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Renormalization for piecewise smooth homeomorphisms on the circle

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Abstract

In this work we study the renormalization operator acting on piecewise smooth homeomorphisms on the circle, that turns out to be essentially the study of Rauzy–Veech renormalizations of generalized interval exchange maps with genus one. In particular we show that renormalizations of such maps with zero mean nonlinearity and satisfying certain smoothness and combinatorial assumptions converge to the set of piecewise affine interval exchange maps.

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1. Introduction and results

One of the most studied classes of dynamical systems are orientation-preserving diffeomorphisms of the circle. It may be classified according to their rotation number $\rho(f)$ which, roughly speaking, measures the average rate of rotation of orbits around the circle. When $\rho(f) \in \mathbb{Q}$ then f has a periodic point and all other orbits will converge to some periodic orbit both in the future and in the past. If $\rho(f)$ is irrational then f has not periodic point and its dynamics depends on the smoothness of f. Denjoy result ensures that if f is C^2 then it is conjugate to the rigid rotation of angle $\rho(f)$. In this context, it is a natural question to ask under what conditions the conjugacy is smooth. Several authors, Herman [4], Yoccoz [17], Khanin and Sinaĭ [7,14], Katznelson and Ornstein [6], have shown that if f is C^3 or $C^{2+\nu}$ and $\rho(f)$ satisfies certain Diophantine condition then the conjugacy will be at least C^1.

A natural generalization of diffeomorphisms of the circle are diffeomorphisms with breaks, i.e., f has jumps in the first derivative on finitely many points. In this setting Khanin and Vul [9] show that for diffeomorphisms with one

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break the renormalization operator converges to a two-dimensional space of the fractional linear transformation. Our first main result generalizes results of Khanin and Vul [9] for finitely many break points. A key combinatorial method in our proof is to consider a piecewise smooth homeomorphism on the circle as a generalized interval exchange transformation.

Let I be an interval and let $\mathcal{A}$ be a finite set (the *alphabet*) with $d \geqslant 2$ elements and $\mathcal{P} = \{I_\alpha\colon\ \alpha \in \mathcal{A}\}$ be an $\mathcal{A}$-*indexed* partition of I into subintervals.[3] We say that the triple $(f, \mathcal{A}, \mathcal{P})$, where $f : I \to I$ is a bijection, is a generalized interval exchange transformation with d intervals (g.i.e.m. with d intervals, for short), if $f|_{I_\alpha}$ is an orientation-preserving homeomorphism for each $\alpha \in \mathcal{A}$. Most of the time we will abuse the notation saying that f is a g.i.e.m. with d intervals. The order of the subintervals in the domain and image constitute the combinatorial data for f, which will be defined explicitly in the next section.

When $f|_{I_\alpha}$ is a translation we say which f is a standard i.e.m. Standard i.e.m. arise naturally as Poincaré return maps of measured foliations and geodesic flows on translation surfaces. But they are also interesting examples of simple dynamical systems with very rich dynamics and have been extensively studied for their own sake. When $d = 2$, by identifying the endpoints of I, standard i.e.m. correspond to rotations of the circle and generalized i.e.m. correspond to circle homeomorphisms.

In another article, Khanin and Sinaĭ [7] show a new proof of M. Herman's theorem. From the viewpoint of the renormalization group approach they show the convergence of the renormalizations of a circle diffeomorphism to the linear fixed point of the renormalization operator for diffeomorphisms of the circle. We use a similar approach to study generalized interval exchange maps of genus one.

1.1. Renormalization: Rauzy–Veech induction

To describe the combinatorial assumptions of our results, we need to introduce the Rauzy–Veech scheme. This is a renormalization scheme. Renormalization group techniques are a very powerful tool in one-dimensional dynamics. For example see Khanin and Vul [9] for circle homeomorphisms and de Melo and van Strien [3] for unimodal maps.

Following the algorithm of Rauzy [12] and Veech [15], for every i.e.m. f without connections, we define the Rauzy–Veech induction by considering the first return maps f_n of f on a decreasing sequence of intervals I^n, with the same left endpoint as I. The map f_n is again generalized i.e.m. with the same alphabet $\mathcal{A}$ but the combinatorial data may be different.

Given two intervals J and U, we will write $J < U$ if their interiors are disjoint and $x < y$ for every $x \in J$ and $y \in U$. This defines a partial order in the set of all intervals. Denote the length of an interval J by $|J|$.

Let $f : I^0 \to I^0$ be a g.i.e.m. with alphabet $\mathcal{A}$ and $\pi_j : \mathcal{A} \to \{1, \ldots, d\}$, with $j = 0, 1$, be bijections such that

$$I_\alpha < I_\beta$$

iff $\pi_0(\alpha) < \pi_0(\beta)$ and

$$f(I_\alpha) < f(I_\beta)$$

iff $\pi_1(\alpha) < \pi_1(\beta)$.

The pair $\pi = \pi(f) = (\pi_0, \pi_1)$ is called the combinatorial data associated to the g.i.e.m. f. We call

$$p = \pi_1 \circ \pi_0^{-1} : \{1, \ldots, d\} \to \{1, \ldots, d\} \tag{1}$$

the *monodromy invariant* of the pair $\pi = (\pi_0, \pi_1)$. When appropriate we will use the notation $p = (p(1)\ p(2)\ \ldots\ p(d))$ for the data combinatorial of f. We also assume that the pair $\pi = (\pi_0, \pi_1)$ is irreducible, i.e., for all $j \in \{1, \ldots, d-1\}$ we have $\pi_1 \circ \pi_0^{-1}(\{1, \ldots, j\}) \neq \{1, \ldots, j\}$.

For each $\varepsilon \in \{0, 1\}$, define $\alpha(\varepsilon) = \pi_\varepsilon^{-1}(d)$. If $|I_{\alpha(0)}| \neq |f(I_{\alpha(1)})|$ we say that f is Rauzy–Veech renormalizable (or simply renormalizable, from now on). If $|I_{\alpha(0)}| > |f(I_{\alpha(1)})|$ we say that the letter $\alpha(0)$ is the winner and the letter $\alpha(1)$ is the loser. We say that f is type 0 renormalizable and we can define a map $\hat{R}(f)$ as the first return map of f to the interval

$$I^1 = I \setminus f(I_{\alpha(1)}).$$

[3] All the subintervals will be bounded, close on the left and open on the right.

Otherwise $|I_{\alpha(0)}| < |f(I_{\alpha(1)})|$, the letter $\alpha(1)$ is the winner and the letter $\alpha(0)$ is the loser, we say that f is type 1 renormalizable and we can define a map $\hat{R}(f)$ as the first return map of f to the interval

$$I^1 = I \setminus I_{\alpha(0)}.$$

We want to see $R(f)$ as a g.i.e.m. To this end we need to associate to this map an $\mathcal{A}$-indexed partition of its domain. We do this in the following way. The subintervals of the $\mathcal{A}$-partition $\mathcal{P}^1$ of I^1 are denoted by I^1_α. If f has type 0, $I^1_\alpha = I_\alpha$. If $\alpha \neq \alpha(0)$, $I^1_{\alpha(0)} = I_{\alpha(0)} \setminus f(I_{\alpha(1)})$ and when f has type 1, $I^1_\alpha = I_\alpha$ if $\alpha \neq \alpha(1), \alpha(0)$, $I^1_{\alpha(1)} = f^{-1}(f(I_{\alpha(1)}) \setminus I_{\alpha(0)})$ and $I^1_{\alpha(0)} = I_{\alpha(1)} \setminus I^1_{\alpha(1)}$. It is easy to see that in both cases (type 0 and 1) we have

$$R(f)(x) = \begin{cases} f^2(x), & \text{if } x \in I^1_{\alpha(1-\varepsilon)}, \\ f(x), & \text{otherwise} \end{cases}$$

and $(R(f), \mathcal{A}, \mathcal{P}^1)$ is a g.i.e.m., called the **Rauzy–Veech renormalization** (or simply renormalization, for short) of f. If f is renormalizable with type $\varepsilon \in \{0, 1\}$ then the combinatorial data $\pi^1 = (\pi^1_0, \pi^1_1)$ of $R(f)$ is given by

$$\pi^1_\varepsilon := \pi_\varepsilon$$

and

$$\pi^1_{1-\varepsilon}(\alpha) = \begin{cases} \pi_{1-\varepsilon}(\alpha), & \text{if } \pi_{1-\varepsilon}(\alpha) \leqslant \pi_{1-\varepsilon}(\alpha(\varepsilon)), \\ \pi_{1-\varepsilon}(\alpha) + 1, & \text{if } \pi_{1-\varepsilon}(\alpha(\varepsilon)) < \pi_{1-\varepsilon}(\alpha) < d, \\ \pi_{1-\varepsilon}(\alpha(\varepsilon)) + 1, & \text{if } \pi_{1-\varepsilon}(\alpha) = d. \end{cases}$$

Since π^1 depends only on π and the type ε, we denote $r_\varepsilon(\pi) = \pi^1$.

A g.i.e.m. is infinitely renormalizable if $R^n(f)$ is well defined, for every $n \in \mathbb{N}$. For every interval of the form $J = [a, b)$ we denote $\partial J = \{a\}$. We say that a g.i.e.m. f has no connection if

$$f^m(\partial I_\alpha) \neq \partial I_\beta \quad \text{for all } m \geqslant 1,\ \alpha, \beta \in \mathcal{A} \text{ with } \pi_0(\beta) \neq 1.$$

This property is invariant under iteration of R. Keane [5] show that no connection condition is a necessary condition for f to be infinitely renormalizable.

Let ε_n be the type of the n-th renormalization, $\alpha_n(\varepsilon_n)$ be the winner and $\alpha_n(1-\varepsilon_n)$ be the loser of the n-th renormalization.

We say that infinitely renormalizable g.i.e.m. f has k-**bounded combinatorics** if for each n and $\beta, \gamma \in \mathcal{A}$ there exist $n_1, p \geqslant 0$, with $|n - n_1| < k$ and $|n - n_1 - p| < k$, such that $\alpha_{n_1}(\varepsilon_{n_1}) = \beta$, $\alpha_{n_1+p}(1 - \varepsilon_{n_1+p}) = \gamma$ and

$$\alpha_{n_1+i}(1 - \varepsilon_{n_1+p}) = \alpha_{n_1+i+1}(\varepsilon_{n_1+i})$$

for every $0 \leqslant i < p$.

We say that a g.i.e.m. $f : I \to I$ has **genus one** by Veech [16] (or belongs to the **rotation class** by Nogueira and Rudolph [11]) if f has at most two discontinuities. Note that every g.i.e.m. with either two or three intervals has genus one. If f is renormalizable and has genus one, it is easy to see that $R(f)$ has genus one.

Given two infinitely renormalizable g.i.e.m. f and g, defined with the same alphabet $\mathcal{A}$, we say that f and g have the **same combinatorics** if $\pi(f) = \pi(g)$ and the n-th renormalization of f and g have the same type, for every $n \in \mathbb{N}$. It follows that $\pi^n(f) = \pi^n(g)$ for every n, where $\pi^n(f)$ is the combinatorial data of the n-th renormalization of f.

Definition 1.1. Let $\mathcal{B}^{2+\nu}_k$, $k \in \mathbb{N}$ and $\nu > 0$, be the set of g.i.e.m. $f : I \to I$ such that

(i) for each $\alpha \in \mathcal{A}$ we can extend f to $\overline{I_\alpha}$ as an orientation-preserving diffeomorphism of class $C^{2+\nu}$;
(ii) the g.i.e.m. f has k-bounded combinatorics;
(iii) the map f has genus one and has no connection.

