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ON G_2-MANIFOLDS

A. Kobotis and Ph.J. Xenos

1. Introduction

Let (M,J,g) be a 2m-dimensional almost Hermitian manifold. We denote by ∇ the natural metric connection on M, and R(X,Y,)Z is the curvature operator associated with ∇. The fundamental 2-form of M is F(X,Y) = = $g(\bar{X},Y)$ for all vector fields X and Y where $\bar{X}$=JX.

An almost Hermitian manifold (M,J,g) is said to be a G_2-manifold, if and only if, for all vector fields X,Y and Z on M holds,

$$\underset{X,Y,Z}{\mathfrak{S}} (\nabla_X F)(Y,Z) = \underset{X,Y,Z}{\mathfrak{S}} (\nabla_{\bar{X}} F)(\bar{Y},Z) \tag{1.1}$$

where by $\mathfrak{S}$ denote the cyclic sum [2].

In the present paper, we shall obtain some properties of a G_2-manifold. In the second paragraph we give some fundamental properties of a G_2-manifold. The curvature identities of a G_2-manifold are given in the third paragraph. The G_2-manifolds with constant holomorphic sectional curvature are studied in the fourth paragraph, from this section and in what follows we consider only G_2-manifolds which are RK-manifolds. In the fifth section, we deal with so called "the Schur's theorem". Finally, the Chern classes of a G_2-manifold are given in the last paragraph.

2. Preliminaries

On an almost Hermitian manifold (M,J,g) we denote by

$$(X,Y,Z,W) = (\nabla_X F)(Y,(\nabla_Z J)W)),$$

$$[X,Y,Z,W] = (\nabla_{(\nabla_X J)Y} F)(Z,W),$$

$$R(X,Y,Z,W) = g(R(X,Y)Z,W).$$

It is well known that the following conditions hold :

$$(\nabla_X\nabla_Y F)(Z,W)-(\nabla_{\nabla_X Y} F)(Z,W) + (\nabla_X\nabla_Y F)(W,Z)-(\nabla_{\nabla_X Y} F)(W,Z)=0, \tag{2.1}$$

$$(\nabla_X\nabla_Y F)(\bar{Z},\bar{W})-(\nabla_{\nabla_X Y} F)(\bar{Z},\bar{W})+(\nabla_X\nabla_Y F)(Z,W)-(\nabla_{\nabla_X Y} F)(Z,W) =$$

$$= (\nabla_Y\nabla_X F)(\bar{Z},\bar{W})-(\nabla_{\nabla_Y X} F)(\bar{Z},\bar{W})+(\nabla_Y\nabla_X F)(Z,W)-(\nabla_{\nabla_Y X} F)(Z,W) =$$

$$= (X,Z,Y,\bar{W}) - (X,W,Y,\bar{Z}), \tag{2.2}$$

$$R(X,Y,\bar{Z},W)+R(X,Y,Z,\bar{W})= (\nabla_X\nabla_Y F)(Z,W) - (\nabla_{\nabla_X Y} F)(Z,W) -$$

$$-(\nabla_Y\nabla_X F)(Z,W)+(\nabla_{\nabla_Y X} F)(Z,W). \tag{2.3}$$

<u>Proposition 2.1</u>. Let M be a G_2-manifold. Then for all vector fields X,Y,Z and W on M, we have :

$$[X,\bar{Y},Z,W] = [X,Y,Z,\bar{W}]+(X,Y,Z,\bar{W})-(X,Y,W,\bar{Z})+(X,Y,\bar{Z},W)-(X,Y,\bar{W},Z),$$

$$[X,\bar{Y},Z,\bar{W}] =-[X,Y,Z,W]-(X,Y,Z,W)+(X,Y,W,Z)+(X,Y,\bar{Z},\bar{W})-(X,Y,\bar{W},\bar{Z}),$$

$$[\bar{X},Y,Z,W] =-[\bar{X},\bar{Y},Z,\bar{W}]-(\bar{X},Y,Z,W)+(\bar{X},Y,W,Z)+(\bar{X},Y,\bar{Z},\bar{W})-(\bar{X},Y,\bar{W},\bar{Z}),$$

$$[\bar{X},Y,Z,\bar{W}] = [\bar{X},\bar{Y},Z,W]+(\bar{X},\bar{Y},Z,W)-(\bar{X},\bar{Y},W,Z)-(\bar{X},Y,\bar{Z},W)+(\bar{X},Y,\bar{W},Z).$$

Proof. The above conditions are proved using the definition-relation (1.1) of a G_2-manifold.

