in \{1, \ldots, m\} \) fixed, the constants \( M_i \) (\( i \in \{1, \ldots, n\} \)) by
\[
M_i := \begin{cases} 
1 & \text{if } i > \ell \\
\delta / |c_1| & \text{if } i = \ell \\
|c_i / c_1|^k & \text{if } i < \ell
\end{cases}
\]
and use the following three steps.

**Step 1.** For each \( j \in \{0, 1, \ldots, n\}, i \in \{1, \ldots, n\} \) we have
\[
\|\Phi_j(1 - \left(\frac{x}{c_i}\right)^k)\|_{\mathcal{V}^{j+1} B_\ell} \leq \begin{cases} 
1 & \text{if } \ell = 0, j = 0 \\
\delta^{-j} (\frac{\delta}{|c_1|})^k & \text{if } \ell = 0, j > 0 \\
\delta^{-j} M_i & \text{if } \ell > 0.
\end{cases}
\]
Proof.

a. The case \( j = 0 \). Then for \( x \in B_{\ell} \) we have
- if \( i > \ell \) then \( |1 - \left( \frac{x}{c_{i}} \right)^{k}| = 1 \)
- if \( i = \ell \) then \( |1 - \left( \frac{x}{c_{i}} \right)^{k}| = \frac{|c_{i} - x|^{k}}{|c_{i}|^{k}} \leq \frac{\delta}{|c_{i}|} \leq \delta \)
- if \( i < \ell \) then \( |1 - \left( \frac{x}{c_{i}} \right)^{k}| = \left| \frac{x}{c_{i}} \right|^{k} = \frac{\delta}{|c_{i}|} \)
and the statement follows.

b. The case \( j > 0 \). Then \( \Phi_{j}(1) = 0 \) so that

\[
\Phi_{j}(1 - \left( \frac{X}{c_{i}} \right)^{k}) = \frac{1}{c_{i}^{k}} \Phi_{j}(X^{k})
\]

Let \( (x_{1}, \ldots, x_{j+1}) \in \nabla^{j+1} B_{\ell} \). By the Product Rule 1.1, \( \Phi_{j}(X^{k})(x_{1}, \ldots, x_{j+1}) \) is a sum of terms of the form \( \prod_{s=1}^{k} (\Phi_{j,s}(X))(x_{s}) \). Such a term is 0 if one of the \( j_{s} \) is \( > 1 \), so we only have to deal with \( j_{s} = 0 \) (then \( \Phi_{j,s}(X) = X \)) or \( j_{s} = 1 \) (then \( \Phi_{j,s}(X) = 1 \)). The latter case occurs \( j \) times (as \( \sum_{s=1}^{k} j_{s} = j \)) and it follows that

\[
\prod_{s=1}^{k} (\Phi_{j,s}(X))(x_{s}) \text{ is a product of } k-j \text{ distinct terms taken from } \{x_{1}, \ldots, x_{j+1}\} \text{ (observe that, indeed, } j < k \text{ since } j \leq n < k \), so its absolute value is } \leq |c_{i}|^{k-j}.
\]
It follows that \( \|\Phi_{j}(1 - (\frac{X}{c_{i}})^{k})\|_{\nabla^{j+1} B_{\ell}} \leq |c_{i}|^{k-j}/|c_{i}|^{k} \) from which we conclude
- if \( \ell = 0 \) : \( |c_{i}|^{k-j}/|c_{i}|^{k} \leq \delta^{k-j}/|c_{i}|^{k} = \delta^{-j}(\delta/|c_{i}|)^{k} \),
- if \( i > \ell > 0 \) : \( |c_{i}|^{k-j}/|c_{i}|^{k} \leq |c_{i}^{-j}| < \delta^{-j} = \delta^{-j}M_{i} \)
- if \( i = \ell > 0 \) : \( |c_{i}|^{k-j}/|c_{i}|^{k} \leq |c_{i}^{-j}| \leq |c_{i}^{-j}| = \delta^{-j}(\frac{\delta}{|c_{i}|})^{j} \leq \delta^{-j}M_{i} \)
- if \( i < \ell \) : \( |c_{i}|^{k-j}/|c_{i}|^{k} \leq |c_{i}^{-j}|^{k} \leq \delta^{-j}M_{i} \)
and step 1 is proved.

**Step 2.** For each \( j \in \{0,1,\ldots,n\} \), \( i \in \{1,\ldots,n\} \) we have

\[
\|\Phi_{j}(1 - (\frac{X}{c_{i}})^{k})^{t_{i}}\|_{\nabla^{j+1} B_{\ell}} \leq \begin{cases} 
1 & \text{if } \ell = 0, j = 0 \\
\delta^{-i}(\frac{\delta}{|c_{i}|})^{k} & \text{if } \ell = 0, j > 0 \\
\delta^{-j}M_{i}^{t_{i}} & \text{if } \ell > 0 
\end{cases}
\]

