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Propagation of low regularity for solutions of nonlinear PDEs on a Riemannian manifold with a sub-Laplacian structure

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Abstract

Following Bernicot (2012) [7], we introduce a notion of paraproducts associated to a semigroup. We do not use Fourier transform arguments and the background manifold is doubling, endowed with a sub-Laplacian structure. Our main result is a paralinearization theorem in a non-Euclidean framework, with an application to the propagation of regularity for some nonlinear PDEs.

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The theory of paradifferential calculus was introduced by Bony in [8] and developed by many others, particularly Meyer in [28]. This tool is quite powerful in nonlinear analysis. The key idea relies on Meyer's formula for a nonlinarity $F(f)$ as $M(x,D)f + R$ where F is smooth in its argument(s), f belongs to a Hölder or Sobolev space, $M(x,D)$ is a pseudo-differential operator (depending on f) of type $(1,1)$ and R is more regular than f and $F(f)$. This operation is called "paralinearization".

Such an approach has given many important results (or improvements of existing results): Moser estimates, elliptic regularity estimates, Kato–Ponce inequalities, ... and is the basis of microlocal analysis.

The notion of paradifferential operators is built on appropriate functional calculus and symbolic representation, available on the Euclidean space. The Fourier transform is crucial for this point of view to study and define symbolic classes. That is why this approach cannot be extended to Riemannian manifolds. More recently, Ivanovici and Planchon have already extended this theory in the context of a self-adjoint semigroup (on a manifold) in [24, Appendix A], where paraproducts are built by a C^∞ functional calculus.

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However, for the last years, numerous works deal with nonlinear PDEs on manifolds, involving differential operators which may be non-self-adjoint. So it seems important to try to extend this tool of "paralinearization" in a non-Euclidean situation and without requiring self-adjointness of the semigroup.

First, on specific situations, namely on a Carnot group, it is possible to define a suitable Fourier transform, involving irreducible representations. In this context, we can also define the notion of symbols and so of pseudo-differential calculus (see the survey [4] of Bahouri, Fermanian-Kammerer and Gallagher for Heisenberg groups and [20] of Gallagher and Sire for more general Carnot groups).

The aim of the present work is to define another suitable notion of paralinearization on a manifold, without requiring use of Fourier transform. Since (nonlinear) PDEs on a manifold usually requires vector fields, we work on a manifold having a sub-Riemannian structure. To define a suitable paralinearization, we use paraproducts defined via the heat semigroup (introduced by Bernicot in [7], independently by Frey in [17,18] and already used by Badr, Bernicot and Russ in [3] to get Leibniz-type estimates and algebra properties for Sobolev spaces) and look for a paralinearization result. The semigroup is not assumed to be self-adjoint, so we aim to only use sectorial holomorphic functional calculus (more precisely we only use the heat semigroup and its time-derivatives). However, a new phenomenon appears due to the lack of flexibility of the method; the classical paralinearization result holds only for low regularity, which appeared already in [24, Appendix A].

Since we are motivated by applications to nonlinear PDEs involving vector fields, we present our result in the context of a sub-Laplacian operator: a finite sum of square of vector fields. Indeed, such operators naturally appear and a systematic study of their properties has begun in recent years and still attract much attention. So we consider $L = -\sum_i X_i^2$ associated to $X = (X_1, \dots, X_\kappa)$ a finite collection of vector fields on a doubling Riemannian manifold M. One of the most famous results concerning such operators is due to Hörmander in [22,23], where a condition is assumed to ensure the hypoellipticity of L. More precisely, if X satisfies the Hörmander condition (i.e. the collection at each point $x \in M$ $((X_i)_i, ([X_i, X_j])_{i,j}, \dots)$, span the whole tangent space T_xM) then L is locally hypoelliptic, which means that if u is a solution of $L(u) \in C^\infty$ then $u \in C^\infty$. Note that such a result can be seen as a first result of what we expect. Indeed paradifferential calculus aims to obtain regularity for u, a solution of a nonlinear PDE involving L.

To consider a nonlinear PDE, we have first to describe a way to linearize it. This is our first result. First we define and study Bessel-type Sobolev spaces (see Section 3) and then the notion of paraproduct Π (Section 4) associated to L. Then we prove the following result:

Theorem 0.1. *Assume that the manifold satisfies a Poincaré inequality, that the operator L generates a holomorphic semigroup with pointwise estimates on its kernel and that the Riesz transforms are L^p-bounded* (*Assumtion* 1.11). *Consider $p \in (1, \infty)$, $s \in (d/p, 1)$ and $f \in W^{s+\epsilon,p}$ for some $\epsilon > 0$* (*as small as we want*). *Then for every smooth function $F \in C^\infty(\mathbb{R})$ with $F(0) = 0$,*

$$F(f) = \Pi_{F'(f)}(f) + w \qquad (1)$$

with $w \in W^{2s-d/p,p}$.

The precise assumptions will be given later. We point out that we do not directly assume that the collection X satisfies the Hörmander condition; however this information is in some sense encoded in the pointwise estimates for the heat kernel (see [25,31]).

With respect to the well-known paralinearization results (existing in the Euclidean setting), the first point is that we have only a gain of regularity at order $s - d/p - \epsilon$ and the main difference is that this result is only proved for $s < 1$. This condition is in some sense inherent to our method and has already appeared in [3,24]: Fourier transform allows to use an "exact" spectral decomposition although the paraproduct algorithm brings some error-terms which are difficult to estimate. This assumption may seem very strong; we will explain in Theorem 5.4 how it is possible to get larger regularity $s > 1$ modifying the definition of the paraproduct (then involving the higher-order derivatives of the nonlinearity F).

As in the Euclidean situation, we are able to obtain some applications concerning propagation of the regularity for solutions of nonlinear PDEs. See Section 6 for a detailed statement: for $u \in W^{s+1,p}$ a solution of a nonlinear PDE, $F(u, X_1u, \dots, X_\kappa u) \in C^\infty$, then u is more regular where the linearized equation can be inverted.

1. Preliminaries: Riemannian structure with a sub-Laplacian operator

In this section, we aim to describe the framework and the required assumptions we will use. Let us precise the main hypothesis about the manifold M and the operator L.

1.1. Structure of doubling Riemannian manifold

In all this paper, M denotes a complete Riemannian manifold. We write μ for the Riemannian measure on M, ∇ for the Riemannian gradient, $|\cdot|$ for the length on the tangent space (forgetting the subscript x for simplicity) and $\|\cdot\|_{L^p}$ for the norm on $L^p := L^p(M,\mu)$, $1 \leqslant p \leqslant +\infty$. We denote by $B(x,r)$ the open ball of center $x \in M$ and radius $r > 0$.

1.1.1. The doubling property

Definition 1.1 *(Doubling property).* Let M be a Riemannian manifold. One says that M satisfies the doubling property (D) if there exists a constant $C > 0$, such that for all $x \in M$, $r > 0$ we have

$$\mu\big(B(x,2r)\big) \leqslant C\mu\big(B(x,r)\big). \tag{D}$$

Lemma 1.2. *Let M be a Riemannian manifold satisfying (D) and let $d := \log_2 C$. Then for all $x, y \in M$ and $\theta \geqslant 1$*

$$\mu\big(B(x,\theta R)\big) \leqslant C\theta^d \mu\big(B(x,R)\big). \tag{2}$$

There also exist c and $N \geqslant 0$, so that for all $x, y \in M$ and $r > 0$

$$\mu\big(B(y,r)\big) \leqslant c\left(1 + \frac{d(x,y)}{r}\right)^N \mu\big(B(x,r)\big). \tag{3}$$

For example, if M is the Euclidean space $M = \mathbb{R}^d$ then $N = 0$ and $c = 1$.

Observe that if M satisfies (D) then

$$\operatorname{diam}(M) < \infty \quad \Leftrightarrow \quad \mu(M) < \infty \quad \big(\text{see [1]}\big).$$

Therefore if M is a non-compact Riemannian manifold satisfying (D) then $\mu(M) = \infty$.

Theorem 1.3 *(Maximal theorem). (See [11].) Let M be a Riemannian manifold satisfying (D). Denote by $\mathcal{M}$ the uncentered Hardy–Littlewood maximal function over open balls of M defined by*

$$\mathcal{M}f(x) := \sup_{\substack{Q \text{ ball} \\ x \in Q}} |f|_Q$$

where $f_E := ⨍_E f\,d\mu := \frac{1}{\mu(E)}\int_E f\,d\mu$. Then for every $p \in (1,\infty]$, $\mathcal{M}$ is L^p bounded and moreover of weak type $(1,1)$.

Consequently for $s \in (0,\infty)$, the operator $\mathcal{M}_s$ defined by

$$\mathcal{M}_s f(x) := \big[\mathcal{M}\big(|f|^s\big)(x)\big]^{1/s}$$

is of weak type (s,s) and L^p bounded for all $p \in (s,\infty]$.

The doubling property allows us to control the growth of volume of balls. However, it can be interesting to have a lower bound too. So we will make the following assumption:

Assumption 1.4. We assume that there exists a constant $c > 0$ such that for all $x \in M$

$$\mu\big(B(x,1)\big) \geqslant c. \tag{4}$$

Due to the homogeneous type of the manifold M, this is equivalent to a control from below of the volume of balls

$$\mu\big(B(x,r)\big) \gtrsim r^d \tag{MV_d}$$

for all $0 < r \leqslant 1$.

1.1.2. Poincaré inequality

Definition 1.5 *(Poincaré inequality on M).* We say that a complete Riemannian manifold M admits *a Poincaré inequality* (P_q) for some $q \in [1,\infty)$ if there exists a constant $C > 0$ such that, for every function $f \in W^{1,q}_{loc}(M)$ and every ball Q of M of radius $r > 0$, we have

$$\left(\fint_Q |f - f_Q|^q \, d\mu\right)^{1/q} \leqslant Cr\left(\fint_Q |\nabla f|^q \, d\mu\right)^{1/q}. \tag{P_q}$$

Remark 1.6. By density of $C_0^\infty(M)$ in $W^{1,q}_{loc}(M)$, we can replace $W^{1,q}_{loc}(M)$ by $C_0^\infty(M)$.

Assumption 1.7. We assume that the considered manifold satisfies a Poincaré inequality (P_1).

Remark 1.8. Indeed we could just assume a Poincaré inequality (P_σ) for some $\sigma < 2$ and all of our results will remain true for Lebesgue exponents bigger than σ.