Proposition 2.2. On a G_2-manifold M, for all vector fields X,Y,Z and W we have :

$$\begin{aligned}
&[Y,X,Z,\bar{W}] - [X,Y,Z,\bar{W}] + [\bar{X},\bar{Y},Z,\bar{W}] - [\bar{Y},\bar{X},Z,\bar{W}] + \\
&+ [W,Z,X,\bar{Y}] - [Z,W,X,\bar{Y}] - [\bar{W},\bar{Z},X,\bar{Y}] + [\bar{Z},\bar{W},X,\bar{Y}] + \\
&+ [W,X,Y,\bar{Z}] - [X,W,Y,\bar{Z}] - [\bar{W},\bar{X},Y,\bar{Z}] + [\bar{X},\bar{W},Y,\bar{Z}] + \\
&+ [X,Z,Y,\bar{W}] - [Z,X,Y,\bar{W}] - [\bar{X},\bar{Z},Y,\bar{W}] + [\bar{Z},\bar{X},Y,\bar{W}] + \\
&+ [Y,W,X,\bar{Z}] - [W,Y,X,\bar{Z}] + [\bar{W},\bar{Y},X,\bar{Z}] - [\bar{Y},\bar{W},X,\bar{Z}] + \\
&+ [Z,Y,X,\bar{W}] - [Y,Z,X,\bar{W}] - [\bar{Z},\bar{Y},X,\bar{W}] + [\bar{Y},\bar{Z},X,\bar{W}] = 0 \qquad (2.4)
\end{aligned}$$

$$\begin{aligned}
&[X,Y,Z,W] - [Y,X,Z,W] - [\bar{X},\bar{Y},Z,W] + [\bar{Y},\bar{X},Z,W] + \\
&+ [Z,W,X,Y] - [W,Z,X,Y] + [\bar{W},\bar{Z},X,Y] - [\bar{Z},\bar{W},X,Y] + \\
&+ [X,W,Y,Z] - [W,X,Y,Z] + [\bar{W},\bar{X},Y,Z] - [\bar{X},\bar{W},Y,Z] + \\
&+ [Z,X,Y,W] - [X,Z,Y,W] + [\bar{X},\bar{Z},Y,W] - [\bar{Z},\bar{X},Y,W] + \\
&+ [W,Y,X,Z] - [Y,W,X,Z] - [\bar{W},\bar{Y},X,Z] + [\bar{Y},\bar{W},X,Z] + \\
&+ [Y,Z,X,W] - [Z,Y,X,W] + [\bar{Z},\bar{Y},X,W] - [\bar{Y},\bar{Z},X,W] - \\
&-(W,X,Y,Z)+(W,X,Z,Y)+(W,Y,X,Z)-(W,Y,Z,X)-(W,Z,X,Y)+(W,Z,Y,X)+ \\
&+(X,Y,Z,W)+(X,W,Y,Z)-(X,W,Z,Y)-(X,Z,Y,W)+(Z,X,Y,W)-(Z,W,Y,X)+ \\
&+(\bar{W},\bar{X},Y,Z)-(\bar{W},\bar{X},Z,Y)-(\bar{W},\bar{Y},X,Z)+(\bar{W},\bar{Y},Z,X)+(\bar{W},\bar{Z},X,Y)-(\bar{W},\bar{Z},Y,X)- \\
&-(\bar{X},\bar{Y},Z,W)-(\bar{X},\bar{W},Y,Z)+(\bar{X},\bar{W},Z,Y)+(\bar{X},\bar{Z},Y,W)-(\bar{Z},\bar{X},Y,W)+(\bar{Z},\bar{W},Y,X)+
\end{aligned}$$

$$+(W,X,\bar{Y},\bar{Z})-(W,X,\bar{Z},\bar{Y})-(W,Y,\bar{X},\bar{Z})+(W,Y,\bar{Z},\bar{X})+(W,Z,\bar{X},\bar{Y})-(W,Z,\bar{Y},\bar{X})-$$

$$-(X,Y,\bar{Z},\bar{W})-(X,W,\bar{Y},\bar{Z})+(X,W,\bar{Z},\bar{Y})+(X,Z,\bar{Y},\bar{W})-(Z,X,\bar{Y},\bar{W})+(Z,W,\bar{Y},\bar{X})-$$

$$-(\bar{W},X,\bar{Y},Z)+(\bar{W},X,\bar{Z},Y)+(\bar{W},Y,\bar{X},Z)-(\bar{W},Y,\bar{Z},X)-(\bar{W},Z,\bar{X},Y)+(\bar{W},Z,\bar{Y},X)+$$

$$+(\bar{X},Y,\bar{Z},W)+(\bar{X},W,\bar{Y},Z)-(\bar{X},W,\bar{Z},Y)-(\bar{X},Z,\bar{Y},W)+(\bar{Z},X,\bar{Y},W)-(\bar{Z},W,\bar{Y},X)=0 \quad (2.5)$$

<u>Proof</u>. On a G_2-manifold it holds (1.1) and the relation :

$$(\nabla_X F)(\bar{Y},Z)+(\nabla_{\bar{Y}} F)(Z,X)+(\nabla_Z F)(X,\bar{Y})+(\nabla_{\bar{X}} F)(Y,Z)+(\nabla_Y F)(\bar{Z},X)+(\nabla_{\bar{Z}} F)(X,Y)=0.$$

We take the covariant derivative of the above relations and making use of the condition :

$$(\nabla^2_{WX} F)(Y,Z)-(\nabla^2_{XW} F)(Y,Z)= -R(W,X,\bar{Y},Z)- R(W,X,Y,\bar{Z}) ,$$

we obtain (2.4). If we substitute JX for X in (2.4) we obtain (2.5).