Let H be a non-degenerate interval, let $g : H \to \mathbb{R}$ be a diffeomorphism and let $J \subset H$ be an interval. We define the *Zoom* of g in H, denoted by $\mathcal{Z}_H(g)$, the transformation $\mathcal{Z}_H(g) = A_1 \circ g \circ A_2$, where A_1 and A_2 are orientation-preserving affine maps, which send $[0,1]$ into H and $g(H)$ into $[0,1]$ respectively. Consider the set $C^2([0,1],\mathbb{R})$ of all C^2 functions $g:[0,1]\to\mathbb{R}$ with the usual norm

$$d_{C^2}(f,g) := \sum_{i=0}^{2} \sup_{x\in[0,1]} \left|D^{(i)}f(x) - D^{(i)}g(x)\right|,$$

where $D^{(i)}f$ and $D^{(i)}g$ denote the i-th derivative of f and g respectively.

Denote by $\mathcal{M}$ the set of Möbius transformations $M:[0,1]\to[0,1]$ such that $M(0)=0$ and $M(1)=1$. Note that $\mathcal{M}$ is a one-dimensional real Lie group. Indeed any element $M\in\mathcal{M}$ has the form

$$M = M_N(x) = \frac{xe^{\frac{-N}{2}}}{1+x(e^{\frac{-N}{2}}-1)} \tag{2}$$

for some $N\in\mathbb{R}$ and $M_{N_1}\circ M_{N_2} = M_{N_1+N_2}$. Moreover M_N is the unique Möbius transformation M which sends $[0,1]$ onto $[0,1]$, $M(0)=0$, $M(1)=1$, and

$$\int_0^1 \frac{D^2M(x)}{DM(x)}\,dx = N.$$

1.2. Main results

Theorem 1. *Let $f\in\mathcal{B}_k^{2+\nu}$. Then there are $C=C(f)>0$ and $0<\lambda=\lambda(k)<1$ such that*

$$\mathrm{d}_{C^2}\big(\mathcal{Z}_{I_\alpha^n}\big(R^n(f)\big), M_{N_\alpha^n}\big) \leqslant C\lambda^n$$

for all $\alpha\in\mathcal{A}$. Here

$$N_\alpha^n = \int_{I_\alpha^n} \frac{D^2R^n(f)(x)}{DR^n(f)(x)}\,dx.$$

In particular

$$\mathrm{d}_{C^2}\big(\mathcal{Z}_{I_\alpha^n}\big(R^n(f)\big), \mathcal{M}\big) \leqslant C\lambda^n.$$

We can say more about the mean nonlinearities N_α^n. Denote by $q_\alpha^n\in\mathbb{N}$ the first return time of the interval I_α^n to the interval I^n, i.e., $\hat{R}^n(f)|_{I_\alpha^n} = f^{q_\alpha^n}$, for some $q_\alpha^n\in\mathbb{N}$.

Theorem 2. *Let $f\in\mathcal{B}_k^{2+\nu}$. Then there are $C=C(f)>0$ and $0<\lambda=\lambda(k)<1$ such that*

$$\left|N_\alpha^n - \frac{\sum_{i=1}^{q_\alpha^n}|f^i(I_\alpha^n)|}{|I|}\int \frac{D^2f(x)}{Df(x)}\,dx\right| \leqslant C\lambda^{\sqrt{n}}. \tag{3}$$

In particular if

$$\int_{[0,1]} \frac{D^2f(x)}{Df(x)}\,dx = 0$$

then $|N_\alpha^n| < C\lambda^{\sqrt{n}}$.

The rate of convergence obtained in (3) is enough for our purposes. In the case of circle diffeomorphisms Khanin and Teplinsky [8] obtained an exponential rate using a different approach.

Our third result is an almost direct consequence of Theorems 1 and 2.

Theorem 3. *Let $f \in \mathcal{B}_k^{2+\nu}$ such that*

$$\int\limits_{[0,1]} \frac{D^2 f(x)}{Df(x)}\,dx = 0.$$

Then there are $C = C(k) > 0$ and $0 < \lambda = \lambda(k) < 1$ such that

$$\left|\mathcal{Z}_{I_\alpha^n}\big(R^n(f)\big) - \mathrm{Id}\right|_{C^2} \leqslant C \cdot \lambda^{\sqrt{n}} \quad \textit{for all } \alpha \in \mathcal{A}.$$

The structure of this paper is as follows. In Section 2 we describe general results on compositions of diffeomorphisms of class $C^{2+\nu}$. In Section 3 we study renormalization of generalized interval exchange maps of genus one and prove Theorem 1. In Section 4 we codify the dynamics of f using a specially crafted symbolic dynamics to obtain finer geometric properties of the partitions associated with renormalizations of f and we finally prove Theorems 2 and 3.

This is the first of a series of two papers based on the Ph.D. Thesis of the first author Cunha [1]. In the second work [2] we continue our study of the renormalization operator for generalized interval exchange transformations of genus one and its consequences, particularly the rigidity (universality) phenomena in the setting of piecewise smooth homeomorphisms on the circle.

2. Comparing compositions of $C^{2+\nu}$ maps with Möbius maps

In this section, we show some results about composition of $C^{2+\nu}$-diffeomorphisms, comparing these compositions with Möbius maps. Let $f:[a,b] \to [f(a), f(b)]$ be a C^2 orientation-preserving diffeomorphism. Define the *nonlinearity function* $n_f : [a,b] \to \mathbb{R}$ by

$$n_f(x) = \frac{D^2 f(x)}{Df(x)} = D\big(\ln Df(x)\big).$$

Notice that

$$n_{f\circ g}(x) = n_f\big(g(x)\big)Dg(x) + n_f(x),$$

consequently if f_i are C^2-diffeomorphisms such that $f = f_n \circ \cdots \circ f_1$ is defined in $[a,b]$ we have

$$\int\limits_{[a,b]} \frac{D^2 f(x)}{Df(x)}\,dx = \sum_{i=1}^{n} \int\limits_{f^{i-1}[a,b]} \frac{D^2 f_i(x)}{Df_i(x)}\,dx. \tag{4}$$

If $[a,b] = [0,1]$ we define

$$N_f = \int\limits_{[0,1]} \frac{D^2 f(x)}{Df(x)}\,dx.$$

The nonlinearity n_f defines f up to its domain and image. Indeed, given a continuous function $n:[0,1] \to \mathbb{R}$ there is unique C^2-diffeomorphism $f:[0,1] \to [0,1]$ such that $f(0) = 0$, $f(1) = 1$ and $n_f = n$. Indeed, see Martens [10]

$$f(x) = \frac{\int_0^x \exp(\int_0^z n(y)\,dy)\,dz}{\int_0^1 \exp(\int_0^z n(y)\,dy)\,dz}. \tag{5}$$

Let $f:[0,1] \to [f(0), f(1)]$ be a C^2 orientation-preserving diffeomorphism. If $[a,b] \subset [0,1]$, let $\tilde{f} = \mathcal{Z}_{[a,b]}(f)$ be the Zoom of f in $[a,b]$. Then

$$n_{\mathcal{Z}(f)}(x) = (b-a) \cdot n_f\big(a + x(b-a)\big). \tag{6}$$

Suppose

$$\left|n_f(x) - n_f(y)\right| \leqslant C_0 \cdot |x-y|^\nu \tag{7}$$

for $x, y \in [0,1]$ and

$$|n_f|_{C^0[0,1]} := \sup_{x\in[0,1]} \{|n_f(x)|\} \leqslant C_1.$$

Then by (7) and (6) we have that

$$\left|n_{\mathcal{Z}(f)}(x) - n_{\mathcal{Z}(f)}(y)\right| \leqslant C_0 \cdot \delta^{1+\nu} \quad \text{and} \quad |n_{\mathcal{Z}(f)}|_{C^0[0,1]} \leqslant C_1 \cdot \delta, \tag{8}$$

with $x, y \in [0,1]$ and $\delta = b - a$. Note that

$$N_{Z(f)} = \int_0^1 (b-a)\cdot n_f\big(a + x(b-a)\big)\,dx = \int_{[a,b]} \frac{D^2 f(x)}{Df(x)}\,dx. \tag{9}$$

Proposition 2.1. *Let $f:[0,1] \to [f(0), f(1)]$ be an orientation-preserving diffeomorphism of class $C^{2+\nu}$, $[a,b] \subset [0,1]$ and define $\tilde f = Z_{[a,b]} f$. Then*

$$\mathrm{d}_{C^2}(\tilde f, M_{N_{\tilde f}}) = \mathrm{O}\big(\delta^{1+\nu}\big),$$

where $\delta = b - a$.

Before we prove Proposition 2.1 we prove the following lemma:

Lemma 2.2. *Let $N \in \mathbb{R}$. Let $f_N : [0,1] \mapsto [0,1]$ be a diffeomorphism such that $n_f(x) = N$ for all $x \in [0,1]$, $f(0) = 0$, $f(1) = 1$. Then*

$$\mathrm{d}_{C^2}(f_N, M_N) = \mathrm{O}\big(N^2\big).$$

Proof. By (5) we have

$$f_N(x) = \frac{\int_0^x e^{Nz}\,dz}{\int_0^1 e^{Nz}\,dz} = \frac{e^{Nx}-1}{e^N - 1}.$$

Therefore,

$$\begin{aligned}
\left|f_N(x) - M_N(x)\right| &= \left|\frac{e^{Nx}-1}{e^N-1} - \frac{xe^{\frac{-N}{2}}}{1 + x(e^{\frac{-N}{2}} - 1)}\right| \\
&= \left|\frac{Nx + \frac{N^2x^2}{2} + \mathrm{O}(N^3)}{N + \frac{N^2}{2} + \mathrm{O}(N^3)} - \frac{x(1 - \frac{N}{2} + \mathrm{O}(N^2))}{1 + x(-\frac{N}{2} + \mathrm{O}(N^2))}\right| \\
&= \left|x\left(1 + \frac{Nx}{2} - \frac{N}{2}\right) + \mathrm{O}\big(N^2\big) - x\left(1 + \frac{Nx}{2} - \frac{N}{2}\right) + \mathrm{O}\big(N^2\big)\right| \\
&= \mathrm{O}\big(N^2\big),
\end{aligned}$$

$$\begin{aligned}
\left|Df_N(x) - DM_N(x)\right| &= \left|\frac{Ne^{Nx}}{e^N - 1} - \frac{e^{\frac{-N}{2}}}{[1 + x(e^{\frac{-N}{2}} - 1)]^2}\right| \\
&= \left|\frac{(1 + Nx + \mathrm{O}(N^2))}{(1 + \frac{N}{2} + \mathrm{O}(N^2))} - e^{-\frac{N}{2}}\left(1 + \frac{Nx}{2} + \mathrm{O}\big(N^2\big)\right)^2\right| \\
&= \left|(1 + Nx)\left(1 - \frac{N}{2}\right) + \mathrm{O}\big(N^2\big) - e^{-\frac{N}{2}}\big(1 + Nx + \mathrm{O}\big(N^2\big)\big)\right| \\
&= \left|1 - \frac{N}{2} + Nx + \mathrm{O}\big(N^2\big) - 1 - \frac{N}{2} + Nx + \mathrm{O}\big(N^2\big)\right| \\
&= \mathrm{O}\big(N^2\big),
\end{aligned}$$

$$\begin{aligned}\left|D^2 f_N(x) - D^2 M_N(x)\right| &= \left|\frac{N^2 e^{Nx}}{e^N - 1} - \frac{-2e^{\frac{-N}{2}}(e^{-\frac{N}{2}} - 1)}{[1 + x(e^{\frac{-N}{2}} - 1)]^3}\right| \\ &= \left|N + \mathrm{O}(N^2) - (N + \mathrm{O}(N^2))\left(1 + \frac{Nx}{2} + \mathrm{O}(N^2)\right)^3\right| \\ &= \mathrm{O}(N^2). \qquad \square\end{aligned}$$