**Proof.** The case \( j = 0 \) follows directly from Step 1, part a, so assume \( j > 0 \). By the Product Rule 1.1 applied to \( h_{s} = 1 - (\frac{X}{c_{i}})^{k} \) for all \( s \in \{1,\ldots,t_{i}\} \) we have for \( (x_{1}, \ldots, x_{j+1}) \in \nabla^{j+1} B_{\ell} \) that \( \Phi_{j}(1 - (\frac{X}{c_{i}})^{k})^{t_{i}}(x_{1}, \ldots, x_{j+1}) \) is a sum of terms of the form

\[ (5) \quad \prod_{s=1}^{t_{i}} \Phi_{j,s}(1 - (\frac{X}{c_{i}})^{k})(x_{s}) \]
where \( j_1 + \cdots + j_s = j \). If \( \ell = 0 \) it follows from Step 1 that the absolute value of (5) is
\[
\leq \prod \delta^{-j_i}(\frac{\delta}{|c_i|})^k
\]
where the product is taken over all \( s \) in the nonempty set \( \Gamma := \{ s \in \{1, \ldots, t_i\} : j_s > 0 \} \), so the product is
\[
\leq \delta^{-j}(\frac{\delta}{|c_i|})^k
\]
If \( \ell > 0 \) it follows from Step 1 that the absolute value of (5) is
\[
\leq \prod_{i=1}^{t_i} \delta^{-j_i} M_i = \delta^{-j} M_i^{t_i}.
\]
The statement of Step 2 follows.

\textbf{Step 3.} Proof of (4). Again, the Product Rule 1.1, now applied to \( h_i = (1 - (\frac{x_i}{c_i})^k)_{t_i} \) for \( i \in \{1, \ldots, m\} \) tells us that for \((x_1, \ldots, x_{n+1}) \in \nabla^{n+1} B_\ell \) the expression
\[
\Phi_n P(x_1, \ldots, x_{n+1})
\]
is a sum of terms of the form
\[
(6) \quad \prod_{i=1}^{m} \Phi_n \left(1 - \left(\frac{x_i}{c_i}\right)^k\right)^{t_i}(z_s)
\]
where \( n_1 + \cdots + n_m = n \). If \( \ell = 0 \) we have by Step 2 that the absolute value of (6) is
\[
\leq \prod \delta^{-n_i}(\frac{\delta}{|c_i|})^k
\]
where the product is taken over \( i \) in the nonempty set \( \Gamma := \{ i : n_i \neq 0 \} \), so the product is
\[
\leq \delta^{-n}(\frac{\delta}{|c_i|})^{k+1} \leq \delta^{-n}(\frac{\delta}{|c_i|})^k \leq \delta^{-n} \cdot \varepsilon \delta^n = \varepsilon,
\]
where we used the assumption \( \delta/|c_i|^k \leq \varepsilon \delta^n \). We see that \( |\Phi_n P(x_1, \ldots, x_{n+1})| \leq \varepsilon \) if \((x_1, \ldots, x_n) \in B_0\). Now let \( \ell > 0 \). By Step 2 we have that the absolute value of (6) is
\[
\leq \prod \delta^{-n_i} M_i^{t_i} = \delta^{-n} |\frac{x_1}{c_1}|^{k_1} \cdots |\frac{x_m}{c_m}|^{k_m} (\frac{\delta}{|c_i|})^{t_i}
\]
which is \( \leq \delta^{-n} \varepsilon \delta^n \) by (1). This proves (4) and the Lemma.

\textbf{Corollary 1.3.} For every locally constant \( f : X \to K \), for every \( n \in \mathbb{N}_0 \) and \( \varepsilon > 0 \) there exists a polynomial function \( P : K \to K \) such that \( \|f - P\|_{n,X} \leq \varepsilon \). 

\textbf{Proof.} There exist a \( \delta \in (0,1) \), pairwise disjoint 'closed' balls \( B_1, \ldots, B_m \) of radius \( \delta \) covering \( X \) and \( \lambda_1, \ldots, \lambda_m \in K \) such that
\[
f(x) = \sum_{i=1}^{m} \lambda_i \xi_{B_i}(x) \quad (x \in X)
\]
By Lemma 1.2 there exist polynomials \( P_1, \ldots, P_m \) such that \( \|\xi_{B_i} - P_i\|_{n,X} \leq \varepsilon (|\lambda_i| + 1)^{-1} \) for each \( i \in \{1, \ldots, m\} \). Then \( P := \sum \lambda_i P_i \) is a polynomial function and \( \|f - P\|_{n,X} \leq \max_i \|\lambda_i (\xi_{B_i} - P_i)\|_{n,X} \leq \max_i |\lambda_i| (|\lambda_i| + 1)^{-1} \leq \varepsilon \).

\textbf{Theorem 1.4.} (\( C^n \)-\text{Weierstrass Theorem}) For each \( n \in \mathbb{N}_0 \), \( f \in C^n(X \to K) \) and \( \varepsilon > 0 \) there exists a polynomial function \( P : K \to K \) such that \( \|f - P\|_{n,X} \leq \varepsilon \).

