Available online at www.sciencedirect.com

SciVerse ScienceDirect

Ann. I. H. Poincaré – AN 30 (2013) 917–934

ANNALES DE L'INSTITUT HENRI POINCARÉ ANALYSE NON LINÉAIRE

www.elsevier.com/locate/anihpc

Improved interpolation inequalities, relative entropy and fast diffusion equations

Jean Dolbeault [a,*], Giuseppe Toscani [b]

[a] *Ceremade (UMR CNRS no. 7534), Université Paris-Dauphine, Place de Lattre de Tassigny, F-75775 Paris Cédex 16, France*
[b] *University of Pavia Department of Mathematics, Via Ferrata 1, 27100 Pavia, Italy*

Received 4 December 2012; received in revised form 21 December 2012; accepted 22 December 2012

Available online 11 January 2013

Abstract

We consider a family of Gagliardo–Nirenberg–Sobolev interpolation inequalities which interpolate between Sobolev's inequality and the logarithmic Sobolev inequality, with optimal constants. The difference of the two terms in the interpolation inequalities (written with optimal constant) measures a distance to the manifold of the optimal functions. We give an explicit estimate of the remainder term and establish an improved inequality, with explicit norms and fully detailed constants. Our approach is based on nonlinear evolution equations and improved entropy–entropy production estimates along the associated flow. Optimizing a relative entropy functional with respect to a scaling parameter, or handling properly second moment estimates, turns out to be the central technical issue. This is a new method in the theory of nonlinear evolution equations, which can be interpreted as the best fit of the solution in the asymptotic regime among all asymptotic profiles.

MSC: 26D10; 46E35; 35K55

Keywords: Gagliardo–Nirenberg–Sobolev inequalities; Improved inequalities; Manifold of optimal functions; Entropy–entropy production method; Fast diffusion equation; Barenblatt solutions; Second moment; Intermediate asymptotics; Sharp rates; Optimal constants

1. Introduction and main results

Consider the following sub-family of the Gagliardo–Nirenberg–Sobolev inequalities

$$\|f\|_{2p} \leqslant \mathcal{C}_{p,d}^{\mathrm{GN}} \|\nabla f\|_2^\theta \|f\|_{p+1}^{1-\theta} \tag{1}$$

with $\theta = \theta(p) := \frac{p-1}{p} \frac{d}{d+2-p(d-2)}$, $1 < p \leqslant \frac{d}{d-2}$ if $d \geqslant 3$ and $1 < p < \infty$ if $d = 2$. Such an inequality holds for any smooth function f with sufficient decay at infinity and, by density, for any function $f \in \mathrm{L}^{p+1}(\mathbb{R}^d)$ such that ∇f is square integrable. We shall assume that $\mathcal{C}_{p,d}^{\mathrm{GN}}$ is the best possible constant in (1). In [16], it has been established that equality holds in (1) if $f = F_p$ with

$$F_p(x) = \left(1 + |x|^2\right)^{-\frac{1}{p-1}} \quad \forall x \in \mathbb{R}^d \tag{2}$$

* Corresponding author.
E-mail addresses: dolbeaul@ceremade.dauphine.fr (J. Dolbeault), giuseppe.toscani@unipv.it (G. Toscani).

0294-1449/$ – see front matter
http://dx.doi.org/10.1016/j.anihpc.2012.12.004

and that all extremal functions are equal to F_p up to a multiplication by a constant, a translation and a scaling. See Appendix A for an expression of $\mathcal{C}^{\mathrm{GN}}_{p,d}$. If $d \geqslant 3$, the limit case $p = d/(d-2)$ corresponds to Sobolev's inequality and one recovers the optimal functions found by T. Aubin and G. Talenti in [3,23]. When $p \to 1$, the inequality becomes an equality, so that we may differentiate both sides with respect to p and recover the euclidean logarithmic Sobolev inequality in optimal scale invariant form (see [20,25,16] for details).

It is rather straightforward to observe that inequality (1) can be rewritten, in a non-scale invariant form, as a *non-homogeneous Gagliardo–Nirenberg–Sobolev inequality*: for any $f \in \mathrm{L}^{p+1} \cap \mathcal{D}^{1,2}(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} |\nabla f|^2\,dx + \int_{\mathbb{R}^d} |f|^{p+1}\,dx \geqslant \mathsf{K}_{p,d}\left(\int_{\mathbb{R}^d} |f|^{2p}\,dx\right)^{\gamma} \tag{3}$$

with

$$\gamma = \gamma(p,d) := \frac{d+2-p(d-2)}{d-p(d-4)}. \tag{4}$$

The optimal constant $\mathsf{K}_{p,d}$ can easily be related with $\mathcal{C}^{\mathrm{GN}}_{p,d}$. Indeed, by optimizing the left hand side of (3) written for $f_\lambda(x) := \lambda^{d/(2p)} f(\lambda x)$ for any $x \in \mathbb{R}^d$, with respect to $\lambda > 0$, one recovers that (3) and (1) are equivalent. The detailed relation between $\mathsf{K}_{p,d}$ and $\mathcal{C}^{\mathrm{GN}}_{p,d}$ can be found in Section 7.

Define now

$$C_M := \left(\frac{M_*}{M}\right)^{\frac{2(p-1)}{d-p(d-4)}}, \quad M_* := \int_{\mathbb{R}^d} \left(1+|x|^2\right)^{-\frac{2p}{p-1}}\,dx = \pi^{\frac{d}{2}} \frac{\Gamma\left(\frac{d-p(d-4)}{2(p-1)}\right)}{\Gamma\left(\frac{2p}{p-1}\right)}.$$

Consider next a generic, non-negative optimal function,

$$f^{(p)}_{M,y,\sigma}(x) := \sigma^{-\frac{d}{4p}}\left(C_M + \frac{1}{\sigma}|x-y|^2\right)^{-\frac{1}{p-1}} \quad \forall x \in \mathbb{R}^d$$

and let us define the manifold of the optimal functions as

$$\mathfrak{M}^{(p)}_d := \left\{f^{(p)}_{M,y,\sigma}\colon (M,y,\sigma) \in \mathcal{M}_d\right\}.$$

We shall measure the distance to $\mathfrak{M}^{(p)}_d$ with the functional

$$\mathcal{R}^{(p)}[f] := \inf_{g \in \mathfrak{M}^{(p)}_d} \int_{\mathbb{R}^d} \left[g^{1-p}\left(|f|^{2p} - g^{2p}\right) - \frac{2p}{p+1}\left(|f|^{p+1} - g^{p+1}\right)\right] dx.$$

To simplify our statement, we will introduce a normalization constraint and assume that $f \in \mathrm{L}^{2p}(\mathbb{R}^2, (1+|x|^2)\,dx)$ is such that

$$\frac{\int_{\mathbb{R}^d} |x|^2 |f|^{2p}\,dx}{(\int_{\mathbb{R}^d} |f|^{2p}\,dx)^{\gamma}} = \frac{d(p-1)\sigma_* M_*^{\gamma-1}}{d+2-p(d-2)}, \quad \sigma_*(p) := \left(4\frac{d+2-p(d-2)}{(p-1)^2(p+1)}\right)^{\frac{4p}{d-p(d-4)}}. \tag{5}$$

Such a condition is not restrictive, as it is always possible to cover the general case by rescaling the inequality, but significantly simplifies the expressions. As we shall see in the proof, the only goal is to fix $\sigma = 1$.

Our main result goes as follows.

Theorem 1. *Let $d \geqslant 2$, $p > 1$ and assume that $p < d/(d-2)$ if $d \geqslant 3$. For any $f \in \mathrm{L}^{p+1} \cap \mathcal{D}^{1,2}(\mathbb{R}^d)$ such that condition (5) holds, we have*

$$\int_{\mathbb{R}^d} |\nabla f|^2\,dx + \int_{\mathbb{R}^d} |f|^{p+1}\,dx - \mathsf{K}_{p,d}\left(\int_{\mathbb{R}^d} |f|^{2p}\,dx\right)^{\gamma} \geqslant \mathsf{C}_{p,d} \frac{(\mathcal{R}^{(p)}[f])^2}{(\int_{\mathbb{R}^d} |f|^{2p}\,dx)^{\gamma}}$$

where γ is given by (4).

Although we do not know its optimal value, we are able to give an expression of $C_{p;d}$ (see Appendix A), which is such that

$$\lim_{p\to 1_+} \mathsf{C}_{p,d} = 0 \quad \text{and} \quad \lim_{p\to d/(d-2)_-} \mathsf{C}_{p,d} = 0.$$

The space $\mathrm{L}^{p+1} \cap \mathcal{D}^{1,2}(\mathbb{R}^d)$ is the natural space for Gagliardo–Nirenberg inequalities as it can be characterized as the completion of the space of smooth functions with compact support with respect to the norm $\|\cdot\|$ such that $\|f\|^2 = \|\nabla f\|_2^2 + \|f\|_{p+1}^2$. In this paper, we shall also use the notations $\|f\|_{p,q} := (\int_{\mathbb{R}^d} |x|^p |f|^q\,dx)^{1/q}$, so that $\|f\|_q = \|f\|_{0,q}$.

Under condition (5), we shall deduce from Theorem 4 that

$$\mathcal{R}^{(p)}[f] \geqslant \mathsf{C}_{\mathrm{CK}} \|f\|_{2p}^{2p(\gamma-2)} \inf_{g\in\mathfrak{M}_d^{(p)}} \big\||f|^{2p} - g^{2p}\big\|_1^2 \tag{6}$$

with $\delta = d + 2 - p(d+6)$ for some constant C_{CK} whose expression is given in Section 3, Eq. (13). Putting this estimate together with the result of Theorem 1, with

$$\mathfrak{C}_{p,d} := \mathsf{C}_{d,p} {\mathsf{C}_{\mathrm{CK}}}^2,$$

we obtain the following estimate.