3. Curvature Identities

On a G_2-manifold

$$(\nabla_X \nabla_Y F)(Z,W) - (\nabla_{\nabla_X Y} F)(Z,W) \quad (3.1)$$

is a linear combination of terms of the form :

$$(A,B,C,D),\ (\bar{A},\bar{B},C,D),\ (A,B,\bar{C},\bar{D}),\ (\bar{A},B,\bar{C},D) ,$$

$$(\bar{A},B,C,D),\ (A,B,\bar{C},D),\ (A,B,C,\bar{D}),\ (\bar{A},B,\bar{C},\bar{D}) ,$$

$$[A,B,C,D],\ [A,B,C,\bar{D}],\ [\bar{A},\bar{B},C,D],\ [\bar{A},\bar{B},C,\bar{D}],$$

where A,B,C and D are some combinations of vector fields X,Y,Z and W on M.

parameters.

Applying (2.1), (2.2), (2.3), (2.4), (2.5) and the first Bianchi identity the number of the parameters is reduced to eight.

Let $\{E_1,\dots,E_m,JE_1,\dots,JE_m\}$ be an orthonormal local frame field on M. We denote by r and r^* the Ricci tensor and the Ricci *tensor respectively. The Ricci *tensor r^* is defined by

$$r^*(x,y) \to \text{ trace of } \ (z \to R(Jz,x)Jy)\ ,$$

for $x,y,z \in T_p M$, where $T_p M$ is the tangent space at $p \in M$.

Using (3.2) we obtain an equation which gives $r^*(X,Y) - r(X,Y)$. In this relation we substitute X,Y for Y,X and subtracting last two equations we have a relation giving $r^*(X,Y) - r^*(\bar{X},\bar{Y})$.

From the last equation substituting JX,JY for X,Y respectively and adding these two equations we can determine other three parameters.

By virtue of the above calculations we can state the following assertion.

Theorem 3.1. If M is a G_2-manifold and R its curvature operator, then for arbitrary vector fields X,Y,Z and W on M we have :

$$R(X,Y,\bar{Z},\bar{W}) - R(X,Y,Z,W) =$$

$$= a_1 \Big[(W,Z,X,\bar{Y}) - (W,Z,Y,\bar{X}) + (X,Y,Z,\bar{W}) + (Z,W,Y,\bar{X}) -$$

$$-2(\bar{W},X,Z,Y) + 2(\bar{W},Y,Z,X) - (\bar{W},Z,X,Y) + (\bar{W},Z,Y,X) + (\bar{X},Y,Z,W) - (\bar{Z},W,Y,X) -$$

$$-2(W,X,\bar{Z},Y) + 2(W,Y,\bar{Z},X) - (W,Z,\bar{X},Y) + (W,Z,\bar{Y},X) + (X,Y,\bar{Z},W) - (Z,W,\bar{Y},X) -$$

$$-(\bar{W},Z,\bar{X},\bar{Y}) + (\bar{W},Z,\bar{Y},\bar{X}) - (\bar{X},Y,\bar{Z},\bar{W}) - (\bar{Z},W,\bar{Y},\bar{X}) \Big] +$$