1.2. Framework of sub-Laplacian operator

We will only consider operators L which are sub-Laplacians, which means: there exists $X = \{X_k\}_{k=1,\dots,\kappa}$ a finite family of real-valued vector fields (so X_k is defined on M and $X_k(x) \in TM_x$) such that

$$L = -\sum_{k=1}^{\kappa} X_k^2. \tag{5}$$

We identify the X_k's with the first-order differential operators acting on Lipschitz functions defined on M by the formula

$$X_k f(x) = X_k(x) \cdot \nabla f(x),$$

and we set $Xf = (X_1 f, X_2 f, \dots, X_\kappa f)$ and

$$\left|Xf(x)\right| = \left(\sum_{k=1}^{\kappa} \left|X_k f(x)\right|^2\right)^{1/2}, \quad x \in M.$$

Let us point out that the operator X_k is self-adjoint on $L^2(M, d\mu)$ if and only if the vector field X_k satisfies $\operatorname{div}(X_k) = 0$.

We define also the higher-order differential operators as follows: for $I \subset \{1,\dots,\kappa\}^k$, we set

$$X_I := \prod_{i \in I} X_i.$$

We assume the following:

Assumption 1.9. For every subset I, the I-th local-Riesz transforms $\mathcal{R}_I := X_I(1+L)^{-|I|/2}$ and $\overline{\mathcal{R}_I} := (1+L)^{-|I|/2} X_I$ are bounded on L^p for every $p \in (1,\infty)$.

Remark 1.10. It is easy to check that this last assumption is implied by the boundedness of each local-Riesz transform $\mathcal{R}_i$ and $\overline{\mathcal{R}_i}$ in Sobolev spaces $W^{k,p}$ for every $p \in (1,\infty)$ and $k \in \mathbb{N}$. Indeed for $I = \{i_1,\dots,i_n\}$, we have

$$\|\overline{\mathcal{R}_I} f\|_{L^p} \leqslant \left\|\overline{\mathcal{R}_{i_1}}(X_{i_2} \cdots X_{i_n} f)\right\|_{W^{|I|-1,p}} \lesssim \|\overline{\mathcal{R}_{i_2,\dots,i_n}} f\|_{L^p}.$$

Repeating this reasoning, we obtain that the Sobolev boundedness of the Riesz transforms imply the previous assumption.

1.3. Framework of heat semigroup

Let us recall the framework of [14,15], applied to our sub-Laplacian operator L.

Let $\omega \in [0, \pi/2)$. We define the closed sector in the complex plane $\mathbb{C}$ by

$$S_\omega := \{z \in \mathbb{C},\ |\arg(z)| \leqslant \omega\} \cup \{0\}$$

and denote the interior of S_ω by S_ω^0. We set $H_\infty(S_\omega^0)$ for the set of bounded holomorphic functions b on S_ω^0, equipped with the norm

$$\|b\|_{H_\infty(S_\omega^0)} := \|b\|_{L^\infty(S_\omega^0)}.$$

We assume that L is injective and of type ω on L^2, for some $\omega \in [0, \pi/2)$, which means that L is closed and its spectrum $\sigma(L) \subset S_\omega$ and for each $\nu > \omega$, there exists a constant c_ν such that

$$\|(L-\lambda)^{-1}\|_{L^2 \to L^2} \leqslant c_\nu |\lambda|^{-1}$$

for all $\lambda \notin S_\nu$.

As a consequence, we know that L is densely defined on L^2, i.e. its domain

$$\mathcal{D}(L) := \{f \in L^2,\ L(f) \in L^2\}$$

is dense in L^2.

In particular, it is well known that $-L$ generates a holomorphic semigroup and we refer the reader to [14] and [27] for more details concerning holomorphic calculus of such operators.

Let us now detail some other assumptions we make on the semigroup:

Assumption 1.11. There exists $\delta > 1$ with:

- For every $z \in S_{\pi/2-\omega}$, the linear operator e^{-zL} is given by a kernel p_z satisfying
$$|p_z(x,y)| \lesssim \frac{1}{\mu(B(x,|z|^{1/2}))}\left(1 + \frac{d(x,y)}{|z|^{1/2}}\right)^{-d-2N-\delta} \tag{6}$$
where d is the homogeneous dimension of the space (see (2)) and N is the other dimension parameter (see (3)); $N \geqslant 0$ could be equal to 0.
- The operator L has a bounded H_∞-calculus on L^2. That is, there exists c_ν such that for $b \in H_\infty(S_\nu^0)$, we can define $b(L)$ as an L^2-bounded linear operator and
$$\|b(L)\|_{L^2 \to L^2} \leqslant c_\nu \|b\|_\infty. \tag{7}$$
- The Riesz transform $\mathcal{R} := \nabla L^{-1/2}$ is bounded on L^p for every $p \in (1, \infty)$.

We note that assuming (6), the semigroup e^{-tL} initially defined on L^2, can be extended to L^p for every $p \in [1, \infty]$.

Remark 1.12. The bounded H_∞-calculus on L^2 allows us to deduce some extra properties (see [15] and [27]):

- Due to the Cauchy formula for complex differentiation, pointwise estimate (6) still holds for the kernel of $(tL)^k e^{-tL}$ with $t > 0$.
- For any holomorphic function $\psi \in H(S_\nu^0)$ such that for some $s > 0$, $|\psi(z)| \lesssim \frac{|z|^s}{1+|z|^{2s}}$, the quadratic functional
$$f \to \left(\int_0^\infty |\psi(tL)f|^2 \frac{dt}{t}\right)^{1/2}$$
is L^2-bounded.

Remark 1.13. It follows from the L^2-boundedness of the Riesz transform that for every integer $k \geqslant 0$ the quadratic functional

$$f \to \left(\int_0^\infty \left| t^{1/2} \nabla (tL)^k e^{-tL}(f) \right|^2 \frac{dt}{t} \right)^{1/2} \tag{8}$$

is bounded on L^2.

The spectral analysis associated to L relies on a suitable Calderón reproducing formula:

Proposition 1.14. *(See [9, Thm. 2.3].) Since L is sectorial on L^2 (and so densely defined on L^2), Assumption 1.11 yields the following spectral decomposition: for every $f \in L^p$, $p \in [1, \infty]$ and n an integer*

$$f = \frac{1}{(n-1)!} \int_0^\infty (tL)^n e^{-tL}(f) \frac{dt}{t},$$

where the integral strongly converges in L^p.

This proposition guarantees us a rigorous sense of such spectral decomposition, which we later use in this work. Moreover, let us define the set

$$\mathcal{S} := \left\{ f \in L^1 \cap L^\infty,\ \forall n \geqslant 1,\ L^n(f) \in L^1 \cap L^\infty \right\}. \tag{9}$$

Then for every function $f \in L^1 \cap L^\infty$, the pointwise estimate of the heat kernel and its time-derivative imply that for every $t > 0$ and every integer $p \geqslant 0$, $L^p e^{-tL}(f) \in \mathcal{S}$. So we have the following corollary:

Corollary 1.15. *The set $\mathcal{S}$ is dense into L^p for every $p \in (1, \infty)$.*

About square functions, we have the following proposition:

Proposition 1.16. *Under these assumptions, we know that the quadratic functionals in Remark 1.12 or in (8) are L^p-bounded for every $p \in (1, \infty)$.*

Proof. We mainly follow the arguments developed in [2, Chapter 6, Theorem 6.1]. The results proved there are very general, even if they are only written in the context of operators having a divergence form. Indeed, this theorem states that the considered functionals (appearing in Remark 1.12) are L^p-bounded for every exponent p belonging to $(1, \infty)$ the range for which the semigroup e^{-tL} is L^p-bounded (and has off-diagonal decay). Composing with ℓ^2-valued inequalities of the Riesz transform for $p \in (1, \infty)$ (due to its boundedness), we obtain the L^p boundedness for the functionals of the form (8). These arguments are developed in [2] using Gaussian-type estimates, which means that (6) is supposed with exponentially decreasing kernels.

Let us explain why such decay is not necessary and (6) is sufficient. Let T be one of the square functions in Remark 1.12. The proof of [2, point 1, Theorem 6.1] relies on Theorems 1.1 and 1.2 there, which generalize usual Calderón–Zygmund theory. Note that Theorem 1.1 was improved by [6, Theorem 5.5] as soon as we have (6). Using this new version in [6] (when $p < 2$) and [2, Theorem 1.2] (when $p > 2$), the proposition is then reduced to the proof of the following inequalities

$$\left(\fint_{Q_1} \left| T\left(1 - e^{-tL}\right) f \right|^2 d\mu \right)^{\frac{1}{2}} \lesssim \left(1 + \frac{d(Q_1, Q_2)}{t^{\frac{1}{2}}} \right)^{-d-\epsilon} \left(\fint_{Q_2} |f|^2 \, d\mu \right)^{\frac{1}{2}} \tag{10}$$

for some $\epsilon > 0$, every balls Q_1, Q_2 of radius t and every function $f \in L^2(Q_2)$.

A careful examination of Step 3 in the proof of [2, point 1, Theorem 6.1] allows us to conclude (10) as soon as we have enough decay in (6) with $\delta > 0$. □

From now on, we will consider a doubling Riemannian manifold M satisfying Poincaré inequality (P_1), lower bound of the volume (Assumption 1.4) and a structure of sub-Laplacian generating a semigroup satisfying Assumption 1.11 and with bounded Riesz transforms (Assumption 1.9).

2. Examples of such situations

In this section, we would like to give examples of situations where all these assumptions are satisfied. First we give examples of Riemannian structure. Once the Riemannian structure is defined, we can consider any sum of square of vector fields for L.

2.1. Examples of Riemannian structure

2.1.1. Carnot–Caratheodory spaces

Let Ω be an open connected subset of $\mathbb{R}^d$ and $Y = \{Y_k\}_{k=1}^{\kappa}$ a family of real-valued, infinitely differentiable vector fields.

Definition 2.1. Let Ω and Y be as above. Y is said to satisfy Hörmander's condition in Ω if the family of commutators of vector fields in Y $(Y_i, [Y_i, Y_j], \dots)$ span $\mathbb{R}^d$ at every point of Ω.

Suppose that $Y = \{Y_k\}_{k=1}^{M}$ satisfies Hörmander's condition in Ω. Let C_Y be the family of absolutely continuous curves $\zeta : [a,b] \to \Omega$, $a \leqslant b$, such that there exist measurable functions $c_j(t)$, $a \leqslant t \leqslant b$, $j = 1, \dots, M$, satisfying $\sum_{j=1}^{M} c_j(t)^2 \leqslant 1$ and $\zeta'(t) = \sum_{j=1}^{M} c_j(t) Y_j(\zeta(t))$ for almost every $t \in [a,b]$. If $x, y \in \Omega$ define

$$\rho(x,y) = \inf\bigl\{T > 0\text{: there exists } \zeta \in C_Y \text{ with } \zeta(0) = x \text{ and } \zeta(T) = 1\bigr\}.$$

The function ρ is in fact a metric in Ω called the Carnot–Carathéodory metric associated to Y. This allows us to equip the space Ω with a sub-Riemannian structure. Then every Y_i are by definition self-adjoint.