Corollary 2. *Under the same assumptions as in Theorem 1, we have*

$$\int_{\mathbb{R}^d} |\nabla f|^2\,dx + \int_{\mathbb{R}^d} |f|^{p+1}\,dx - \mathsf{K}_{p,d}\bigg(\int_{\mathbb{R}^d} |f|^{2p}\,dx\bigg)^{\gamma} \geqslant \mathfrak{C}_{p,d} \|f\|_{2p}^{2p(\gamma-4)} \inf_{g\in\mathfrak{M}_d(p)} \big\||f|^{2p} - g^{2p}\big\|_1^4.$$

The critical case $p = d/(d-2)$ corresponding to Sobolev's inequality raises a number of difficulties which are not under control at this stage. However, results which have been obtained in such a critical case, by different methods, are the main motivation for the present paper.

In [9, Question (c), p. 75], H. Brezis and E. Lieb asked the question of what kind of distance to $\mathfrak{M}_d^{(p)}$ is controlled by the difference of the two terms in the critical Sobolev inequality written with an optimal constant. Some partial answers have been provided over the years, of which we can list the following ones. First G. Bianchi and H. Egnell gave in [5] a result based on the concentration-compactness method, which determines a non-constructive estimate for a distance to the set of optimal functions. In [15], A. Cianchi, N. Fusco, F. Maggi and A. Pratelli established an improved inequality using symmetrization methods. Also see [14] for an overview of various results based on such methods. Recently another type of improvement, which relates Sobolev's inequality to the Hardy–Littlewood–Sobolev inequalities, has been established in [17], based on the flow of a nonlinear diffusion equation, in the regime of extinction in finite time. Theorem 1 does not provide an answer in the critical case, but gives an improvement with fully explicit constants in the subcritical regime. Our method of proof enlightens a new aspect of the problem. Indeed, Theorem 1 shows that the difference of the two terms in the critical Sobolev inequality provides a better control under the additional information that $\|f\|_{2,2p}$ is finite. Such a condition disappears in the setting of Corollary 2.

In this paper, our goal is to establish an improvement of Gagliardo–Nirenberg inequalities based on the flow of the *fast diffusion equation* in the regime of convergence towards Barenblatt self-similar profiles, with an explicit measure of the distance to the set of optimal functions. Our approach is based on a *relative entropy* functional. The method relies on a recent paper [19], which is itself based on a long series of studies on intermediate asymptotics of the fast diffusion equation, and on the entropy–entropy production method introduced in [4,2] in the linear case and later extended to nonlinear diffusions: see [21,22,16,12,11]. In this setting, having a finite second moment is crucial. Let us give some explanations.

Consider the fast diffusion equation with exponent m given in terms of the exponent p of Theorem 1 by

$$p = \frac{1}{2m-1} \quad \Longleftrightarrow \quad m = \frac{p+1}{2p}. \tag{7}$$

More specifically, for $m \in (0,1)$, we shall consider the solutions of

$$\frac{\partial u}{\partial t} + \nabla\cdot\big[u\big(\eta\nabla u^{m-1} - 2x\big)\big] = 0 \quad t>0,\ x\in\mathbb{R}^d \tag{8}$$

with initial datum $u(t=0,\cdot)=u_0$. Here η is a positive parameter which does not depend on t. Let u_∞ be the unique stationary solution such that $M=\int_{\mathbb{R}^d} u\,dx=\int_{\mathbb{R}^d} u_\infty\,dx$. It is given by

$$u_\infty(x)=\left(K+\frac{1}{\eta}|x|^2\right)^{\frac{1}{m-1}}\quad \forall x\in\mathbb{R}^d$$

for some positive constant K which is uniquely determined by M. The following exponents are associated with the fast diffusion equation (8) and will be used all over this paper:

$$m_c:=\frac{d-2}{d},\qquad m_1:=\frac{d-1}{d}\quad\text{and}\quad \tilde{m}_1:=\frac{d}{d+2}.$$

To the critical exponent $2p=2d/(d-2)$ for Sobolev's inequality corresponds the critical exponent m_1 for the fast diffusion equation. For $d\geqslant 3$, the condition $p\in(1,d/(d-2))$ in Theorem 1 is equivalent to $m\in(m_1,1)$ while for $d=2$, $p\in(1,\infty)$ means $m\in(1/2,1)$.

It has been established in [21,22] that the *relative entropy* (or *free energy*)

$$\mathcal{F}[u|u_\infty]:=\frac{1}{m-1}\int_{\mathbb{R}^d}\left[u^m-u_\infty^m-m u_\infty^{m-1}(u-u_\infty)\right]dx$$

decays according to

$$\frac{d}{dt}\mathcal{F}\big[u(\cdot,t)|u_\infty\big]=-\mathcal{I}\big[u(\cdot,t)|u_\infty\big]$$

if u is a solution of (8), where

$$\mathcal{I}\big[u(\cdot,t)|u_\infty\big]:=\eta\frac{m}{1-m}\int_{\mathbb{R}^d} u\left|\nabla u^{m-1}-\nabla u_\infty^{m-1}\right|^2 dx$$

is the *entropy production term* or *relative Fisher information*. If $m\in[m_1,1)$, according to [16], these two functionals are related by a Gagliardo–Nirenberg interpolation inequality, namely

$$\mathcal{F}[u|u_\infty]\leqslant\frac{1}{4}\mathcal{I}[u|u_\infty].\tag{9}$$

We shall give a concise proof of this inequality in the next section (see Remark 1) based on the entropy–entropy production method, which amounts to relate $\frac{d}{dt}\mathcal{I}[u(\cdot,t)|u_\infty]$ and $\mathcal{I}[u(\cdot,t)|u_\infty]$. We shall later replace the diffusion parameter η in (8) by a time-dependent coefficient $\sigma(t)$, which is itself computed using the second moment of u, $\int_{\mathbb{R}^d}|x|^2u(x,t)\,dx$. By doing so, we will be able to capture the *best matching* Barenblatt solution and get improved decay rates in the entropy–entropy production inequality. Elementary estimates allow to rephrase these improved rates into improved functional inequalities for f such that $|f|^{2p}=u$, for any $p\in(1,d/(d-2))$, as in Theorem 1.

This paper is organized as follows. In Section 2, we apply the entropy–entropy production method to the fast diffusion equation as in [11]. The key computation, without justifications for the integrations by parts, is reproduced here since we need it later in Section 6, in the case of a time-dependent diffusion coefficient. Next, in Section 3, we establish a new estimate of Csiszár–Kullback type. By requiring a condition on the second moment, we are able to produce a new estimate which was not known before, namely to directly control the difference of the solution with a Barenblatt solution in $L^1(\mathbb{R}^d)$.

Second moment estimates are the key of a recent paper and we shall primarily refer to [19] in which the asymptotic behavior of the solutions of the fast diffusion equation was studied. In Section 4 we recall the main results that were proved in [19], and that are also needed in the present paper.

With these preliminaries in hand, an improved entropy–entropy production inequality is established in Section 5, which is at the core of our paper. It is known since [16] that entropy–entropy production inequalities amount to optimal Gagliardo–Nirenberg–Sobolev inequalities. Such a rephrasing of our result in a more standard form of functional inequalities is done in Section 6, which contains the proof of Theorem 1. Further observations have been collected in Section 7. One of the striking results of our approach is that all constants can be explicitly computed. This is somewhat technical although not really difficult. To make the reading easier, explicit computations have been collected in Appendix A.

2. The entropy–entropy production method

Consider a solution $u = u(x,t)$ of Eq. (8) and define

$$z(x,t) := \eta \nabla u^{m-1} - 2x$$

so that Eq. (8) can be rewritten as

$$\frac{\partial u}{\partial t} + \nabla \cdot (uz) = 0.$$

To keep notations compact, we shall use the following conventions. If $A = (A_{ij})_{i,j=1}^d$ and $B = (B_{ij})_{i,j=1}^d$ are two matrices, let $A : B = \sum_{i,j=1}^d A_{ij} B_{ij}$ and $|A|^2 = A : A$. If a and b take values in $\mathbb{R}^d$, we adopt the definitions:

$$a \cdot b = \sum_{i=1}^d a_i b_i, \qquad \nabla \cdot a = \sum_{i=1}^d \frac{\partial a_i}{\partial x_i}, \qquad a \otimes b = (a_i b_j)_{i,j=1}^d, \qquad \nabla \otimes a = \left(\frac{\partial a_j}{\partial x_i}\right)_{i,j=1}^d.$$

Later we will need a version of the entropy–entropy production method in case of a time-dependent diffusion coefficient. Before doing so, let us recall the key computation of the standard method. With the above notations, it is straightforward to check that

$$\frac{\partial z}{\partial t} = \eta(1-m)\nabla\big(u^{m-2}\nabla \cdot (uz)\big) \quad \text{and} \quad \nabla \otimes z = \eta \nabla \otimes \nabla u^{m-1} - 2\,\mathrm{Id}.$$

With these definitions, the time-derivative of $\frac{1-m}{m}\eta \mathcal{I}[u|u_\infty] = \int_{\mathbb{R}^d} u|z|^2\,dx$ can be computed as

$$\frac{d}{dt}\int_{\mathbb{R}^d} u|z|^2\,dx = \int_{\mathbb{R}^d} \frac{\partial u}{\partial t}|z|^2\,dx + 2\int_{\mathbb{R}^d} uz \cdot \frac{\partial z}{\partial t}\,dx.$$