$$+ b_1 \Big[(W,X,Y,Z) - (W,Y,X,Z) + (X,W,Z,Y) - (Z,X,Y,W) +$$

$$\begin{aligned}
&+ (\bar{W},\bar{X},Y,Z)-(\bar{W},\bar{Y},X,Z)+(\bar{X},\bar{Y},Z,W)-(\bar{Z},\bar{X},Y,W) - \\
&- (W,Z,\bar{X},\bar{Y})+(W,Z,\bar{Y},\bar{X})+(X,W,\bar{Z},\bar{Y})-(Z,W,\bar{Y},\bar{X}) + \\
&+ (\bar{W},Z,\bar{X},Y)-(\bar{W},Z,\bar{Y},X)-(\bar{X},Y,\bar{Z},W)+(\bar{Z},W,\bar{Y},X)\Big] + \\
+ b_2\Big[&(W,Z,X,Y)-(W,Z,Y,X)-(X,Y,Z,W)+(Z,W,Y,X) - \\
&- (\bar{W},\bar{Z},X,Y)+(\bar{W},\bar{Z},Y,X)+(\bar{X},\bar{Y},Z,W)-(\bar{Z},\bar{W},Y,X) - \\
&- (W,Z,\bar{X},\bar{Y})+(W,Z,\bar{Y},\bar{X})+(X,Y,\bar{Z},\bar{W})-(Z,W,\bar{Y},\bar{X}) + \\
&+ (\bar{W},Z,\bar{X},Y)-(\bar{W},Z,\bar{Y},X)-(\bar{X},Y,\bar{Z},W)+(\bar{Z},W,\bar{Y},X)\Big] + \\
+ b_3\Big[&(W,X,Z,Y)-(W,Y,Z,X)-(\bar{X},\bar{Y},Z,W)+(W,Z,\bar{X},\bar{Y}) - \\
&- (W,Z,\bar{Y},\bar{X})+(Z,W,\bar{Y},\bar{X})-(\bar{W},X,\bar{Z},Y)+(\bar{W},Y,\bar{Z},X) - \\
&-(\bar{W},Z,\bar{X},Y)+(\bar{W},Z,\bar{Y},X)+(\bar{X},Y,\bar{Z},W)-(\bar{Z},W,\bar{Y},X)\Big] + \\
+ b_4\Big[&-(\bar{X},\bar{Y},Z,W)+(\bar{X},\bar{W},Z,Y)+(W,X,\bar{Y},\bar{Z})-(W,Y,\bar{X},\bar{Z}) + \\
&+ (W,Z,\bar{X},\bar{Y})-(W,Z,\bar{Y},\bar{X})-(Z,X,\bar{Y},\bar{W})+(Z,W,\bar{Y},\bar{X}) + \\
&+ (\bar{W},X,\bar{Y},Z)-(\bar{W},Y,\bar{X},Z)-(\bar{W},Z,\bar{X},Y)+(\bar{W},Z,\bar{Y},X) + \\
&+ (\bar{X},Y,\bar{Z},W)+(\bar{X},W,\bar{Z},Y)-(\bar{Z},X,\bar{Y},W)-(\bar{Z},W,\bar{Y},X)\Big] .
\end{aligned} \tag{3.2}$$

$$\begin{aligned}
&R(\bar{X},\bar{Y},\bar{Z},\bar{W}) - R(X,Y,Z,W) = \\
= 2a_1 \Big[&-(\bar{W},X,Z,Y)+(\bar{W},Y,Z,X)-(\bar{W},Z,X,Y)+(\bar{W},Z,Y,X) + \\
&+(\bar{X},Y,Z,W)-(\bar{X},W,Y,Z)+(\bar{X},Z,Y,W)-(\bar{Z},W,Y,X) - \\
&-(W,X,\bar{Z},Y)+(W,Y,\bar{Z},X)-(W,Z,\bar{X},Y)+(W,Z,\bar{Y},X) + \\
&+(X,Y,\bar{Z},W)-(X,W,\bar{Y},Z)+(X,Z,\bar{Y},W)-(Z,W,\bar{Y},X)\Big] + \\
+ b_3 \Big[&(W,X,Z,Y)-(W,Y,Z,X)+(W,Z,X,Y)-(W,Z,Y,X) - \\
&-(X,Y,Z,W)+(X,W,Y,Z)-(X,Z,Y,W)+(Z,W,Y,X) -
\end{aligned}$$

$$
\begin{aligned}
&-(\bar{W},X,\bar{Z},Y)+(\bar{W},Y,\bar{Z},X)-(\bar{W},Z,\bar{X},Y)+(\bar{W},Z,\bar{Y},X)\;+\\
&+(\bar{X},Y,\bar{Z},W)-(\bar{X},W,\bar{Y},Z)+(\bar{X},Z,\bar{Y},W)-(\bar{Z},W,\bar{Y},X)\Big]\;+\\
+(b_1-b_4)\;&\Big[(W,X,Y,Z)-(W,Y,X,Z)-(W,Z,X,Y)+(W,Z,Y,X)\;+\\
&+(X,Y,Z,W)+(X,W,Z,Y)-(Z,X,Y,W)-(Z,W,Y,X)\;-\\
&-(\bar{W},X,\bar{Y},Z)+(\bar{W},Y,\bar{X},Z)+(\bar{W},Z,\bar{X},Y)-(\bar{W},Z,\bar{Y},X)\;-\\
&-(\bar{X},Y,\bar{Z},W)-(\bar{X},W,\bar{Z},Y)+(\bar{Z},X,\bar{Y},W)+(\bar{Z},W,\bar{Y},X)\Big]\;. \qquad (3.3)
\end{aligned}
$$

4. Curvature tensors on G_2-manifolds with constant holomorphic sectional curvature

We consider in what follows only G_2-manifolds which are RK-manifolds (i.e. $R(\bar{X},\bar{Y},\bar{Z},\bar{W})= R(X,Y,Z,W)$) without mentioning it always explicitly. Because of (3.2) and (3.3) we have :

