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Annales mathématiques Blaise Pascal, tome 1, n° 1 (1994), p. 27-42

<http://www.numdam.org/item?id=AMBP_1994__1_1_27_0>
ON $G_2$-MANIFOLDS

A. Kobotis and Ph.J. Xenos

1. Introduction

Let $(M,J,g)$ be a $2m$-dimensional almost Hermitian manifold. We denote by $\nabla$ the natural metric connection on $M$, and $R(X,Y,Z)$ is the curvature operator associated with $\nabla$. The fundamental 2-form of $M$ is $F(X,Y) = g(\bar{X},Y)$ for all vector fields $X$ and $Y$ where $\bar{X} = JX$.

An almost Hermitian manifold $(M,J,g)$ is said to be a $G_2$-manifold, if and only if, for all vector fields $X,Y$ and $Z$ on $M$ holds,

$$\sum_{X,Y,Z} (\nabla_X F)(Y,Z) = \sum_{X,Y,Z} (\nabla_{\bar{X}} F)(\bar{Y},Z)$$

(1.1)

where by $\sum$ denote the cyclic sum [2].

In the present paper, we shall obtain some properties of a $G_2$-manifold. In the second paragraph we give some fundamental properties of a $G_2$-manifold. The curvature identities of a $G_2$-manifold are given in the third paragraph. The $G_2$-manifolds with constant holomorphic sectional curvature are studied in the fourth paragraph, from this section and in what follows we consider only $G_2$-manifolds which are RK-manifolds. In the fifth section, we deal with so called "the Schur's theorem". Finally, the Chern classes of a $G_2$-manifold are given in the last paragraph.
2. Preliminaries

On an almost Hermitian manifold \((M, J, g)\) we denote by

\[ (X, Y, Z, W) = (\nabla_X F)(Y, (\nabla_Z J) W), \]

\[ [X, Y, Z, W] = (\nabla F)(Z, W), \]

\[ (\nabla_X J) Y \]

\[ R(X, Y, Z, W) = g(R(X, Y)Z, W). \]

It is well known that the following conditions hold:

\[
(\nabla_X \nabla_Y F)(Z, W) - (\nabla F)(Z, W) + (\nabla_X \nabla_Y F)(W, Z) - (\nabla F)(W, Z) = 0,
\]

\[ (\nabla_X \nabla_Y F)(\bar{Z}, \bar{W}) - (\nabla F)(\bar{Z}, \bar{W}) + (\nabla_X \nabla_Y F)(Z, W) - (\nabla F)(Z, W) = \]

\[ = (\nabla_Y \nabla_X F)(\bar{Z}, \bar{W}) - (\nabla F)(\bar{Z}, \bar{W}) + (\nabla_Y \nabla_X F)(Z, W) - (\nabla F)(Z, W) = \]

\[ = (X, Z, Y, \bar{W}) - (X, W, Y, \bar{Z}) , \]

\[ R(X, Y, Z, W) + R(X, Y, Z, \bar{W}) = (\nabla_X \nabla_Y F)(Z, W) - (\nabla F)(Z, W) - \]

\[ - (\nabla_Y \nabla_X F)(Z, W) + (\nabla F)(Z, W). \]

Proposition 2.1. Let \(M\) be a \(G_2\)-manifold. Then for all vector fields \(X, Y, Z\) and \(W\) on \(M\), we have:

\[ [X, Y, Z, W] = [X, Y, Z, \bar{W}] + (X, Y, Z, \bar{W}) - (X, Y, W, \bar{Z}) + (X, Y, \bar{Z}, W) - (X, Y, \bar{W}, Z) , \]

\[ [X, Y, Z, \bar{W}] = -[X, Y, Z, W] - (X, Y, Z, W) + (X, Y, W, Z) + (X, Y, \bar{Z}, W) - (X, Y, \bar{W}, Z) , \]

\[ [\bar{X}, Y, Z, W] = -[\bar{X}, Y, Z, \bar{W}] - (\bar{X}, Y, Z, W) + (\bar{X}, Y, W, Z) + (\bar{X}, Y, \bar{Z}, W) - (\bar{X}, Y, \bar{W}, Z) , \]
\[
\]