The first term can be evaluated by

$$\begin{aligned}
\int_{\mathbb{R}^d} \frac{\partial u}{\partial t}|z|^2\,dx &= -\int_{\mathbb{R}^d} \nabla \cdot (uz)|z|^2\,dx \\
&= 2\int_{\mathbb{R}^d} uz \otimes z : \nabla \otimes z\,dx \\
&= 2\eta \int_{\mathbb{R}^d} uz \otimes z : \nabla \otimes \nabla u^{m-1}\,dx - 4\int_{\mathbb{R}^d} u|z|^2\,dx \\
&= 2\eta(1-m)\int_{\mathbb{R}^d} u^{m-2}\nabla u \otimes \nabla : (uz \otimes z)\,dx - 4\int_{\mathbb{R}^d} u|z|^2\,dx \\
&= 2\eta(1-m)\int_{\mathbb{R}^d} u^{m-2}(\nabla u \cdot z)^2\,dx + 2\eta(1-m)\int_{\mathbb{R}^d} u^{m-1}(\nabla u \cdot z)(\nabla \cdot z)\,dx \\
&\quad + 2\eta(1-m)\int_{\mathbb{R}^d} u^{m-1}(z \otimes \nabla u) : (\nabla \otimes z)\,dx - 4\int_{\mathbb{R}^d} u|z|^2\,dx.
\end{aligned}$$

The second term can be evaluated by

$$\begin{aligned}
2\int_{\mathbb{R}^d} uz \cdot \frac{\partial z}{\partial t}\,dx &= 2\eta(1-m)\int_{\mathbb{R}^d} (uz \cdot \nabla)\big(u^{m-2}\nabla \cdot (uz)\big)\,dx \\
&= -2\eta(1-m)\int_{\mathbb{R}^d} u^{m-2}\big(\nabla \cdot (uz)\big)^2\,dx
\end{aligned}$$

$$= -2\eta(1-m)\int_{\mathbb{R}^d}\left[u^m(\nabla\cdot z)^2+2u^{m-1}(\nabla u\cdot z)(\nabla\cdot z)+u^{m-2}(\nabla u\cdot z)^2\right]dx.$$

Summarizing, we have found that

$$\int_{\mathbb{R}^d}\frac{\partial u}{\partial t}|z|^2\,dx+4\int_{\mathbb{R}^d}u|z|^2\,dx=-2\eta(1-m)\int_{\mathbb{R}^d}u^{m-2}\left[u^2(\nabla\cdot z)^2+u(\nabla u\cdot z)(\nabla\cdot z)-u(z\otimes\nabla u):(\nabla\otimes z)\right]dx.$$

Using the fact that

$$\frac{\partial^2 z^j}{\partial x_i\partial x_j}=\frac{\partial^2 z^i}{\partial x_j^2},$$

we obtain that

$$\int_{\mathbb{R}^d}u^{m-1}(\nabla u\cdot z)(\nabla\cdot z)\,dx=-\frac{1}{m}\int_{\mathbb{R}^d}u^m(\nabla\cdot z)^2\,dx-\frac{1}{m}\int_{\mathbb{R}^d}u^m\sum_{i,j=1}^d z^i\frac{\partial^2 z^j}{\partial x_i\partial x_j}\,dx$$

and

$$-\int_{\mathbb{R}^d}u^{m-1}(z\otimes\nabla u):(\nabla\otimes z)\,dx=\frac{1}{m}\int_{\mathbb{R}^d}u^m|\nabla z|^2\,dx+\frac{1}{m}\int_{\mathbb{R}^d}u^m\sum_{i,j=1}^d z^i\frac{\partial^2 z^i}{\partial x_j^2}\,dx$$

can be combined to give

$$\int_{\mathbb{R}^d}u^{m-2}\left[u(\nabla u\cdot z)(\nabla\cdot z)-u\nabla u\otimes z:\nabla\otimes z\right]dx=-\frac{1}{m}\int_{\mathbb{R}^d}u^m(\nabla\cdot z)^2\,dx+\frac{1}{m}\int_{\mathbb{R}^d}u^m|\nabla z|^2\,dx.$$

This shows that

$$\frac{d}{dt}\int_{\mathbb{R}^d}u|z|^2\,dx+4\int_{\mathbb{R}^d}u|z|^2\,dx=-2\eta\frac{1-m}{m}\int_{\mathbb{R}^d}u^m\left(|\nabla z|^2-(1-m)(\nabla\cdot z)^2\right)dx. \tag{10}$$

By the arithmetic geometric inequality, we know that

$$|\nabla z|^2-(1-m)(\nabla\cdot z)^2\geqslant 0$$

if $1-m\leqslant 1/d$, that is, if $m\geqslant m_1$. Altogether, we have formally established the following result.

Proposition 3. *Let $d\geqslant 1$, $m\in(m_1,1)$ and assume that u is a non-negative solution of (8) with initial datum u_0 in $\mathrm{L}^1(\mathbb{R}^d)$ such that u_0^m and $x\mapsto|x|^2u_0$ are both integrable on $\mathbb{R}^d$. With the above defined notations, we get that*

$$\frac{d}{dt}\mathcal{I}\left[u(\cdot,t)|u_\infty\right]\leqslant -4\mathcal{I}\left[u(\cdot,t)|u_\infty\right]\quad\forall t>0.$$

The proof of such a result requires to justify that all integrations by parts make sense. We refer to [12,13] for a proof in the porous medium case ($m>1$) and to [11] for $m_1\leqslant m<1$. The case $m=1$ was covered long ago in [4].

Remark 1. Proposition 3 provides a proof of (9). Indeed, with a Gronwall estimate, we first get that

$$\mathcal{I}\left[u(\cdot,t)|u_\infty\right]\leqslant\mathcal{I}[u_0|u_\infty]e^{-4t}\quad\forall t\geqslant 0$$

if $\mathcal{I}[u_0|u_\infty]$ is finite. Since $\mathcal{I}[u(\cdot,t)|u_\infty]$ is non-negative, we know that

$$\lim_{t\to\infty}\mathcal{I}\left[u(\cdot,t)|u_\infty\right]=0,$$

which proves the convergence of $u(\cdot,t)$ to u_∞ as $t\to\infty$. As a consequence, we also have $\lim_{t\to\infty}\mathcal{F}[u(\cdot,t)|u_\infty]=0$ and since

$$\frac{d}{dt}\left(\mathcal{I}\left[u(\cdot,t)|u_\infty\right]-4\mathcal{F}\left[u(\cdot,t)|u_\infty\right]\right)=\frac{d}{dt}\mathcal{I}\left[u(\cdot,t)|u_\infty\right]+4\mathcal{I}\left[u(\cdot,t)|u_\infty\right]\leqslant 0,$$

an integration with respect to t on $(0,\infty)$ shows that

$$\mathcal{I}[u_0|u_\infty]-4\mathcal{F}[u_0|u_\infty]\geqslant 0,$$

which is precisely (9) written for $u=u_0$.

3. A Csiszár–Kullback inequality

Let $m\in(\tilde m_1,1)$ with $\tilde m_1=\frac{d}{d+2}$ and consider the relative entropy

$$\mathcal{F}_\sigma[u]:=\frac{1}{m-1}\int_{\mathbb{R}^d}\left[u^m-B_\sigma^m-m\,B_\sigma^{m-1}(u-B_\sigma)\right]dx$$

for some Barenblatt function

$$B_\sigma(x):=\sigma^{-\frac{d}{2}}\left(C_M+\frac{1}{\sigma}|x|^2\right)^{\frac{1}{m-1}}\quad\forall x\in\mathbb{R}^d \tag{11}$$

where σ is a positive constant and C_M is chosen such that $\|B_\sigma\|_1=M>0$. With p and m related by (7), the definition of C_M coincides with the one of Section 1. See details in Appendix A.

Theorem 4. *Let $d\geqslant 1$, $m\in(\tilde m_1,1)$ and assume that u is a non-negative function in $\mathrm{L}^1(\mathbb{R}^d)$ such that u^m and $x\mapsto|x|^2u$ are both integrable on $\mathbb{R}^d$. If $\|u\|_1=M$ and $\int_{\mathbb{R}^d}|x|^2u\,dx=\int_{\mathbb{R}^d}|x|^2B_\sigma\,dx$, then*

$$\frac{\mathcal{F}_\sigma[u]}{\sigma^{\frac{d}{2}(1-m)}}\geqslant\frac{m}{8\int_{\mathbb{R}^d}B_1^m\,dx}\left(C_M\|u-B_\sigma\|_1+\frac{1}{\sigma}\int_{\mathbb{R}^d}|x|^2|u-B_\sigma|\,dx\right)^2.$$

Notice that the condition $\int_{\mathbb{R}^d}|x|^2u\,dx=\int_{\mathbb{R}^d}|x|^2B_\sigma\,dx$ is explicit and determines σ uniquely:

$$\sigma=\frac{1}{K_M}\int_{\mathbb{R}^d}|x|^2u\,dx\quad\text{with } K_M:=\int_{\mathbb{R}^d}|x|^2B_1\,dx.$$

For further details, see Lemma 5 and (20) below, and Appendix A for detailed expressions of K_M and $\int_{\mathbb{R}^d}B_1^m\,dx$. With this choice of σ, since $B_\sigma^{m-1}=\sigma^{\frac{d}{2}(1-m)}C_M+\sigma^{\frac{d}{2}(m_c-m)}|x|^2$, we remark that $\int_{\mathbb{R}^d}B_\sigma^{m-1}(u-B_\sigma)\,dx=0$ so that the relative entropy reduces to

$$\mathcal{F}_\sigma[u]:=\frac{1}{m-1}\int_{\mathbb{R}^d}\left[u^m-B_\sigma^m\right]dx.$$

Proof of Theorem 4. Let $v:=u/B_\sigma$ and $d\mu_\sigma:=B_\sigma^m\,dx$. With these notations, we observe that

$$\begin{aligned}\int_{\mathbb{R}^d}(v-1)\,d\mu_\sigma&=\int_{\mathbb{R}^d}B_\sigma^{m-1}(u-B_\sigma)\,dx\\&=\sigma^{\frac{d}{2}(1-m)}C_M\int_{\mathbb{R}^d}(u-B_\sigma)\,dx+\sigma^{\frac{d}{2}(m_c-m)}\int_{\mathbb{R}^d}|x|^2(u-B_\sigma)\,dx=0.\end{aligned}$$