Proof of Proposition 2.1. By Eq. (8) we have

$$|N_{\tilde f}| \leqslant |n_f|_{C^0[0,1]}\delta.$$

To simplify the notation, denote $\tilde N = N_{\tilde f}$. First note that

$$\mathrm{d}_{C^2}(\tilde f, M_{\tilde N}) \leqslant \mathrm{d}_{C^2}(\tilde f, f_{\tilde N}) + \mathrm{d}_{C^2}(f_{\tilde N}, M_{\tilde N}).$$

In view of Lemma 2.2, we only need to estimate the first term of the right-hand side. For this note that $\tilde N = \int_0^1 n_{\tilde f}(s)\,ds = n_{\tilde f}(\theta)$ for some $\theta \in [0,1]$. If

$$\left|n_f(x) - n_f(y)\right| \leqslant C_0|x-y|^\nu,$$

then by Eq. (8) we have $|n_{\tilde f}(x) - \tilde N| = |n_{\tilde f}(x) - n_{\tilde f}(\theta)| \leqslant C_0 \cdot \delta^{1+\nu}$, so $n_{\tilde f}(x) = \tilde N + \mathrm{O}(\delta^{1+\nu})$. Then by (5) we obtain

$$\tilde f(x) = x\left(1 - \frac{\tilde N}{2} + \frac{\tilde N}{2}x + \mathrm{O}(\delta^{1+\nu})\right); \tag{10}$$

$$D\tilde f(x) = 1 + \tilde N x - \frac{\tilde N}{2} + \mathrm{O}(\delta^{1+\nu}); \tag{11}$$

$$D^2\tilde f(x) = \tilde N + \mathrm{O}(\delta^{1+\nu}). \tag{12}$$

Using the estimates for $f_{\tilde N}$, $Df_{\tilde N}$ and $D^2 f_{\tilde N}$ similar to those in the proof of Lemma 2.2, the proof is complete. $\square$

From now on let $f_i : [0,1] \to [0,1]$, $i \in \mathbb{N}$, be orientation-preserving diffeomorphisms of class $C^{2+\nu}$, with $f_i(0) = 0$, $f_i(1) = 1$, and such that there exist $C_0, C_1 > 0$ satisfying

$$\left|n_{f_i}(x) - n_{f_i}(y)\right| \leqslant C_0 \cdot |x-y|^\nu \tag{13}$$

and

$$|n_{f_i}|_{C^0[0,1]} \leqslant C_1$$

for every $i \in \mathbb{N}$. Let $[a_i, b_i] \subset [0,1]$, $\delta_i = b_i - a_i$, and $\tilde f_i = \mathcal{Z}_{[a_i,b_i]}(f_i)$, $M_i = M_{N_{\tilde f_i}}$,

$$\tilde f_1^n = \tilde f_n \circ \tilde f_{n-1} \circ \cdots \circ \tilde f_1 \quad \text{and} \quad M_1^n = M_n \circ M_{n-1} \circ \cdots \circ M_1.$$

The following proposition is the main result of this section. It compares the compositions of $\tilde f_i$'s and M_i's.

Proposition 2.3. *(See also [9].) Let f_i be as above. Then for every $C_2 > 0$ there exists $C_3 > 0$ with the following property. If $\sum_{i=1}^n \delta_i \leqslant C_2$, then*

$$\left|\tilde f_1^n - M_1^n\right|_{C^2} \leqslant C_3 \cdot \left(\max_{1\leqslant j\leqslant n} \delta_j\right)^\nu.$$

The proof of Theorem 1 in Khanin and Vul [9] is the main motivation to Proposition 2.3. Since Proposition 2.3 is not stated explicitly in the paper cited above in its full generality, we include the full argument for the sake of completeness. Before we prove this proposition, we need some lemmas.

Lemma 2.4. *There is $C_4 = C_4(C_1, C_2) > 0$ such that*

$$e^{-C_4} \leqslant D\tilde{f}_1^n(x) \leqslant e^{C_4},$$

for all $x \in [0,1]$ and for all $n \geqslant 0$.

Proof.

$$\begin{aligned}
\ln \frac{D\tilde{f}_1^n(x)}{D\tilde{f}_1^n(y)} &= \ln \frac{D\tilde{f}_n(\tilde{f}_1^{n-1}(x)) \cdot D\tilde{f}_{n-1}(\tilde{f}_1^{n-2}(x)) \cdots D\tilde{f}_1(x)}{D\tilde{f}_n(\tilde{f}_1^{n-1}(y)) \cdot D\tilde{f}_{n-1}(\tilde{f}_1^{n-2}(y)) \cdots D\tilde{f}_1(y)} \\
&= \sum_{j=1}^{n} \ln D\tilde{f}_j\big(\tilde{f}_1^{j-1}(x)\big) - \ln D\tilde{f}_j\big(\tilde{f}_1^{j-1}(y)\big) \\
&= \sum_{j=1}^{n} \int_{\tilde{f}_1^{j-1}(y)}^{\tilde{f}_1^{j-1}(x)} \frac{D^2\tilde{f}_j(s)}{D\tilde{f}_j(s)}\,ds \\
&= \sum_{j=1}^{n} \frac{D^2\tilde{f}_j(z_{j-1})}{D\tilde{f}_j(z_{j-1})} \big|\tilde{f}_1^{j-1}(x) - \tilde{f}_1^{j-1}(y)\big|,
\end{aligned}$$

for some $z_{j-1} \in [\tilde{f}_1^{j-1}(y), \tilde{f}_1^{j-1}(x)]$.

Therefore by (8) we have

$$\left|\ln \frac{D\tilde{f}_1^n(x)}{D\tilde{f}_1^n(y)}\right| \leqslant \sum_{j=1}^{n} \left|\frac{D^2\tilde{f}_j(z_{j-1})}{D\tilde{f}_j(z_{j-1})}\right| \leqslant C_1 \cdot \sum_{j=1}^{n} \delta_j \leqslant C_1 C_2 = C_4.$$

Taking $y \in [0,1]$ such that $D\tilde{f}_1^n(y) = 1$ we have the result. □

Lemma 2.5. *There is $C_5 = C_5(C_1, C_2) > 0$ such that*

$$\big|D^2\tilde{f}_1^n(x)\big| \leqslant C_5,$$

for all $x \in [0,1]$ and for all $n \geqslant 0$.

Proof. Note that

$$\begin{aligned}
\big|D^2\tilde{f}_1^n(x)\big| &= \left|\frac{D^2\tilde{f}_1^n(x)}{D\tilde{f}_1^n(x)}\right| \cdot \big|D\tilde{f}_1^n(x)\big| \\
&= \left|\sum_{j=1}^{n} \frac{D^2\tilde{f}_j(\tilde{f}_1^{j-1}(x))}{D\tilde{f}_j(\tilde{f}_1^{j-1}(x))} \cdot D\tilde{f}_1^{j-1}(x)\right| \cdot \big|D\tilde{f}_1^n(x)\big| \\
&\leqslant e^{2C_4} \sum_{j=1}^{n} \left|\frac{D^2\tilde{f}_j(\tilde{f}_1^{j-1}(x))}{D\tilde{f}_j(\tilde{f}_1^{j-1}(x))}\right| \\
&\leqslant e^{2C_4} \cdot C_1 \cdot \sum_{j=1}^{n} \delta_j \leqslant e^{2C_4} \cdot C_1 C_2 = C_5. \qquad \square
\end{aligned}$$

Lemma 2.6. *There are $C_6 = C_6(C_1, C_2)$, $C_7 = (C_1, C_2) > 0$ such that*

$$e^{-C_6} \leqslant \big|DM_1^n(x)\big| \leqslant e^{C_6}, \qquad \big|D^2M_1^n(x)\big| \leqslant C_7, \qquad \big|D^3M_1^n(x)\big| \leqslant C_8$$

for all $x \in [0,1]$ and for all $n \geqslant 0$.

Proof. Since $\mathcal{M}$ is a commutative Lie group, we have $M_1^n = M_N$, where $N = \sum_{i=1}^n N_{\tilde{f}_i}$. By Eq. (8) we have $|N_{\tilde{f}_i}| < C_1\delta_i$, so

$$|N| \leqslant C_1 \cdot C_2.$$

One can easily use Eq. (2) to obtain estimates for $D^i M_1^n$, $i = 1, 2, 3$. □

Proof of Proposition 2.3. We write

$$\tilde{f}_1^n - M_1^n = \sum_{i=1}^n M_{i+1}^n \circ \tilde{f}_1^i - M_i^n \circ \tilde{f}_1^{i-1},$$

where $M_{n+1}^n = f_1^0 = \mathrm{Id}$. Then

$$\begin{aligned}
\left|\tilde{f}_1^n(x) - M_1^n(x)\right| &\leqslant \sum_{i=1}^n \left|M_{i+1}^n\left(\tilde{f}_i \circ \tilde{f}_1^{i-1}\right)(x) - M_{i+1}^n\left(M_i \circ \tilde{f}_1^{i-1}\right)(x)\right| \\
&\leqslant e^{C_6} \cdot \sum_{i=1}^n \left|\tilde{f}_i \circ \tilde{f}_1^{i-1}(x) - M_i \circ \tilde{f}_1^{i-1}(x)\right| \\
&\leqslant e^{C_6} \cdot C \cdot \sum_{i=1}^n \delta_i^{1+\nu} \\
&\leqslant e^{C_6} \cdot C \cdot \left(\max_{1\leqslant i\leqslant n} \delta_i\right)^\nu \sum_i \delta_i \\
&\leqslant e^{C_6} \cdot C \cdot C_2 \cdot \left(\max_{1\leqslant i\leqslant n} \delta_i\right)^\nu,
\end{aligned}$$

where $C > 0$ is the constant given by Proposition 2.1;

$$\begin{aligned}
\left|D\tilde{f}_1^n(x) - DM_1^n(x)\right| &= \Bigg|\sum_{i=1}^n \Big[DM_{i+1}^n\left(\tilde{f}_i \circ \tilde{f}_1^{i-1}(x)\right) \cdot D\tilde{f}_i\left(\tilde{f}_1^{i-1}(x)\right) \\
&\qquad - DM_{i+1}^n\left(M_i \circ \tilde{f}_1^{i-1}(x)\right) \cdot DM_i\left(\tilde{f}_1^{i-1}(x)\right)\Big] D\tilde{f}_1^{i-1}(x)\Bigg| \\
&\leqslant e^{C_4} \cdot \sum_{i=1}^n \Big|DM_{i+1}^n\left(\tilde{f}_i \circ \tilde{f}_1^{i-1}(x)\right) \cdot D\tilde{f}_i\left(\tilde{f}_1^{i-1}(x)\right) \\
&\qquad - DM_{i+1}^n\left(M_i \circ \tilde{f}_1^{i-1}(x)\right) \cdot DM_i\left(\tilde{f}_1^{i-1}(x)\right)\Big|.
\end{aligned}$$