\textbf{Proof.} There is by Proposition 0.4 a local polynomial \( g : K \to K \) with \( \|f - g\|_{n,X} \leq \varepsilon \). This \( g \) has the form \( g = \sum_{i=1}^{m} Q_i h_i \) where \( Q_1, \ldots, Q_m \) are polynomials and \( h_1, \ldots, h_m \)

\[
\text{where } j_1 + \cdots + j_s = j. \text{ If } \ell = 0 \text{ it follows from Step 1 that the absolute value of (5) is}
\leq \prod \delta^{-j_i}(\frac{\delta}{|c_i|})^k \text{ where the product is taken over all } s \text{ in the nonempty set } \Gamma := \{ s \in \{1, \ldots, t_i\} : j_s > 0 \}, \text{ so the product is}
\leq \delta^{-j}(\frac{\delta}{|c_i|})^k \leq \delta^{-j}(\frac{\delta}{|c_i|})^k. \text{ If } \ell > 0 \text{ it}
follows from Step 1 that the absolute value of (5) is}
\leq \prod_{i=1}^{t_i} \delta^{-j_i} M_i = \delta^{-j} M_i^{t_i}. \text{ The statement of Step 2 follows.}

\textbf{Step 3.} Proof of (4). Again, the Product Rule 1.1, now applied to \( h_i = (1 - (\frac{x_i}{c_i})^k)_{t_i} \) for \( i \in \{1, \ldots, m\} \) tells us that for \((x_1, \ldots, x_{n+1}) \in \nabla^{n+1} B_\ell \) the expression
\[
\Phi_n P(x_1, \ldots, x_{n+1})
\]
is a sum of terms of the form
\[
(6) \quad \prod_{i=1}^{m} \Phi_n \left(1 - \left(\frac{x_i}{c_i}\right)^k\right)^{t_i}(z_s)
\]
where \( n_1 + \cdots + n_m = n \). If \( \ell = 0 \) we have by Step 2 that the absolute value of (6) is
\[
\leq \prod \delta^{-n_i}(\frac{\delta}{|c_i|})^k
\]
where the product is taken over \( i \) in the nonempty set \( \Gamma := \{ i : n_i \neq 0 \} \), so the product is
\[
\leq \delta^{-n}(\frac{\delta}{|c_i|})^{k+1} \leq \delta^{-n}(\frac{\delta}{|c_i|})^k \leq \delta^{-n} \cdot \varepsilon \delta^n = \varepsilon,
\]
where we used the assumption \( \delta/|c_i|^k \leq \varepsilon \delta^n \). We see that \( |\Phi_n P(x_1, \ldots, x_{n+1})| \leq \varepsilon \) if \((x_1, \ldots, x_n) \in B_0\). Now let \( \ell > 0 \). By Step 2 we have that the absolute value of (6) is
\[
\leq \prod \delta^{-n_i} M_i^{t_i} = \delta^{-n} |\frac{x_1}{c_1}|^{k_1} \cdots |\frac{x_m}{c_m}|^{k_m} (\frac{\delta}{|c_i|})^{t_i}
\]
which is \( \leq \delta^{-n} \varepsilon \delta^n \) by (1). This proves (4) and the Lemma.

\textbf{Corollary 1.3.} For every locally constant \( f : X \to K \), for every \( n \in \mathbb{N}_0 \) and \( \varepsilon > 0 \) there exists a polynomial function \( P : K \to K \) such that \( \|f - P\|_{n,X} \leq \varepsilon \).

\textbf{Proof.} There exist a \( \delta \in (0,1) \), pairwise disjoint 'closed' balls \( B_1, \ldots, B_m \) of radius \( \delta \) covering \( X \) and \( \lambda_1, \ldots, \lambda_m \in K \) such that
\[
f(x) = \sum_{i=1}^{m} \lambda_i \xi_{B_i}(x) \quad (x \in X)
\]
By Lemma 1.2 there exist polynomials \( P_1, \ldots, P_m \) such that \( \|\xi_{B_i} - P_i\|_{n,X} \leq \varepsilon (|\lambda_i| + 1)^{-1} \) for each \( i \in \{1, \ldots, m\} \). Then \( P := \sum \lambda_i P_i \) is a polynomial function and \( \|f - P\|_{n,X} \leq \max_i \|\lambda_i (\xi_{B_i} - P_i)\|_{n,X} \leq \max_i |\lambda_i| (|\lambda_i| + 1)^{-1} \leq \varepsilon \).

\textbf{Theorem 1.4.} (\( C^n \)-Weierstrass Theorem) For each \( n \in \mathbb{N}_0 \), \( f \in C^n(X \to K) \) and \( \varepsilon > 0 \) there exists a polynomial function \( P : K \to K \) such that \( \|f - P\|_{n,X} \leq \varepsilon \).

\textbf{Proof.} There is by Proposition 0.4 a local polynomial \( g : K \to K \) with \( \|f - g\|_{n,X} \leq \varepsilon \). This \( g \) has the form \( g = \sum_{i=1}^{m} Q_i h_i \) where \( Q_1, \ldots, Q_m \) are polynomials and \( h_1, \ldots, h_m \)
are locally constant. By Corollary 1.3 we can find polynomials $P_1, \ldots, P_m$ for which $\|h_i - P_i\|_{n,X} \leq \varepsilon(\|Q_i\|_{n,X} + 1)^{-1}$ for each $i$. Then $P := \sum_{i=1}^{m} Q_i P_i$ is a polynomial and $\|g - P\|_{n,X} \leq \varepsilon$. It follows that $\|f - P\|_{n,X} \leq \max(\|f - g\|_{n,X}, \|g - P\|_{n,X}) \leq \varepsilon$.