Thus

$$\int_{\mathbb{R}^d}(v-1)\,d\mu_\sigma=\int_{v>1}(v-1)\,d\mu_\sigma-\int_{v<1}(1-v)\,d\mu_\sigma=0,$$

which, coupled with

$$\int_{v>1}(v-1)\,d\mu_\sigma+\int_{v<1}(1-v)\,d\mu_\sigma=\int_{\mathbb{R}^d}|v-1|\,d\mu_\sigma,$$

implies

$$\int_{\mathbb{R}^d} |u - B_\sigma| B_\sigma^{m-1}\, dx = \int_{\mathbb{R}^d} |v-1|\, d\mu_\sigma = 2 \int_{v<1} |v-1|\, d\mu_\sigma .$$

On the other hand, a Taylor expansion shows that

$$\mathcal{F}_\sigma[u] = \frac{1}{m-1} \int_{\mathbb{R}^d} \left[v^m - 1 - m(v-1) \right] d\mu_\sigma = \frac{m}{2} \int_{\mathbb{R}^d} \xi^{m-2} |v-1|^2\, d\mu_\sigma$$

for some function ξ taking values in the interval $(\min\{1,v\}, \max\{1,v\})$, thus giving the lower bound

$$\mathcal{F}_\sigma[u] \geqslant \frac{m}{2} \int_{v<1} \xi^{m-2} |v-1|^2\, d\mu_\sigma \geqslant \frac{m}{2} \int_{v<1} |v-1|^2\, d\mu_\sigma .$$

Using the Cauchy–Schwarz inequality, we get

$$\left(\int_{v<1} |v-1|\, d\mu_\sigma \right)^2 = \left(\int_{v<1} |v-1| B_\sigma^{\frac{m}{2}} B_\sigma^{\frac{m}{2}}\, dx \right)^2 \leqslant \int_{v<1} |v-1|^2\, d\mu_\sigma \int_{\mathbb{R}^d} B_\sigma^m\, dx$$

and finally obtain that

$$\mathcal{F}_\sigma[u] \geqslant \frac{m}{2} \frac{(\int_{v<1} |v-1|\, d\mu_\sigma)^2}{\int_{\mathbb{R}^d} B_\sigma^m\, dx} = \frac{m}{8} \frac{(\int_{\mathbb{R}^d} |u - B_\sigma| B_\sigma^{m-1}\, dx)^2}{\int_{\mathbb{R}^d} B_\sigma^m\, dx},$$

which concludes the proof. □

Notice that the inequality of Theorem 4 can be rewritten in terms of $|f|^{2p} = u$ and $g^{2p} = B_\sigma$ with $p = 1/(2m-1)$. See Appendix A for the computation of $\int_{\mathbb{R}^d} B_\sigma^m\, dx, \sigma, C_M$ and K_M in terms of $\int_{\mathbb{R}^d} |x|^2 u\, dx$ and M_*. In the framework of Corollary 2, we observe that condition (5) can be rephrased as

$$\sigma = \frac{1}{K_M} \|f\|_{2,2p}^{2p} = \frac{1}{K_1} \frac{\|f\|_{2,2p}^{2p}}{\|f\|_{2p}^{2p\gamma}} = \sigma_* . \tag{12}$$

Altogether we find in such a case that

$$\mathcal{R}^{(p)}[f] = \frac{p-1}{p+1} \mathcal{F}_{\sigma_*}[u] \geqslant \mathsf{C}_{\mathrm{CK}} \big\| |f|^{2p} - |g|^{2p} \big\|_1^2$$

with

$$\mathsf{C}_{\mathrm{CK}} = \frac{p-1}{p+1} \frac{d+2-p(d-2)}{32p} \sigma_*^{d\frac{p-1}{4p}} M_*^{1-\gamma} . \tag{13}$$

Remark 2. Various other estimates can be derived, based on second order Taylor expansions. For instance, as in [16], we can write that

$$\mathcal{F}_\sigma[u] = \int_{\mathbb{R}^d} \left[\psi\big(v^m\big) - \psi(1) - \psi'(1)\big(v^m - 1\big) \right] d\mu_\sigma$$

with $v := u/B_\sigma$ and $\psi(s) := \frac{m}{1-m} s^{1/m}$, and get

$$\mathcal{F}_\sigma[u] \geqslant \frac{1}{m} 2^{-2m} \frac{\|v^m - 1\|_{\mathrm{L}^{1/m}(\mathbb{R}^d, d\mu_\sigma)}^2}{\max\{\|v^m\|_{\mathrm{L}^{1/m}(\mathbb{R}^d, d\mu_\sigma)}, \|1\|_{\mathrm{L}^{1/m}(\mathbb{R}^d, d\mu_\sigma)}\}^{2-\frac{1}{m}}} .$$

Using $\|v^m\|_{\mathrm{L}^{1/m}(\mathbb{R}^d, d\mu_\sigma)} = \|1\|_{\mathrm{L}^{1/m}(\mathbb{R}^d, d\mu_\sigma)} = \|B_\sigma^m\|_1^m$ and

$$\int_{\mathbb{R}^d} \left|u^m - B_\sigma^m\right| dx = \int_{\mathbb{R}^d} \left|u^m - B_\sigma^m\right| B_\sigma^{m(m-1)} B_\sigma^{m(1-m)}\, dx$$
$$\leqslant \left\|v^m - 1\right\|_{\mathrm{L}^{1/m}(\mathbb{R}^d, d\mu_\sigma)} \left\|B_\sigma^m\right\|_1^{1-m}$$

by the Cauchy–Schwarz inequality, we find

$$\mathcal{F}_\sigma[u] \geqslant \frac{\|u^m - B_\sigma^m\|_1^2}{m 2^{2m} \|B_\sigma^m\|_1}.$$

With $f = u^{m-\frac{1}{2}}$, this also gives another estimate of Csiszár–Kullback type, namely

$$\mathcal{R}^{(p)}[f] \geqslant \frac{\kappa_{p,d}}{\|f\|_{2,2p}^{\frac{d}{2}(p-1)} \|f\|_{2p}^{\frac{1}{2}(d+2-p(d-2))}} \inf_{g \in \mathfrak{M}_d^{(p)}} \left\| |f|^{p+1} - g^{p+1} \right\|_1^2,$$

for some positive constant $\kappa_{p,d}$, which is valid for any $p \in (1, \infty)$ if $d = 2$ and any $p \in (1, \frac{d}{d-2}]$ if $d \geqslant 3$. Also see [24,12,10,18] for further results on Csiszár–Kullback type inequalities corresponding to entropies associated with porous media and fast diffusion equations.

4. Recent results on the optimal matching by Barenblatt solutions

Consider on $\mathbb{R}^d$ the fast diffusion equation with harmonic confining potential given by

$$\frac{\partial u}{\partial t} + \nabla \cdot \left[u\left(\sigma^{\frac{d}{2}(m-m_c)} \nabla u^{m-1} - 2x\right)\right] = 0, \quad t > 0,\ x \in \mathbb{R}^d, \tag{14}$$

with initial datum u_0. Here σ is a function of t. Let us summarize some results obtained in [19] and the strategy of their proofs.

Result 1. At any time $t > 0$, we can choose the *best matching Barenblatt* as follows. Consider a given function u and optimize $\lambda \mapsto \mathcal{F}_\lambda[u]$.

Lemma 5. *For any given $u \in \mathrm{L}^1_+(\mathbb{R}^d)$ such that u^m and $|x|^2 u$ are both integrable, if $m \in (\tilde{m}_1, 1)$, there is a unique $\lambda = \lambda^* > 0$ which minimizes $\lambda \mapsto \mathcal{F}_\lambda[u]$, and it is explicitly given by*

$$\lambda^* = \frac{1}{K_M} \int_{\mathbb{R}^d} |x|^2 u\, dx$$

where $K_M = \int_{\mathbb{R}^d} |x|^2 B_1\, dx$. For $\lambda = \lambda^*$, the Barenblatt profile B_λ satisfies

$$\int_{\mathbb{R}^d} |x|^2 B_\lambda\, dx = \int_{\mathbb{R}^d} |x|^2 u\, dx.$$

As a consequence, we know that

$$\frac{d}{d\lambda}\left(\mathcal{F}_\lambda[u]\right)_{\lambda=\lambda_*} = 0.$$

Of course, if u is a solution of (14), the value of λ in Lemma 5 may depend on t. Now we choose $\sigma(t) = \lambda(t)$, i.e.,

$$\sigma(t) = \frac{1}{K_M} \int_{\mathbb{R}^d} |x|^2 u(x,t)\, dx \quad \forall t \geqslant 0. \tag{15}$$

This makes (14) a non-local equation.

Result 2. With the above choice, if we consider a solution of (14) and compute the time derivative of the relative entropy, we find that

$$\frac{d}{dt}\mathcal{F}_{\sigma(t)}\big[u(\cdot,t)\big]=\sigma'(t)\bigg(\frac{d}{d\sigma}\mathcal{F}_\sigma[u]\bigg)_{|\sigma=\sigma(t)}+\frac{m}{m-1}\int_{\mathbb{R}^d}\big(u^{m-1}-B_{\sigma(t)}^{m-1}\big)\frac{\partial u}{\partial t}\,dx.$$

However, as a consequence of the choice (15) and of Lemma 5, we know that

$$\bigg(\frac{d}{d\sigma}\mathcal{F}_\sigma[u]\bigg)_{|\sigma=\sigma(t)}=0,$$

and finally obtain

$$\frac{d}{dt}\mathcal{F}_{\sigma(t)}\big[u(\cdot,t)\big]=-\frac{m\sigma(t)^{\frac{d}{2}(m-m_c)}}{1-m}\int_{\mathbb{R}^d}u\big|\nabla\big[u^{m-1}-B_{\sigma(t)}^{m-1}\big]\big|^2\,dx. \tag{16}$$

The computation then goes as in [7,8] (also see [21,22,16] for details). With our choice of σ, we gain an additional orthogonality condition which is useful for improving the rates of convergence (see [19, Theorem 1]) in the asymptotic regime $t\to\infty$, compared to the results of [8] (also see below).