Now, add and subtract the term $DM_{i+1}^n(\tilde{f}_i \circ \tilde{f}_1^{i-1}(x)) \cdot DM_i(\tilde{f}_1^{i-1}(x))$ in the above expression to obtain

$$\begin{aligned}
\left|D\tilde{f}_1^n(x) - DM_1^n(x)\right| &\leqslant e^{C_4} \cdot e^{C_6} \cdot \sum_{i=1}^n \left|D\tilde{f}_i\left(\tilde{f}_1^{i-1}(x)\right) - DM_i\left(\tilde{f}_1^{i-1}(x)\right)\right| \\
&\quad + e^{C_4} \cdot e^{C_6} \cdot \sum_{i=1}^n \left|DM_{i+1}^n\left(\tilde{f}_i \circ \tilde{f}_1^{i-1}(x)\right) - DM_{i+1}^n\left(M_i \circ \tilde{f}_1^{i-1}(x)\right)\right| \\
&\leqslant e^{C_4} \cdot e^{C_6} \cdot C \cdot \sum_{i=1}^n \delta_i^{1+\nu} + e^{C_4} \cdot e^{C_6} \cdot C_7 \cdot C \cdot \sum_{i=1}^n \delta_i^{1+\nu} \\
&\leqslant e^{C_4} \cdot e^{C_6} \cdot (1 + C_7) \cdot C \cdot C_2 \cdot \left(\max_{1\leqslant i\leqslant n} \delta_i\right)^\nu.
\end{aligned}$$

Now note that

$$D^2\tilde{f}_1^n(x) - D^2M_1^n(x) = \text{(I)} + \text{(II)} + \text{(III)},$$

where

$$(\mathrm{I}) = \sum_{i=1}^{n} D^2 M_{i+1}^n\big(\tilde f_i \circ \tilde f_1^{i-1}(x)\big) \cdot \big(D\tilde f_i\big(\tilde f_1^{i-1}(x)\big)\big)^2 \cdot \big(D\tilde f_1^{i-1}(x)\big)^2$$
$$- \sum_{i=1}^{n} D^2 M_{i+1}^n\big(M_i \circ \tilde f_1^{i-1}(x)\big) \cdot \big(DM_i\big(\tilde f_1^{i-1}(x)\big)\big)^2 \cdot \big(D\tilde f_1^{i-1}(x)\big)^2,$$
$$(\mathrm{II}) = \sum_{i=1}^{n} DM_{i+1}^n\big(\tilde f_i \circ \tilde f_1^{i-1}(x)\big) \cdot D^2\tilde f_i\big(\tilde f_1^{i-1}(x)\big) \cdot \big(D\tilde f_1^{i-1}(x)\big)^2$$
$$- \sum_{i=1}^{n} DM_{i+1}^n\big(M_i \circ \tilde f_1^{i-1}(x)\big) \cdot D^2 M_i\big(\tilde f_1^{i-1}(x)\big) \cdot \big(D\tilde f_1^{i-1}(x)\big)^2$$

and

$$(\mathrm{III}) = \sum_{i=1}^{n} DM_{i+1}^n\big(\tilde f_i \circ \tilde f_1^{i-1}(x)\big) \cdot D\tilde f_i\big(\tilde f_1^{i-1}(x)\big) \cdot D^2\tilde f_1^{i-1}(x)$$
$$- \sum_{i=1}^{n} DM_{i+1}^n\big(M_i \circ \tilde f_1^{i-1}(x)\big) \cdot DM_i\big(\tilde f_1^{i-1}(x)\big) \cdot D^2\tilde f_1^{i-1}(x).$$

In (I) we first add and subtract the term

$$\sum_{i=1}^{n} D^2 M_{i+1}^n\big(\tilde f_i \circ \tilde f_1^{i-1}(x)\big) \cdot \big(DM_i\big(\tilde f_1^{i-1}(x)\big)\big)^2 \cdot \big(D\tilde f_1^{i-1}(x)\big)^2$$

and then we use Lemmas 2.4 and 2.6 with estimates for the first derivative of $\tilde f_1^n$ and M_1^n, to obtain

$$\big|(\mathrm{I})\big| \leqslant 2 \cdot \max\{C_9, C_{10}\} \cdot \Big(\max_{1\leqslant i\leqslant n} \delta_i\Big)^{\nu}, \tag{14}$$

where $C_9 = C_2 \cdot C_7 \cdot C \cdot e^{2C_4}(e^{C_4} + e^{C_6})$ and $C_{10} = C \cdot C_2 \cdot C_8 \cdot e^{2C_4} \cdot e^{C_6}$.

In (II), we first add and subtract the term

$$\sum_{i=1}^{n} DM_{i+1}^n\big(M_i \circ \tilde f_1^{i-1}(x)\big) \cdot D^2\tilde f_i\big(\tilde f_1^{i-1}(x)\big) \cdot \big(D\tilde f_1^{i-1}(x)\big)^2$$

and then we use Lemmas 2.4 and 2.6 with estimates for the first derivative of $\tilde f_1^n$ and M_1^n, to obtain

$$\big|(\mathrm{II})\big| \leqslant 2 \cdot \max\{C_{11}, C_{12}\} \cdot \Big(\max_{1\leqslant i\leqslant n} \delta_i\Big)^{\nu}, \tag{15}$$

where $C_{11} = C \cdot C_2 \cdot C_5 \cdot C_7 \cdot e^{2C_4}$ and $C_{12} = C \cdot C_2 \cdot e^{2C_4 + C_6}$.

Finally we add and subtract the expression

$$\sum_{i=1}^{n} DM_{i+1}^n\big(\tilde f_i \circ \tilde f_1^{i-1}(x)\big) \cdot DM_i\big(\tilde f_1^{i-1}(x)\big) \cdot D^2\tilde f_1^{i-1}(x)$$

in (III) and use again Lemmas 2.4 and 2.6, obtaining

$$\big|(\mathrm{III})\big| \leqslant 2 \cdot \max\{C_{13}, C_{14}\} \cdot \Big(\max_{1\leqslant i\leqslant n} \delta_i\Big)^{\nu}, \tag{16}$$

where $C_{13} = C \cdot C_2 \cdot C_5 \cdot e^{C_6}$ and $C_{14} = C \cdot C_2 \cdot C_5 \cdot C_7 \cdot e^{C_6}$.

Taking $C_3 = 6 \cdot \max\{C_9, C_{10}, C_{11}, C_{12}, C_{13}, C_{14}\}$ we get the result. □

3. Renormalization for genus one g.i.e.m.

Let $f \in \mathcal{B}_k^{2+\nu}$ be a g.i.e.m. with d intervals. If f has only one discontinuity then if we identify the endpoints of its domain I we obtain a piecewise smooth homeomorphism of the circle with irrational rotation. So we can apply the Denjoy Theorem for these maps, proving the nonexistence of wandering intervals. That is a main difference between genus on g.i.e.m. and those with higher genus (even piecewise affine g.i.e.m. with higher genus can have wandering intervals). In this section we will study this relation between genus one g.i.e.m. and piecewise homeomorphisms of the circle in more detail. The combinatorial analysis next somehow extends the results of Nogueira and Rudolph [11].

For the sake of simplify the notation, assume that f has only one discontinuity. Note that f can be seen as a g.i.e.m. in $f \in \mathcal{B}_k^{0+1}$ with *two* intervals. Indeed, if $I_\alpha = (c_\alpha, d_\alpha)$, where d_α is the unique point of discontinuity of f, define

$$J_A := \overline{\bigcup_{\pi_0(\beta)\leqslant\pi_0(\alpha)} I_\beta},$$
$$J_B := \overline{\bigcup_{\pi_0(\beta)>\pi_0(\alpha)} I_\beta}.$$

Then $(f, \{A, B\}, \{J_A, J_B\})$ is a g.i.e.m. with two intervals. We can either renormalize as a g.i.e.m. with d intervals, denoted by

$$R_d(f),\ R_d^2(f),\ R_d^3(f),\ \ldots$$

or as a g.i.e.m. with two intervals, denoted

$$R_2(f),\ R_2^2(f),\ R_2^3(f),\ \ldots.$$

We call R_d^i the i-th d-renormalization of f and R_2^i the i-th 2-renormalization of f. If we see f as a homeomorphism of the circle then we can do the usual renormalization of the circle. This sequence of renormalizations, denoted $R_{\mathrm{rot}}^i(f)$ turns out to be just an acceleration of the Rauzy–Veech induction consisting in the subsequence of 2-renormalizations $R_2^{n_i}(f)$ defined in the following way: $n_0 = 0$ and n_{i+1} is the first $n > n_i$ whose type is distinct from the type of the n_i-th 2-renormalization.

The relation between the d and 2-renormalizations is given by the following proposition.

Proposition 3.1. *Let f be a genus one g.i.e.m. with d intervals in $\mathcal{B}_k$ with only one discontinuity, where $\pi_1(\alpha_0) = 1$ and $\pi_0(\alpha_1) = 1$. One of the two cases occurs*

(A) *We have*

$$\overline{\bigcup_{\pi_1(\beta)\geqslant\pi_1(\alpha_1)} f(I_\beta)} \subset \overline{\bigcup_{\pi_0(\beta)\geqslant\pi_0(\alpha_0)} I_\beta}.$$

Then f is 2-renormalizable of type 0 and $R_2(f) = R_d^n(f)$, where n is the first d-renormalization where the letter α_* wins from letter α_0. Here α_* is such that $f(c_{\alpha_0}) \in I_{\alpha_*}$.

(B) *We have*

$$\overline{\bigcup_{\pi_0(\beta)\geqslant\pi_0(\alpha_0)} I_\beta} \subset \overline{\bigcup_{\pi_1(\beta)\geqslant\pi_1(\alpha_1)} f(I_\beta)}.$$

Then f is 2-renormalizable of type 1 and $R_2(f) = R_d^n(f)$, where n is the first d-renormalization where the letter α_1 wins from letter α_*. Here α_* is such that $c_{\alpha_*} \in f(I_{\alpha_1})$.

Proof. We are going to prove the claim (A). The proof of the claim (B) is similar. It is easy to see that when the letter α_* wins from the letter α_0 for the first time, it wins with type 0. Using the notation defined in Eq. (1) the Rauzy–Veech algorithm is given by

$$\big(p(1), p(2), \ldots, p(s), \ldots, p(d)\big) \xrightarrow{0} \big(p(2), \ldots, p(1), p(s), \ldots, p(d)\big), \tag{17}$$

where s is such that $p(s) = 1$, and

$$\big(p(1), \ldots, p(r), \ldots, p(d)\big) \xrightarrow{1} \big(p(1), \ldots, p(r), p(d), \ldots\big), \tag{18}$$

where r is such that $p(r) = d$.

As by assumption f has only one discontinuity we have that $p = (k \ldots d\ 1 \ldots k-1)$, where $\pi_1(\alpha_1) = k$.

We assert that iterating the algorithm N times, with $N = s + r \leqslant n - 1$, where n is such that the letter α_* wins from the letter α_0 for the first time, we obtain that

$$p^N = (k+s, \ldots, d, k-r, \ldots, k+s-1, 1, \ldots, k-r-1), \tag{19}$$

where $0 \leqslant s \leqslant d-k$ and $0 \leqslant r \leqslant k-1$ are such that

$$s = \#\{\varepsilon_m = 0\colon\ 0 \leqslant m \leqslant N\} \quad \text{and} \quad r = \#\{\varepsilon_m = 1\colon\ 0 \leqslant m \leqslant N\}.$$

For $N = 1$ the assertion is true because $s = 0$ and $r = 1$ or $s = 1$ and $r = 0$. Assume that the formula (19) holds for $N - 1$. Then by formulas (17) and (18) the assertion holds for N.