**Proof.** The above conditions are proved using the definition-relation (1.1) of a $G_2$-manifold.

**Proposition 2.2.** On a $G_2$-manifold $M$, for all vector fields $X, Y, Z$ and $W$ we have:

\[
\begin{align*}
\end{align*}
\]
Proof. On a $G_2$-manifold it holds (1.1) and the relation:

$$\nabla_X F(Y,Z) + \nabla_Y F(Z,X) + \nabla_Z F(X,Y) = 0$$

We take the covariant derivative of the above relations and making use of the condition:

$$\nabla_{XW} F(Y,Z) - \nabla_{WX} F(Y,Z) = -R(W,X,Y,Z) - R(W,X,Y,Z)$$

we obtain (2.4). If we substitute $JX$ for $X$ in (2.4) we obtain (2.5).

3. Curvature Identities

On a $G_2$-manifold

$$\nabla_X F(Y,Z) - \nabla_Y F(Z,X)$$

is a linear combination of terms of the form:

\begin{align*}
(A,B,C,D), & \quad (\tilde{A},\tilde{B},C,D), \quad (A,B,\tilde{C},D), \quad (\tilde{A},B,\tilde{C},D), \\
(\tilde{A},B,C,D), & \quad (A,B,\tilde{C},D), \quad (A,B,C,\tilde{D}), \quad (\tilde{A},B,\tilde{C},\tilde{D}), \\
[A,B,C,D], & \quad [A,B,\tilde{C},D], \quad [\tilde{A},B,C,D], \quad [\tilde{A},B,\tilde{C},D],
\end{align*}

where $A, B, C$ and $D$ are some combinations of vector fields $X, Y, Z$ and $W$ on $M$. 

\begin{align*}
+(W,X,\tilde{Y},Z)-(W,X,\tilde{Y},Z)+(W,Y,\tilde{Z},X)+(W,Z,\tilde{X},Y)-(W,Z,\tilde{Y},X)-
-(X,Y,\tilde{Z},\tilde{W})-(X,W,\tilde{Y},Z)+(X,W,\tilde{Z},\tilde{Y})+(X,Z,\tilde{Y},\tilde{W})-(Z,X,\tilde{Y},\tilde{W})+(Z,W,\tilde{Y},X)-
-(\tilde{W},X,\tilde{Y},Z)+(\tilde{W},X,\tilde{Z},\tilde{Y})+(\tilde{W},Y,\tilde{X},Z)-(\tilde{W},Y,\tilde{Z},X)+(\tilde{W},Z,\tilde{X},Y)+(\tilde{W},Z,\tilde{Y},X)+
+(X,Y,\tilde{Z},\tilde{W})+(X,W,\tilde{Y},Z)-(X,W,\tilde{Z},\tilde{Y})+(X,Z,\tilde{Y},\tilde{W})-(Z,W,\tilde{Y},X)=0 \quad (2.5)
\end{align*}
parameters.

Applying (2.1), (2.2), (2.3), (2.4), (2.5) and the first Bianchi identity the number of the parameters is reduced to eight.

Let \( \{E_1, \ldots, E_m, JE_1, \ldots, JE_m\} \) be an orthonormal local frame field on \( M \). We denote by \( r \) and \( r^* \) the Ricci tensor and the Ricci *tensor respectively. The Ricci *tensor \( r^* \) is defined by

\[
r^*(x,y) = \text{trace of } (z \to R(Jz,x)Jy),
\]

for \( x, y, z \in T_p M \), where \( T_p M \) is the tangent space at \( p \in M \).

Using (3.2) we obtain an equation which gives \( r^*(X,Y) - r(X,Y) \). In this relation we substitute \( X, Y \) for \( Y, X \) and subtracting last two equations we have a relation giving \( r^*(X,Y) - r^*(X,Y) \).

From the last equation substituting \( JX, JY \) for \( X, Y \) respectively and adding these two equations we can determine other three parameters.

By virtue of the above calculations we can state the following assertion.