**Remarks.**

1. In the case where $X = \mathbb{Z}_p$, $K \supset \mathbb{Q}_p$ the above Theorem 1.4 is not new: The Mahler base $e_0, e_1, \ldots$ of $C(\mathbb{Z}_p \to K)$ defined by $e_m(x) = \left(\frac{x}{m}\right)$ is proved in [3], §54 to be a Schauder base for $C^n(\mathbb{Z}_p \to K)$, for each $n$.

2. It follows directly from Theorem 1.4 that the polynomial functions $X \to K$ form a dense subset of $C^\infty(X \to K)$.

2. A WEIERSTRASS-STONE THEOREM FOR $C^n$-FUNCTIONS

For this Theorem (2.10) we will need the continuity of $g \mapsto g \circ f$ in the $C^n$-topologies (Proposition 2.5). To prove it we need some technical lemmas that are in the spirit of [3], §77.

Let $n \in \mathbb{N}$. For a function $h : \nabla^n X \to K$ we define $\Delta h : \nabla^{n+1} X \to K$ by the formula

$$\Delta h(x_1, x_2, \ldots, x_{n+1}) = \frac{h(x_1, x_3, x_4, \ldots, x_{n+1}) - h(x_2, x_3, \ldots, x_{n+1})}{x_1 - x_2}$$

We have the following product rule.

**Lemma 2.1. (Product Rule).** Let $n \in \mathbb{N}$, let $h, t : \nabla^n X \to K$. Then for all $(x_1, x_2, \ldots, x_{n+1}) \in \nabla^{n+1} X$ we have $\Delta(ht)(x_1, x_2, \ldots, x_{n+1}) = h(x_2, x_3, \ldots x_{n+1}) \Delta t(x_1, x_2, \ldots, x_{n+1}) + t(x_1, x_3, \ldots, x_{n+1}) \Delta h(x_1, x_2, \ldots, x_{n+1})$.

**Proof.** Straightforward.

**Lemma 2.2.** Let $f : X \to K$, $n \in \mathbb{N}_0$. Let $S_n$ be the set of the following functions defined on $\nabla^{n+1} X$.

$$\begin{align*}
(x_1, \ldots, x_{n+1}) &\mapsto \Phi_1 f(x_{i_1}, x_{i_2}) & (1 \leq i_1 < i_2 \leq n + 1) \\
(x_1, \ldots, x_{n+1}) &\mapsto \Phi_2 f(x_{i_1}, x_{i_2}, x_{i_3}) & (1 \leq i_1 < i_2 < i_3 \leq n + 1) \\
& \vdots \\
(x_1, \ldots, x_{n+1}) &\mapsto \Phi_n f(x_1, \ldots, x_{n+1}).
\end{align*}$$

For $k \in \mathbb{N}$, let $R^n_k$ be the additive group generated by $S_n, S_n^2, \ldots, S_n^k$ where, for each $j \in \{1, \ldots, k\}$, $S_n^j$ is the product set $\{h_1 h_2 \ldots h_j : h_i \in S_n \text{ for each } i \in \{1, \ldots, j\}\}$. Then, for all $k, n \in \mathbb{N}$, $\Delta R^n_k \subseteq R^n_{k+1}$. 

Proof. We use induction with respect to \( k \). For the case \( k = 1 \) it suffices to prove \( h \in S_n \Rightarrow \Delta h \in R^1_{n+1} \). Then \( h \) has the form

\[
(x_1, \ldots, x_{n+1}) \mapsto \Phi_j f(x_{i_1}, x_{i_2}, \ldots, x_{i_{j+1}})
\]

for some \( j \in \{2, 3, \ldots, n+1\} \) and so

\[
\Delta h(x_1, x_2, \ldots, x_{n+1}) = \frac{h(x_1, x_3, \ldots, x_{n+2}) - h(x_2, x_3, \ldots, x_{n+2})}{x_1 - x_2}
\]

vanishes if \( i_1 > 1 \) (and then \( \Delta h \) is the null function), while if \( i_1 = 1 \) it equals

\[
\frac{\Phi_j f(x_1, x_{i_2}, \ldots, x_{i_{j+1}+1}) - \Phi_j f(x_2, x_{i_2}, \ldots, x_{i_{j+1}+1})}{x_1 - x_2}
\]

and it follows that \( \Delta h \in S_{n+1} \subset R^1_{n+1} \). For the induction step assume \( \Delta R^{k-1}_{n} \subset R^{k-1}_{n+1} \); it suffices to prove that \( \Delta S^k_n \subset R^k_{n+1} \). So let \( h \in S^k_n \) and write \( h = h_1 H \), where \( h_1 \in S_n, H \in S^{k-1}_n \). By the Product Rule 2.1 we have

\[
\Delta h(x_1, \ldots, x_{n+2}) = h_1(x_2, x_3, \ldots, x_{n+2}) \Delta H(x_1, x_2, \ldots, x_{n+2}) + H(x_1, x_3, \ldots, x_{n+2}) \Delta h_1(x_1, x_2, \ldots, x_{n+2}).
\]