2.1.2. Lie groups

Let $M = G$ be a unimodular connected Lie group endowed with its Haar measure $d\mu = dx$ and assume that it has a polynomial volume growth. Recall that "unimodular" means that dx is both left-invariant and right-invariant. Denote by $\mathcal{L}$ the Lie algebra of G. Consider a family $Y = \{Y_1, \dots, Y_M\}$ of left-invariant vector fields on G satisfying the Hörmander condition, which means that the Lie algebra generated by the Y_i's is $\mathcal{L}$. By "left-invariant" one means that, for any $g \in G$ and any $f \in C_0^\infty(G)$, $Y(\tau_g f) = \tau_g(Yf)$, where τ_g is the left-translation operator. As previously, we can build the Carnot–Carathéodory metric on G. The left-invariance of the Y_i's implies the left-invariance of the distance d. So that for every r, the volume of the ball $B(x,r)$ does not depend on $x \in G$ and also will be denoted $V(r)$. It is well known (see [21,29]) that (G,d) is then a space of homogeneous type. Particular cases are Carnot groups, where the vector fields are given by a Jacobian basis of its Lie algebra and satisfy Hörmander condition. In this situation, two cases may occur: either the manifold is doubling or the volume of the balls admit an exponential growth [21]. For example, nilpotents Lie groups satisfy the doubling property [13].

We refer the reader to [30, Thm. 5.14] and [12, Section 3, Appendix 1] where properties of the heat semigroup are studied: in particular the heat semigroup e^{-tL} satisfies Gaussian upper-bounds and Assumption 1.11 on the higher-order Riesz transforms (Assumption 1.9) is satisfied too.

2.1.3. Carnot groups

A nilpotent Lie group is called Carnot group, if it admits a stratification. A stratification on a Lie group G (whose $\mathfrak{g}$ is its Lie algebra) is a collection of linear subspaces $V_1, \dots, V_r$ of $\mathfrak{g}$ such that

$$\mathfrak{g} = V_1 \oplus \cdots \oplus V_r$$

which satisfy $[V_1, V_i] = V_{i+1}$ for $i = 1, \dots, r-1$ and $[V_1, V_r] = 0$. By $[V_1, V_i]$, we denote the subspace of $\mathfrak{g}$ generated by the elements $[X,Y]$ where $X \in V_1$ and $Y \in V_i$. Consider n_i the dimension of V_i, $d := n_1 + \cdots + n_r$ and dilations $\{\delta_\lambda\}_{\lambda>0}$ of the form

$$\delta_\lambda(x) = \bigl(\lambda x^{(1)}, \lambda^2 x^{(2)}, \dots, \lambda^r x^{(r)}\bigr), \quad x^{(i)} \in V_i.$$

The couple $\mathbb{G} = (G, \delta_\lambda)$ is called a homogeneous Carnot group (of step r and n_1 generators) if δ_λ is an automorphism of G for every $\lambda > 0$ and if the first n_1 elements of the Jacobian basis of $\mathfrak{g}$, say $Z_1, \dots, Z_{n_1}$, satisfy

$$\operatorname{rank}\bigl(\mathrm{Lie}[Z_1, \dots, Z_{n_1}](x)\bigr) = d, \quad \text{for all } x \in G, \tag{11}$$

where $\mathrm{Lie}[Z_1, \dots, Z_{n_1}]$ is the Lie algebra generated by the vector fields $Z_1, \dots, Z_{n_1}$. The number $Q := \sum_{i=1}^{r} i n_i$ is called the homogeneous dimension of $\mathbb{G}$.

For example the Heisenberg group $\mathcal{H}^d$ is a Carnot group of dimension $Q = 2d + 2$. We refer the reader to [20] for an introduction of pseudo-differential operators in this context using a kind of Fourier transform involving irreductible representations and to [4] for a complete work about pseudo-differential calculus on Heisenberg groups.

2.1.4. Riemannian manifolds with a bounded geometry

We shall say that a Riemannian manifold M has a bounded geometry if

- the curvature tensor and all its derivatives are bounded,
- Ricci curvature is bounded from below,
- and M has a positive injectivity radius.

In such situations, Assumption 1.11 and Poincaré inequality are satisfied (see [12] and [32]).

2.2. Examples of nonlinear PDEs

Let $X = (X_1, \dots, X_\kappa)$ be a collection of vector fields satisfying Hörmander condition. Then, it is well known that the associated heat semigroup has a heat kernel with Gaussian bounds. This was proved by Varopoulos in the context of Carnot groups [33] and then extended by Fefferman, Jerison and Sánchez-Calle [31,16,25] without an underlying group structure.

If we consider a collection of vector fields $X \subset Y$, where Y was the one used to build the Riemannian structure (see the last paragraphs) then by construction the vector fields X_i are self-adjoint and so is L. In this case, we can use the C^∞ functional calculus as done in [24]. If the collection X is independent of Y then there is no reason for L to be self-adjoint and we are restricted to only use holomorphic sectorial functional calculus as we aim to do here.

A standard PDE (whose numerous works deal with) to which our paralinearization result would apply is the following quasilinear wave equation on a Riemannian manifold:

$$\partial_t^2 u + Lu = |Xu|^2 F(u) \tag{12}$$

where $F \in C^\infty(\mathbb{R})$. Then the Strichartz estimates which can be found in [26] together with the paralinearization theorem (Theorem 7.4) allow to prove local well-posedness in H^s Sobolev spaces with s large enough.

Other classical PDEs coming from fluids mechanic have a quadratic nonlinearity, which could be studied with the paralinearization, for example Navier–Stokes equation, Euler equation with kinematic viscosity and more generally Boussinesq systems.

3. The scale of Sobolev spaces

We use the Bessel-type Sobolev spaces, adapted to the operator L:

Definition 3.1. For $p \in (1, \infty)$ and $s \geqslant 0$, we define the Sobolev space $W^{s,p} = W_L^{s,p}$ as

$$W^{s,p} = W_L^{s,p} := \bigl\{ f \in L^p,\ (1 + L)^{s/2}(f) \in L^p \bigr\}.$$

More precisely, since the subspace $\mathcal{S}$ defined in (9) is dense in every Lebesgue space, $W^{s,p}$ denotes the closure of $\mathcal{S}$ relatively to the norm

$$\|f\|_{W^{s,p}} := \|f\|_{L^p} + \bigl\|(1 + L)^{s/2}(f)\bigr\|_{L^p}.$$

Consequently the Sobolev spaces are Banach spaces and have a common dense subspace $\mathcal{S}$, on which one can do a priori computations, involving spectral decomposition or square functions.

First, we have this characterization:

Proposition 3.2. *For all $p \in (1,\infty)$ and $s>0$, we have the following equivalence: for every $f \in \mathcal{S}$*

$$\|f\|_{L^p} + \left\|L^{s/2}(f)\right\|_{L^p} \simeq \left\|(1+L)^{s/2} f\right\|_{L^p}.$$

Proof. This result is well known and can be proved using bounded holomorphic functional calculus in L^p. For the sake of completeness, we detail a proof using the previous quadratic functionals. Set $\alpha = s/2$ and write $\alpha = k+\theta$ with $k \in \mathbb{N}$ and $\theta \in [0,1)$. We decompose $(1+L)^\alpha$ with the semigroup as follows

$$\begin{aligned}(1+L)^\alpha f &= \int_0^\infty e^{-t} e^{-tL}(1+L)t^{1-\theta}\frac{dt}{t}(1+L)^k(f) \\ &= \int_0^\infty e^{-t}e^{-tL}t^{1-\theta}\frac{dt}{t}(1+L)^k(f) + \int_0^\infty e^{-t}e^{-tL}(tL)^{1-\theta}\frac{dt}{t}L^\theta(1+L)^k(f).\end{aligned}$$

The first integral operator is easily bounded on L^p since the semigroup e^{-tL} is uniformly bounded. The second integral operator is bounded using duality:

$$\begin{aligned}\left\langle \int_0^\infty e^{-t}e^{-tL}(tL)^{1-\theta}(u)\frac{dt}{t}, g\right\rangle &= \int_0^\infty e^{-t}\left\langle e^{-tL/2}(tL)^{\frac{1-\theta}{2}}(u), e^{-tL^*/2}\left(tL^*\right)^{\frac{1-\theta}{2}} g\right\rangle \frac{dt}{t} \\ &\leqslant \int \left(\int_0^\infty \left|e^{-tL/2}(tL)^{\frac{1-\theta}{2}}(u)\right|^2 \frac{dt}{t}\right)^{1/2} \left(\int_0^\infty \left|e^{-tL^*/2}\left(tL^*\right)^{\frac{1-\theta}{2}}(g)\right|^2\frac{dt}{t}\right)^{1/2} d\mu.\end{aligned}$$

Since $(1-\alpha)/2 > 0$, then the two quadratic functionals are bounded in L^p and $L^{p'}$ (by Proposition 1.16) and that concludes the proof of

$$\left\|(1+L)^\alpha f\right\|_{L^p} \lesssim \left\|(1+L)^k f\right\|_{L^p} + \left\|L^\theta(1+L)^k(f)\right\|_{L^p}.$$

Then, developing $(1+L)^k$, we have a finite sum of $\|L^z(f)\|_{L^p}$ with $z \in [0,\alpha]$. We decompose

$$L^z(f) = \int_0^\infty e^{-tL}(tL)^\alpha(f)t^{-z}\frac{dt}{t} = \int_0^1 e^{-tL}(tL)^\alpha(f)t^{-z}\frac{dt}{t} + \int_1^\infty e^{-tL}(tL)^\alpha t^{-z}\frac{dt}{t}.$$

The first quantity in L^p is controlled by $\|L^\alpha(f)\|_{L^p}$ and the second one by $\|f\|_{L^p}$, which concludes the proof of

$$\left\|(1+L)^{s/2} f\right\|_{L^p} \lesssim \|f\|_{L^p} + \left\|L^{s/2}(f)\right\|_{L^p}.$$

Let us now check the reverse inequality. As previously, for $u=0$ or $u=\alpha$ we write

$$L^u f = \int_0^\infty e^{-t(1+L)}(1+L)L^u t^{1+\alpha}\frac{dt}{t}(1+L)^\alpha f.$$

By similar arguments as above, the operator $\int_0^\infty e^{-t(1+L)}(1+L)L^u t^{1+\alpha}\frac{dt}{t}$ is easily bounded on L^p (splitting the integral for $t \leqslant 1$ and $t \geqslant 1$) and we can also conclude to

$$\left\|L^u(f)\right\|_{L^p} \lesssim \left\|(1+L)^\alpha f\right\|_{L^p},$$

which ends the proof. □

Corollary 3.3. *For all $p \in (1,\infty)$ and $0 \leqslant t \leqslant s$, we have the following inequality*

$$\left\|L^t f\right\|_{L^p} \lesssim \left\|(1+L)^s f\right\|_{L^p} \simeq \|f\|_{W^{2s,p}}.$$

Let us then describe classical Sobolev embeddings in this setting (see [3] for a more general framework):

Proposition 3.4. *Under Assumption 1.4 (lower bound on the ball-volumes), let $s \geqslant t \geqslant 0$ be fixed and take $p \leqslant q$ such that*

$$\frac{1}{q} - \frac{t}{d} > \frac{1}{p} - \frac{s}{d}.$$

Then, we have the continuous embedding

$$W^{s,p} \hookrightarrow W^{t,q}.$$

We refer the reader to [3, Proposition 3.3] for a precise proof. The proof is based on a spectral decomposition. Then we write the resolvent with the semigroup and then use the off-diagonal estimates (here the pointwise estimates on the heat kernel).