Result 3. Now let us state one more result of [19] which is of interest for the present paper.

Lemma 6. *With the above notations, if u and σ are defined respectively by* (14) *and* (15)*, then the function $t\mapsto\sigma(t)$ is positive, decreasing, with $\sigma_\infty:=\lim_{t\to\infty}\sigma(t)>0$ and*

$$\sigma'(t)=-2d\frac{(1-m)^2}{mK_M}\sigma^{\frac{d}{2}(m-m_c)}\mathcal{F}_{\sigma(t)}\big[u(\cdot,t)\big]\leqslant 0. \tag{17}$$

The main difficulty is to establish that σ_∞ is positive. This can be done with an appropriate change of variables which reduces (14) to the case where σ does not depend on t. In [19], a proof has been given, based on asymptotic results for the fast diffusion equation that were established in [16,7,6,8]. An alternative proof will be given in Remark 3, below.

5. The scaled entropy–entropy production inequality

Consider the relative Fisher information

$$\mathcal{I}_\sigma[u]:=\sigma^{\frac{d}{2}(m-m_c)}\frac{m}{1-m}\int_{\mathbb{R}^d}u\big|\nabla u^{m-1}-\nabla B_\sigma^{m-1}\big|^2\,dx.$$

By applying (9) with $u_\infty=B_1$ and $\eta=1$ to $x\mapsto\sigma^{d/2}u(\sqrt{\sigma}x)$ and using the fact that $B_1(x)=\sigma^{d/2}B_\sigma(\sqrt{\sigma}x)$, we get the inequality

$$\mathcal{F}_\sigma[u]\leqslant\frac{1}{4}\mathcal{I}_\sigma[u].$$

Now, if σ is time-dependent as in Section 4, we have the following relations.

Lemma 7. *If u is a solution of* (14) *with $\sigma(t)=\frac{1}{K_M}\int_{\mathbb{R}^d}|x|^2u(x,t)\,dx$, then σ satisfies* (17)*. Moreover, for any $t\geqslant 0$, we have*

$$\frac{d}{dt}\mathcal{F}_{\sigma(t)}\big[u(\cdot,t)\big]=-\mathcal{I}_{\sigma(t)}\big[u(\cdot,t)\big] \tag{18}$$

and

$$\frac{d}{dt}\mathcal{I}_{\sigma(t)}\big[u(\cdot,t)\big]\leqslant-\bigg[4+\frac{1}{2}(m-m_c)(m-m_1)\,d^2\frac{|\sigma'(t)|}{\sigma(t)}\bigg]\mathcal{I}_{\sigma(t)}\big[u(\cdot,t)\big]. \tag{19}$$

Proof. Eqs. (17) and (18) have already been stated respectively in Lemma 6 and in (16). They are recalled here only for the convenience of the reader. It remains to prove (19).

For any given $\sigma = \sigma(t)$, Proposition 3 gives

$$\frac{d}{dt}\mathcal{I}_{\sigma(t)}\big[u(\cdot,t)\big] = \left(\frac{d}{dt}\mathcal{I}_\lambda\big[u(\cdot,t)\big]\right)_{|\lambda=\sigma(t)} + \sigma'(t)\left(\frac{d}{d\lambda}\mathcal{I}_\lambda[u]\right)_{|\lambda=\sigma(t)}$$
$$\leqslant -4\mathcal{I}_{\sigma(t)}\big[u(\cdot,t)\big] + \sigma'(t)\left(\frac{d}{d\lambda}\mathcal{I}_\lambda[u]\right)_{|\lambda=\sigma(t)}.$$

Owing to the definition of $\mathcal{I}_\lambda$, we obtain

$$\frac{d}{d\lambda}\mathcal{I}_\lambda[u] = \frac{d}{2}(m-m_c)\frac{1}{\lambda}\mathcal{I}_\lambda[u] - \frac{m}{1-m}\lambda^{\frac{d}{2}(m-m_c)}\int_{\mathbb{R}^d} 2u\big(\nabla u^{m-1} - \nabla B_\lambda^{m-1}\big)\cdot\frac{d}{d\lambda}\big(\nabla B_\lambda^{m-1}\big)\,dx.$$

By definition (11), $\nabla B_\lambda^{m-1}(x) = 2x\lambda^{-\frac{d}{2}(m-m_c)}$, which implies

$$\lambda^{\frac{d}{2}(m-m_c)}\frac{d}{d\lambda}\big(\nabla B_\lambda^{m-1}\big) = -\frac{d}{\lambda}(m-m_c)x.$$

Substituting this expression into the above computation and integrating by parts, we conclude with the equality

$$\frac{d}{d\lambda}\mathcal{I}_\lambda[u] = \frac{d}{2}(m-m_c)\frac{1}{\lambda}\mathcal{I}_\lambda[u] - \frac{2d}{\lambda}(m-m_c)\left[\frac{2m\lambda^{-\frac{d}{2}(m-m_c)}}{1-m}\int_{\mathbb{R}^d}|x|^2u\,dx - d\int_{\mathbb{R}^d}u^m\,dx\right].$$

A simple computation shows that

$$d\int_{\mathbb{R}^d}B_1^m\,dx = -\int_{\mathbb{R}^d}x\cdot\nabla B_1^m\,dx = \frac{2m}{1-m}\int_{\mathbb{R}^d}|x|^2B_1\,dx = \frac{2m}{1-m}K_M \tag{20}$$

and, as a consequence, if $\lambda = \sigma = \frac{1}{K_M}\int_{\mathbb{R}^d}|x|^2u\,dx$, then

$$\frac{2m\lambda^{-\frac{d}{2}(m-m_c)}}{1-m}\int_{\mathbb{R}^d}|x|^2u\,dx = d\int_{\mathbb{R}^d}B_\lambda^m\,dx,$$

and finally

$$\frac{d}{d\lambda}\mathcal{I}_\lambda[u] = \frac{d}{2\lambda}(m-m_c)\big(\mathcal{I}_\lambda[u] - 4d(1-m)\mathcal{F}_\lambda[u]\big). \tag{21}$$

Altogether, we have found that

$$\frac{d}{dt}\big(\mathcal{I}_{\sigma(t)}\big[u(\cdot,t)\big]\big) + 4\mathcal{I}_{\sigma(t)}\big[u(\cdot,t)\big] \leqslant \frac{d}{2}(m-m_c)\frac{\sigma'(t)}{\sigma(t)}\big(\mathcal{I}_{\sigma(t)}[u] - 4d(1-m)\mathcal{F}_{\sigma(t)}[u]\big).$$

The last term of the right hand side is non-positive because by (17) we know that $\sigma'(t) \leqslant 0$ and

$$\mathcal{I}_{\sigma(t)}[u] - 4d(1-m)\mathcal{F}_{\sigma(t)}[u] = d(1-m)\big(\mathcal{I}_{\sigma(t)}[u] - 4\mathcal{F}_{\sigma(t)}[u]\big) + d(m-m_1)\mathcal{I}_{\sigma(t)}[u]$$
$$\geqslant d(m-m_1)\mathcal{I}_{\sigma(t)}[u] \geqslant 0.$$

This implies (19). □

To avoid carrying heavy notations, let us write

$$f(t) := \mathcal{F}_{\sigma(t)}\big[u(\cdot,t)\big] \quad\text{and}\quad j(t) := \mathcal{J}_{\sigma(t)}\big[u(\cdot,t)\big]$$

and denote $f(0)$, $j(0)$ and $\sigma(0)$ respectively by f_0, j_0 and σ_0. Estimates (17), (18) and (19) can be rewritten as

$$\begin{cases} f' = -j \leqslant 0, \\ \sigma' = -\kappa_1\sigma^{\frac{d}{2}(m-m_c)}f \leqslant 0, \\ j' + 4j \leqslant \kappa_2 j\dfrac{\sigma'}{\sigma} \end{cases} \tag{22}$$

where the constants κ_i, $i = 1, 2$, are given by

$$\kappa_1 := 2d\,\frac{(1-m)^2}{m K_M} \quad \text{and} \quad \kappa_2 := \frac{1}{2}(m-m_c)(m-m_1)d^2.$$

Using the fact that $\lim_{t\to\infty} f(t) = \lim_{t\to\infty} f(t) = 0$, as in the proof of Proposition 3, we find that $j(t) - 4f(t) \geqslant 0$ and $f(t) \leqslant f_0 e^{-4t}$ for any $t \geqslant 0$.