The fact that \( h_1 \in S_n \) makes

\[
(x_1, x_2, \ldots, x_{n+2}) \mapsto h_1(x_1, x_3, \ldots, x_{n+2})
\]

into an element of \( S_{n+1} \). Similarly, since \( H \in S^{k-1}_n \), the function

\[
(x_1, x_2, \ldots, x_{n+2}) \mapsto H(x_2, x_3, \ldots, x_{n+2})
\]

is in \( S^{k-1}_{n+1} \). By our first induction step, \( \Delta h_1 \in R^1_{n+1} \) and by the induction hypothesis \( \Delta H \in R^{k-1}_{n+1} \). Hence,

\[
\Delta h \in S_{n+1} R^1_{n+1} + S^{k-1}_{n+1} R^1_{n+1} \\
\subset R^1_{n+1} R^1_{n+1} + R^{k-1}_{n+1} R^1_{n+1} \subset R^k_{n+1}.
\]

Lemma 2.3. Let \( f, n, S_n, k, R^k_n \) be as in the previous lemma. Let \( f(X) \subset Y \subset K \) where \( Y \) has no isolated points. Let \( g : Y \to K \) be a \( C^n \)-function. Let \( B_n \) be the set of the following functions defined on \( \nabla^{n+1} X \):

\[
(x_1, \ldots, x_{n+1}) \mapsto \Phi_1 g(f(x_{i_1}), f(x_{i_2})) \quad (1 \leq i_1 < i_2 \leq n + 1)
\]

\[
(x_1, \ldots, x_{n+1}) \mapsto \Phi_2 g(f(x_{i_1}), f(x_{i_2}), f(x_{i_3})) \quad (1 \leq i_1 < i_2 < i_3 \leq n + 1)
\]

\[
\vdots
\]

\[
(x_1, \ldots, x_{n+1}) \mapsto \Phi_n g(f(x_1), f(x_2), \ldots, f(x_{n+1})).
\]
Let $A_n$ be the additive group generated by $B_nR^n$. Then

$$\Delta A_n \subset A_{n+1}.$$ 

**Proof.** We prove: $h \in B_nR^n \Rightarrow \Delta h \in A_{n+1}$. Write $h = br$ where $b \in B_n$, $r \in R^n$. By the Product Rule 2.1 we have for all $(x_1, x_2, \ldots, x_{n+2}) \in \nabla^{n+2}X$

$$\Delta h(x_1, x_2, \ldots, x_{n+2}) = b(x_2, x_3, \ldots, x_{n+2})\Delta r(x_1, x_2, \ldots, x_{n+2}) + r(x_1, x_2, \ldots, x_{n+2})\Delta b(x_1, x_2, \ldots, x_{n+2}).$$

We have:

(i) $b \in B_n$ so $(x_1, \ldots, x_{n+2}) \mapsto b(x_2, x_3, \ldots, x_{n+1})$ is in $B_{n+1}$.

(ii) $r \in R^n$ so $(x_1, \ldots, x_{n+2}) \mapsto r(x_1, x_3, \ldots, x_{n+2})$ is in $R_{n+1}$ (in the previous proof we had $r \in S_n^{1}$, the map $(x_1, \ldots, x_{n+2}) \mapsto r(x_1, x_3, \ldots, x_{n+1})$ is in $S_n^{1}$, and (ii) follows from this).

(iii) $r \in R^n$ so $\Delta r \in R^{n+1}$ (Previous Lemma).

(iv) $b$ has the form

$$(x_1, x_2, \ldots, x_{n+1}) \mapsto \Phi_j g(f(x_i), \ldots, f(x_{i+1}))$$

for some $j \in \{2, \ldots, n+1\}$ and so

$$\Delta b(x_1, x_2, \ldots, x_{n+2}) = \frac{b(x_1, x_3, x_4, \ldots, x_{n+2}) - b(x_2, x_3, \ldots, x_{n+2})}{x_1 - x_2}$$

vanishes if $i_1 > 1$ (and then $\Delta b$ is the null function), while if $i_1 = 1$ it equals

$$\frac{\Phi_j g(f(x_1), f(x_{i_2} + 1), \ldots, f(x_{i_{j+1} + 1})) - \Phi_j g(f(x_2), f(x_{i_2} + 1), \ldots, f(x_{i_{j+1} + 1}))}{x_1 - x_2}$$

$$= \Phi_{j+1} g(f(x_1), f(x_2), f(x_{i_2} + 1), \ldots, f(x_{i_{j+1} + 1}))\Phi_1 f(x_1, x_2).$$

(if $f(x_1) = f(x_2)$ we have 0 at both sides). So we see that $\Delta b \in B_{n+1}R_{n+1}^{1}$.

Combining (i) - (iv) we get $\Delta h \in B_{n+1}R_{n+1}^{1} + R^n_{n+1}B_{n+1}R_{n+1}^{1} \subset B_{n+1}R_{n+1}^{n+1} + B_{n+1} \cdot R_{n+1}^{n+1} \subset A_{n+1}$.