Corollary 3.5. *Under the previous assumption, $W^{s,p} \hookrightarrow L^\infty$ as soon as*

$$s > \frac{d}{p}.$$

We now recall a result of [3], where a characterization of Sobolev spaces is obtained, involving some fractional functionals.

Proposition 3.6. *(See [3, Thm. 5.2].) Under Poincaré inequality (P_1), for $s \in (0,1)$ we have the following characterization: a function $f \in L^p$ belongs to $W^{s,p}$ if and only if*

$$S_s^{\rho,loc} f(x) = \left(\int_0^1 \left[\frac{1}{r^s} \left(\frac{1}{\mu(B(x,r))} \int_{B(x,r)} \left| f(y) - f(x) \right|^\rho d\mu(y) \right)^{1/\rho} \right]^2 \frac{dr}{r} \right)^{\frac{1}{2}}$$

belong to L^p, for some $\rho < \min(2,p)$.

This characterization can be extended for $s > 1$, using the sub-Laplacian structure. Indeed, we have this first lemma:

Lemma 3.7. *For every integer k and $p \in (1,\infty)$,*

$$\|f\|_{W^{k,p}} \simeq \sum_{I \subset \{1,\dots,\kappa\}^k} \left\| X_I(f) \right\|_{L^p}.$$

Proof. As pointed out in [12], this is consequence of Assumption 1.9 about the local Riesz transforms. Indeed, for $k \geqslant 1$ and I a subset, we have assumed that the I-th Riesz transform $\mathcal{R}_I$ are bounded on L^p, which is equivalent to

$$\left\| X_I(f) \right\|_{L^p} \lesssim \|f\|_{W^{|I|/2,p}}.$$

Moreover, writing the Riesz transforms and the resolvent (which are all bounded on L^p) as follows

$$(1+L)^{1/2} = (1+L)^{-1/2}(1+L) = (1+L)^{-1/2} - \sum_{i=1}^{\kappa} (1+L)^{-1/2} X_i^2 = (1+L)^{-1/2} - \sum_{i=1}^{\kappa} \overline{\mathcal{R}_i} X_i,$$

we conclude to the reverse inequality and so we have proved the desired result for $k = 1$. We let the details for $k \geqslant 2$ to the reader, the reasoning being exactly the same (writing a finite sum of higher-order Riesz transforms ...). □

We also deduce the following characterization (see Proposition 19 in [12]):

Proposition 3.8. *Let $s := k + t > 1$ (with k an integer and $t \in (0,1)$), then*

$$f \in W^{s,p} \iff f \in L^p \text{ and } \forall I \subset \{1,\dots,\kappa\}^k,\ X_I(f) \in W^{t,p}$$
$$\iff f \in L^p \text{ and } \forall I \subset \{1,\dots,\kappa\}^k,\ S_t^\rho\big(X_I(f)\big) \in L^p.$$

We also deduce the following chain rule (see Theorem 22 in [12] for another proof by induction on k):

Proposition 3.9. *If $F \in C^\infty$ with $F(0)=0$ and let $s := k+t > \frac{d}{p}$ (with $p \in (1,\infty)$, k an integer and $t \in (0,1)$). Then*

$$\|F(f)\|_{W^{s,p}} \lesssim \|f\|_{W^{s,p}} + \|f\|_{W^{s,p}}^k.$$

If $F(0) \neq 0$, we still have such inequalities with localized Sobolev spaces.

Proof. We use the previous characterization of the Sobolev space with S_t^ρ for $s=k+t$. First using the differentiation rule, it comes $X_i(F(f)) = X_i(f)F'(f)$, then $X_jX_i(F(f)) = X_jX_i(f)F'(f) + X_i(f)X_j(f)F''(f)$. By iterating the reasoning, for $I \subset \{1,\dots,\kappa\}^k$, estimating $X_I(F(f))$ in $W^{t,p}$ is reduced to estimate quantities such as

$$h := \left[\prod_{\alpha=1}^{l} X_{i_\alpha}\right](f)F^{(n)}(f)$$

where $i_\alpha \subset I$, $n \leqslant k$ and $\sum |i_\alpha| = |I| \leqslant k$. Then for x, y, we have

$$\begin{aligned} \left|h(x)-h(y)\right| &\leqslant \sum_\beta \left|X_{i_\beta}(f)(x) - X_{i_\beta}(f)(y)\right| \prod_{\alpha\neq\beta} \sup_{z=x,y}\left|X_{i_\alpha}(f)(z)\right| \left\|F^{(n)}(f)\right\|_{L^\infty} \\ &\quad + \prod_\alpha \sup_{z=x,y}\left|X_{i_\alpha}(f)(z)\right|\left|F^{(n)}(f)(x) - F^{(n)}(f)(y)\right|. \end{aligned}$$

Since $\rho \leqslant p$ let us choose exponents ρ_α, p_α such that

$$\frac{1}{\rho} = \sum_\alpha \frac{1}{\rho_\alpha}, \quad \rho_\alpha \leqslant p_\alpha$$

and

$$\frac{1}{p} = \sum_\alpha \frac{1}{p_\alpha}.$$

Moreover we require that

$$\frac{1}{p_\alpha} - \frac{|i_\alpha|+t}{d} > \frac{1}{p} - \frac{s}{d}. \tag{13}$$

This is possible since $\sum_\alpha |i_\alpha| = |I| \leqslant s-t$ and $s > d/p$ (indeed we let the reader check that $p_\alpha = \frac{|I|+t}{|i_\alpha|+t}p$ is a good choice). Moreover, we chose exponents $\overline{\rho_\alpha}$, $\overline{\rho}$, $\overline{p_\alpha}$ and $\overline{p}$ such that

$$\frac{1}{\rho} = \sum_\alpha \frac{1}{\overline{\rho_\alpha}} + \frac{1}{\overline{\rho}}, \quad \overline{\rho_\alpha} \leqslant \overline{p_\alpha}$$

and $\overline{\rho} \leqslant \overline{\rho} \leqslant \overline{p}$ with

$$\frac{1}{p} = \sum_\alpha \frac{1}{\overline{p_\alpha}} + \frac{1}{\overline{p}}.$$

As previously, we require (13) with $\overline{p_\alpha}$ instead of p_α and

$$\frac{1}{\overline{p}} > \frac{1}{p} - \frac{s}{d}. \tag{14}$$

Such exponents can be chosen by perturbing the previous construction with a small parameter since $s > d/p$. By Hölder inequality, we deduce that

$$\begin{aligned} S_t^\rho(h) &\lesssim \sum_\beta S_t^{\rho_\beta}\left(X_{i_\beta}(f)\right) \prod_{\alpha\neq\beta} \mathcal{M}_{\rho_\alpha}\left[X_{i_\alpha}(f)\right] \left\|F^{(n)}(f)\right\|_{L^\infty} \\ &\quad + \prod_\alpha \mathcal{M}_{\overline{\rho_\alpha}}\left[X_{i_\alpha}(f)\right] S_t^{\overline{\rho}}\left(F^{(n)}(f)\right). \end{aligned}$$

Since F is supposed to be bounded in C^∞, then $F^{(n)}$ is Lipschitz and so, we finally obtain

$$S_t^\rho(h) \lesssim \sum_\beta S_t^{\rho_\beta}\big(X_{i_\beta}(f)\big)\prod_{\alpha\neq\beta}\mathcal{M}_{\rho_\alpha}\big[X_{i_\alpha}(f)\big]\big\|F^{(n)}(f)\big\|_{L^\infty} + \prod_\alpha \mathcal{M}_{\overline{\rho_\alpha}}\big[X_{i_\alpha}(f)\big]S_t^{\bar\rho}(f).$$

Then applying Hölder inequality, we get

$$\big\|S_t^\rho(h)\big\|_{L^p} \lesssim \sum_\beta \big\|S_t^{\rho_\beta}\big(X_{i_\beta}(f)\big)\big\|_{L^{p_\beta}}\prod_{\alpha\neq\beta}\big\|\mathcal{M}_{\rho_\alpha}\big[X_{i_\alpha}(f)\big]\big\|_{L^{p_\alpha}}\big\|F^{(n)}(f)\big\|_{L^\infty} + \prod_\alpha \big\|\mathcal{M}_{\overline{\rho_\alpha}}\big[X_{i_\alpha}(f)\big]\big\|_{L^{\overline{p_\alpha}}}\big\|S_t^{\bar\rho}(f)\big\|_{L^{\bar p}}.$$

By (13) with Sobolev embeddings (Proposition 3.4), we have

$$\big\|\mathcal{M}_{\rho_\alpha}\big[X_{i_\alpha}(f)\big]\big\|_{L^{p_\alpha}} \lesssim \|f\|_{W^{|i_\alpha|,p_\alpha}} \lesssim \|f\|_{W^{s,p}}$$

and

$$\big\|S_t^{\rho_\beta}\big(X_{i_\beta}(f)\big)\big\|_{L^{p_\beta}} \lesssim \|f\|_{W^{|i_\beta|+t,p_\beta}} \lesssim \|f\|_{W^{s,p}}.$$

So with (13) and (14), we finally obtain

$$\big\|S_t^\rho(h)\big\|_{L^p} \lesssim \|f\|_{W^{s,p}} + \|f\|_{W^{s,p}}^k,$$

where we used $s > d/p$ and the Sobolev embedding $W^{s,p} \subset L^\infty$ with the smoothness of F to control $\|F^{(n)}(f)\|_{L^\infty}$.

Since $F(0) = 0$ and F is Lipschitz, we also deduce that $F(f)$ belongs to L^p, which allows us to get the expected result

$$\big\|F(f)\big\|_{W^{s,p}} \lesssim \big\|F(f)\big\|_{L^p} + \sum_I \big\|S_t^\rho\big(F(f)\big)\big\|_{L^p} \lesssim \|f\|_{W^{s,p}} + \|f\|_{W^{s,p}}^k. \qquad \square$$

Remark 3.10. If $F \in C^\infty$ with $F(0) = 0$ and $s > d/p$ then we obtain

$$\big\|F(f)\big\|_{W^{s,p}} \lesssim \|f\|_{W^{s,p}} + \|f\|_{W^{s,p}}^k.$$

It is sufficient to assume that F is locally bounded in C^∞ and then the implicit constant will depend on $\|f\|_{L^\infty}$. Indeed, using Sobolev embedding, we know that as soon as $s > d/p$, $W^{s,p}$ is continuously embedded in L^∞.