Remark 3. The decay of σ can be estimated by

$$-\frac{d}{dt}\big(\sigma^{\frac{d}{2}(1-m)}\big) = \frac{d}{2}(1-m)\kappa_1 f \leqslant \frac{d}{2}(1-m)\kappa_1 f_0 e^{-4t},$$

thus showing that $\sigma_\infty^{\frac{d}{2}(1-m)} \geqslant \sigma_0^{\frac{d}{2}(1-m)} - \frac{d^2(1-m)^3}{4mK_M} f_0$. Since u_0 and B_{σ_0} have the same mass and second moment, we know that

$$f_0 = \frac{1}{1-m}\int_{\mathbb{R}^d} \big(B_{\sigma_0}^m - u_0^m\big)\,dx = \frac{2mK_M}{d(1-m)^2}\sigma_0^{\frac{d}{2}(1-m)} - \frac{1}{(1-m)}\int_{\mathbb{R}^d} u_0^m\,dx.$$

Hence we end up with the positive lower bound

$$\sigma_\infty^{\frac{d}{2}(1-m)} \geqslant \frac{d}{2}(m-m_c)\sigma_0^{\frac{d}{2}(1-m)} + \frac{d^2(1-m)^2}{4mK_M}\int_{\mathbb{R}^d} u_0^m\,dx.$$

From (22) we get the estimates $\sigma(t) \leqslant \sigma_0$ for any $t \geqslant 0$ and

$$j' - 4f' = j' + 4j \leqslant \kappa_2 j\frac{\sigma'}{\sigma} = \kappa_1\kappa_2\sigma^{-\frac{d}{2}(1-m)} f f' \leqslant \kappa_1\kappa_2\sigma_0^{-\frac{d}{2}(1-m)} f f'.$$

Integrating from 0 to ∞ with respect to t and taking into account the fact that $\lim_{t\to\infty} f(t) = \lim_{t\to\infty} f(t) = 0$, we get

$$-j_0 + 4f_0 \leqslant -\frac{1}{2}\kappa_1\kappa_2\sigma_0^{-\frac{d}{2}(1-m)} f_0^2.$$

By rewriting this estimate in terms of $\mathcal{F}_{\sigma_0}[u_0] = f_0$, $\mathcal{I}_{\sigma_0}[u_0] = j_0$ and after omitting the index 0, we have achieved our key estimate, which can be written using

$$C_{m,d} := \frac{d^3}{2mK_M}(m-m_c)(m-m_1)(1-m)^2$$

as follows.

Theorem 8. *Let $d \geqslant 1$, $m \in (m_1, 1)$ and assume that u is a non-negative function in $\mathrm{L}^1(\mathbb{R}^d)$ such that u^m and $x \mapsto |x|^2 u$ are both integrable on $\mathbb{R}^d$. Let $\sigma = \frac{1}{K_M}\int_{\mathbb{R}^d} |x|^2 u(x)\,dx$ where $M = \int_{\mathbb{R}^d} u(x)\,dx$. Then the following inequality holds*

$$4\mathcal{F}_\sigma[u] + C_{m,d}\frac{(\mathcal{F}_\sigma[u])^2}{\sigma^{\frac{d}{2}(1-m)}} \leqslant \mathcal{I}_\sigma[u]. \tag{23}$$

Recall that $K_M = K_1 M^\gamma$, with $\gamma = \frac{(d+2)m-d}{d(m-m_c)}$. See Appendix A for details. Notice that this definition of γ is compatible with the one of Theorem 1 if $p = 1/(2m-1)$.

Remark 4. If we do not drop any term in the proof of Proposition 3 and Lemma 7, an ODE can be obtained for j, based on (10) and (21) and we can replace (22) by a system of coupled ODEs that reads

$$\begin{cases} f' = -j \leqslant 0, \\ \sigma' = -\kappa_1\sigma^{\frac{d}{2}(m-m_c)} f \leqslant 0, \\ j' + 4j = d(1-m)(j-4f)\dfrac{\sigma'}{\sigma} + \kappa_2 j\dfrac{\sigma'}{\sigma} - \mathsf{r} \end{cases}$$

where $\mathsf{r} := 2\sigma^{d(m-m_c)}\int_{\mathbb{R}^d} u^m(|\nabla z|^2 - (1-m)(\nabla\cdot z)^2)\,dx \geqslant 0$ and $z := \sigma^{\frac{d}{2}(m-m_c)}\nabla u^{m-1} - 2x$.

It is then clear that the estimates $\sigma \leqslant \sigma_0$ and $j' + 4j \leqslant \kappa_2 j \frac{\sigma'}{\sigma}$, which have been used for the proof of Theorem 8, are not optimal.

6. Proofs of Theorem 1 and Corollary 2

Let us start by rephrasing Theorem 8 in terms of $f = u^{m-1/2}$. Assume that

$$M = \int_{\mathbb{R}^d} u\,dx = \int_{\mathbb{R}^d} |f|^{2p}\,dx \quad \text{and} \quad \sigma = \frac{1}{K_M}\int_{\mathbb{R}^d} |x|^2 u\,dx = \int_{\mathbb{R}^d} |x|^2|f|^{2p}\,dx$$

where $p = 1/(2m-1)$ and using the notation $f^{(p)}_{M,0,\sigma} \in \mathfrak{M}^{(p)}_d$ defined in Section 1, consider the functional

$$\mathsf{R}^{(p)}[f] := -\frac{2p}{p+1}\int_{\mathbb{R}^d} \left[|f|^{p+1} - \left(f^{(p)}_{M,0,\sigma}\right)^{p+1}\right]dx.$$

In preparation for the proof of Theorem 1, we can state the following result.

Corollary 9. *Let $d \geqslant 2$, $p > 1$ and assume that $p < d/(d-2)$ if $d \geqslant 3$. For any $f \in \mathrm{L}^{p+1} \cap \mathcal{D}^{1,2}(\mathbb{R}^d)$ such that condition (5) holds, we have*

$$\int_{\mathbb{R}^d} |\nabla f|^2\,dx + \int_{\mathbb{R}^d} |f|^{p+1}\,dx - \mathsf{K}_{p,d}\left(\int_{\mathbb{R}^d} |f|^{2p}\,dx\right)^{\gamma} \geqslant \mathsf{C}_{p,d}\frac{(\mathsf{R}^{(p)}[f])^2}{(\int_{\mathbb{R}^d} |f|^{2p}\,dx)^{\gamma}}$$

where γ is given by (4).

This results is slightly more precise than the one given in Theorem 1, as we simply measure the distance to a special function in $\mathfrak{M}^{(p)}_d$, the one with the same mass and second moment, centered at 0. The constant $\mathsf{C}_{p,d}$ is the same as in Theorem 1: see Appendix A for its expression.

Proof of Corollary 9. By expanding the square in $\mathcal{I}_\sigma[u]$ and collecting the terms with the ones of $\mathcal{F}_\sigma[u]$, we find that

$$\frac{1}{4}\mathcal{I}_\sigma[u] - \mathcal{F}_\sigma[u] = \frac{m(1-m)}{(2m-1)^2}\sigma^{\frac{d}{2}(m-m_c)}\int_{\mathbb{R}^d} \left|\nabla u^{m-\frac{1}{2}}\right|^2 dx + d\frac{m-m_1}{1-m}\int_{\mathbb{R}^d} u^m\,dx + \frac{1}{1-m}\left(mK_M\sigma^{\frac{d}{2}(1-m)} - \int_{\mathbb{R}^d} B^m_\sigma\,dx\right).$$

The last term of the right hand side can be rewritten as

$$\frac{1}{1-m}\left(mK_M\sigma^{\frac{d}{2}(1-m)} - \int_{\mathbb{R}^d} B^m_\sigma\,dx\right) = -\frac{m}{1-m}\frac{d(m-m_c)}{(d+2)m-d}\sigma^{\frac{d}{2}(1-m)}C_1M^\gamma$$

with $\gamma = \frac{(d+2)m-d}{d(m-m_c)}$ (as in the previous section) and $C_1 = M_*^{1-\gamma}$ (see Appendix A for details). Consequently inequality (23) can be equivalently rewritten as

$$\frac{m(1-m)}{(2m-1)^2}\sigma^{\frac{d}{2}(m-m_c)}\int_{\mathbb{R}^d} \left|\nabla u^{m-\frac{1}{2}}\right|^2 dx + d\frac{m-m_1}{1-m}\int_{\mathbb{R}^d} u^m\,dx$$
$$\geqslant \frac{m}{1-m}\frac{d(m-m_c)}{(d+2)m-d}\sigma^{\frac{d}{2}(1-m)}C_1M^\gamma + \frac{d^3(m-m_c)(m-m_1)(1-m)^2}{8mK_1}\frac{(\mathcal{F}_\sigma[u])^2}{M^\gamma\sigma^{\frac{d}{2}(1-m)}}. \tag{24}$$

This inequality is invariant under scaling and homogeneous. As already noticed in (12), condition (5) means $\sigma = \sigma_*$, that is $\frac{m(1-m)}{(2m-1)^2}\sigma^{\frac{d}{2}(m-m_c)} = d\frac{m-m_1}{1-m}$. Using the explicit expressions that can be found in Appendix A and reexpressing all quantities in terms of $p = \frac{1}{2m-1}$ completes the proof of Corollary 9. See Appendix A for an expression of $\mathsf{C}_{p,d}$. □

Proof of Theorem 1. It is itself a simple consequence of Corollary 9.

Let us consider the relative entropy with respect to a general Barenblatt function, not even necessarily normalized with respect to its mass. For a given function $u \in \mathrm{L}^1_+(\mathbb{R}^d)$ with $u^m \in \mathrm{L}^1(\mathbb{R}^d)$ and $|x|^2 u \in \mathrm{L}^1(\mathbb{R}^d)$, we can consider on $(0,\infty) \times \mathbb{R}^d \times (0,\infty)$ the function h defined by

$$h(C,y,\sigma) = \frac{1}{m-1} \int_{\mathbb{R}^d} \left[u^m - B_{C,y,\sigma}^m - m B_{C,y,\sigma}^{m-1} (u - B_{C,y,\sigma}) \right] dx$$

where $B_{C,y,\sigma}$ is a general Barenblatt function

$$B_{C,y,\sigma}(x) := \sigma^{-\frac{d}{2}} \left(C + \frac{1}{\sigma} |x-y|^2 \right)^{\frac{1}{m-1}} \quad \forall x \in \mathbb{R}^d.$$

An elementary computation shows that

$$\frac{\partial h}{\partial C} = \frac{m \sigma^{\frac{d}{2}(1-m)}}{1-m} \int_{\mathbb{R}^d} (u - B_{C,y,\sigma})\, dx,$$

$$\nabla_y h = \frac{2m\sigma^{-\frac{d}{2}(m-m_c)}}{1-m} \int_{\mathbb{R}^d} (x-y)(u - B_{C,y,\sigma})\, dx,$$

$$\frac{\partial h}{\partial \sigma} = m \frac{d}{2} \sigma^{-\frac{d}{2}(m-m_c)} \left[C \int_{\mathbb{R}^d} (u - B_{C,y,\sigma})\, dx - \frac{m-m_c}{1-m} \frac{1}{\sigma} \int_{\mathbb{R}^d} |x-y|^2 (u - B_{C,y,\sigma})\, dx \right].$$

Optimizing with respect to C fixes $C = C_M$, with $M = \int_{\mathbb{R}^d} u\, dx$. Once $C = C_M$ is assumed, optimizing with respect to σ amounts to choose it such that $\int_{\mathbb{R}^d} |x|^2 B_{C,y,\sigma}\, dx = \int_{\mathbb{R}^d} |x-y|^2 u\, dx$ as it has been shown in Lemma 5.

