S.D. Bajpai

Generating function and orthogonality property of a class of polynomials occurring in quantum mechanics

Annales mathématiques Blaise Pascal, tome 1, n° 1 (1994), p. 21-26

<http://www.numdam.org/item?id=AMBP_1994__1_1_21_0>
ABSTRACT: In this paper, we present a generating function and an orthogonality property of a class of polynomials occurring in quantum mechanics.

Key words: Generating function, Orthogonality property, Hermite Polynomials, Quantum mechanics.

AMS (MOS): Subject classification: 33C25, 81

INTRODUCTION: The object of this paper is to present a generating function and an orthogonality property of the polynomials $\text{$_1F_1$}(-n; b+3/2; x^2)$, which occurs in the radical wave function of isotropic harmonic oscillator [4, p. 36, (6.60)].

The generating function for the polynomials $\text{$_1F_1$}(-n; b+3/2; x^2)$ has been obtained as a particular case of the generating function of $B$-polynomials, which has recently been defined by the author [2]. We obtain the orthogonality property of the polynomials $\text{$_1F_1$}(-n; b+3/2; x^2)$ as a bonus in our attempts to establish an orthogonality property of $B$-polynomials. We shall use the symbol $H_b^n(x)$ to denote the polynomials $\text{$_1F_1$}(-n; b+3/2; x^2)$.
It is interesting to note that the polynomials \( H_n^k(x) \) appear to lead to the generalization of the Hermite polynomials \( H_n(x) \) \[5, \text{p. 380, (25)}\].

We visualize at least three orthogonality properties of the \( B \)-polynomials for different weight functions on different intervals. However, we have not been successful to establish any of them. The proofs are difficult in view of the general nature of \( B \)-polynomials.

In what follows for sake of brevity, the symbol \( a_p \) is used to denote \( a_1, ..., a_p \), the symbol \( 1 - a_p - m \) is used to denote \( 1 - a_1 - m, ..., 1 - a_p - m \) and the notation \( \prod_{j=1}^{p}(a_j)_m \) stands for the product \( (a_1)_m \cdots (a_p)_m \). Further, the expression

$$
{}_pF_q \left[ \begin{array}{c} a_p; x \\ b_q \end{array} \right] = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n x^n}{(b_1)_n \cdots (b_q)_n n!} 
$$

(1.0)

is known as the generalized hypergeometric series or generalized hypergeometric function. Here \( p \) and \( q \) are positive integers or zero, and we assume that the variable \( z \), the numerator parameters \( a_1, ..., a_p \) and the denominator parameters \( b_1, ..., b_q \) take on complex values, provided that no \( b_j (j = 1, ..., q) \) is zero or a negative integer.

Recently \[2\], we have defined the \( B \)-polynomials:

$$
B_m(x) = \frac{\prod_{j=1}^{p}(a_j)_m}{\prod_{j=1}^{q}(b_j)_m} \cdot \frac{1}{1 + q + 1} s + p \left[ \binom{c_r}{s} x \right] \binom{1 - b_q - m, -m}{d_s, 1 - a_p - m} (x)^m, 
$$

(1.1)

by means of the generating function:

$$
{}_pF_q \left[ \begin{array}{c} a_p; \alpha t \\ b_q \end{array} \right] = \sum_{m=0}^{\infty} \frac{(\alpha t)^m}{m!} \frac{\prod_{j=1}^{p}(a_j)_m}{\prod_{j=1}^{q}(b_j)_m} 
$$

(1.2)

The generating function of the polynomials \( H_n^k(x) \):

In (1.2), putting \( \alpha = \beta = 1, p = q = r = 0, s = 1, d_1 = b + 3/2 \), and setting \( t^2 \) for \( t \) and \(-x^2 \) for \( x \), we obtain the generating function for \( \binom{1}{1} F_1 \left( -m; b + 3/2; x^2 \right) \):
Generating function and orthogonality

In (1.3), setting \( {}_0F_0(-; -; t^2) = e^{t^2} \), we have

\[
>_0F_0(-; -; t^2)_0F_1(-; b + 3/2; -t^2 x^2) = \sum_{m=0}^{\infty} \frac{t^{2m}}{m!} F_1(-m; b + 3/2; x^2) \tag{1.3}
\]