**Theorem 3.1.** If \( M \) is a \( G_2 \)-manifold and \( R \) its curvature operator, then for arbitrary vector fields \( X, Y, Z \) and \( W \) on \( M \) we have:

\[
R(X,Y,Z,W) - R(X,Y,Z,W) = \\
= a_1 \left[ (W,Z,X,Y)-(W,Z,Y,X)+(X,Y,Z,W)+(Z,W,Y,X) - \\
-2(\tilde{W},X,Z,Y)-(\tilde{W},Z,X,Y)-(\tilde{W},Z,X,Y) - \\
\right]
\]
+ (\tilde{W}, \tilde{X}, Y, Z) - (\tilde{W}, \tilde{Y}, X, Z) + (\tilde{X}, \tilde{Y}, Z, W) - (\tilde{Z}, \tilde{X}, Y, W) -
- (W, Z, \tilde{X}, \tilde{Y}) + (W, Z, \tilde{Y}, X) + (X, W, \tilde{Z}, \tilde{Y}) - (Z, W, \tilde{Y}, X) +
+ (\tilde{W}, Z, \tilde{X}, Y) - (\tilde{W}, Z, \tilde{Y}, X) - (\tilde{X}, Y, Z, W) + (\tilde{Z}, W, \tilde{Y}, X)
+ b_2 \left[ (W, Z, X, Y) - (W, Z, Y, X) - (X, Y, Z, W) + (Z, W, Y, X) -
- (\tilde{W}, Z, X, Y) + (\tilde{W}, Z, Y, X) + (\tilde{X}, Y, Z, W) - (\tilde{Z}, \tilde{W}, Y, X) -
- (W, Z, \tilde{X}, Y) + (W, Z, \tilde{Y}, X) + (X, Y, \tilde{Z}, W) - (Z, W, \tilde{Y}, X) +
+ (\tilde{W}, Z, \tilde{X}, Y) - (\tilde{W}, Z, \tilde{Y}, X) - (\tilde{X}, Y, Z, W) + (\tilde{Z}, W, \tilde{Y}, X) \right]
+ b_3 \left[ (W, X, Z, Y) - (W, Y, Z, X) - (X, \tilde{Y}, Z, W) + (W, Z, \tilde{X}, \tilde{Y}) -
- (W, Z, \tilde{Y}, X) + (Z, W, \tilde{Y}, X) - (\tilde{W}, X, \tilde{Z}, Y) + (\tilde{W}, Y, \tilde{Z}, X) -
- (\tilde{W}, Z, \tilde{X}, Y) + (\tilde{W}, Z, \tilde{Y}, X) + (\tilde{X}, Y, \tilde{Z}, W) - (\tilde{Z}, W, \tilde{Y}, X) \right]
+ b_4 \left[ - (\tilde{X}, \tilde{Y}, Z, W) + (\tilde{X}, \tilde{W}, Z, Y) + (W, X, \tilde{Y}, \tilde{Z}) - (W, Y, \tilde{X}, \tilde{Z}) +
+ (W, Z, \tilde{X}, \tilde{Y}) - (W, Z, \tilde{Y}, X) - (Z, X, \tilde{Y}, W) + (Z, W, \tilde{Y}, X) +
+ (\tilde{W}, X, \tilde{Y}, Z) - (\tilde{W}, Y, \tilde{X}, Z) - (\tilde{W}, Z, \tilde{X}, Y) + (\tilde{W}, Z, \tilde{Y}, X) +
+ (\tilde{X}, X, \tilde{Y}, Z, W) + (X, W, Z, Y) - (\tilde{Z}, X, Y, W) - (\tilde{Z}, W, \tilde{Y}, X) \right]. (3.2)

R(X, \tilde{Y}, Z, W) - R(X, Y, Z, W) =

= 2a_1 \left[ - (\tilde{W}, X, Z, Y) + (\tilde{W}, Y, Z, X) - (\tilde{W}, Z, X, Y) + (\tilde{W}, Z, Y, X) +
+ (\tilde{X}, Y, Z, W) - (\tilde{X}, W, Y, Z) + (\tilde{X}, Z, Y, W) - (\tilde{Z}, W, Y, X) -
- (W, X, \tilde{Z}, Y) + (W, Y, \tilde{Z}, X) - (W, Z, \tilde{X}, Y) + (W, Z, \tilde{Y}, X) +
+ (X, Y, \tilde{Z}, W) - (X, W, \tilde{Y}, Z) + (X, Z, \tilde{Y}, W) - (Z, W, \tilde{Y}, X) \right]
+ b_3 \left[ (W, X, Z, Y) - (W, Y, Z, X) + (W, Z, X, Y) - (W, Z, Y, X) -
- (X, Y, Z, W) + (X, W, Y, Z) - (X, Z, Y, W) + (Z, W, Y, X) -