This completes the proof of Theorem 1, since $\mathcal{R}^{(p)}[f] \geqslant \mathsf{R}^{(p)}[f]$ by definition of $\mathcal{R}^{(p)}$ (see Section 1). Notice that optimizing on y amounts to fix the center of mass of the Barenblatt function to be the same as the one of u. This is however required neither in the proof of Corollary 9 nor in the one of Theorem 1. □

Proof of Corollary 2. It is a straightforward consequence of Theorem 1 and of the Csiszár–Kullback inequality (6) when $f \in \mathcal{D}^{1,2}(\mathbb{R}^d)$ is such that $\|f\|_{2,2p}$ is finite. However, $\|f\|_{2,2p}$ does not enter in the inequality. Since smooth functions with compact support (for which $\|f\|_{2,2p}$ is obviously finite) are dense $\mathcal{D}^{1,2}(\mathbb{R}^d)$, the inequality therefore holds without restriction, by density. □

7. Concluding remarks

Let us conclude this paper with a few remarks. First of all, notice that Theorem 4 gives a stronger information than Theorem 1, as not only the $\mathrm{L}^1(\mathbb{R}^d, dx)$ norm is controlled, but also a stronger norm involving the second moment, properly scaled.

No condition is imposed on the location of the center of mass, which simply has to satisfy $(\int_{\mathbb{R}^d} x u\, dx)^2 \leqslant \int_{\mathbb{R}^d} u\, dx \int_{\mathbb{R}^d} |x|^2 u\, dx = \sigma M K_M$ according to the Cauchy–Schwarz inequality. Hence in the definition of $\mathcal{R}[f]$ and $\mathcal{R}^{(p)}[f]$ (in Theorem 1) as well as in Corollary 2, the result holds without optimizing on $y \in \mathbb{R}^d$. In [8,19], improved asymptotic rates were obtained by fixing the center of mass in order to kill the linear mode associated to the translation invariance of the Barenblatt functions. Here this is not required since, as $t \to \infty$, the squared relative entropy is simply of higher order. Our improvement is better when the relative entropy is large, and is clearly not optimal for large values of t.

Our approach differs from the one of G. Bianchi and H. Egnell in [5] and the one of A. Cianchi, N. Fusco, F. Maggi and A. Pratelli [15]. It gives fully explicit constants in the subcritical regime. The norms involved in the corrective term are not of the same nature.

Let us list a series of remarks which help for the understanding of our results.

(i) *Scaling properties of the Barenblatt profiles.* Consider the scaling $\lambda \mapsto u_\lambda$ with $u_\lambda(x) := \lambda^d u(\lambda x)$ for any $x \in \mathbb{R}^d$. Then we have

$$\sigma_\lambda := \frac{1}{K_M} \int_{\mathbb{R}^d} |x|^2 u_\lambda \, dx = \frac{1}{\lambda^2} \frac{1}{K_M} \int_{\mathbb{R}^d} |x|^2 u \, dx = \frac{\sigma}{\lambda^2}$$

and may observe that

$$B_{\sigma_\lambda}(x) = \lambda^d B_\sigma(\lambda x).$$

As a consequence, we find that $\mathcal{F}_\sigma[u_\lambda] = \lambda^{d(m-1)} \mathcal{F}_\sigma[u]$.

(ii) *Homogeneity properties of the Barenblatt profiles.* Similarly notice that for any $m \in (m_1, 1)$, we have $C_M = C_1 M^{-\frac{2(1-m)}{d(m-m_c)}}$ and $K_M = K_1 M^{1-\frac{2(1-m)}{d(m-m_c)}}$. Let $u_\lambda := \lambda u$ and denote by B_{σ_λ} the corresponding best matching Barenblatt function. Using the fact that $\|u_\lambda\|_1 = \lambda M$ if $\|u\|_1 = M$ and observing that

$$K_{\lambda M} = K_M \lambda^{1-\frac{2(1-m)}{d(m-m_c)}} \quad \text{and} \quad \int_{\mathbb{R}^d} |x|^2 u_\lambda \, dx = \lambda \int_{\mathbb{R}^d} |x|^2 u \, dx,$$

we find

$$\sigma_\lambda = \frac{1}{K_{\lambda M}} \int_{\mathbb{R}^d} |x|^2 u_\lambda \, dx = \lambda^{\frac{2(1-m)}{d(m-m_c)}} \sigma.$$

Since $C_{\lambda M} = \lambda^{-\frac{2(1-m)}{d(m-m_c)}} C_M$, we find that

$$B_{\sigma_\lambda}(x) = \left(\lambda^{\frac{2(1-m)}{d(m-m_c)}} \sigma\right)^{-\frac{d}{2}} \left(\lambda^{-\frac{2(1-m)}{d(m-m_c)}} C_M + \frac{|x|^2}{\lambda^{\frac{2(1-m)}{d(m-m_c)}} \sigma}\right)^{\frac{1}{m-1}} = \lambda B_\sigma(x).$$

As a consequence, we find that $\mathcal{F}_\sigma[u_\lambda] = \lambda^m \mathcal{F}_\sigma[u]$.

(iii) *The $m = 1$ limit.* As $m \to 1$, which also corresponds to $p \to 1$, we observe that the constant $\mathsf{C}_{p,d}$ in Theorem 1 has a finite limit. Hence we get no improvement by dividing the improved Gagliardo–Nirenberg inequality by $(p-1)$ and passing to the limit $p \to 1_+$, since $\mathcal{R}^{(p)}[f] = O(p-1)$. By doing so, we simply recover the logarithmic Sobolev inequality as in [16].

This is consistent with the fact that, as $m \to 1_-$, we have $C_{m,d} \sim (1-m)^2$, $\sigma = O(K_M^{-1}) = O(1-m)$ and, since

$$B_\sigma(x) \sim B_0(x) := M \left(\frac{d M}{2\pi \int_{\mathbb{R}^d} |x|^2 u \, dx}\right)^{\frac{d}{2}} \exp\left(-\frac{d}{2} \frac{M}{\int_{\mathbb{R}^d} |x|^2 u \, dx} |x|^2\right),$$

we also get that $\mathcal{F}_\sigma[u] \sim \int_{\mathbb{R}^d} u \log(\frac{u}{B_0}) \, dx$. Hence, in Theorem 8, the additional term in (23) is of the order of $1-m$ and disappears when passing to the limit $m \to 1_-$.

Acknowledgements

The authors acknowledge support both by the ANR projects CBDif-Fr and EVOL (J.D.) and by MIUR project “Optimal mass transportation, geometrical and functional inequalities with applications” (G.T.). The warm hospitality of the Department of Mathematics of the University of Pavia, where this work has been partially done, is kindly acknowledged. J.D. thanks Michael Loss for enlightening discussions on the entropy–entropy production method. Both authors thank D. Matthes for his careful reading of an earlier version of the manuscript, which has resulted in a significant improvement of our results.

Appendix A. Computation of the constants

Let us recall first some useful formulae. The surface of the $d-1$ dimensional unit sphere $\mathbb{S}^{d-1}$ is given by $|\mathbb{S}^{d-1}| = 2\pi^{d/2}/\Gamma(d/2)$. Using the integral representation of Euler's Beta function (see [1, 6.2.1 p. 258]), we have

$$\int_{\mathbb{R}^d} \left(1+|x|^2\right)^{-a} dx = \pi^{\frac{d}{2}} \frac{\Gamma(a-\frac{d}{2})}{\Gamma(a)}.$$

With this formula in hand, various quantities associated with *Barenblatt functions* can be computed. Applied to the function $B(x) := (1+|x|^2)^{\frac{1}{m-1}}$, $x \in \mathbb{R}^d$, we find that

$$M_* := \int_{\mathbb{R}^d} B\,dx = \pi^{\frac{d}{2}} \frac{\Gamma\left(\frac{d(m-m_c)}{2(1-m)}\right)}{\Gamma\left(\frac{1}{1-m}\right)}.$$

Notice that when $M = M_*$, $B = B_1$ with the notation (11) of Section 3. As a consequence, for $B_1(x) = (C_M + |x|^2)^{\frac{1}{m-1}}$, a simple change of variables shows that

$$M := \int_{\mathbb{R}^d} B_1\,dx = \int_{\mathbb{R}^d} \left(C_M+|x|^2\right)^{\frac{1}{m-1}} dx = M_* C_M^{-\frac{d(m-m_c)}{2(1-m)}},$$

which determines the value of C_M, namely

$$C_M = \left(\frac{M_*}{M}\right)^{\frac{2(1-m)}{d(m-m_c)}}.$$

A useful equivalent formula is $C_M = C_1 M^{-\frac{2(1-m)}{d(m-m_c)}}$ where $C_1 = M_*^{\frac{2(1-m)}{d(m-m_c)}}$.