$$
\begin{aligned}
&R(\bar{X},\bar{Y},Z,W)- R(X,Y,Z,W) = R(X,Y,\bar{Z},\bar{W}) - R(X,Y,Z,W) =\\
&= b_1\Big[(W,X,Y,Z) - (W,Y,X,Z) + (X,W,Z,Y) - (Z,X,Y,W)\;+\\
&\quad +(\bar{W},\bar{X},Y,Z) - (\bar{W},\bar{Y},X,Z) + (\bar{X},\bar{W},Z,Y) - (\bar{Z},\bar{X},Y,W)\;+\\
&\quad +(W,X,\bar{Y},\bar{Z}) - (W,Y,\bar{X},\bar{Z}) + (X,W,\bar{Z},\bar{Y}) - (Z,X,\bar{Y},\bar{W})\;+\\
&\quad +(\bar{W},X,\bar{Y},Z) - (\bar{W},Y,\bar{X},Z) + (\bar{X},W,\bar{Z},Y) - (\bar{Z},X,\bar{Y},W)\Big]\;+\\
&+ b_2\Big[(W,Z,X,Y) - (W,Z,Y,X) - (X,Y,Z,W) + (Z,W,Y,X)\;-\\
&\quad -(\bar{W},\bar{Z},X,Y) + (\bar{W},\bar{Z},Y,X) + (\bar{X},\bar{Y},Z,W) - (\bar{Z},\bar{W},Y,X)\;-\\
&\quad -(W,Z,\bar{X},\bar{Y}) + (W,Z,\bar{Y},\bar{X}) + (X,Y,\bar{Z},\bar{W}) - (Z,W,\bar{Y},\bar{X})\;+\\
&\quad +(\bar{W},Z,\bar{X},Y) - (\bar{W},Z,\bar{Y},X) + (\bar{X},Y,\bar{Z},W) - (\bar{Z},W,\bar{Y},X)\Big]\;. \qquad (4.1)
\end{aligned}
$$

We put

$$\lambda(x,y) = R(x,y,x,y) - R(\bar{x},\bar{y},x,y)$$

and

$$Q(x) = R(x,\bar{x},x,\bar{x}) = H(x)\,\|x\|^4 ,$$

for $x,y \in T_pM$. L.Vanhecke in [5] proved the following :

Proposition 4.1. Let M be an almost Hermitian manifold and $x,y \in T_pM$. Then :

$$32R(x,y,x,y) = 3Q(x+\bar{y}) + 3Q(x-\bar{y}) - Q(x+y) - Q(x-y) - 4Q(x) - $$
$$- 4Q(y) + 2\left[13\lambda(x,y) - 3\lambda(\bar{x},\bar{y}) + \lambda(\bar{x},y) + \lambda(x,\bar{y})\right] .$$

We assume that the holomorphic sectional curvature H(x) is constant c(p) for all $x \in T_pM$ at each point p of M. Since $Q(x) = c(p)\,\|x\|^4$, we have from the proposition 4.1:

$$16R(x,y,xy) = 4c(p)\left[\|x\|^2\ \|y\|^2 - g^2(x,y) + 3g^2(x,\bar{y})\right] +$$
$$+ 13\lambda(x,y) - 3\lambda(\bar{x},\bar{y}) + \lambda(\bar{x},y) + \lambda(x,\bar{y}).$$

By linearizing the above equation and using (4.1) we obtain the following:

Proposition 4.2. Let M be a G_2-manifold of pointwise constant holomorphic sectional curvature c(p). If X,Y,Z and W are arbitrary vector fields, then the curvature tensor is given by :

$$4R(X,Y,Z,W) = c(p)\left[g(X,Z)g(Y,W) - g(X,W)g(Y,Z) + g(X,\bar{Z})g(Y,\bar{W}) - \right.$$
$$\left. - g(X,\bar{W})g(Y,\bar{Z}) + 2g(X,\bar{Y})g(Z,\bar{W})\right] -$$