**Corollary 2.4.** With the notations as in the previous lemma we have $\Phi_n(g \circ f) \in A_n$ ($n \in N$).

**Proof.** We proceed by induction on $n$. For the case $n = 1$ we write, for $(x_1, x_2) \in \nabla^2X$,

$$\Phi_1(g \circ f)(x_1, x_2) = (x_1 - x_2)^{-1}\left(g(f(x_1)) - g(f(x_2))\right) = \Phi_1 g(f(x_1), f(x_2))\Phi_1 f(x_1, x_2).$$
Hence, \( \Phi_1(g \circ f) \in B_1 S_1 \subset B_1 R_1 \subset A_1 \). To prove the step \( n \to n+1 \) observe that by the induction hypothesis, \( \Phi_n(g \circ f) \in A_n \). By Lemma 2.3, \( \Phi_{n+1}(g \circ f) = \Delta \Phi_n(g \circ f) \in A_{n+1} \).

**Remark.** From Corollary 2.4 it follows easily that the composition of two \( C^n \)-functions is again a \( C^n \)-function, a result that already was obtained in [3], 77.5.

**Proposition 2.5.** (Continuity of \( g \mapsto g \circ f \)) Let \( n \in \mathbb{N}_0 \), let \( f \in C^n(X \to K) \) and let \( g \in C^n(Y \to K) \) where \( Y \) has no isolated points, \( Y \supset f(X) \). Then \( \|g \circ f\|_{n,X} \leq \|g\|_{n,Y} \max_{0 \leq j \leq n} \|f\|^j_{j,X} \).

**Proof.** We may assume \( \|g\|_{n,Y} < \infty \). It suffices to prove \( \|\Phi_0(g \circ f)\|_{n+1,X} \leq \|g\|_{n,Y} \|f\|_{n,X} \). Now \( \|\Phi_0(g \circ f)\|_{n+1,X} = \max_{x \in X} |g(f(x))| \leq \|g\|_{n,Y} \|f\|_{n,X} \) which proves the case \( n = 0 \). For \( n \geq 1 \) we apply Corollary 2.4 which says that \( \Phi_n(g \circ f) \in A_n \) i.e. \( \Phi_n(g \circ f) \) is a sum of functions in \( B_n S^n \). By the definition of \( B_n \) we have

\[
(*) \quad h \in B_n \Rightarrow \|h\|_{n+1,X} \leq \|g\|_{n,Y}
\]

Similarly

\[
k \in S_n \Rightarrow \|k\|_{n+1,X} \leq \max_{1 \leq i \leq n} \|\Phi_i f\|_{n+1,X} \leq \|f\|_{n,X}
\]

so that

\[
(**) \quad k \in S^n \Rightarrow \|k\|_{n+1,X} \leq \|f\|_{n,X}
\]

Combination of (*) and (**) yields \( \|\Phi_n(g \circ f)\|_{n+1,X} \leq \|g\|_{n,Y} \|f\|_{n,X} \).

Proposition 2.5 enables us to prove

**Proposition 2.6.** Let \( n \in \mathbb{N}_0 \) and let \( A \) be a closed subalgebra of \( C^n(X \to K) \). Suppose \( A \) separates the points of \( X \) and contains the constant functions. Then \( A \) contains all locally constant functions \( X \to K \).

**Proof.** 1. We first prove that \( f \in A, U \subset K \), \( U \) clopen implies \( \xi_{f^{-1}(U)} \in A \). In fact, \( f(X) \) is compact so there exist a \( \delta \in (0,1) \) and finitely many disjoint balls \( B_1, \ldots, B_m \) of radius \( \delta \) covering \( f(X) \) where, say, \( B_1, \ldots, B_q \) lie in \( U \), and \( B_{q+1}, \ldots, B_m \) are in \( K \backslash U \). Let \( \varepsilon > 0 \). By the Key Lemma 1.2 there exists, for each \( i \in \{1, \ldots, m\} \) a polynomial \( P_i \) such that \( \|\xi_{B_i} - P_i\|_{n,B} < \varepsilon \), where \( B := \bigcup_{i=1}^m B_i \). Then \( P := \sum_{i=1}^q P_i \) is a polynomial and

\[
\|P - \xi_U\|_{n,B} = \|P - \xi_{B^0}\|_{n,B} = \| \sum_{i=1}^q (P_i - \xi_{B_i})\|_{n,B} < \varepsilon,
\]

where \( B^0 := \bigcup_{i=1}^q B_i \).

By Proposition 2.5

\[
\|(P - \xi_U) \circ f\|_{n,X} \leq \|P - \xi_U\|_{n,B} \max_{0 \leq j \leq n} \|f\|^j_{j,X} \leq \varepsilon \max_{0 \leq j \leq n} \|f\|^j_{j,X}
\]
and we see that there exists a sequence $P_1, P_2, \ldots$ of polynomials such that $\|P_k \circ f - \xi U \circ f\|_n, X \to 0$. Since $A$ is an algebra with an identity we have $P_k \circ f \in A$ for all $k$. Then $\xi_{f^{-1}(U)} = \xi U \circ f = \lim_{k \to \infty} P_k \circ f \in A$.