We know that $\varepsilon_n = 0$ and that $p^{n-1}(1) = d$. So

$$p = p^0 = (k \ldots d\ 1 \ldots k-1) \xrightarrow{\varepsilon_0} \cdots \xrightarrow{\varepsilon_{n-1}} p^{n-1} = (d\ j \ldots d-1\ 1 \ldots j-1).$$

Then $p^n = (j \ldots d-1\ d\ 1 \ldots j-1)$ which completes the proof. □

Corollary 3.2. *Let f be a genus one g.i.e.m. with d intervals in $\mathcal{B}_k$ with only one discontinuity. Then there exists a sequence $m_i < m_{i+1}$ such that $m_{i+1} - m_i < d$ and*

$$R_2^i(f) = R_d^{m_i}(f).$$

The next result gives us a relationship between $R_{\mathrm{rot}}(f)$, $R_2(f)$ and $R(f)$.

Proposition 3.3. *Let f be a g.i.e.m. such that $\gamma(f)$ is k-bounded. Then for all $i \geqslant 0$*

$$R_{\mathrm{rot}}^i(f) = R_2^{k_i}(f) = R_d^{n_{k_i}}(f).$$

Proof. The first equality follows by definition of R_{rot} and R_2. The second equality follows by Proposition 3.1. □

3.1. Bounded geometry for maps in $\mathcal{B}_k^{2+\nu}$

A classical result on the circle homeomorphisms of class P (absolutely continuous homeomorphisms on the circle with bounded variation derivative) is the following lemma, whose demonstration can be found in Herman [4].

Lemma 3.4. *Let f be a g.i.e.m. with genus. Let n_0 be the first n such that $R_d^{n_0} f$ has only one discontinuity and define n_i such that $R_d^{n_i} f = R_{\mathrm{rot}}^i(R_d^{n_0} f)$. Then for all $i \geqslant 0$ and $\alpha \in \mathcal{A}$,*

$$\exp(-V) \leqslant DR_d^{n_i} f(x) \leqslant \exp(V) \quad \textit{for all } x \in I_\alpha^{n_i},$$

where $V = \mathrm{Var}(\log Df)$.

Lemma 3.5. *Let $f \in \mathcal{B}_k^{2+\mu}$. There is $C_{16} > 0$ such that*

$$\exp(-C_{16}V) \leqslant DR_d^n f(x) \leqslant \exp(C_{16}V) \quad \textit{for all } x \in I^n,\ n \in \mathbb{N}.$$

Proof. Because f has k-bounded combinatorics, there exists C with the following property: Let $i \geqslant 0$ be such that $n_{k_i} \leqslant n < n_{k_{i+1}}$. Then for every $\alpha \in \mathcal{A}$ there exists $a \leqslant C$ such that $(R_d^n f)(x) = (R_d^{n_{k_i}} f)^a(x)$, for $x \in I_\alpha^n$. Now the lemma follows from Lemma 3.4. □

Lemma 3.6 *(Non-collapsing domains). Let $f \in \mathcal{B}_k^{2+\mu}$. There is $C_{17} > 1$ such that*

$$\frac{1}{C_{17}} \leqslant \frac{|I_\alpha^n|}{|I_\beta^n|} \leqslant C_{17}, \quad \text{for all } \alpha, \beta \in \mathcal{A} \text{ and } n \geqslant 0.$$

Proof. Note that by Lemma 3.5 we have that for all $\alpha \in \mathcal{A}$

$$\exp(-C_{16}V) \leqslant \frac{|R^n(f)(I_\alpha^n)|}{|I_\alpha^n|} \leqslant \exp(C_{16}V). \tag{20}$$

We claim that if the letter $\alpha_\star$ is the winner in the n-level then

$$\left|I_{\alpha_\star}^n\right|, \left|R^n(f)\left(I_{\alpha_\star}^n\right)\right| \leqslant \frac{k+1-\exp(-(2^k+1)C_{16}V)}{k+1}\left|I^n\right|.$$

Indeed, otherwise by (20)

$$\left|I_\alpha^n\right|, \left|R^n(f)\left(I_\alpha^n\right)\right| < \frac{\exp(-(2^k+1)C_{16}V)|I^n|}{k+1} < \frac{\min\{|I_{\alpha_\star}^n|, |R^n(f)(I_{\alpha_\star}^n)|\}}{\exp(2^k C_{16}V)k}$$

for every $\alpha \neq \alpha_\star$. As a consequence the letter $\alpha_\star$ will be the winner for at least k consecutive times, which contradicts $f \in \mathcal{B}_k$. So there exists $\delta \in (0,1)$ such that

$$\frac{|I^{n+1}|}{|I^n|} \geqslant 1-\delta \tag{21}$$

for every n. Note that by (20) we have

$$\left|I_\alpha^{n+1}\right| \leqslant \exp(C_{16}V)\left|I_\alpha^n\right|,$$

for every n and α. Moreover if $\alpha_\star$ is the winner and $\beta_\star$ is the loser at the n-th level we have

$$\left|I_{\beta_\star}^{n+1}\right| \leqslant \exp(2C_{16}V)\left|I_{\alpha_\star}^n\right|.$$

So fix $\alpha, \beta \in \mathcal{A}$. Since β loses transitively from α after at most k renormalizations, we obtain

$$\left|I_\beta^{n+k}\right| \leqslant \exp(2kC_{16}V)\left|I_\alpha^n\right|, \tag{22}$$

for every n, α and β. We claim that

$$\left|I_\alpha^n\right| \geqslant \frac{(1-\delta)^k \exp(-2kC_{16}V)}{d}\left|I^n\right|.$$

Indeed, otherwise by (22)

$$\left|I^{n+k}\right| = \sum_\beta \left|I_\beta^n\right| \leqslant (1-\delta)^k\left|I^n\right|,$$

which contradicts (21). □

Lemma 3.7 *(Non-collapsing images). Let $f \in \mathcal{B}_k^{2+\mu}$. There is a constant $C_{18} > 1$ such that*

$$\frac{1}{C_{18}} \leqslant \frac{|R^n(f)(I_\alpha^n)|}{|R^n(f)(I_\beta^n)|} \leqslant C_{18}, \quad \text{for all } \alpha, \beta \in \mathcal{A} \text{ and } n \geqslant 0.$$

Proof. Follows directly from Lemmas 3.5 and 3.6. □

Proposition 3.8. *Let $f \in \mathcal{B}_k^{2+\mu}$ and let $\alpha, \beta \in \mathcal{A}$ be the winner and loser letters of $R^n(f)$, respectively. Then there is $0 < \lambda_1 < \lambda_2 < 1$, such that*

$$\lambda_1 < \frac{|(R^n f)^{1-\varepsilon}(I_\beta^n)|}{|(R^n f)^\varepsilon(I_\alpha^n)|} < \lambda_2, \tag{23}$$

where $\varepsilon \in \{0,1\}$ is the type of $R^n(f)$.

Proof. If the quotient in (23) is too close to 0 then $(R^n f)^{1-\varepsilon}(I^n_\beta)$ is very small compared to I^n, which contradicts either Lemma 3.6 or Lemma 3.7. If the quotient in (23) is too close to 1 then $|(R^{n+1} f)^{\varepsilon}(I^{n+1}_\alpha)| = |(R^n f)^{\varepsilon}(I^n_\alpha)| - |(R^n f)^{1-\varepsilon}(I^n_\beta)|$ is very small compared to I^{n+1}, what again contradicts either Lemma 3.6 or Lemma 3.7. $\square$

By definition of renormalization operator we know that

$$[0,1) = \bigvee_{\alpha\in\mathcal{A}} \bigvee_{i=0}^{q^\alpha_n - 1} f^i\big(I^n_\alpha\big),$$

where $\bigvee$ means disjoint union. Thus the elements $f^i(I^n_\alpha)$ for all $\alpha \in \mathcal{A}$ and for all $0 \leqslant i \leqslant q^\alpha_n - 1$ form a partition which we denote by $\mathcal{P}^n$. The norm of $\mathcal{P}^n$ is defined by

$$\big|\mathcal{P}^n\big| = \max_{\substack{\alpha\in\mathcal{A}\\ 0\leqslant i\leqslant q^\alpha_n-1}} \big\{\big|f^i\big(I^n_\alpha\big)\big|\big\}.$$

The next result says that $|\mathcal{P}^n|$ tends to zero exponentially fast.

Proposition 3.9. *Let $f \in \mathcal{B}^{2+\nu}_k$. Then for n sufficiently large there is $\lambda = \lambda(\lambda_1, \lambda_2)$ with $0 < \lambda < 1$ such that*

$$\big|\mathcal{P}^{n+k}\big| \leqslant \lambda \cdot \big|\mathcal{P}^n\big|.$$

Proof. Let $f^{i_{n+k}}(I^{n+k}_\alpha) \in \mathcal{P}^{n+k}$. There are $\alpha_j \in \mathcal{A}$ and $0 \leqslant i_j \leqslant q^{\alpha_j}_j - 1$ for $n \leqslant j \leqslant n+k$ such that

$$f^{i_n}\big(I^n_{\alpha_n}\big) \supset \cdots \supset f^{i_j}\big(I^j_{\alpha_j}\big) \supset f^{i_{j+1}}\big(I^{j+1}_{\alpha_{j+1}}\big) \cdots \supset f^{i_{n+k}}\big(I^{n+k}_{\alpha_{n+k}}\big).$$

We claim that there exists j_0 such that α_{j_0} is the winner in the j_0-th level. Indeed if $\alpha = \alpha_n = \cdots = \alpha_{n+k}$, let j_0 be a level between levels n and $n+k$ such that α wins. Such level exists because $f \in \mathcal{B}_k$. Otherwise there exists j_0 such that $\alpha_{j_0+1} \neq \alpha_{j_0}$. This is only possible if α_{j_0} is the winner and α_{j_0+1} the loser in the j_0-th level. By Proposition 3.8 and Lemma 3.5 we have

$$\frac{|f^{i_{n+k}}(I^{n+k}_{\alpha_{n+k}})|}{|f^{i_n}(I^n_{\alpha_n})|} \leqslant \frac{|f^{i_{j_0+1}}(I^{j_0+1}_{\alpha_{j_0+1}})|}{|f^{i_{j_0}}(I^{j_0}_{\alpha_{j_0}})|} \leqslant \lambda < 1$$

for some λ that depends only on $V = \mathrm{Var}(\log Df)$, λ_1 and λ_2. $\square$

Proof of Theorem 1. Note that

$$Z_{I^n_\alpha}\big(R^n(f)\big) = Z_{f^{q^n_\alpha-1}(I^n_\alpha)}(f) \circ \cdots \circ Z_{f(I^n_\alpha)}(f) \circ Z_{I^n_\alpha}(f).$$

The intervals $f^i(I^n_\alpha)$, $i = 0, \ldots, q^n_\alpha - 1$, belong to the partition $\mathcal{P}^n$. In particular

$$\sum_{i=0}^{q^n_\alpha-1} \big|f^i\big(I^n_\alpha\big)\big| \leqslant 1$$

and by Proposition 3.9

$$\sup_{0\leqslant i<q^n_\alpha} \big|f^i\big(I^n_\alpha\big)\big| \leqslant \big|\mathcal{P}^n\big| \leqslant \lambda^{n/k-1}.$$

So we can apply Proposition 2.3 to obtain that

$$\big|Z_{I^n_\alpha}\big(R^n(f)\big) - M^n_1\big|_{C^2} \leqslant C_3 \cdot \lambda^{\mu(n/k-1)}.$$

Recall that $M_a \circ M_b = M_{a+b}$. So

$$M^n_1 = M_N,$$

where (see (9) and (4))

$$N = \sum_{i=0}^{q^n_\alpha-1} N_{Z_{f^i(I^n_\alpha)}(f)} = \sum_{i=0}^{q^n_\alpha-1} \int_{f^i(I^n_\alpha)} \frac{D^2 f(x)}{Df(x)}\,dx = \int_{I^n_\alpha} \frac{D^2 R^n(f)(x)}{DR^n(f)(x)}\,dx \quad \forall \alpha \in \mathcal{A}. \qquad \square$$

4. Symbolic representation

To prove Theorems 2 and 3 we need a finer understanding of the geometry of the partitions generated by the sequence of renormalizations. To this end we will introduce a certain symbolic representation for the dynamics, that is somehow a generalization of the symbolic representation introduced by Sinaĭ and Khanin [14]. Consider the set of *letters*

$$\mathcal{L} = \{(\alpha, \varepsilon, n) \text{ s.t. } \alpha \in \mathcal{A},\ \varepsilon \in \{0, 1\},\ n \in \mathbb{N}\}.$$

Define $\pi_3(\alpha, \varepsilon, n) = n$, $\pi_2(\alpha, \varepsilon, n) = \varepsilon$. We will use the notation a_i for $a_i \in \mathcal{L}$ such that $\pi_n(a_i) = i$.