$$
\begin{aligned}
&-b_1 \Big[2(W,X,Y,Z)+(W,X,Z,Y)-2(W,Y,X,Z)-(W,Y,Z,X)-(W,Z,X,Y)+(W,Z,Y,X)+\\
&+(X,Y,Z,W)+(X,W,Y,Z)+2(X,W,Z,Y)-(X,Z,Y,W)-2(Z,X,Y,W)-(Z,W,Y,X)+\\
&+2(\bar W,\bar X,Y,Z)+(\bar W,\bar X,Z,Y)-2(\bar W,\bar Y,X,Z)-(\bar W,\bar Y,Z,X)-(\bar W,\bar Z,X,Y)+(\bar W,\bar Z,Y,X)+\\
&+(\bar X,\bar Y,Z,W)-(\bar X,\bar W,Y,Z)+2(\bar X,\bar W,Z,Y)-(\bar X,\bar Z,Y,W)-2(\bar Z,\bar X,Y,W)-(\bar Z,\bar W,Y,X)+\\
&+2(W,X,\bar Y,\bar Z)+(W,X,\bar Z,\bar Y)-2(W,Y,\bar X,\bar Z)-(W,Y,\bar Z,\bar X)-(W,Z,\bar X,\bar Y)+(W,Z,\bar Y,\bar X)+\\
&+(X,Y,\bar Z,\bar W)+(X,W,\bar Y,\bar Z)+2(X,W,\bar Z,\bar Y)-(X,Z,\bar Y,\bar W)-2(Z,X,\bar Y,\bar W)-(Z,W,\bar Y,\bar X)+\\
&+2(\bar W,X,\bar Y,Z)+(\bar W,X,\bar Z,Y)-2(\bar W,Y,\bar X,Z)-(\bar W,Y,\bar Z,X)-(\bar W,Z,\bar X,Y)+(\bar W,Z,\bar Y,X)+\\
&+(\bar X,Y,\bar Z,W)+(\bar X,W,\bar Y,Z)+2(\bar X,W,\bar Z,Y)-(\bar X,Z,\bar Y,W)-2(\bar Z,X,\bar Y,W)-(\bar Z,W,\bar Y,X)\Big]-\\
&-b_2\Big[-(W,X,Y,Z)+(W,X,Z,Y)+(W,Y,X,Z)-(W,Y,Z,X)+2(W,Z,X,Y)-2(W,Z,Y,X)-\\
&-2(X,Y,Z,W)+(X,W,Y,Z)-(X,W,Z,Y)-(X,Z,Y,W)+(Z,X,Y,W)+2(Z,W,Y,X)+\\
&+(\bar W,\bar X,Y,Z)-(\bar W,\bar X,Z,Y)-(\bar W,\bar Y,X,Z)+(\bar W,\bar Y,Z,X)-2(\bar W,\bar Z,X,Y)+2(\bar W,\bar Z,Y,X)+\\
&+2(\bar X,\bar Y,Z,W)-(\bar X,\bar W,Y,Z)+(\bar X,\bar W,Z,Y)+(\bar X,\bar Z,Y,W)-(\bar Z,\bar X,Y,W)-2(\bar Z,\bar W,Y,X)+\\
&+(W,X,\bar Y,\bar Z)-(W,X,\bar Z,\bar Y)-(W,Y,\bar X,\bar Z)+(W,Y,\bar Z,\bar X)-2(W,Z,\bar X,\bar Y)+2(W,Z,\bar Y,\bar X)+\\
&+2(X,Y,\bar Z,\bar W)-(X,W,\bar Y,\bar Z)+(X,W,\bar Z,\bar Y)+(X,Z,\bar Y,\bar W)-(Z,X,\bar Y,\bar W)-2(Z,W,\bar Y,\bar X)-\\
&-(\bar W,X,\bar Y,Z)+(\bar W,X,\bar Z,Y)+(\bar W,Y,\bar X,Z)-(\bar W,Y,\bar Z,X)+2(\bar W,Z,\bar X,Y)-2(\bar W,Z,\bar Y,X)-\\
&-2(\bar X,Y,\bar Z,W)+(\bar X,W,\bar Y,Z)-(\bar X,W,\bar Z,Y)-(\bar X,Z,\bar Y,W)+(\bar Z,X,\bar Y,W)+2(\bar Z,W,\bar Y,X)\Big]. \quad (4.2)
\end{aligned}
$$

5. On Schur's theorem

In this paragraph, we shall consider the following problem :"Let M be an almost Hermitian manifold of pointwise constant holomorphic sectional curvature $c(p)$. When is c a constant function?".

Gray and Vanhecke have proved an interesting theorem ([1],thm.4.7). We shall make a slightly different approach to this problem.

Lemma 5.1. Let M be an RK-manifold. Then :

$$2(\nabla_{E_j} r^*)(X,E_j) - \nabla_X s^* =$$

$$= \sum_i \Big[2R(X,E_i,(\nabla_{E_j} J)E_j,\bar{E}_i) + 2R(X,\bar{E}_j,\nabla_{E_j} E_i,\bar{E}_i) - \\ -R(E_j,\bar{E}_j,\nabla_X E_i,\bar{E}_i)\Big] , \tag{5.1}$$

for a vector field X on M.

Proof. By definition we have

$$r^*(X,Y) = \sum_i R(X,E_i,\bar{Y},\bar{E}_i)$$

and

$$s^* = \sum_j r^*(E_j,E_j) = \sum_{i,j} R(E_j,E_i,\bar{E}_j,\bar{E}_i).$$

By the standard calculation,

$$(\nabla_V r^*)(X,Y) = \sum_i \Big[(\nabla_V R)(X,E_i,\bar{Y},\bar{E}_i)+R(X,E_i,(\nabla_V J)Y,\bar{E}_i)+ \\ +R(X,\nabla_V E_i,\bar{Y},\bar{E}_i)+R(X,E_i,\bar{Y},\overline{\nabla_V E_i})\Big] , \tag{5.2}$$

and using the second Bianchi identity,

$$\nabla_X s^* = \sum_{i,j} \Big[2(\nabla_{E_j} R)(X,E_i,\bar{E}_j,\bar{E}_i)+R(E_j,\bar{E}_j,\nabla_X E_i,\bar{E}_i)\Big] . \tag{5.3}$$

From (5.2) and (5.3) we obtain (5.1).