2. Now consider

$$B := \{V \subset X, \xi_V \in A\}.$$ 

It is very easy to see that $B$ is a ring of clopen subsets of $X$ and that $B$ covers $X$. To show that $B$ separates the points of $X$ let $x \in X, y \in X, x \neq y$. Then there is an $f \in A$ for which $f(x) \neq f(y)$. Set $U := \{\lambda \in K : |\lambda - f(x)| < |f(x) - f(y)|\}$. Then $U$ is clopen in $K$. By the first part of the proof, $f^{-1}(U) \in B$. But $x \in f^{-1}(U)$ whereas $y \notin f^{-1}(U)$.

By [1], Exercise 2.H $B$ is the ring of all clopens of $X$. It follows easily that all locally constant functions are in $A$.

To arrive at the Weierstrass-Stone Theorem 2.10 we need a final technical lemma.

**Lemma 2.7.** Let $a_1, \ldots, a_m \in X$, let $\delta_1, \ldots, \delta_m$ be in $(0, 1)$ such that $B(a_1, \delta_1), \ldots, B(a_m, \delta_m)$ form a disjoint covering of $X$. Let $n \in \mathbb{N}_0$, $h \in C^n(X \to K)$ and suppose $D_j h(a_i) = 0$ and $|D^n_{x_j} h(x_1, \ldots, x_{n+j})| \leq \varepsilon$ for all $i \in \{1, \ldots, m\}, x_1, \ldots, x_{n+1} \in B(a_i, \delta_i) \cap X, j \in \{0, 1, \ldots, n\}$. Then $\|h\|_n, X \leq \varepsilon$.

**Proof.** We first prove that $\|h\|_n, X \leq \varepsilon$ (see Proposition 0.4(iii)). Let $i \in \{1, \ldots, m\}$. Set $B_i = B(a_i, \delta_i)$. By Taylor's formula (Proposition 0.3(iv)) we have for $x \in X \cap B_i$:

$$|h(x)| = \left| \sum_{s=0}^{n-1} \left( x - a_i \right)^s D_s h(a_i) + \left( x - a_i \right)^n \rho_1 h(x, a_i) \right| \leq |x - a_i|^n |\Phi_n h(x, a_i, \ldots, a_i)| \leq \delta_i^n \varepsilon.$$

Similarly we have for $j \in \{0, \ldots, n-1\}$ and $x \in X \cap B_i$:

$$|D_j h(x)| = \left| \sum_{t=0}^{n-1-j} \left( x - a_i \right)^t D_t D_j h(a_i) + \left( x - a_i \right)^{n-j} \rho_1 \rho_1(D_j h)(x, a_i) \right| \leq \delta_i^{n-j} \varepsilon.$$

It follows that $\|h\|_X, \|D_1 h\|_X, \ldots, \|D_{n-1} h\|_X$ are all $\leq \varepsilon$. Now let $x, y \in X$. If $x, y$ are in the same $B_i$ then $|\rho_1 h(x, y)| = |\Phi_n h(x, y, y, \ldots, y)| \leq \varepsilon$ by assumption. If $x \in B_i$, $y \in B_s$ and $i \neq s$ then $|x - y| \geq \delta := \max(\delta_i, \delta_s)$ and by Taylor's formula

$$h(x) = \sum_{t=0}^{n-1} \left( x - y \right)^t D_t h(y) + \left( x - y \right)^n \rho_1 h(x, y)$$

we obtain, using (*),

$$|\rho_1 h(x, y)| \leq \frac{|h(x) - h(y)|}{|x - y|^n} \vee \frac{|D_1 h(y)|}{|x - y|^{n-1}} \vee \ldots \vee \frac{|D_{n-1} h(y)|}{|x - y|} \leq \frac{\delta^n \varepsilon}{\delta^n} \vee \frac{\delta_{n-1}^{n-1} \varepsilon}{\delta_{n-1}} \vee \ldots \vee \frac{\delta_s \varepsilon}{\delta} \leq \varepsilon.$$
and we have proved \( \|h\|_{n,X}^\alpha \leq \varepsilon \).

Now to prove that even \( \|h\|_{n,X} \leq \varepsilon \) observe that by Proposition 0.4(iii)

\[
\|h\|_{n,X} = \|h\|_{n,X}^\alpha \vee \|D_1 h\|_{n-1,X}^\alpha \vee \cdots \vee \|D_n h\|_{0,X}^\alpha.
\]

To prove, for example, that \( \|D_1 h\|_{n-1,X}^\alpha \leq \varepsilon \) we observe that \( D_1 h \in C^{n-1}(X \to K) \) and that for \( i \in \{1, \ldots, m\} \) and \( j \in \{0, 1, \ldots, n-2\} \) we have \( D_j D_1 h(a_i) = (j+1)D_{j+1} h(a_i) = 0 \) and for all \( x_1, \ldots, x_n \in B(a_i, \delta_i) \) and \( j \in \{0, 1, \ldots, n-2\} \)

\[
|\overline{\mathbb{F}}_{n-1-j} D_j(D_1 h)(x_1, \ldots, x_{n-j})| = |(j+1)| \overline{\mathbb{F}}_{n-1-j} D_{j+1} h(x_1, \ldots, x_{n-j})| \leq \varepsilon
\]

by assumption. So the conditions of our Lemma (with \( D_1 h, n-1 \) in place of \( h, n \) respectively) are satisfied and by the first part of the proof we may conclude that \( \|D_1 h\|_{n-1,X}^\alpha \leq \varepsilon \). In a similar way we prove that \( \|D_2 h\|_{n-2,X}^\alpha \leq \varepsilon, \ldots, \|D_n f\|_{0,X}^\alpha \leq \varepsilon \)

and it follows that \( \|h\|_{n,X} \leq \varepsilon \).