In this section we construct a symbolic representation for the dynamics of a g.i.e.m. $f \in \mathcal{B}_k$. For each n we consider the partition of [0, 1] given by

$$\tilde{\mathcal{P}}^n = \{f^i(I^n_\alpha) \text{ s.t. } \alpha \in \mathcal{A} \text{ and } 1 \leqslant i \leqslant q^\alpha_n\}.$$

Let

$$\Lambda = [0, 1] \setminus \bigcup_n \bigcup_{J \in \tilde{\mathcal{P}}^n} \partial J.$$

We will define a function

$$s : \Lambda \to \mathcal{L}^{\mathbb{N}}$$

in the following way. We have $s(x) = (a_i)_{i \in \mathbb{N}}$, $a_i \in \mathcal{L}$. If $i = 0$ then $x \in f(I^0_\beta)$ for some β and we define $a_0 = (\beta, 0, 0)$. If $i > 0$, let

$$k_{i-1}(x) := \min\{k \geqslant 0 \text{ s.t. } f^k(x) \in I^{i-1}\}.$$

Then either $f^{k_{i-1}}(x) \in I^i$, so $f^{k_{i-1}}(x) \in f_i(I^i_\beta)$ for some β and we define $a_i = (\beta, 0, i)$, or $f^{k_{i-1}}(x) \in I^{i-1} \setminus I^i$, so $f_{i-1}(f^{k_{i-1}}(x)) \in f_i(I^i_\beta)$ for some β and we define $a_i = (\beta, 1, i)$. Note that in any case $f^{k_i(x)}(x) \in I^i_\beta$ and $\pi_2(a_i) = 0$ if and only if $k_i(x) = k_{i-1}(x)$.

4.1. Admissible cylinders and their properties

Given a finite subset $S = \{n_1, n_2, \ldots, n_k\} \subset \mathbb{N}$, with $\#S = k$ and a finite sequence $a_{n_1}, \ldots, a_{n_k} \in \mathcal{L}$, with $\pi_3(a_{n_i}) = n_i$ we can consider the *word*

$$\omega = a_{n_k} a_{n_{k-1}} \ldots a_{n_1}.$$

For each word we can define the cylinder

$$[\omega] = [a_{n_k} a_{n_{k-1}} \ldots a_{n_1}] = \overline{\{x \in \Lambda \text{ s.t. } s_{n_i}(x) = a_{n_i},\ 1 \leqslant i \leqslant k\}}.$$

If this cylinder is not empty we will say that the word ω is *admissible*. Indeed we can give a definition of admissible words just in terms of the combinatorial data of the g.i.e.m. f.

We claim that the set whose elements are the closures of intervals in $\tilde{\mathcal{P}}^n$ is exactly the set of admissible cylinders of the form $[a_n a_{n-1} \ldots a_0]$. Indeed when $n = 0$ we have $\tilde{\mathcal{P}}^0 = \{f(I_\alpha)\}_{\alpha \in \mathcal{A}}$. Then

$$\overline{f(I_\alpha)} = [(\alpha, 0, 0)].$$

Suppose by induction that we have verified that the set of all elements $\overline{f^i(I^n_\alpha)}$, $1 \leqslant i \leqslant q^n_\alpha$, is the set of admissible cylinders of the form $[a_n a_{n-1} \ldots a_0]$. Recall that $\alpha^n(\varepsilon), \alpha^n(1-\varepsilon) \in \mathcal{A}$ are the winner and loser, respectively, if f_n has type ε. There are three cases:

- If $\alpha \neq \alpha^n(1-\varepsilon)$ then $I^{n+1}_\alpha \subset I^n_\alpha$ and $q^\alpha_{n+1} = q^\alpha_n$. So

  $$\overline{f^i(I^{n+1}_\alpha)} = [a_{n+1} a_n \ldots a_0]$$

  for every $1 \leqslant i \leqslant q^\alpha_{n+1} = q^\alpha_n$, where $a_{n+1} = (\alpha, 0, n+1)$ and $\overline{f^i(I^n_\alpha)} = [a_n \ldots a_0]$.

- If $\alpha = \alpha^n(1-\varepsilon)$ and $1 \leqslant i \leqslant q_n^{\alpha^n(\varepsilon)}$ we have
$$\overline{f^{i+(1-\varepsilon)q_n^{\alpha^n(1-\varepsilon)}}\big(I^{n+1}_{\alpha^n(1-\varepsilon)}\big)} = [a_{n+1}a_n \dots a_0]$$
where $a_{n+1} = (\alpha^n(1-\varepsilon), \varepsilon, n+1)$ and $\overline{f^i(I^n_{\alpha^n(\varepsilon)})} = [a_n \dots a_0]$.
- If $\alpha = \alpha^n(1-\varepsilon)$ and $1 \leqslant j \leqslant q_n^{\alpha^n(1-\varepsilon)}$ we have
$$\overline{f^{j+\varepsilon q_n^{\alpha^n(\varepsilon)}}\big(I^{n+1}_{\alpha^n(1-\varepsilon)}\big)} = [a_{n+1}a_n \dots a_0],$$
where $a_{n+1} = (\alpha^n(1-\varepsilon), 1-\varepsilon, n+1)$ and $\overline{f^j(I^n_{\alpha^n(1-\varepsilon)})} = [a_n \dots a_0]$.

As a consequence, for any admissible word of the form $a_n \dots a_0$, with $a_n = (\alpha, \chi, n)$ we have that the first entry times $k_1, k_2, \dots, k_n$ are constant functions in $[a_n \dots a_0]$ and $f^{k_n}[a_n \dots a_0] = f_n(I^n_\alpha)$.

Denote by $\ell(a_n \dots a_0)$ the Lebesgue measure of the cylinder $[a_n \dots a_0]$. The proof of Theorem 2 will be based on the ergodic properties of the sequence of random variables $a_n = (\alpha_n, \chi_n, n)$, $\alpha_n \in \mathcal{A}$ and $\chi_n \in \{0,1\}$ with respect to the Lebesgue measure.

If the word $a_{n-1} \dots a_0$ is admissible we can consider the conditional probabilities
$$\ell(a_n | a_{n-1} \dots a_0) = \frac{\ell(a_n \dots a_0)}{\ell(a_{n-1} \dots a_0)}.$$

Lemma 4.1. *Let*
$$\omega_1 = a'_0 \dots a'_{n-1} a_n \dots a_{n+k},$$
$$\tilde{\omega}_1 = a''_0 \dots a''_{n-1} a_n \dots a_{n+k}$$
be admissible words. Denote $\omega_2 = a'_0 \dots a'_{n-1}$ *and* $\tilde{\omega}_2 = a''_0 \dots a''_{n-1}$. *Then*:

A. *Indeed* $\pi_1(a'_{n-1}) = \pi_1(a''_{n-1}) =: \beta$ *and there exist* $1 \leqslant i, j \leqslant q_\beta^{n-1}$ *such that* $[\omega_2] = \overline{f^i(I^{n-1}_\beta)}$ *and* $[\tilde{\omega}_2] = \overline{f^j(I^{n-1}_\beta)}$.
B. *In particular* $r = j - i$ *is the unique* $r \in \mathbb{Z}$ *such that* f^r *is a diffeomorphism on* $[\omega_2]$ *and* $f^r([\omega_2]) = [\tilde{\omega}_2]$.
C. *The integer* r *is also the unique integer such that* f^r *is a diffeomorphism on* $[\omega_1]$ *and* $f^r([\omega_1]) = [\tilde{\omega}_1]$.

Proof. The uniqueness of r follows from the fact that f does not have periodic points. Indeed if f^{r_1} and f^{r_2}, $r_1 < r_2$, are diffeomorphisms on $[\omega_i]$ and $f^r([\omega_i]) = [\tilde{\omega}_i]$ for some $i \in \{1,2\}$ then the points in $\partial[\omega_i]$ are fixed points of $f^{r_2 - r_1}$, which is a contradiction. It remains to show the existence of r. We will prove this by induction on k. Suppose $k = 0$. Denote $a_n = (\alpha, \chi, n)$. Let ε be the type of f_{n-1}. By the previous discussion about the partitions $\tilde{\mathcal{P}}^n$, there are three cases.

Case (i). If $\alpha \neq \alpha^{n-1}(1-\varepsilon)$ then $\chi = 0$ and
$$\big[a_n a'_{n-1} \dots a'_0\big] = \overline{f^i\big(I^n_\alpha\big)},$$
$$\big[a_n a''_{n-1} \dots a''_0\big] = \overline{f^j\big(I^n_\alpha\big)},$$
for some $1 \leqslant i, j \leqslant q_n^\alpha = q_{n-1}^\alpha$, with $\overline{f^i(I^{n-1}_\alpha)} = [a'_{n-1} \dots a'_0]$ and $\overline{f^j(I^{n-1}_\alpha)} = [a''_{n-1} \dots a''_0]$. In particular $\alpha = \pi_1(a'_{n-1}) = \pi_1(a''_{n-1})$. So take $r = i - j$.

Case (ii). If $\alpha = \alpha^{n-1}(1-\varepsilon)$ and $\chi = \varepsilon$ then
$$\overline{f^{i+(1-\varepsilon)q^\alpha_{n-1}}\big(I^n_\alpha\big)} = \big[a_n a'_{n-1} \dots a'_0\big],$$
$$\overline{f^{j+(1-\varepsilon)q^\alpha_{n-1}}\big(I^n_\alpha\big)} = \big[a_n a''_{n-1} \dots a''_0\big],$$
for some $1 \leqslant i, j \leqslant q_{n-1}^{\alpha^{n-1}(\varepsilon)}$ with $[a'_{n-1} \dots a'_0] = \overline{f^i(I^{n-1}_{\alpha^{n-1}(\varepsilon)})}$ and $[a''_{n-1} \dots a''_0] = \overline{f^j(I^{n-1}_{\alpha^{n-1}(\varepsilon)})}$. In particular $\alpha^{n-1}(\varepsilon) = \pi_1(a'_{n-1}) = \pi_1(a''_{n-1})$. So take $r = i - j$.