4. Paraproducts associated to a semigroup

Our aim is to describe a kind of "paralinearization" result. In the Euclidean case, this is performed by using paraproducts (defined with the help of Fourier transform). Here, we cannot use such powerful tools, so we require other kind of paraproducts, defined in terms of a semigroup. These paraproducts were introduced by the first author in [7], already used in [3] and more recently extended in [17,18]. Let us recall these definitions.

4.1. Definitions and spectral decomposition

We consider a sub-Laplacian operator L satisfying the assumptions of the previous sections. We write for convenience c_0 for a suitably chosen constant, $\psi(x) = c_0 x^N e^{-x}(1 - e^{-x})$ and so

$$\psi_t(L) := c_0(tL)^N e^{-tL}\big(1 - e^{-tL}\big),$$

with a large enough integer $N > d/2$. Let ϕ be the function

$$\phi(x) := -c_0\int_x^\infty y^N e^{-y}\big(1 - e^{-y}\big)\,dy,$$

$$\widetilde{\phi}(x) := -c_0\int_x^\infty y^{N-1} e^{-y}\big(1 - e^{-y}\big)\,dy,$$

and set $\phi_t(L) := \phi(tL)$. Then we get a "spectral" decomposition of the identity as follows (choosing the appropriate constant c_0)

$$f = -\int_0^\infty \phi'(tL) f \frac{dt}{t}.$$

So for two smooth functions, we have

$$fg := -\int_{s,u,v>0} \phi'(sL)\big[\phi'(uL) f \phi'(vL) g\big] \frac{ds\,du\,dv}{suv}.$$

Since $\phi'(x) = \psi(x) := c_0 x^N e^{-x}(1-e^{-x})$ and $x(\widetilde{\phi})'(x) = \phi'(x)$, it comes that (by integrating according to $t := \min\{s,u,v\}$)

$$\begin{aligned} fg := &-\int_0^\infty \psi(tL)\big[\widetilde{\phi}(tL) f \widetilde{\phi}(tL) g\big]\frac{dt}{t} - \int_0^\infty \widetilde{\phi}(tL)\big[\psi(tL) f \widetilde{\phi}(tL) g\big]\frac{dt}{t} \\ &- \int_0^\infty \widetilde{\phi}(tL)\big[\widetilde{\phi}(tL) f \psi(tL) g\big]\frac{dt}{t}. \end{aligned} \tag{15}$$

Let us now focus on the first term in (15):

$$I(f,g) = \int_0^\infty \psi(tL)\big[\widetilde{\phi}(tL) f \widetilde{\phi}(tL) g\big]\frac{dt}{t}.$$

Since $N \gg 1$, let us write $\psi(z) = z\widetilde{\psi}$ with $\widetilde{\psi}$ (still vanishing at 0 and at infinity). Then using the structure of the sub-Laplacian L, the following algebra rule holds

$$L(fg) = L(f)g + fL(g) - 2\langle Xf \cdot Xg\rangle,$$

where X is the collection of vector fields $Xf := (X_1 f, \dots, X_\kappa f)$. Hence, we get

$$\begin{aligned} I(f,g) &= \int_0^\infty \widetilde{\psi}(tL)(tL)\big[\widetilde{\phi}(tL) f \widetilde{\phi}(tL) g\big]\frac{dt}{t} \\ &= \int_0^\infty \widetilde{\psi}(tL)\big[tL\widetilde{\phi}(tL) f \widetilde{\phi}(tL) g\big]\frac{dt}{t} + \int_0^\infty \widetilde{\psi}(tL)\big[\widetilde{\phi}(tL) f tL\widetilde{\phi}(tL) g\big]\frac{dt}{t} \\ &\quad - 2\int_0^\infty \widetilde{\psi}(tL) t\big\langle X\widetilde{\phi}(tL) f \cdot X\widetilde{\phi}(tL) g\big\rangle\frac{dt}{t}. \end{aligned}$$

Combining with (15), we define the paraproduct as follows:

Definition 4.1. With the previous notations, we define the paraproduct of f by g, by

$$\begin{aligned} \Pi_g(f) := &-\int_0^\infty \widetilde{\psi}(tL)\big[tL\widetilde{\phi}(tL) f \widetilde{\phi}(tL) g\big]\frac{dt}{t} \\ &- \int_0^\infty \widetilde{\phi}(tL)\big[\psi(tL) f \widetilde{\phi}(tL) g\big]\frac{dt}{t}. \end{aligned}$$

Remark 4.2. We first want to point out the difference with the initial definition in [7]. There, a general semigroup was considered and the previous operation on the term I can be performed by computing the "carré du champ" introduced by Bakry and Émery (see [5] for details)

$$\Gamma(f,g) := L(fg) - L(f)g - fL(g)$$

instead of the vector field X. However in [7], the paraproducts were only defined by the second term. This new definition comes from the following observation: considering the quantity $I(f,g)$ and distributing the Laplacian as we have done (writing the "carré du champ"), three terms show up. The term $\widetilde{\psi}(tL)[tL\widetilde{\phi}(tL)f\widetilde{\phi}(tL)g]$ has the same regularity properties as $\widetilde{\phi}(tL)[\psi(tL)f\widetilde{\phi}(tL)g]$ (in the sense that $tL\widetilde{\phi}(tL)$ can be considered as $\psi(tL)$). This also legitimates to add this extra term in the definition of the paraproducts.

This new paraproduct is the "maximal" (in a certain sense) part of the product fg, where the regularity is given by the regularity of f.

The following decomposition comes naturally:

Corollary 4.3. *Let f, g be two smooth functions, then we have*

$$fg = \Pi_g(f) + \Pi_f(g) + \mathtt{Rest}(f,g)$$

where the "rest" is given by

$$\mathtt{Rest}(f,g) := 2\int_0^\infty \widetilde{\psi}(tL)\big\langle t^{1/2}X\widetilde{\phi}(tL)f, t^{1/2}X\widetilde{\phi}(tL)g\big\rangle \frac{dt}{t}.$$

4.2. Boundedness of paraproducts in Sobolev and Lebesgue spaces

Concerning estimates on these paraproducts in Lebesgue spaces, we refer to [7]:

Theorem 4.4 *(Boundedness in Lebesgue spaces). For $p, q \in (1,\infty]$ with $0 < \frac{1}{r} := \frac{1}{p} + \frac{1}{q}$ then*

$$(f,g) \to \Pi_g(f)$$

is bounded from $L^p \times L^q$ into L^r.

Let us now describe boundedness in the scale of Sobolev spaces.

Theorem 4.5 *(Boundedness in Sobolev spaces). For $p, q, r \in (1,\infty)$ with $\frac{1}{r} := \frac{1}{p} + \frac{1}{q}$ and $s \in (0; 2N-4)$ then*

$$(f,g) \to \Pi_g(f)$$

is bounded from $W^{s,p} \times L^q$ into $W^{s,r}$.

Proof. It is sufficient to prove the following homogeneous estimates: for every $\beta \in [0, N-2)$

$$\big\| L^\beta \Pi_g(f)\big\|_{L^r} \lesssim \big\| L^\beta(f)\big\|_{L^p} \|g\|_{L^q}.$$

By density, we may only focus on functions $f, g \in \mathcal{S}$ (see (9)). For $\beta = 0$, this is the previous theorem so it remains to check for $\beta \in (0, N-2)$. We recall that

$$\Pi_g(f) = -\int_0^\infty \widetilde{\psi}(tL)\big[tL\widetilde{\phi}(tL)f\widetilde{\phi}(tL)g\big]\frac{dt}{t}$$
$$- \int_0^\infty \widetilde{\phi}(tL)\big[\psi(tL)f\widetilde{\phi}(tL)g\big]\frac{dt}{t},$$

giving rise to two quantities, $\Pi_g^1(f)$ and $\Pi_g^2(f)$. Indeed, applying L^β to the paraproduct $\Pi_g^2(f)$, it yields

$$L^\beta \Pi_g^2(f) = \int_0^\infty L^\beta \widetilde{\phi}(tL)\big[\psi(tL)f\widetilde{\phi}(tL)g\big]\frac{dt}{t}$$
$$= \int_0^\infty \overline{\psi}(tL)\big[t^{-\beta}\psi(tL)f\widetilde{\phi}(tL)g\big]\frac{dt}{t}$$
$$= \int_0^\infty \overline{\psi}(tL)\big[\widetilde{\psi}(tL)L^\beta f\widetilde{\phi}(tL)g\big]\frac{dt}{t},$$

where we set $\overline{\psi}(z) = z^\beta\phi(z)$ and $\widetilde{\psi}(z) = z^{-\beta}\psi(z)$. So if the integer N in ϕ and ψ is taken sufficiently large, then $\overline{\psi}$ and $\widetilde{\psi}$ are still holomorphic functions with vanishing properties at 0 and at infinity. As a consequence, we get

$$L^\beta \Pi_g(f) = \overline{\Pi}_g\big(L^\beta f\big)$$

with the new paraproduct $\overline{\Pi}$ built with $\overline{\psi}$ and $\widetilde{\psi}$. We also apply the classical reasoning aiming to estimate this paraproduct. By duality, for any function $h \in \mathcal{S} \subset L^{r'}$ we have (since Proposition 1.14)

$$\big\langle L^\beta \Pi_g^2(f), h\big\rangle = \int \int_0^\infty \overline{\psi}\big(tL^*\big)h\widetilde{\psi}(tL)\big(L^\beta f\big)\widetilde{\phi}(tL)g\frac{dt}{t}\,d\mu$$
$$\leqslant \int \left(\int_0^\infty \big|\overline{\psi}\big(tL^*\big)h\big|^2\frac{dt}{t}\right)^{1/2}\left(\int_0^\infty \big|\widetilde{\psi}(tL)\big(L^\beta f\big)\big|^2\frac{dt}{t}\right)^{1/2}\sup_t\big|\widetilde{\phi}(tL)g\big|\,d\mu.$$

From the pointwise decay on the semigroup (6), we know that

$$\sup_t\big|\widetilde{\phi}(tL)g(x)\big| \leqslant \mathcal{M}(g)(x)$$

and so by Hölder inequality

$$\big|\big\langle L^\beta \Pi_g^2(f), h\big\rangle\big| \lesssim \left\|\left(\int_0^\infty \big|\overline{\psi}\big(tL^*\big)h\big|^2\frac{dt}{t}\right)^{1/2}\right\|_{L^{r'}}\left\|\left(\int_0^\infty \big|\widetilde{\psi}(tL)\big(L^\beta f\big)\big|^2\frac{dt}{t}\right)^{1/2}\right\|_{L^p}\|\mathcal{M}g\|_{L^q}.$$

Since $\overline{\psi}$ and $\widetilde{\psi}$ are holomorphic functions vanishing at 0 and having fast decay at infinity, we know from (1.16) that the two square functions are bounded on Lebesgue spaces. We also conclude the proof by duality, since it comes

$$\big|\big\langle L^\beta \Pi_g^2(f), h\big\rangle\big| \lesssim \|h\|_{L^{r'}}\big\|L^\beta f\big\|_{L^p}\|g\|_{L^q}.$$

We let the reader check that the same arguments still holds for the first part $\Pi_g^1(f)$ and so the proof is also finished. □

5. Paralinearization theorem

Theorem 5.1. *Consider $s \in (d/p, 1)$ and $f \in W^{s+\epsilon,p}$ for some $\epsilon > 0$. Then for every smooth function $F \in C^\infty(\mathbb{R})$ with $F(0) = 0$,*

$$F(f) = \Pi_{F'(f)}(f) + w \tag{16}$$

with $w \in W^{2s-d/p,p}$.