In (1.3), setting \( {}_0F_0(-; -; t^2) = e^{t^2} \), we have

\[
>_0F_0(-; -; t^2)_0F_1(-; b + 3/2; -t^2 x^2) = (tx)^{b+1/4} \Gamma(b + 3/2) J_{b+1/2}(2\sqrt{tx}) \text{ and } F_1(-m; b + 3/2; x^2) = H_m^b(x),
\]

we have

\[
e^{t^2} (tx)^{b+1/4} \Gamma(b + 3/2) J_{b+1/2}(2\sqrt{tx}) = \sum_{m=0}^{\infty} \frac{t^{2m}}{m!} H_m^b(x) \tag{1.4}
\]

The following formulae are required in the proofs:

**The integral:**

\[
\int_{-\infty}^{\infty} x^{2u} e^{-z^2} pF_q \left[ \begin{array}{c} a_p \; z^2 \\ b_q \end{array} \right] dx
\]

\[
= \Gamma(u + 1/2) pF_q \left[ \begin{array}{c} a_p \; z^2 \\ b_q \end{array} \right], \tag{1.5}
\]

where \( p < q + 1 \) (or \( p = q + 1 \) and \( |z| < 1 \)), \( u = 0, 1, 2, \ldots \).

The integral (1.5) can easily be established by expressing the hypergeometric function in the integrand as \([1, \text{p. 322}, (10.1)]\) and interchanging the order of integration and summation, which is justified due to the absolute convergence of the integral and summation involved in the process, and evaluating the inner-integral with the help of the following integral:

\[
\int_{-\infty}^{\infty} x^{2n} e^{-z^2} dx = \Gamma(n + 1/2), \; n = 0, 1, 2, \ldots \tag{1.6}
\]

**The integral:**

\[
\int_{-\infty}^{\infty} x^{2u} e^{-z^2} pF_q \left[ \begin{array}{c} a_p \; z^2 \\ b_q \end{array} \right] F_s \left[ \begin{array}{c} c_r \; y^2 \\ d_s \end{array} \right] dx
\]

\[
= \sum_{m=0}^{\infty} \frac{y^m}{m!} \Gamma(m + u + 1/2) pF_q \left[ \begin{array}{c} a_p, m + u + 1/2 \; x \\ b_q \end{array} \right], \tag{1.7}
\]

where in addition to the conditions of (1.5), \( r < s + 1 \) (or \( r = s + 1 \) and \( |y| < 1 \)).

To derive (1.7), we use the series representation of \( {}_rF_s \), interchange the order of integration and summation and evaluate the resulting integral with the help of (1.5).

The Vandermonde's theorem \([3, \text{p. 110}, (4.1.2)]\) :
The modified form of the relation [1, p. 308, (9.37)]:

\[ H_{2n}(x) = (-1)^n(2n)(1/2)_n F_1 \left[ \begin{array}{c} -n; x^2 \\ 1/2 \end{array} \right] \]  
\[ (1.9) \]

The modified form of the relation [1, p. 312, (6)]:

\[ H_{2n+1}(x) = (-1)^n 2^{2n+1}(3/2)_n F_1 \left[ \begin{array}{c} -n; x^2 \\ 3/2 \end{array} \right] \]  
\[ (1.10) \]

The Legendre duplication formula [1, p. 58, (2.24)]

\[ 2^{2x-1} \Gamma(x) \Gamma(x+1/2) = \sqrt{\pi} \Gamma(2x) \]  
\[ (1.11) \]

The following well known relations [1, pp. 275, 323]:

\[ _0 F_0 (-; -; x) = e^x \]  
\[ (1.12) \]

\[ _0 F_1 \left[ \begin{array}{c} -; - x^2/4 \\ 1/2 \end{array} \right] = \cos x \]  
\[ (1.13) \]

\[ _0 F_1 \left[ \begin{array}{c} -; - x^2/4 \\ 3/2 \end{array} \right] = \sin x \]  
\[ (1.14) \]

\[ (-k)_n = \begin{cases} 0, & n > k \\ (k = 1, 2, 3, ...) \\ (-1)^n n!, & k = n \end{cases} \]  
\[ (1.15) \]