Lemma 5.2. Let M be an almost Hermitian manifold. Then :

$$2(\nabla_{E_j} r)(X,E_j) - \nabla_X s = 2\sum_i \Big[R(X,\nabla_{E_j} E_i,E_j,E_i) +$$

$$+R(X,E_i,E_j,\nabla_{E_j} E_i) - R(E_j,\nabla_X E_i,E_j,E_i)\Big] , \qquad (5.4)$$

for a vector field X on M.

Proof. Direct computations give :

$$(\nabla_V r)(X,Y) = \sum_i \Big[(\nabla_V R)(X,E_i,Y,E_i)+R(X,\nabla_V E_i,Y,E_i)+R(X,E_i,Y,\nabla_V E_i)\Big], \quad (5.5)$$

$$\nabla_X s = 2\sum_{i,j} \Big[(\nabla_{E_j} R)(X,E_i,E_j,E_i)+R(E_j,\nabla_X E_i,E_j,E_i)\Big] . \qquad (5.6)$$

The equation (5.4) can easily come from (5.5) and (5.6).

By the definition of Ricci tensor and Ricci *tensor we obtain :

Proposition 5.3. If M is a 2m-dimensional almost Hermitian manifold of pointwise constant holomorphic sectional curvature c(p), then :

$$s + 3s^* = 4m(m+1)\, c(p). \qquad (5.7)$$

If M is an RK-manifold, then :

$$r(x,y) + 3r^*(x,y) = 2(m+1)c(p)\, g(x,y) , \qquad (5.8)$$

for $x,y \in T_p M$.

Now, we shall prove the Schur's theorem for almost Hermitian manifolds under some conditions.

Theorem 5.4. Let M be a connected RK-manifold of pointwise holomorphic sectional curvature c(p), with dim $M \geq 4$. If

$$2\left[R(X,Y,X,Z)+R(X,Z,Z,W)\right]+3\left[R(X,\bar{X},Y,\bar{Z})-2R(Z,\bar{X},W,\bar{Z})\right]=0, \tag{5.9}$$

for arbitrary vector fields X,Y,Z and W, then c is a constant function.

Proof. Differentiating (5.7) with respect to arbitrary E_i, we have

$$\nabla_{E_i}s + 3\nabla_{E_i}s^* = 4m(m+1)\nabla_{E_i}c . \tag{5.10}$$

Making use of (5.1), (5.4), (5.8) and (5.9) we obtain :

$$\nabla_{E_i}s + 3\nabla_{E_i}s^* = 2\sum_j\left[(\nabla_{E_j}r)(E_i,E_j)+3(\nabla_{E_j}r^*)(E_i,E_j)\right]=$$

$$= 4(m+1)\nabla_{E_i}c . \tag{5.11}$$

By (5.10) and (5.11)

$$4(m^2-1)\nabla_{E_i}c = 0$$

from which it follows that c is a constant function.

In particular we have :

Proposition 5.5. If M is a connected G_2-manifold of pointwise holomorphic sectional curvature $c(p)$, with $\dim M \geq 4$, then c is a constant function if M is an F-space (in the sence of [4]).

6. The Chern classes of a G_2-manifold.

On an almost Hermitian manifold M always there is a connection D adapted to g and J ([3]). Denote by S the curvature tensor of D, i.e.

$$S(X,Y,Z)= D_{[X,Y]}Z- [D_X,D_Y]Z, \quad S(X,Y,Z,W)= g(S(X,Y,Z),W) .$$

The importance of S is that because $(D_XJ)Y= 0$ where $D_XY= \frac{1}{2}(\nabla_XY-$ $-J\nabla_XJY)$. It is possible to express the Chern classes in terms of S.

Let M be a compact G_2-manifold and $\{E_1,\dots,E_m,\bar{E}_1,\dots,\bar{E}_m\}$ be a local frame field. Then

$$\det (\delta_{ij}- \Xi_{ij}/ 2\pi \sqrt{-1}) = \sum_{i=1}^{n} \gamma_i , \tag{6.1}$$

is a globaly defined closed form which represents the total Chern class of M via de Rham's theorem, where

$$\begin{aligned}
\Xi_{ij}(X,Y)= {} & R(X,Y,E_i,E_j)+ \frac{1}{4}\Big[(X,E_i,Y,E_j)- (X,E_j,Y,E_i)\Big] + \\
& + \frac{b_1}{2}\Big[-(E_i,X,Y,E_j)+(E_i,Y,X,E_j)+(E_j,X,Y,E_i)-(E_j,Y,X,E_i)- \\
& -(\bar{E}_i,\bar{X},Y,E_j)+(\bar{E}_i,\bar{Y},X,E_j)+(\bar{E}_j,\bar{X},Y,E_i)-(\bar{E}_j,\bar{Y},X,E_i)- \\
& -(E_i,X,\bar{Y},\bar{E}_j)+(E_i,Y,\bar{X},\bar{E}_j)+(E_j,X,\bar{Y},\bar{E}_i)-(E_j,Y,\bar{X},\bar{E}_i)- \\
& -(\bar{E}_i,X,\bar{Y},E_j)+(\bar{E}_i,Y,\bar{X},E_j)+(\bar{E}_j,X,\bar{Y},E_i)-(\bar{E}_j,Y,\bar{X},E_i)\Big] + \\
& + \frac{b_2}{2}\Big[-(E_i,E_j,X,Y)+(E_i,E_j,Y,X)+(E_j,E_i,X,Y)-(E_j,E_i,Y,X)+ \\
& +(\bar{E}_i,\bar{E}_j,X,Y)-(\bar{E}_i,\bar{E}_j,Y,X)-(\bar{E}_j,\bar{E}_i,X,Y)+(\bar{E}_j,\bar{E}_i,Y,X)+ \\
& +(E_i,E_j,\bar{X},\bar{Y})-(E_i,E_j,\bar{Y},\bar{X})-(E_j,E_i,\bar{X},\bar{Y})+(E_j,E_i,\bar{Y},\bar{X})-
\end{aligned}$$