We follow the proof in [10,28,8].

Proof of Theorem 5.1. Let us refer the reader to the operators $\phi(tL)$ and $\psi(tL)$, defined in Section 4.1: $\psi(x) = c_0 x^N e^{-x}(1-e^{-x})$, ϕ is its primitive vanishing at infinity. Let us write $\widetilde{\psi}(z) = z^{-1}\psi(z)$ and $\widetilde{\phi}$ its primitive vanishing at infinity. Moreover, these functions are normalized by the suitable constant c_0 such that $\widetilde{\phi}(0) = 1$.

First, since L is densely defined in L^2, Eq. (6) yields that $(e^{-tL})_{t>0}$ converges to the identity operator when $t \to 0$, in L^p (see the proof of [9, Theorem 2.3]). So by commuting with the Bessel potential $(1+L)^{\frac{s+\epsilon}{2}}$ it follows that for $f \in W^{s+\epsilon,p}$

$$f = \lim_{t\to 0} \widetilde{\phi}(tL)(f) \in W^{s+\epsilon,p}$$

and so we decompose

$$F(f) = \widetilde{\phi}(L)F\big(\widetilde{\phi}(L)f\big) - \int_0^1 \frac{d}{dt}\widetilde{\phi}(tL)F\big(\widetilde{\phi}(tL)f\big)\,dt.$$

Since

$$\begin{aligned} t\frac{d}{dt}\widetilde{\phi}(tL)F\big(\widetilde{\phi}(tL)f\big) &= tL\widetilde{\phi}'(tL)F\big(\widetilde{\phi}(tL)f\big) + \widetilde{\phi}(tL)\big[\big(tL\widetilde{\phi}'(tL)f\big)F'\big(\widetilde{\phi}(tL)f\big)\big] \\ &= \phi'(tL)F\big(\widetilde{\phi}(tL)f\big) + \widetilde{\phi}(tL)\big[\big(\phi'(tL)f\big)F'\big(\widetilde{\phi}(tL)f\big)\big], \end{aligned}$$

we get

$$\begin{aligned} F(f) &= \widetilde{\phi}(L)F\big(\widetilde{\phi}(L)f\big) - \int_0^1 \widetilde{\phi}'(tL)tL\big[F\big(\widetilde{\phi}(tL)f\big)\big] + \widetilde{\phi}(tL)\big[\big(\phi'(tL)f\big)F'\big(\widetilde{\phi}(tL)f\big)\big]\frac{dt}{t} \\ &= \widetilde{\phi}(L)F\big(\widetilde{\phi}(L)f\big) - \int_0^1 \widetilde{\phi}'(tL)\big[F''\big(\widetilde{\phi}(tL)f\big)\big|t^{1/2}X\widetilde{\phi}(tL)f\big|^2 + F'\big(\widetilde{\phi}(tL)f\big)tL\widetilde{\phi}(tL)f\big] \\ &\quad + \widetilde{\phi}(tL)\big[\big(\phi'(tL)f\big)F'\big(\widetilde{\phi}(tL)f\big)\big]\frac{dt}{t}, \end{aligned}$$

where we used the differentiation rule for the composition with the vector fields $X = (X_1, \ldots, X_\kappa)$. We also set

$$w := I + II + III + IV + V$$

with

$$\begin{aligned} I &:= \widetilde{\phi}(L)F\big(\widetilde{\phi}(L)f\big), \\ II &:= -\int_0^1 \widetilde{\phi}'(tL)\big[F''\big(\widetilde{\phi}(tL)f\big)\big|t^{1/2}X\widetilde{\phi}(tL)f\big|^2\big]\frac{dt}{t}, \\ III &:= \int_0^1 \widetilde{\phi}'(tL)\big[\big(\widetilde{\phi}(tL)F'(f) - F'\big(\widetilde{\phi}(tL)f\big)\big)tL\widetilde{\phi}(tL)f\big]\frac{dt}{t}, \\ IV &:= \int_0^1 \widetilde{\phi}(tL)\big[\big(\phi'(tL)f\big)\big(\widetilde{\phi}(tL)F'(f) - F'\big(\widetilde{\phi}(tL)f\big)\big)\big]\frac{dt}{t}, \end{aligned}$$

and

$$V := \int_1^\infty \widetilde{\psi}(tL)\big[tL\widetilde{\phi}(tL)f\widetilde{\phi}(tL)F'(f)\big]\frac{dt}{t} + \int_1^\infty \widetilde{\phi}(tL)\big[\psi(tL)f\widetilde{\phi}(tL)F'(f)\big]\frac{dt}{t}$$

in order that (16) is satisfied. It remains to check that each term belongs to $W^{2s-d/p,p}$.

Step 1: Term I.

Since $f \in W^{s+\epsilon,p}$ then $\widetilde{\phi}(L)f$ belongs to $W^{\rho,p}$ for every $\rho \geqslant s+\epsilon$ and so Proposition 3.9 yields that

$$\big\|\widetilde{\phi}(L)F\big(\widetilde{\phi}(L)f\big)\big\|_{W^{2s-d/p,p}} \lesssim \|f\|_{W^{s+\epsilon,p}}.$$

Step 2: Term V.

We only treat the first term in V (the second one can be similarly estimated). Using duality, we have with some $g \in L^{p'}$ and for $\alpha \in \{0, 2s-d/p\}$ since $\alpha \geqslant 0$ and $t \geqslant 1$

$$\begin{aligned}
\big\|L^{\alpha/2}V\big\|_{L^p} &\leqslant \int \int_1^\infty \big|(tL)^{\alpha/2}\widetilde{\psi}\big(tL^*\big)g\big|\big|tL\widetilde{\phi}(tL)f\widetilde{\phi}(tL)F'(f)\big|\frac{dt\,d\mu}{t} \\
&\lesssim \left\|\left(\int_1^\infty \big|tL\widetilde{\phi}(tL)f\widetilde{\phi}(tL)F'(f)\big|^2\frac{dt}{t}\right)^{1/2}\right\|_{L^p} \\
&\lesssim \|f\|_{L^p}\sup_{t\geqslant 1}\big\|\widetilde{\phi}(tL)F'(f)\big\|_{L^\infty},
\end{aligned}$$

where we used the boundedness of the square function. Then we conclude since $\widetilde{\phi}(tL)F'(f)$ is uniformly bounded by $\|F'(f)\|_{L^\infty}$ which is controlled by $\|f\|_{W^{s,p}}$ (due to Sobolev embedding with $s > d/p$ and Proposition 3.9).

Indeed our problem is to gain some extra regularity (from s to $2s-d/p$) so the main difficulty relies on the study of the "high frequencies" and not on the lower ones.

Step 3: Term II.

By duality and previous arguments, we get

$$\|II\|_{W^{2s-d/p,p}} \lesssim \left\|\left(\int_0^1 t^{-2s+d/p}\big|t^{1/2}X\widetilde{\phi}(tL)f\big|^4\frac{dt}{t}\right)^{1/2}\right\|_{L^p} + \left\|\left(\int_0^1 \big|t^{1/2}X\widetilde{\phi}(tL)f\big|^4\frac{dt}{t}\right)^{1/2}\right\|_{L^p}$$

where we decomposed the norm in its homogeneous and inhomogeneous parts and then used uniform boundedness of $F''(\widetilde{\phi}(tL)f)$. Since (using L^p-boundedness of the Riesz transforms, see Assumption 1.9 and Sobolev embedding)

$$\begin{aligned}
\big\|X\widetilde{\phi}(tL)f\big\|_{L^\infty} &\lesssim \big\|X\widetilde{\phi}(tL)f\big\|_{W^{d/p+\epsilon,p}} \\
&\lesssim \big\|L^{1/2}\widetilde{\phi}(tL)f\big\|_{L^p} + \big\|L^{1/2+d/2p+\epsilon/2}\widetilde{\phi}(tL)f\big\|_{L^p} \\
&\lesssim t^{s/2-1/2}\|f\|_{W^{s,p}} + t^{s/2-d/2p-\epsilon/2-1/2}\|f\|_{W^{s,p}} \\
&\lesssim t^{s/2-d/2p-\epsilon/2-1/2}\|f\|_{W^{s,p}}
\end{aligned}$$

where we used $t < 1$. Finally it comes,

$$\begin{aligned}
\|II\|_{W^{2s-d/p,p}} &\lesssim \left\|\left(\int_0^1 t^{-s-\epsilon}\big|t^{1/2}X\widetilde{\phi}(tL)f\big|^2\frac{dt}{t}\right)^{1/2}\right\|_{L^p} \\
&\lesssim \left\|\left(\int_0^1 \big|(tL)^{1/2-(s+\epsilon)/2}\widetilde{\phi}(tL)L^{(s+\epsilon)/2}f\big|^2\frac{dt}{t}\right)^{1/2}\right\|_{L^p} \\
&\lesssim \|f\|_{W^{s+\epsilon,p}},
\end{aligned}$$

where we used $s < 1$ and the boundedness of the square function.

Step 4: Terms III and IV.