Case (iii). If $\alpha = \alpha^{n-1}(1-\varepsilon)$ and $\chi = 1-\varepsilon$ then

$$\overline{f^{i+\varepsilon q_{n-1}^{\alpha^{n-1}(\varepsilon)}}\bigl(I_\alpha^n\bigr)} = \bigl[a_n a'_{n-1}\dots a'_0\bigr],$$
$$\overline{f^{j+\varepsilon q_{n-1}^{\alpha^{n-1}(\varepsilon)}}\bigl(I_\alpha^n\bigr)} = \bigl[a_n a''_{n-1}\dots a''_0\bigr],$$

for some $1 \leqslant i, j \leqslant q_{n-1}^\alpha$ with $[a'_{n-1}\dots a'_0] = \overline{f^i(I_\alpha^{n-1})}$ and $[a''_{n-1}\dots a''_0] = \overline{f^j(I_\alpha^{n-1})}$. Again we have $\alpha = \pi_1(a'_{n-1}) = \pi_1(a''_{n-1})$. Take $r = i - j$.

This completes the proof for $k = 0$. Suppose by induction we have proved the statement for $k-1 \geqslant 0$. By the case $k = 0$ there exists a unique r such that $f^r[a'_0\dots a'_{n-1}a_n\dots a_{n+k-1}] = [a''_0\dots a''_{n-1}a_n\dots a_{n+k-1}]$ in a diffeomorphic way, and moreover r is the unique r such that $f^r[a'_0\dots a'_{n-1}a_n\dots a_{n+k}] = [a''_0\dots a''_{n-1}a_n\dots a_{n+k}]$. By induction assumption there exists a unique r' such that $f^r[a'_0\dots a'_{n-1}] = [a''_0\dots a''_{n-1}]$ and moreover r' is the unique integer such that

$$f^r\bigl[a'_0\dots a'_{n-1}a_n\dots a_{n+k-1}\bigr] = \bigl[a''_0\dots a''_{n-1}a_n\dots a_{n+k-1}\bigr].$$

So $r = r'$. This completes the proof. □

Lemma 4.2. *There are constants $C_{19} > 0$ and $0 < \lambda_3 = \lambda_3(\lambda) < 1$ such that*

$$e^{-C_{19}\lambda_3^s} \leqslant \frac{\ell(a_n|a_{n-1}\dots a_{n-s}a''_{n-s-1}\dots a''_0)}{\ell(a_n|a_{n-1}\dots a_{n-s}a'_{n-s-1}\dots a'_0)} \leqslant e^{C_{19}\lambda_3^s},$$

provided both words are admissible.

Proof. Let

$$f^{i_3}\bigl(I_\alpha^{n-s}\bigr) \subset f^{i_2}\bigl(I_\beta^n\bigr) \subset f^{i_1}\bigl(I_\gamma^{n+1}\bigr),$$

with $1 \leqslant i_3 \leqslant q_\alpha^{n-s}$, be the intervals corresponding to words

$$\bigl(a'_0,\dots,a'_{n-s-1},a_{n-s},\dots,a_n\bigr),\ \bigl(a'_0,\dots,a'_{n-s-1},a_{n-s},\dots,a_{n-1}\bigr),\ \bigl(a'_0,\dots,a'_{n-s-1}\bigr),$$

respectively. By Lemma 4.1 there is $j \in \mathbb{Z}$ with $1 \leqslant i_3 + j \leqslant q_{n-s}^{\pi_1(a_{n-s})}$, such that

$$f^{i_3+j}\bigl(I_\alpha^{n-s}\bigr) \subset f^{i_2+j}\bigl(I_\beta^n\bigr) \subset f^{i_1+j}\bigl(I_\gamma^{n+1}\bigr)$$

are the intervals corresponding to words

$$\bigl(a''_0,\dots,a''_{n-s-1},a_{n-s},\dots,a_n\bigr),\ \bigl(a''_0,\dots,a''_{n-s-1},a_{n-s},\dots,a_{n-1}\bigr) \quad \text{and} \quad \bigl(a''_0,\dots,a''_{n-s-1}\bigr),$$

respectively. Denote

$$\rho_k := \frac{|f^{i_2}(I_\beta^{n+1})|\cdot|f^{i_1+k}(I_\gamma^n)|}{|f^{i_1}(I_\gamma^n)|\cdot|f^{i_2+k}(I_\beta^{n+1})|}, \quad 0 \leqslant k \leqslant j.$$

We have

$$\begin{aligned}
\rho_{k+1} &:= \frac{\int_{f^{i_1+k}(I_\gamma^n)} Df(s)\,ds}{\int_{f^{i_2+k}(I_\beta^{n+1})} Df(s)\,ds}\cdot\frac{|f^{i_2}(I_\beta^{n+1})|}{|f^{i_1}(I_\gamma^n)|}\\
&= \frac{Df(s_{i_1+k})|f^{i_1+k}(I_\gamma^n)|}{Df(s_{i_2+k})|f^{i_2+k}(I_\beta^{n+1})|}\cdot\frac{|f^{i_2}(I_\beta^{n+1})|}{|f^{i_1}(I_\gamma^n)|}\\
&= \frac{Df(s_{i_1+k})}{Df(s_{i_2+k})}\cdot\rho_k,
\end{aligned}$$

where $s_{i_1+k} \in f^{i_1+k}(I_\gamma^n)$ and $s_{i_2+k} \in f^{i_2+k}(I_\beta^{n+1})$. Furthermore,

$$\exp\{-C_1 \cdot |f^{i_1+k}(I_\gamma^n)|\} \leqslant \frac{Df(s_{i_1+k})}{Df(s_{i_2+k})} \leqslant \exp\{C_1 \cdot |f^{i_1+k}(I_\gamma^n)|\}. \tag{24}$$

Then by (24) we have

$$\exp\left\{-C_1 \sum_{t=0}^{j-1} |f^{i_1+t}(I_\gamma^n)|\right\} \leqslant \rho_j \leqslant \exp\left\{C_1 \sum_{t=0}^{j-1} |f^{i_1+t}(I_\gamma^n)|\right\}.$$

However by Proposition 3.9,

$$\sum_{t=0}^{j-1} |f^{i_1+t}(I_\gamma^n)| = \sum_{t=0}^{j-1} |f^{i_3+t}(I_\alpha^{n-s})| \cdot \frac{|f^{i_1+t}(I_\gamma^n)|}{|f^{i_3+t}(I_\alpha^{n-s})|} \leqslant C_{20} \cdot \lambda_3^s,$$

where $C_{20} = C_{20}(C_{17}, C_{18}, \lambda) > 0$. Taking $C_{19} = C_1 \cdot C_{20}$ we obtain the result. □

Lemma 4.3. *There exists $C_{21} = C_{21}(C_{19}, \lambda) > 0$ such that for all n, m,*

$$e^{-C_{21}} \leqslant \frac{\ell(a_{n+m}, \dots, a_n | a'_{n-1}, \dots, a'_0)}{\ell(a_{n+m}, \dots, a_n | a''_{n-1}, \dots, a''_0)} \leqslant e^{C_{21}},$$

provided both words $(a'_0, \dots, a'_{n-1}, a_n, \dots, a_{n+m})$ and $(a''_0, \dots, a''_{n-1}, a_n, \dots, a_{n+m})$ are admissible.

Proof. The proof follows easily from Lemma 4.2 and the equations

$$\ell(a_{n+m}, \dots, a_n | a'_{n-1}, \dots, a'_0) = \prod_{i=0}^{m} \ell(a_{n+i} | a_{n+i-1}, \dots, a'_{n-1}, \dots, a'_0),$$
$$\ell(a_{n+m}, \dots, a_n | a''_{n-1}, \dots, a''_0) = \prod_{i=0}^{m} \ell(a_{n+i} | a_{n+i-1}, \dots, a''_{n-1}, \dots, a''_0). \qquad \square$$

Lemma 4.4. *Let $a_k \dots a_n$, $k \leqslant n$, and $a'_n a'_{n+1} \dots a'_{n+m}$ be two admissible words, such that $a_n = (\alpha, \chi, n)$, $a'_n = (\alpha, \chi', n)$. Then*

$$a_k \dots a_n a'_{n+1} \dots a'_{n+m}$$

is admissible.

Proof. Since $a_k \dots a_n$ is admissible then there exist $a_0, a_1, \dots, a_{k-1}$ such that $a_0 \dots a_{k-1} a_k \dots a_n$ is admissible. If $a'_n a'_{n+1} \dots a'_{n+m}$ is admissible then there exists an admissible word with the form $a'_0 \dots a'_n a'_{n+1} \dots a'_{n+m}$. Note that the functions $k_1(x) = k'_1$, $k_2(x) = k'_2$, $\dots$, $k_{n+m}(x) = k'_{n+m}$ are constant in $[a'_0 \dots a'_n a'_{n+1} \dots a'_{n+m}]$ and

$$f^{k'_n}[a'_0 \dots a'_n a'_{n+1} \dots a'_{n+m}] \subset f_n(I_\alpha^n).$$

The functions $k_1(x) = k_1$, $k_2(x) = k_2$, $\dots$, $k_n(x) = k_n$ are constant in $[a_0 \dots a_n]$ and

$$f^{k_n}[a_0 \dots a_n] = f_n(I_\alpha^n).$$

In particular every x in the nonempty set

$$J = f^{-k_n} f^{k'_n}[a'_0 \dots a'_n a'_{n+1} \dots a'_{n+m}] \cap \Lambda \subset [a_0 \dots a_n]$$

belongs to the cylinder $[a_0 \dots a_n a'_{n+1} \dots a'_{n+m}]$. Indeed, since $x \in [a_0 \dots a_n]$ we have $s_i(x) = a_i$, $k_i(x) = k_i$, for $0 \leqslant i \leqslant n$. Note that $f^{k_n}(x) = f^{k'_n}(y)$, for some $y \in [a'_0 \dots a'_n a'_{n+1} \dots a'_{n+m}] \cap \Lambda$. Then

$$k_i(x) = k_i(y) - k'_n + k_n = k'_i - k'_n + k_n$$

for $n \leqslant i \leqslant n+m$, since

$$f^{k'_i - k'_n + k_n}(x) = f^{k'_i}(y) \in I^i,$$

and moreover if $k_n \leqslant j < k'_i - k'_n + k_n$ then

$$f^j(x) = f^{j-k_n+k'_n}(y) \notin I^i,$$

since $0 \leqslant j - k_n + k'_n < k'_i = k_i(y)$ and if $j < k_n$ then

$$f^j(x) \notin I^i,$$

because $j < k_n \leqslant k_i(x)$. This implies that $s_i(x) = a'_i$ for $n < i \leqslant n+m$. □

Lemma 4.5. *Let $\alpha, \beta \in \mathcal{A}$. For each n there is an admissible word $a_n \dots a_{n+k}$ with $\pi_1(a_n) = \beta$, $\pi_1(a_{n+k}) = \alpha$.*

Proof. Firstly we claim that if $\alpha \in \mathcal{A}$ does not lose in the n-th level then there is an admissible word $b_n b_{n+1}$ such that $\pi_1(b_n) = \pi_1(b_{n+1}) = \alpha$. Indeed in this case $f_{n+1}(I_\alpha^{n+1}) \subset f_n(I_\alpha^n)$. This implies that the word $(\alpha, 0, n)(\alpha, 0, n+1)$ is admissible.