$$
\begin{aligned}
&-(\bar{E}_i,E_j,\bar{X},Y)+(\bar{E}_i,E_j,\bar{Y},X)+(\bar{E}_j,E_i,\bar{X},Y)-(\bar{E}_j,E_i,\bar{Y},X) \; - \\
&- \sqrt{-1}\left[R(X,Y,E_i,\bar{E}_j)+\frac{1}{4}(X,E_i,Y,\bar{E}_j)+\frac{1}{4}(X,E_j,Y,\bar{E}_i)\right] + \\
&+ \sqrt{-1}\,\frac{b_1}{2}\Big[(E_i,X,Y,\bar{E}_j)-(E_i,Y,X,\bar{E}_j)-(E_j,X,Y,\bar{E}_i)+(E_j,Y,X,\bar{E}_i) \; + \\
&+(\bar{E}_i,X,Y,E_j)-(\bar{E}_i,Y,X,E_j)-(\bar{E}_j,X,Y,E_i)+(\bar{E}_j,Y,X,E_i) \; - \\
&-(E_i,X,\bar{Y},E_j)+(E_i,Y,\bar{X},E_j)+(E_j,X,\bar{Y},E_i)-(E_j,Y,\bar{X},E_i) + \\
&+(\bar{E}_i,X,\bar{Y},\bar{E}_j)-(\bar{E}_i,Y,\bar{X},\bar{E}_j)-(\bar{E}_j,X,\bar{Y},\bar{E}_i)+(\bar{E}_j,Y,\bar{X},\bar{E}_i)\Big] + \\
&+ \sqrt{-1}\,\frac{b_2}{2}\Big[-(E_i,E_j,X,\bar{Y})+(E_i,E_j,Y,\bar{X})+(E_j,E_i,X,\bar{Y})-(E_j,E_i,Y,\bar{X}) \; + \\
&+(\bar{E}_i,E_j,X,Y)-(\bar{E}_i,E_j,Y,X)-(\bar{E}_j,E_i,X,Y)+(\bar{E}_j,E_i,Y,X) \; - \\
&-(E_i,E_j,\bar{X},Y)+(E_i,E_j,\bar{Y},X)+(E_j,E_i,\bar{X},Y)-(E_j,E_i,\bar{Y},X) \; - \\
&-(\bar{E}_i,E_j,\bar{X},\bar{Y})+(\bar{E}_i,E_j,\bar{Y},\bar{X})+(\bar{E}_j,E_i,\bar{X},\bar{Y})-(\bar{E}_j,E_i,\bar{Y},\bar{X})\Big] .
\end{aligned}
$$

We can state the following assertion.

<u>Proposition 6.1.</u> If M is a compact G_2-manifold, then the total Chern class is given by (6.1).

<u>Corollary 6.2.</u> Let M be a compact G_2-manifold. The first Chern class γ_1 of M is given by

$$\gamma_1(X,Y) = \frac{1}{4\pi}\left[-2r^*(X,\bar{Y}) + \sum_{i=1}^{m} (X,E_i,Y,\bar{E}_i)\right]$$

for arbitrary vector fields X and Y, where $\{E_i,\ldots,E_m,\bar{E}_1,\ldots,\bar{E}_m\}$ is a local frame field on M.

REFERENCES

[1] A.Gray and L. Vanhecke, Almost Hermitian manifolds with constant holomorphic sectional curvature, Časopis pro pěstováni mat. 104 (1979), 170-179.

[2] L.Hervella and E. Vidal, Nouvelles géométries pseudo-Kählériennes G_1 et G_2, C.R.Acad. Sci.Paris, I Math. t.283, Série A(1976), 115-121.

[3] S.Kobayashi- K.Nomizu, Foundations of differential geometry, J.Wiley, New York, 1969.

[4] S. Sawaki and K. Sekigawa, Almost Hermitian manifolds with constant holomorphic sectional curvature, J.Differential Geom. 9(1974), 123--134.

[5] L.Vanhecke, Some almost Hermitian manifolds with constant holomorphic sectional curvature, J.Differential Geom.12(1977),461-471.

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