For these terms, we follow the reasoning of the Appendix of [10]. Using the Mean value Theorem, we have

$$\big|\widetilde{\phi}(tL)F'(f) - F'\big(\widetilde{\phi}(tL)f\big)\big| \leqslant \big|\big(\widetilde{\phi}(tL)-I\big)F'(f)\big| + \big|\big(\widetilde{\phi}(tL)-I\big)f\big|.$$

So using similar arguments as previously, we get (with $h = F'(f)$ and $h = f$)

$$\begin{aligned}
\|III\|_{W^{2s-d/p,p}} &\lesssim \left\| \left(\int_0^1 \left|\widetilde{\phi}(tL)F'(f) - F'\big(\widetilde{\phi}(tL)f\big)\right|^2 \left|tL\widetilde{\phi}(tL)f\right|^2 t^{-s+d/(2p)} \frac{dt}{t} \right)^{1/2} \right\|_{L^p} \\
&\lesssim \left\| \left(\int_0^1 \left|\big(\widetilde{\phi}(tL) - I\big)h\right|^2 \left|tL\widetilde{\phi}(tL)f\right|^2 t^{-2s+d/p} \frac{dt}{t} \right)^{1/2} \right\|_{L^p} \\
&\lesssim \left\| \left(\int_0^1 \left|\big(\widetilde{\phi}(tL) - I\big)h\right|^2 t^{-s} \frac{dt}{t} \right)^{1/2} \right\|_{L^p} \\
&\lesssim \left\| \left(\int_0^1 \left|\frac{\widetilde{\phi}(tL) - I}{(tL)^s} L^s h\right|^2 \frac{dt}{t} \right)^{1/2} \right\|_{L^p} \\
&\lesssim \left\|L^s h\right\|_{L^p} \lesssim \|h\|_{W^{s,p}}
\end{aligned}$$

where we used $s < 2$, which yields

$$\left\|tL\widetilde{\phi}(tL)f\right\|_{L^\infty} \leqslant \left\|(tL)^{1-s/2}\widetilde{\phi}(tL)(tL)^{s/2}f\right\|_{L^\infty} \lesssim t^{-d/(2p)}\left\|(tL)^{s/2}f\right\|_{L^p} \lesssim t^{-d/(2p)+s/2}\|f\|_{W^{s,p}} \tag{17}$$

and the boundedness of the quadratic functional associated to the function $\frac{\widetilde{\phi}(z)-1}{z^s}$ which is holomorphic and vanishing at 0 and at ∞ (see Proposition 1.16). We conclude the estimate of III since $h = f$ or $h = F'(f)$ belongs to $W^{s,p}$. The term IV is similarly estimated. □

Corollary 5.2. *The diagonal term* Rest *(defined in Corollary 4.3) is bounded from $L^p \times L^q$ into L^r as soon as $p, q \in (1, \infty]$ with $0 < \frac{1}{r} := \frac{1}{p} + \frac{1}{q}$. Moreover for $p \in (1, \infty)$, $\epsilon > 0$ as small as we want and $s > d/p$ (with $s < (N-2)/2$), then*

$$\left\|\mathtt{Rest}(f, g)\right\|_{W^{2s-d/p,p}} \lesssim \|f\|_{W^{s+\epsilon,p}}\|g\|_{W^{s+\epsilon,p}}.$$

Proof. Apply Theorem 5.1 to the quantities $f + g$ and $f - g$ with $F(u) := u^2$. Then the polarization formulas give that $\mathtt{Rest}(f, g)$ has the same regularity has w in Theorem 5.1. □

Remark 5.3. Usually, we have a gain of regularity of order $s - d/p$ for this quantity. Here we have a gain of $s - d/p - \epsilon$ for every $\epsilon > 0$, as small as we want.

The previous result only holds for functions with low regularity since $s \in (\frac{d}{p}, 1)$, which is quite constraining but inherent to the method employed here. We state below an extension for higher-order Sobolev spaces, with a modification of the paraproduct operator.

To legitimate the definition of the paraproducts (just before Definition 4.1 and Remark 4.2), we have developed $(tL)(\widetilde{\phi}(tL)f\widetilde{\phi}(tL)F'(f))$ using the Leibniz rule of the Laplacian $L(fg) = L(f)g - 2\langle X(f), X(g)\rangle + fL(g)$. Now for $M << N$, it is possible to do the same operation and develop powers of the Laplacian. Indeed, there exist multilinear differential operators $(T_j)_j$ such that for smooth functions F and h, we have

$$\begin{aligned}
(L)^M F(h) = &\sum_j T_j\big(h, F'(h), \ldots, F^{(2M-1)}(h)\big) \\
&+ F^{(2M)}(h)\left|t^{1/2}h\right|^{2M}.
\end{aligned}$$

Then, we may define a "higher-order" paraproduct as follows:

$$\Pi^M_{f,F'(f),\dots,F^{(2M-1)}(f)}(f) := -\int_0^\infty \sum_j \widetilde{\psi}(tL)\big[t^M T_j\big(\widetilde{\phi}(tL)f,\dots,\widetilde{\phi}(tL)F^{(2M-1)}(f)\big)\big]\frac{dt}{t}$$
$$-\int_0^\infty \widetilde{\phi}(tL)\big[\psi(tL)f\widetilde{\phi}(tL)g\big]\frac{dt}{t}.$$

Reproducing the previous reasoning, we may prove

Theorem 5.4. *Consider $M << N$, $s \in (\frac{d}{p}, M)$ and $f \in W^{s+\epsilon,p}$ for some $\epsilon > 0$. Then for every smooth function $F \in C^\infty(\mathbb{R})$ with $F(0) = 0$,*

$$F(f) = \Pi^M_{f,F'(f),\dots,F^{(2M-1)}(f)}(f) + w$$

with $w \in W^{2s-d/p,p}$.

We let the detailed proof to the reader. Indeed the rest w should be decomposed as previously. Each term appearing in the decomposition may be bounded, as we did for $M = 1$ since they only involve quantities as $|\widetilde{\phi}(tL)F^{(k)}(f) - F(k)(\widetilde{\phi}(tL)f)|$ and other differential operators on $\widetilde{\phi}(tL)f$. The key idea is that now the multilinearity of the operator T_j will be sufficiently high to involve sufficiently such differential terms, each of them bringing a positive power of t as shown in (17).

As explained in [10] (see its Appendix I.3, Theorem 38), a vector-valued version of the preceding result allows us to prove the following theorem:

Theorem 5.5. *Consider $s \in (d/p, 1)$, $f \in W^{s+k,p}$ and a smooth function $F(x, u_1, \dots u_N) \in C^\infty(M \times \mathbb{R}^N)$ with $F(x, 0, \dots, 0) = 0$. Then by identifying $\{1, \dots, N\}$ with a set of multi-indices $\{\alpha_1, \dots, \alpha_N\}$ (and $|\alpha_i| \leqslant k$), we can build*

$$x \in M \to F\big(x, X_{\alpha_1} f(x), \dots, X_{\alpha_N} f(x)\big) \tag{18}$$

which belongs to $W^{s,p}$. Moreover,

$$F\big(x, X_{\alpha_1} f(x), \dots, X_{\alpha_N} f(x)\big) = \sum_{i=1}^N \Pi_{[\partial_{u_i} F](x, X_{\alpha_1} f(x), \dots, X_{\alpha_N} f(x))}(X_{\alpha_i} f)(x) + w(x) \tag{19}$$

with $w \in W^{2s-d/p,p}$.

6. Propagation of low regularity for solutions of nonlinear PDEs

As in the Euclidean case, paralinearization is a powerful tool to study nonlinear PDEs and to prove the propagation of regularity for solutions of such PDEs. Let us try to present some results in this direction in our setting of Riemannian manifold.

Let us consider a specific case of nonlinear PDEs for simplifying the exposition: let $F(x, u_1, \dots, u_{\kappa+1}) \in C^\infty(M \times \mathbb{R}^{\kappa+1})$ be a smooth function with $F(x, 0, \dots, 0) = 0$. Then by identifying $\{1, \dots, \kappa + 1\}$ with a set of multi-indices $\{0, 1, \dots, \kappa\}$, we deal with the function

$$F(f, Xf) := x \in M \to F\big(x, f(x), X_1 f(x), \dots, X_\kappa f(x)\big) \tag{20}$$

for some function f. That corresponds to the case $N = \kappa + 1$, $k = 1$ with $\alpha_1 = 0$ and $\alpha_i = X_{i-1}$ for $i = 2, \dots, N + 1$ in (20).

Theorem 6.1. *Consider $s \in (d/p, 1)$, $f \in W^{s+1,p}$ and a smooth function (as above) $F(x, u_1, \dots, u_N) \in C^\infty(M \times \mathbb{R}^N)$ with $F(x, 0, \dots, 0) = 0$ and assume that f is a solution of*

$$F(f, Xf)(x) = 0.$$

Consider the vector field

$$\Gamma(x) := \sum_{i=2}^{\kappa+1} [\partial_{u_i} F]\big(x, f(x), X_1 f(x), \dots, X_\kappa f(x)\big) X_i.$$

Then, locally around each point $x_0 \in M$ in "the direction Γ", the solution f has a regularity $W^{s+1+\rho,p}$ for every $\rho > 0$ such that

$$\rho < \min\{1, s - d/p\}.$$

In the sense that

$$U(f) := \sum_{i=2}^{\kappa+1} [\partial_{u_i} F]\big(x, f(x), X_1 f(x), \dots, X_\kappa f(x)\big) L^{(s+\rho)/2} X_i(f) \in L^p.$$

Such results can be seen as a kind of directional "implicit function theorem", where the regularity of $F(f, Xf)$ implies some directional regularity for f (in the suitable direction, where we can regularly "invert" the nonlinear equation).

Proof of Theorem 6.1. The previous paralinearization result yields that

$$\sum_{i=1}^{\kappa} \Pi_{[\partial_{u_{i+1}} F](f,Xf)}\big(X_i(f)\big) \in W^{s+\rho,p},$$

which gives

$$T_F(f) := \sum_{i=1}^{\kappa} \widetilde{\Pi}_{[\partial_{u_{i+1}} F](f,Xf)}\big(L^\alpha X_i f\big) \in L^p,$$

where $\widetilde{\Pi}$ is another paraproduct. Indeed

$$\widetilde{\Pi}_b(a) = -\int_0^\infty (tL)^\alpha \widetilde{\psi}(tL)\big[(tL)^{1-\alpha} \widetilde{\phi}(tL) a \widetilde{\phi}(tL) b\big] \frac{dt}{t}$$
$$- \int_0^\infty (tL)^\alpha \widetilde{\phi}(tL)\big[t^{-\alpha} \psi(tL) a \widetilde{\phi}(tL) b\big] \frac{dt}{t},$$

where we have taken the notations of the definition for the initial paraproduct Π (see Definition 4.1). Then, we want to compare this quantity to the main one: $U(f)$. So let us examine the difference. Since for every constant c, we have

$$cf = \Pi_c(f) = L^\alpha \Pi_c\big(L^{-\alpha} f\big) = \widetilde{\Pi}_c(f),$$

it comes

$$U(f)(x) = \sum_{i=1}^{\kappa} \widetilde{\Pi}_{[\partial_{u_{i+1}} F](f(x),Xf(x))}\big(L^\alpha X_i f\big)(x),$$

hence

$$T_F(f)(x) - U(f)(x) = \sum_{i=1}^{\kappa} \widetilde{\Pi}_{\lambda_{i,x}}\big(X_i L^\alpha f\big)(x)$$

with $\lambda_{i,x}(\cdot) = [\partial_{u_{i+1}} F](f, Xf) - [\partial_{u_{i+1}} F](f(x), Xf(x))$. It remains to check that for each integer i, the function $x \to \widetilde{\Pi}_{\lambda_{i,x}}(X_i(1+L)^\alpha f)(x)$ belongs to L^p. Let us recall that

$$\widetilde{\Pi}_{\lambda_{i,x}}\big(X_i(1+L)^\alpha f\big)(x) = -\int_0^\infty (tL)^\alpha \widetilde{\psi}(tL)\big[(tL)^{1-\alpha}\widetilde{\phi}(tL)X_i L^\alpha f \widetilde{\phi}(tL)\lambda_{i,x}\big](x)\frac{dt}{t} - \int_0^\infty (tL)^\alpha \widetilde{\phi}(tL)\big[t^{-\alpha}\psi(tL)X_i L^\alpha f \widetilde{\phi}(tL)\lambda_{i,x}\big](x)\frac{dt}{t}.$$