**Proposition 2.8.** Let \( n \in \mathbb{N}_0 \) and let \( A \) be a closed subalgebra of \( C^n(X \to K) \) containing the locally constant functions. Let \( g \in C^n(X \to K) \) and suppose for each \( a \in X \) there exists an \( f_a \in A \) with \( D_i g(a) = D_i f_a(a) \) for \( i \in \{0, 1, \ldots, n\} \). Then \( g \in A \).

**Proof.** Let \( \varepsilon > 0 \). For each \( a \in X \), choose an \( f_a \in A \) with \( f_a(a) = g(a), D_1 f_a(a) = D_1 g(a), \ldots, D_n f_a(a) = D_n g(a) \). By continuity there exists a \( \delta_a > 0 \) such that, with \( h_a := f_a - g, \overline{\mathbb{F}}_{n-j} D_j h_a(x_1, \ldots, x_{n-j+1}) \leq \varepsilon \) for all \( j \in \{0, 1, \ldots, n\} \) and \( x_1, \ldots, x_{n-j+1} \in B(a, \delta_a) \). The \( B(a, \delta) \) cover \( X \) and by compactness there exists a finite disjoint subcovering \( B(a_1, \delta_{a_1}), \ldots, B(a_m, \delta_{a_m}) \). Set

\[
f := \sum_{i=1}^m f_{a_i} \chi_{B(a_i, \delta_{a_i})} \cap X
\]

Then, by our assumption on \( A, f \in A \). By Lemma 2.7, applied to \( h := f - g \) and where \( \delta_1, \ldots, \delta_m \) are replaced by \( \delta_{a_1}, \ldots, \delta_{a_m} \) respectively, we then have \( \|f - g\|_{n,X} \leq \varepsilon \). We see that \( g \in \overline{A} = A \).

**Remark.** It follows directly that the local polynomial functions \( X \to K \) form a dense subset of \( C^n(X \to K) \).

**Proposition 2.9.** Let \( n \in \mathbb{N} \) and let \( A \) be a \( K \)-subalgebra of \( C^n(X \to K) \) containing the constant functions. Suppose \( f'(a) \neq 0 \) for some \( f \in A, a \in X \). Then there is a \( g \in A \) with \( g(a) = 0, g'(a) = 1 \) and \( D_2 g(a) = D_3 g(a) = \cdots = D_n g(a) = 0 \).

**Proof.** By considering the function \( f'(a)^{-1}(f - f(a)) \) it follows that we may assume that \( f(a) = 0, f'(a) = 1 \). Then

\[
(*) \quad f = (X - a)h
\]
where \( h \) is continuous, \( h(a) = 1 \). To obtain the statement by induction with respect to \( n \) we only have to consider the induction step \( n - 1 \to n \) and, to prove that, we may assume that 
\[
D_2 f(a) = \cdots = D_{n-1} f(a) = 0.
\]
From (*) we obtain
\[
f^n = (X - a)^n h^n
\]
and by uniqueness of the Taylor expansion of the \( C^n \)-function \( f^n \) we obtain 
\[
f^n(a) = D_1 f^n(a) = \cdots = D_{n-1} f^n(a) = 0 \quad \text{and} \quad D_n f^n(a) = h^n(a) = 1.
\]
We see that \( g := f - D_n f(a) f^n \) is in \( A \) and that \( g(a) = 0 \), \( g'(a) = 1 \), \( D_2 g(a) = \cdots = D_{n-1} g(a) = 0 \) and 
\[
D_n g(a) = D_n f(a) - D_n f(a) D_n f^n(a) = 0.
\]

**Theorem 2.10.** (Weierstrass-Stone Theorem for \( C^n \)-functions). Let \( n \in \mathbb{N}_0 \) and let \( A \) be a closed subalgebra that separates the points of \( A \) and that contains the constant functions. Suppose also that for each \( a \in X \) there exists an \( f \in A \) with \( f'(a) \neq 0 \). Then \( A = C^n(X \to K) \).

**Proof.** By Proposition 2.9, for each \( a \in X \) there exists an \( f \in A \) with \( f(a) = 0 \), \( f'(a) = 1 \), \( D_i f(a) = 0 \) for \( i \in \{2, \ldots, n\} \). The function \( g := X \) satisfies \( g(a) = 0 \), \( g'(a) = 1 \), \( D_i g(a) = 0 \) for \( i \in \{2, \ldots, n\} \) so applying Proposition 2.8 (observe that \( A \) contains the locally constant functions by Proposition 2.6) we obtain that \( X \in A \). But then all polynomials are in \( A \) and \( A = C^n(X \to K) \) by the Weierstrass Theorem 1.4.

**Remarks.**

1. The case \( n = 0 \) yields, at least for those \( X \) that are embeddable into \( K \), the well known Kaplansky Theorem proved in [1], 6.15.

2. We leave it to the reader to establish a \( C^\infty \)-version of Theorem 2.10.

**REFERENCES**