By recalling (20) and observing that

$$\int_{\mathbb{R}^d} B_1^m\,dx = \int_{\mathbb{R}^d} B_1^{m-1} B_1\,dx = \int_{\mathbb{R}^d} \left(C_M+|x|^2\right) B_1\,dx = M C_M + K_M$$

where $K_M := \int_{\mathbb{R}^d} |x|^2 B_1\,dx$, using $M C_M = C_1 M^\gamma$ with $\gamma = \frac{(d+2)m-d}{d(m-m_c)}$, we find that

$$K_M = \frac{d(1-m)}{(d+2)m-d} C_1 M^\gamma \quad \text{and} \quad \int_{\mathbb{R}^d} B_1^m\,dx = \frac{2m}{(d+2)m-d} C_1 M^\gamma. \tag{25}$$

Consider the sub-family of *Gagliardo–Nirenberg–Sobolev inequalities* (1). It has been established in [16, Theorem 1] that optimal functions are all given by (2), up to multiplications by a constant, translations and scalings. This allows to compute $\mathcal{C}_{p,d}^{\rm GN}$. All computations done, we find

$$\mathcal{C}_{p,d}^{\rm GN} = \left(\frac{(p-1)^{p+1}}{(p+1)^{d+1-p(d-1)}}\right)^{\eta} \left(\frac{d+2-p(d-2)}{2(p-1)}\right)^{\frac{1}{2p}} \left(\frac{\Gamma(\frac{p+1}{p-1})}{(2\pi d)^{\frac{d}{2}}\,\Gamma(\frac{p+1}{p-1}-\frac{d}{2})}\right)^{(p-1)\eta}$$

with $1/\eta = p(d+2-p(d-2))$.

It is easy to relate $\mathcal{C}_{p,d}^{\rm GN}$ and $\mathsf{K}_{p,d}$. As in [16], apply (3) to f_λ such that $f_\lambda(x) = \lambda^{\frac{d}{2p}} f(\lambda x)$ for any $x \in \mathbb{R}^d$. With $a := \int_{\mathbb{R}^d} |\nabla f|^2\,dx$, $b := \int_{\mathbb{R}^d} |f|^{p+1}\,dx$, $\alpha := \frac{d}{p} + 2 - d$ and $\beta := d\,\frac{p-1}{2p}$, inequality (3) amounts to

$$a\lambda^\alpha + b\lambda^{-\beta} \geqslant \mathsf{K}_{p,d}\left(\int_{\mathbb{R}^d} |f|^{2p}\,dx\right)^\gamma.$$

Optimizing the left hand side with respect to $\lambda > 0$ shows that

$$\mathsf{K}_{p,d}\left(\mathcal{C}_{p,d}^{\rm GN}\right)^{2p\gamma} = \frac{\alpha+\beta}{\alpha^{\frac{\alpha}{\alpha+\beta}} + \beta^{\frac{\beta}{\alpha+\beta}}}.$$

Let us consider (24). With $p = \frac{1}{2m-1}$, that is, $m = \frac{p+1}{2p}$, and $\mathcal{F}[u] = \frac{m}{1-m}\mathcal{R}^{(p)}[f]$ with $u = f^{2p}$, it is straightforward to check that

$$\mathsf{K}_{p,d} = \left(\frac{2m-1}{1-m}\right)^2 \frac{d(m-m_c)}{(d+2)m-d} \frac{M_*^{1-\gamma}}{\sigma_*^{d(m-m_1)}} = \frac{4}{(p-1)^2} \frac{d-p(d-4)}{d+2-p(d-2)} M_*^{1-\gamma} \sigma_*^{d\frac{p-1}{p}-1}$$

since $u = B_\sigma$ always provides the equality case. Hence, using identity (12), inequality (24) amounts to

$$\int_{\mathbb{R}^d} |\nabla f|^2\,dx + \int_{\mathbb{R}^d} |f|^{p+1}\,dx - \mathsf{K}_{p,d}\left(\int_{\mathbb{R}^d} |f|^{2p}\,dx\right)^\gamma$$
$$\geqslant \frac{(2m-1)^2}{m(1-m)} \sigma_*^{-\frac{d}{2}(m-m_c)} \frac{d^3(m-m_c)(m-m_1)(1-m)^2}{8mK_1} \frac{(\frac{m}{1-m}\mathcal{R}^{(p)}[f])^2}{M^\gamma \sigma_*^{\frac{d}{2}(1-m)}}.$$

Using $K_1 = \frac{d(1-m)}{(d+2)m-d} M_*^{1-\gamma}$ and expressing everything in terms of p, we finally get

$$\mathsf{C}_{p,d} = \frac{(2m-1)^2}{8(1-m)^2}\big((d+2)m-d\big)d^2(m-m_c)(m-m_1)\frac{M_*^{\gamma-1}}{\sigma_*}$$
$$= \frac{(d-p(d-4))(d-p(d-2))(d+2-p(d-2))}{16p^3(p-1)^2} \frac{M_*^{\gamma-1}}{\sigma_*(p)}.$$

References

[1] M. Abramowitz, I.A. Stegun, Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, National Bureau of Standards Appl. Math. Ser., vol. 55, 1964. For sale by the Superintendent of Documents, U.S. Government Printing Office, Washington, DC.

[2] A. Arnold, P. Markowich, G. Toscani, A. Unterreiter, On convex Sobolev inequalities and the rate of convergence to equilibrium for Fokker–Planck type equations, Comm. Partial Differential Equations 26 (2001) 43–100.

[3] T. Aubin, Problèmes isopérimétriques et espaces de Sobolev, J. Differential Geom. 11 (1976) 573–598.

[4] D. Bakry, M. Émery, Hypercontractivité de semi-groupes de diffusion, C. R. Math. Acad. Sci. Paris 299 (1984) 775–778.

[5] G. Bianchi, H. Egnell, A note on the Sobolev inequality, J. Funct. Anal. 100 (1991) 18–24.

[6] A. Blanchet, M. Bonforte, J. Dolbeault, G. Grillo, J.-L. Vázquez, Hardy–Poincaré inequalities and applications to nonlinear diffusions, C. R. Math. 344 (2007) 431–436.

[7] A. Blanchet, M. Bonforte, J. Dolbeault, G. Grillo, J.-L. Vázquez, Asymptotics of the fast diffusion equation via entropy estimates, Arch. Ration. Mech. Anal. 191 (2009) 347–385.

[8] M. Bonforte, J. Dolbeault, G. Grillo, J.L. Vázquez, Sharp rates of decay of solutions to the nonlinear fast diffusion equation via functional inequalities, Proc. Natl. Acad. Sci. 107 (2010) 16459–16464.

[9] H. Brezis, E.H. Lieb, Sobolev inequalities with remainder terms, J. Funct. Anal. 62 (1985) 73–86.

[10] M.J. Cáceres, J.A. Carrillo, J. Dolbeault, Nonlinear stability in L^p for a confined system of charged particles, SIAM J. Math. Anal. 34 (2002) 478–494 (electronic).

[11] J.A. Carrillo, A. Jüngel, P.A. Markowich, G. Toscani, A. Unterreiter, Entropy dissipation methods for degenerate parabolic problems and generalized Sobolev inequalities, Monatsh. Math. 133 (2001) 1–82.

[12] J.A. Carrillo, G. Toscani, Asymptotic L^1-decay of solutions of the porous medium equation to self-similarity, Indiana Univ. Math. J. 49 (2000) 113–142.

[13] J.A. Carrillo, J.L. Vázquez, Fine asymptotics for fast diffusion equations, Comm. Partial Differential Equations 28 (2003) 1023–1056.

[14] A. Cianchi, Quantitative Sobolev and Hardy inequalities, and related symmetrization principles, in: Sobolev Spaces in Mathematics. I, in: Int. Math. Ser. (N. Y.), vol. 8, Springer, New York, 2009, pp. 87–116.

[15] A. Cianchi, N. Fusco, F. Maggi, A. Pratelli, The sharp Sobolev inequality in quantitative form, J. Eur. Math. Soc. (JEMS) 11 (2009) 1105–1139.

[16] M. Del Pino, J. Dolbeault, Best constants for Gagliardo–Nirenberg inequalities and applications to nonlinear diffusions, J. Math. Pures Appl. 9 (81) (2002) 847–875.

[17] J. Dolbeault, Sobolev and Hardy–Littlewood–Sobolev inequalities: duality and fast diffusion, Math. Res. Lett. 18 (2011) 1037–1050.

[18] J. Dolbeault, G. Karch, Large time behaviour of solutions to nonhomogeneous diffusion equations, Banach Center Publ. 74 (2006) 133–147.

[19] J. Dolbeault, G. Toscani, Fast diffusion equations: Matching large time asymptotics by relative entropy methods, Kinet. Relat. Models 4 (2011) 701–716.

[20] L. Gross, Logarithmic Sobolev inequalities, Amer. J. Math. 97 (1975) 1061–1083.

[21] W.I. Newman, A Lyapunov functional for the evolution of solutions to the porous medium equation to self-similarity. I, J. Math. Phys. 25 (1984) 3120–3123.

[22] J. Ralston, A Lyapunov functional for the evolution of solutions to the porous medium equation to self-similarity. II, J. Math. Phys. 25 (1984) 3124–3127.
[23] G. Talenti, Best constant in Sobolev inequality, Ann. Mat. Pura Appl. (4) 110 (1976) 353–372.
[24] A. Unterreiter, A. Arnold, P. Markowich, G. Toscani, On generalized Csiszár–Kullback inequalities, Monatsh. Math. 131 (2000) 235–253.
[25] F.B. Weissler, Logarithmic Sobolev inequalities for the heat-diffusion semigroup, Trans. Amer. Math. Soc. 237 (1978) 255–269.