Let us study only the first term I (the second one being similar):

$$\begin{aligned} I &:= \left|\int_0^\infty (tL)^\alpha \widetilde{\psi}(tL)\big[(tL)^{1-\alpha}\widetilde{\phi}(tL)X_i L^\alpha f \widetilde{\phi}(tL)\lambda_{i,x}\big](x)\frac{dt}{t}\right| \\ &\lesssim \int_0^\infty \int_M \frac{1}{\mu(B(x,t^{-1/2}))}\left(1+\frac{d(x,y)}{t^{-1/2}}\right)^{-d-\delta} \big|(tL)^{1-\alpha}\widetilde{\phi}(tL)X_i L^\alpha f(y)\big|\big|\widetilde{\phi}(tL)\lambda_{i,x}(y)\big|\frac{d\mu(y)\,dt}{t} \\ &\lesssim \int_0^\infty \sum_{j\geqslant 0} 2^{-j\delta} \fint_{C(x,2^j t^{-1/2})} \big|(tL)^{1-\alpha}\widetilde{\phi}(tL)X_i L^\alpha f(y)\big|\big|\widetilde{\phi}(tL)\lambda_{i,x}(y)\big|\frac{d\mu(y)\,dt}{t}. \end{aligned}$$

where we set $C(x,2^j t^{-1/2}) = B(x,2^{j+1}t^{-1/2}) \setminus B(x,2^j t^{-1/2})$ and by convention $|C(x,2^j t^{-1/2})| = |C(x,2^j t^{-1/2})|$. Now, for $y \in B(x,2^j t^{-1/2})$, we have

$$\begin{aligned} \big|\widetilde{\phi}(tL)\lambda_{i,x}(y)\big| &\lesssim \frac{1}{\mu(B(y,t^{1/2}))}\int\left(1+\frac{d(y,z)}{t^{1/2}}\right)^{-d-\delta}\big|[\partial_{u_{i+1}}F](f,Xf)(z) - [\partial_{u_{i+1}}F](f,Xf)(x)\big|\,d\mu(z) \\ &\lesssim \sum_{k\geqslant 0} 2^{jd-\delta k} \fint_{C(y,2^{k+j}t^{1/2})} \big|H(z)-H(x)\big|\,d\mu(z) \\ &\lesssim \sum_{k\geqslant 0} 2^{jd-\delta k} \fint_{\widetilde{C}(x,2^{k+j}t^{1/2})} \big|H(z)-H(x)\big|\,d\mu(z) \end{aligned}$$

with $H := [\partial_{u_{i+1}}F](f,Xf)$ and $\widetilde{C}$ another systems of coronas. So we get

$$\begin{aligned} I &\lesssim \sum_{k,j\geqslant 0} 2^{-k\delta+j(d-\delta)} \int_0^\infty \left(\fint_{B(x,2^j t^{-1/2})} \big|(tL)^{1-\alpha}\widetilde{\phi}(tL)X_i L^\alpha f(y)\big|\,d\mu(y)\right) \\ &\quad \times \left(\fint_{B(x,2^{k+j}t^{1/2})} \big|H(z)-H(x)\big|\,d\mu(z)\right)\frac{dt}{t} \\ &\lesssim \sum_{k,j\geqslant 0} 2^{-k\delta+j(d-\delta)} \int_0^\infty \mathcal{M}\big[t^{1/2}\big|(tL)^{1-\alpha}\widetilde{\phi}(tL)X_i L^\alpha f\big|\big](x)\left(t^{-1/2}\fint_{B(x,2^{k+j}t^{1/2})} \big|H(z)-H(x)\big|\,d\mu(z)\right)\frac{dt}{t}. \end{aligned}$$

Using Cauchy–Schwarz inequality, we also have

$$\begin{aligned} I &\lesssim \sum_{k,j\geqslant 0} 2^{-k\delta+j(d-\delta)} \left(\int_0^\infty \mathcal{M}\big[t^{1/2}(tL)^{1-\alpha}\widetilde{\phi}(tL)X_i L^\alpha f\big](x)^2\frac{dt}{t}\right)^{1/2} \\ &\quad \times \left(\int_0^\infty t^{-1}\left(\fint_{B(x,2^{k+j}t^{1/2})} \big|H(z)-H(x)\big|\,d\mu(z)\right)^2\frac{dt}{t}\right)^{1/2} \end{aligned}$$

$$\lesssim \sum_{k,j\geqslant 0} 2^{-k(\delta-1)-j(d-\delta-1)} \left(\int_0^\infty \mathcal{M}\big[t^{1/2}(tL)^{1-\alpha}\widetilde{\phi}(tL)X_i L^\alpha f\big](x)^2 \frac{dt}{t} \right)^{1/2}$$

$$\times \left(\int_0^\infty t^{-1} \left(\fint_{B(x,t^{1/2})} \big|H(z)-H(x)\big|\,d\mu(z) \right)^2 \frac{dt}{t} \right)^{1/2}$$

$$\lesssim \left(\int_0^\infty \mathcal{M}\big[t^{1/2}(tL)^{1-\alpha}\widetilde{\phi}(tL)X_i (1+L)^\alpha f\big](x)^2 \frac{dt}{t} \right)^{1/2}$$

$$\times \left(\int_0^\infty t^{-1} \Big(\fint_{B(x,t^{1/2})} \big|H(z)-H(x)\big|\,d\mu(z) \Big)^2 \frac{dt}{t} \right)^{1/2},$$

where we have used a change of variables and $\delta > d+1$. So using exponents $q, r > p$ (later chosen) such that $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$, boundedness of the quadratic functional on the one hand and on the other hand Fefferman–Stein inequality for the maximal operator, it comes

$$\|I\|_{L^p} \lesssim \|f\|_{W^{2\alpha,q}} \left\| \left(\int_0^\infty t^{-1} \left(\fint_{B(x,t^{1/2})} \big|H(z)-H(x)\big|\,d\mu(z) \right)^2 \frac{dt}{t} \right)^{1/2} \right\|_{L^r}.$$

Then, using the characterization of Sobolev norms (using this functional, see Proposition 3.6), we conclude to

$$\|I\|_{L^p} \lesssim \|f\|_{W^{2\alpha,q}} \|H\|_{W^{1,r}}.$$

We also chose exponents q, r such that

$$\frac{1}{q} - \frac{2\alpha}{d} > \frac{1}{p} - \frac{s+1}{d} \quad \text{and} \quad \frac{1}{r} - \frac{1}{d} > \frac{1}{p} - \frac{s}{d},$$

which is possible since $\frac{1}{p} < \frac{s-\rho}{d}$ because of the condition on ρ. Then Sobolev embedding (Proposition 3.4) yields that $W^{s+1,p} \hookrightarrow W^{2\alpha,q}$ and $W^{s,p} \hookrightarrow W^{1,r}$. Finally, the proof is also concluded since we obtain

$$\|I\|_{L^p} \lesssim \|f\|_{W^{s+1,p}} \|H\|_{W^{s,p}},$$

which is bounded by $f \in W^{s+1,p}$ (due to $H := [\partial_{u_{i+1}} F](f, Xf)$ with Proposition 3.9). □

We let the reader write the analogous results for higher-order nonlinear PDEs.

Remark 6.2. Let us suppose that the geometry of the manifold allows us to use the following property: For $\alpha > 0$, the commutators $[X_i, (1+L)^\alpha]$ are operators of order 2α, which means that for all $p \in (1, \infty)$ and $s > 0$, $[X_i, (1+L)^\alpha]$ is bounded from $W^{s+2\alpha,p}$ to $W^{s,p}$.

This property holds as soon as we can define a suitable pseudo-differential calculus with symbolic rules: in particular, this is the case of H-type Lie groups, using a notion of Fourier transforms based on irreductible representations, see [4,20].

Under this property, we can commute the vector field X with any power of the Laplacian and so with the same statement than in the previous theorem, we obtain that

$$\Gamma(1-L)^{\frac{s+\rho}{2}} f \in L^p.$$

This new formulation better describes the fact that $f \in W^{s+\rho+1,p}$ along the vector field Γ.

7. Extension of results to Besov spaces

In this section, we aim to explain how the previous results can be extended to Besov spaces (instead of Sobolev spaces).

Definition 7.1. Let $p, l \in (1, \infty)$ and $s \geqslant 0$ then we define the Besov space $B^s_{p,l}$ by its norm:

$$\|f\|_{B^s_{p,l}} := \|e^{-L}(f)\|_{L^p} + \left(\int_0^1 \|t^{-\frac{s}{2}}(tL)^N e^{-tL} f\|^l_{L^p} \frac{dt}{t}\right)^{\frac{1}{l}},$$

where the definition does not depend on the parameter N as soon as $2N > s$ (see [19, Proposition 12]). More precisely, the Besov space is defined as the completion of the space $\mathcal{S}$ relatively to this norm.

We refer the reader to a recent work [9], where the authors present a general study of Besov spaces associated to a semigroup of operator.

Using duality, we then have the following lemma:

Lemma 7.2. *Consider a sequence of functions $(f_u)_{u\in(0,1)}$ and formally define $F := \int_0^1 (uL)^N e^{-uL}(f_u)\frac{du}{u}$ then for $s \in (0, N)$, we have*

$$\|F\|_{B^s_{p,l}} \lesssim \left(\int_0^1 \|u^{-\frac{s}{2}} f_u\|^l_{L^p} \frac{du}{u}\right)^{\frac{1}{l}},$$

as soon as the right-hand side is finite.

Then, we let the reader check that the results can be proved with these spaces, using this technical lemma. It is even more easy, since we have not to deal with quadratic functionals and just estimate each term appearing in the spectral decomposition. We also have

Theorem 7.3 *(Boundedness in Besov spaces). For $p, q, r, l \in (1, \infty)$ with $\frac{1}{r} := \frac{1}{p} + \frac{1}{q}$ and $s \in (0, 1)$ then*

$$(f, g) \to \Pi_g(f)$$

is bounded from $B^s_{p,l} \times L^q$ into $B^s_{p,l}$.

Theorem 7.4. *Consider $s \in (d/p, 1)$ and $f \in B^{s+\epsilon}_{p,l}$ for some $\epsilon > 0$. Then for every smooth function $F \in C^\infty(\mathbb{R})$ with $F(0) = 0$,*

$$F(f) = \Pi_{F'(f)}(f) + w$$

with $w \in B^{2s-d/p}_{p,l}$.

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