- (X, Y, Z, W) + (X, W, Y, Z) - (X, Z, Y, W) + (Z, W, Y, X) -


We consider in what follows only G2-manifolds which are RK-manifolds (i.e. \( R(X,Y,Z,W) = R(X,Y,Z,W) \)) without mentioning it always explicitly. Because of (3.2) and (3.3) we have:

\[
\begin{align*}
-\langle \tilde{W}, X, \tilde{Z}, Y \rangle + \langle \tilde{W}, Y, \tilde{Z}, X \rangle - \langle \tilde{W}, Z, X, Y \rangle + \langle \tilde{W}, Z, Y, X \rangle + \\
+\langle X, Y, Z, W \rangle - \langle X, W, Y, Z \rangle + \langle X, Z, Y, W \rangle - \langle Z, W, Y, X \rangle + \\
+ (b_1 - b_4) \left[ (W, X, Y, Z) - (W, Y, X, Z) - (W, Z, X, Y) + (W, Z, Y, X) + \\
+ (X, Y, Z, W) + (X, W, Z, Y) - (Z, X, Y, W) - (Z, W, Y, X) - \\
- (W, X, Y, Z) + (W, Y, X, Z) + (W, Z, X, Y) - (W, Z, Y, X) - \\
\end{align*}
\] (3.3)

4. Curvature tensors on G2-manifolds with constant holomorphic sectional curvature

We consider in what follows only G2-manifolds which are RK-manifolds (i.e. \( R(X,Y,Z,W) = R(X,Y,Z,W) \)) without mentioning it always explicitly. Because of (3.2) and (3.3) we have:

\[
= b_1 \left[ (W, X, Y, Z) - (W, Y, X, Z) + (X, W, Z, Y) - (Z, X, Y, W) + \\
+ (W, X, Y, Z) - (W, Y, X, Z) + (X, W, Z, Y) - (Z, X, Y, W) + \\
+ (W, X, Y, Z) - (W, Y, X, Z) + (X, W, Z, Y) - (Z, X, Y, W) + \\
+ (W, X, Y, Z) - (W, Y, X, Z) + (X, W, Z, Y) - (Z, X, Y, W) + \\
+ b_2 \left[ (W, Z, X, Y) - (W, Z, Y, X) - (X, Y, Z, W) + (Z, W, Y, X) - \\
- (W, Z, X, Y) + (W, Z, Y, X) + (X, Y, Z, W) - (Z, W, Y, X) - \\
- (W, Z, X, Y) + (W, Z, Y, X) + (X, Y, Z, W) - (Z, W, Y, X) + \\
+ (W, Z, X, Y) - (W, Z, Y, X) + (X, Y, Z, W) - (Z, W, Y, X) \right]. \quad (4.1)
\]

We put
\( \lambda(x,y) = R(x,y,x,y) - R(\bar{x},\bar{y},x,y) \)

and

\[ Q(x) = R(x,\bar{x},x,\bar{x}) = H(x) \|x\|^4, \]

for \( x, y \in T_p M \). L. Vanhecke in [5] proved the following:

**Proposition 4.1.** Let \( M \) be an almost Hermitian manifold and \( x, y \in T_p M \). Then:

\[
32R(x,y,x,y) = 3Q(x+y) + 3Q(x-y) - Q(x+y) - Q(x-y) - 4Q(x) - 4Q(y) + 2 \left[ 13\lambda(x,y) - 3\lambda(\bar{x},\bar{y}) + \lambda(\bar{x},y) + \lambda(x,\bar{y}) \right].
\]

We assume that the holomorphic sectional curvature \( H(x) \) is constant \( c(p) \) for all \( x \in T_p M \) at each point \( p \) of \( M \). Since \( Q(x) = c(p) \|x\|^4 \), we have from the proposition 4.1:

\[
16R(x,y,xy) = 4c(p) \left[ \|x\|^2 \|y\|^2 - g^2(x,y) + 3g^2(x,\bar{y}) \right] + 13\lambda(x,y) - 3\lambda(\bar{x},\bar{y}) + \lambda(\bar{x},y) + \lambda(x,\bar{y}).
\]

By linearizing the above equation and using (4.1) we obtain the following:

**Proposition 4.2.** Let \( M \) be a \( G_2 \)-manifold of pointwise constant holomorphic sectional curvature \( c(p) \). If \( X, Y, Z \) and \( W \) are arbitrary vector fields, then the curvature tensor is given by:

\[
4R(X,Y,Z,W) = c(p) \left[ g(X,Z)g(Y,W) - g(X,W)g(Y,Z) + g(X,\bar{Z})g(Y,\bar{W}) - g(X,\bar{W})g(Y,\bar{Z}) + 2g(X,\bar{Y})g(Z,\bar{W}) \right].
\]
5. On Schur's theorem

In this paragraph, we shall consider the following problem: "Let $M$ be an almost Hermitian manifold of pointwise constant holomorphic sec-
tional curvature $c(p)$. When is $c$ a constant function?"

Gray and Vanhecke have proved an interesting theorem ([1], thm. 4.7). We shall make a slightly different approach to this problem.

**Lemma 5.1.** Let $M$ be an RK-manifold. Then:

$$2(\nabla_{E_j} r^*)(X, E_j) - \nabla_X s^* =$$

$$= \sum_i \left[ 2R(X, E_i, (\nabla_{E_j} J)E_j, E_i) + 2R(X, E_j, \nabla_{E_j} E_i, E_i) - R(E_j, E_i, \nabla_X E_i, E_i) \right], \quad (5.1)$$

for a vector field $X$ on $M$.

**Proof.** By definition we have

$$r^*(X, Y) = \sum_i R(X, E_i, Y, E_i)$$

and

$$s^* = \sum_j r^*(E_j, E_j) = \sum_{i,j} R(E_j, E_i, E_j, E_i).$$

By the standard calculation,

$$(\nabla_Y r^*)(X, Y) = \sum_i \left[ (\nabla_Y R)(X, E_i, Y, E_i) + R(X, E_i, (\nabla_Y J)Y, E_i) + R(X, E_i, Y, (\nabla_Y E_i) + R(X, E_i, Y, (\nabla_Y E_i) \right], \quad (5.2)$$

and using the second Bianchi identity,

$$\nabla_X s^* = \sum_{i,j} \left[ 2(\nabla_{E_j} R)(X, E_i, E_j, E_j) + R(E_j, E_j, \nabla_X E_i, E_i) \right]. \quad (5.3)$$

From (5.2) and (5.3) we obtain (5.1).
Lemma 5.2. Let $M$ be an almost Hermitian manifold. Then:

$$2(\nabla_{E_j} r)(X, E_j) - \nabla^2 s = 2 \sum_i \left[ R(X, \nabla_{E_j} E_i, E_j, E_i) + R(X, E_i, E_j, \nabla_{E_j} E_i) - R(E_j, \nabla_{E_i} E_i, E_j, E_i) \right], \quad (5.4)$$

for a vector field $X$ on $M$.

Proof. Direct computations give:

\[
(\nabla_Y r)(X, Y) = \sum_i \left[ (\nabla_Y R)(X, E_i, Y, E_i) + R(X, \nabla_Y E_i, Y, E_i) + R(X, E_i, Y, \nabla_Y E_i) \right], \quad (5.5)
\]

\[
\nabla_Y s = 2 \sum_{i,j} \left[ (\nabla_{E_i} R)(X, E_j, E_i, E_j) + R(E_j, \nabla_{E_i} E_i, E_j, E_i) \right]. \quad (5.6)
\]

The equation (5.4) can easily come from (5.5) and (5.6).

By the definition of Ricci tensor and Ricci *tensor we obtain:

Proposition 5.3. If $M$ is a $2m$-dimensional almost Hermitian manifold of pointwise constant holomorphic sectional curvature $c(p)$, then:

$$s + 3s^* = 4m(m+1) c(p). \quad (5.7)$$

If $M$ is an RK-manifold, then:

$$r(x, y) + 3r^*(x, y) = 2(m+1)c(p) g(x, y), \quad (5.8)$$

for $x, y \in T_p M$.

Now, we shall prove the Schur's theorem for almost Hermitian manifolds under some conditions.

Theorem 5.4. Let $M$ be a connected RK-manifold of pointwise holomorphic sectional curvature $c(p)$, with dim $M \geq 4$. If
2 \left[ R(X,Y,X,Z) + R(X,Z,Z,W) \right] + 3 \left[ R(X,Y,Z,W) - 2R(Z,X,Y,W) \right] = 0, \quad (5.9)

for arbitrary vector fields \( X, Y, Z \) and \( W \), then \( c \) is a constant function.