2. ORTHOGONALITY PROPERTY OF THE POLYNOMIALS $H_b^n(x)$.

The polynomials $H_b^n(x)$ are orthogonal with weight $x^{2(b+1)} e^{-x^2}$ on the interval $(-\infty, \infty)$, i.e.

\[ \int_{-\infty}^{\infty} x^{2(b+1)} e^{-x^2} H_b^n(x) H_k^n(x) dx = \begin{cases} 0, & k \neq n \\ \frac{\Gamma(b+3/2)_n}{\Gamma(b+3/2)_k}, & k = n \end{cases} \]  
\[ (2.1) \]

where $b = -1, 0, 1, 2, ...$

**Proof.** In (1.7), setting $y = z = 1, u = b + 1, p = q = r = s = 1, a_1 = -n, b_1 = b + 3/2, c_1 = -k, d_1 = b + 3/2$, we have

\[ \int_{-\infty}^{\infty} x^{2(b+1)} e^{-x^2} F_1(\begin{array}{c} -n; b + 3/2; x^2 \\ 1 \end{array}) F_1(\begin{array}{c} -k; b + 3/2; x^2 \end{array}) dx \]

\[ = \sum_{m=0}^{\infty} \frac{(-k)_m}{(b + 3/2)_m} \frac{1}{m!} \Gamma(m + b + 3/2) \frac{\Gamma(m + b + 3/2)_2}{\Gamma(b + 3/2)} F_1 \left[ \begin{array}{c} -n, m + b + 3/2; 1 \\ b + 3/2 \end{array} \right] \]  
\[ (2.2) \]
Now, using the notation $H_n^b(x)$ for $_1F_1(-n; b + 3/2; x^2)$ and Vandermonde's theorem (1.8), (2.2) reduces to the form:

$$\int_{-\infty}^{\infty} x^{2(b+1)}e^{-x^2}H_n^b(x)H_k^b(x)dx$$

$$= \sum_{m=0}^{\infty} \frac{\Gamma(b + 3/2)(-k)_m(-m)_n}{m!(b + 3/2)_n} \quad (2.3)$$

From (1.15), it is evident that all terms of the series (2.3) are zero for $m > k \neq n$ and $m < n \neq k$.

If $k = n = m$, we have

$$\int_{-\infty}^{\infty} x^{2(b+1)}e^{-x^2}\left\{H_n^b(x)\right\}^2 dx = \frac{\Gamma(b + 3/2)n!}{(b + 3/2)_n} \quad (2.4)$$

This proves (2.1)

3. THE POLYNOMIALS $H_n^b(x)$ AND THE HERMITE POLYNOMIALS $H_n(x)$.

(a) Generating functions

(i) In (1.3), putting $b = -1$, and applying (1.9), (1.11), (1.12) and (1.13), it reduces to the generating function [1, p. 174, 2(a)] for the Hermite polynomials.

(ii) In (1.3), setting $b = 0$, and using (1.10), (1.11), (1.12) and (1.14), it yields the generating function [1, p. 174, 2(b)] for the Hermite polynomials.

(b) Orthogonality properties

(i) In (2.1), putting $b = -1$, and applying (1.9), (1.11), (1.12) and (1.13), we obtain the following orthogonality property of the Hermite polynomials:

$$\int_{-\infty}^{\infty} e^{-x^2}H_{2n}(x)H_{2k}(x)dx = \begin{cases} 0, & k \neq n \\ 2^{2n}(2n)!\sqrt{\pi}, & k = n \end{cases} \quad (3.1)$$

(ii) In (2.1), setting $b = 0$, and using (1.10), (1.11), (1.12) and (1.14), it yields the following orthogonality property of the Hermite polynomials:

$$\int_{-\infty}^{\infty} e^{-x^2}H_{2n+1}(x)H_{2k+1}(x)dx = \begin{cases} 0, & k \neq n \\ 2^{2n+1}(2n + 1)!\sqrt{\pi}, & k = n \end{cases} \quad (3.2)$$

From (3.1) and (3.2), the orthogonality property of the Hermite polynomials [1, pp. 170-171, (5.17) - (5.22)] follows.
ACKNOWLEDGEMENT

I wish to express my sincere thanks to the referee for his useful suggestions for the revision of this paper.

REFERENCES