Proof. Differentiating (5.7) with respect to arbitrary \( E_i \), we have

\[
\nabla_{E_i} s + 3 \nabla_{E_i} s^* = 4m(m+1) \nabla_{E_i} c. \quad (5.10)
\]

Making use of (5.1), (5.4), (5.8) and (5.9) we obtain:

\[
\nabla_{E_i} s + 3 \nabla_{E_i} s^* = 2 \sum_j \left[ (\nabla_{E_j} r)(E_i, E_j) + 3(\nabla_{E_j} r^*)(E_i, E_j) \right] = 4(m+1) \nabla_{E_i} c. \quad (5.11)
\]

By (5.10) and (5.11)

\[
4(m^2-1) \nabla_{E_i} c = 0
\]

from which it follows that \( c \) is a constant function.

In particular we have:

Proposition 5.5. If \( M \) is a connected \( G_2 \)-manifold of pointwise holomorphic sectional curvature \( c(p) \), with \( \dim M > 4 \), then \( c \) is a constant function if \( M \) is an F-space (in the sense of [4]).

6. The Chern classes of a \( G_2 \)-manifold.

On an almost Hermitian manifold \( M \) always there is a connection \( D \) adapted to \( g \) and \( J \) ([3]). Denote by \( S \) the curvature tensor of \( D \), i.e.
On $G_2$-manifolds

The importance of $S$ is that because $\langle D_x Y, Z \rangle = 0$, where $D_x Y = \frac{1}{2}(\nabla_x Y - J\nabla_x Y)$. It is possible to express the Chern classes in terms of $S$.

Let $M$ be a compact $G_2$-manifold and $\{E_1, \ldots, E_m, \tilde{E}_1, \ldots, \tilde{E}_m\}$ be a local frame field. Then

$$\det (\delta_{ij} + \frac{n}{2\pi \sqrt{-1}}) = \sum_{i=1}^n v_i,$$ (6.1)

is a globally defined closed form which represents the total Chern class of $M$ via de Rham's theorem, where

$$\Xi_{ij}(X,Y) = R(X,Y,E_i,E_j) + \frac{1}{4} \left[ (X,E_i,Y,E_j) - (X,E_j,Y,E_i) \right] +$$

$$+ \frac{b_1}{2} \left[ - (E_i,X,Y,E_j) + (E_i,Y,X,E_j) + (E_j,X,Y,E_i) - (E_j,Y,X,E_i) -$$

$$- (E_i,X,Y,E_j) + (E_i,Y,X,E_j) + (E_j,X,Y,E_i) - (E_j,Y,X,E_i) -$$

$$- (E_i,X,Y,E_j) + (E_i,Y,X,E_j) + (E_j,X,Y,E_i) - (E_j,Y,X,E_i) -$$

$$- (E_i,X,Y,E_j) + (E_i,Y,X,E_j) + (E_j,X,Y,E_i) - (E_j,Y,X,E_i) \right] +$$

$$+ \frac{b_2}{2} \left[ - (E_i,E_j,X,Y) + (E_i,E_j,Y,X) + (E_j,E_i,X,Y) - (E_j,E_i,Y,X) +$$

$$+ (E_i,E_j,X,Y) - (E_i,E_j,Y,X) - (E_j,E_i,X,Y) + (E_j,E_i,Y,X) +$$

$$+ (E_i,E_j,X,Y) - (E_i,E_j,Y,X) - (E_j,E_i,X,Y) + (E_j,E_i,Y,X) -$$
We can state the following assertion.

**Proposition 6.1.** If $M$ is a compact $G_2$-manifold, then the total Chern class is given by (6.1).

**Corollary 6.2.** Let $M$ be a compact $G_2$-manifold. The first Chern class $\gamma_1$ of $M$ is given by
\[ \mathcal{V}_1(X,Y) = \frac{1}{4n} \left[ -2r^*(X,Y) + \sum_{i=1}^{m} (X,E_i,Y,E_i) \right] \]

for arbitrary vector fields \( X \) and \( Y \), where \( \{E_1, \ldots, E_m, \tilde{E}_1, \ldots, \tilde{E}_m\} \) is a local frame field on \( M \).
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