Second, if α loses and $\beta \in \mathcal{A}$ wins in the n-th level then there are an admissible word $b_n b_{n+1}$ such that $\beta = \pi_1(b_n)$ and $\alpha = \pi_1(b_{n+1})$ and an admissible word $b'_n b'_{n+1}$ such that $\alpha = \pi_1(b'_n) = \pi_1(b'_{n+1})$. Indeed if f_n has type 1 then we have that $I_\alpha^n \subset f_n(I_\beta^n)$, I_α^n is not inside I^{n+1} and enters I^{n+1} after one iteration of f_n, landing in $f_n(I_\alpha^n) = f_{n+1}(I_\alpha^{n+1})$, so the word $(\beta, 0, n)(\alpha, 1, n+1)$ is admissible. Note also that $f_n(I_\alpha^n) = f_{n+1}(I_\alpha^{n+1})$ so $(\alpha, 0, n)(\alpha, 0, n+1)$ is admissible. If f_n has type 0 then we have that $f_{n+1}(I_\alpha^{n+1}) \subset f_n(I_\beta^n)$, so the word $(\beta, 0, n)(\alpha, 0, n+1)$ is admissible. Furthermore $f_n(I_\alpha^n)$ is not inside I^{n+1} and it enters I^{n+1} after one iteration of f_n, landing in $f_{n+1}(I_\alpha^{n+1})$, so the word $(\alpha, 0, n)(\alpha, 1, n+1)$ is admissible.

In particular, using Lemma 4.4 it follows that for every $m \geqslant 0$, $p > 0$ and $\alpha \in \mathcal{A}$ there exists a word $\omega = a_m a_{m+1} \dots a_{m+p}$ such that $\pi_1(a_i) = \alpha$ for every $m \leqslant i \leqslant m+p$.

Now suppose that β wins from α in the $(m-1)$-th renormalization. Then as we saw above $(\beta, 0, m-1)(\alpha, \epsilon_{m-1}, m)$ and ω are admissible. By Lemma 4.4 there exists a word $(\beta, 0, m-1)a'_m a_{m+1} \dots a_{m+p}$ such that $\pi_1(a_{m+p}) = \alpha$.

Finally, since $f \in \mathcal{B}_k$, given $\alpha, \beta \in \mathcal{A}$, there exists a sequence of letters α_i, and levels n_i, $i \leqslant s$, $n \leqslant n_i < n_{i+1} \leqslant n+k$ for every $i < s$, such that $\alpha_0 = \beta$, $\alpha_s = \alpha$ and α_i wins from α_{i+1} in the n_i-th level. So there are admissible words $a^i_{n_i} \dots a^i_{n_{i+1}}$ such that $\pi_1(a^i_{n_i}) = \alpha_i$ and $\pi_1(a^i_{n_{i+1}}) = \alpha_{i+1}$. By Lemma 4.4 there is an admissible word $a_{n_0} \dots a_{n_s}$ such that $\pi_1(a_{n_0}) = \beta$ and $\pi_1(a_{n_s}) = \alpha$.

Since we already proved that there exist admissible words $b_n \dots b_{n_0}$ and $c_{n_s} \dots c_{n+k}$ such that $\pi_1(b_{n_0}) = \alpha$, $\pi_1(b_{n_s}) = \alpha$, $\pi_1(c_{n_s}) = \beta$, $\pi_1(c_{n+k}) = \beta$, by Lemma 4.4 again there exists a word of type

$$b'_n \dots b_{n_0} a_{n_0+1} \dots a_{n_s-1} c_{n_s} \dots c'_{n+k}$$

with $\pi_1(b'_n) = \alpha$ and $\pi_1(c'_{n+k}) = \beta$. □

The proof of the following lemma is simple:

Lemma 4.6. *There exists $C_{22} > 0$ such that for all n, m, and all admissible words $a'_0 \dots a'_{n-k}$, $a''_0 \dots a''_{n-k}$, $a_n \dots a_{n+m}$*

$$e^{-C_{22}} \leqslant \frac{\ell(a_{n+m} \dots a_n | a'_{n-k} \dots a'_0)}{\ell(a_{n+m} \dots a_n | a''_{n-k} \dots a''_0)} \leqslant e^{C_{22}}.$$

Proposition 4.7. *There are $C_{23} > 0$ and $0 < \lambda_4 < 1$ such that*

$$\left|\ell(a_n | a_{n-r} \dots a_0) - \ell(a_n)\right| \leqslant C_{23} \cdot \lambda_4^{\sqrt{r}},$$

where $r = [\frac{n}{2}]$.

Proof. Indeed Proposition 4.7 is a Markov ergodic theorem and it can be proved by the methods of the theory of Markov chains, as in Khanin and Sinaĭ [7] (see also Sinaĭ [13]). Thus, we shall describe only the main steps. Fix an integer m, $m \sim \sqrt{\frac{n}{2}}$, and introduce a new probability measure on the words of the form

$$\tilde{a} = (a_n a_{n-1} \dots a_{n-m+k} a_{n-m} \dots a_{n-2m+k} a_{n-2m} \dots a_{n-3m+k} \dots a_{n-(i-1)m} \dots a_{n-im+k} a_{n-im} \dots a_0)$$

by the formula

$$\ell'(\tilde{a}) = \ell(a_0 \dots a_{n-im})\ell(a_{n-(i-1)m} \dots a_{n-im+3}|a_{n-im} \dots a_0)$$
$$\times \prod_{s=0}^{i-2} \ell(a_{n-sm} \dots a_{n-(s+1)m+k}|a_{n-(s+1)m} \dots a_{n-(s+2)m+k}).$$

Here $i \sim \sqrt{\frac{n}{2}}$. It follows easily from Lemma 4.2 that

$$\exp\bigl(-C_{19} \cdot \lambda_3^m \cdot i\bigr) \leqslant \frac{\ell'(\tilde{a})}{\ell(\tilde{a})} \leqslant \exp\bigl(C_{19} \cdot \lambda_3^m \cdot i\bigr). \tag{25}$$

Lemma 4.6 shows that the Markov transition operator corresponding to ℓ' for the transition to m steps is a contraction for the appropriate Cayley–Hilbert metric, and this contraction is *uniformly* smaller than 1 on each step. Then the usual Ergodic Theorem for Markov chains shows that the conditional probabilities $\ell'(a_n|a_{n-im} \dots a_0)$ asymptotically do not depend on $a_{n-im} \dots a_0$. Due to (25) the same holds for $\ell(a_n|a_{n-im} \dots a_0)$. This gives the desired result. $\square$

Denote by $\ell(\alpha, \star, n)$ the Lebesgue measure of the union $[(\alpha, 0, n)] \cup [(\alpha, 1, n)]$. Note that

$$\ell(\alpha, \star, n) = \frac{\sum_{i=1}^{q_\alpha^n} |f^i(I_\alpha^n)|}{|I|}.$$

Proof of Theorem 2. For simplify the notation we use f_n to denote $R^n(f)$. Let $r = [\frac{n}{2}]$. We rewrite $\int_{I_\alpha^n} \frac{D^2 f_n(s)}{Df_n(s)}\, ds$ in the following way. By the mean value theorem for integrals

$$\int_{I_\alpha^n} \frac{D^2 f_n(s)}{Df_n(s)}\, ds = \sum_{\beta \in \mathcal{A}} \sum_{j=1}^{q_r^\beta} \sum_{f^i(I_\alpha^n) \subset f^j(I_\beta^r)} \int_{f^i(I_\alpha^n)} \frac{D^2 f(s)}{Df(s)}\, ds$$
$$= \sum_{\beta \in \mathcal{A}} \sum_{j=1}^{q_r^\beta} \sum_{f^i(I_\alpha^n) \subset f^j(I_\beta^r)} \frac{D^2 f(x_j^\alpha)}{Df(x_j^\alpha)} \cdot \bigl|f^i\bigl(I_\alpha^n\bigr)\bigr|,$$

where $x_i^\alpha \in f^i(I_\alpha^n)$. In a similar way we can choose $y_j^\beta \in f^j(I_\beta^r)$ such that

$$\int_I \frac{D^2 f(s)}{Df(s)}\, ds = \sum_{\beta \in \mathcal{A}} \sum_{j=1}^{q_r^\beta} \frac{D^2 f(y_j^\beta)}{Df(y_j^\beta)} \cdot \bigl|f^j\bigl(I_\beta^r\bigr)\bigr|.$$

So

$$\int_{I_\alpha^n} \frac{D^2 f_n(s)}{Df_n(s)}\, ds = \sum_{\beta \in \mathcal{A}} \sum_{j=1}^{q_r^\beta} \sum_{f^i(I_\alpha^n) \subset f^j(I_\beta^r)} \left(\frac{D^2 f(x_j)}{Df(x_j)} - \frac{D^2 f(y_j^\beta)}{Df(y_j^\beta)}\right) \cdot \bigl|f^i\bigl(I_\alpha^n\bigr)\bigr|$$
$$+ \sum_{\beta \in \mathcal{A}} \sum_{j=1}^{q_r^\beta} \sum_{f^i(I_\alpha^n) \subset f^j(I_\beta^r)} \frac{D^2 f(y_j^\beta)}{Df(y_j^\beta)} \cdot \bigl|f^i\bigl(I_\alpha^n\bigr)\bigr|.$$

Due to the smooth properties of f the first term is at most $C_{24} \cdot \lambda_6^{\frac{n}{2}\nu}$, where $C_{24} = C_{24}(\lambda, k) > 0$ and $0 < \lambda_6 = \lambda_6(\lambda, k) < 1$. We will now analyze the second term:

$$\sum_{\beta \in \mathcal{A}} \sum_{j=1}^{q_r^\beta} \sum_{f^i(I_\alpha^n) \subset f^j(I_\beta^r)} \frac{D^2 f(y_j^\beta)}{Df(y_j^\beta)} \cdot \bigl|f^i\bigl(I_\alpha^n\bigr)\bigr|$$

$$= \sum_{\beta \in \mathcal{A}} \sum_{j=1}^{q_r^\beta} \frac{D^2 f(y_j^\beta)}{Df(y_j^\beta)} \cdot \left|f^j\left(I_\beta^r\right)\right| \frac{\sum_{f^i(I_\alpha^n) \subset f^j(I_\beta^r)} |f^i(I_\alpha^n)|}{|f^j(I_\beta^r)|}$$

$$= \sum_{\beta \in \mathcal{A}} \sum_{j=1}^{q_r^\beta} \frac{D^2 f(y_j^\beta)}{Df(y_j^\beta)} \cdot \left|f^j\left(I_\beta^r\right)\right| \cdot \left[\ell\left((\alpha, \star, n)\right|\left[f^j\left(I_\beta^r\right)\right]\right) - \ell\left((\alpha, \star, n)\right)\right]$$

$$+ \ell\left((\alpha, \star, n)\right) \cdot \sum_{\beta \in \mathcal{A}} \sum_{j=1}^{q_r^\beta} \frac{D^2 f(y_j^\beta)}{Df(y_j^\beta)} \cdot \left|f^j\left(I_\beta^r\right)\right|$$

$$= (\mathrm{IV}) + (\mathrm{V}).$$

By Proposition 4.7 we have that $(\mathrm{IV}) = \mathrm{O}(\lambda_4^{\sqrt{\frac{n}{2}}})$. Now observe that (V) is a Riemann sum for the integral $\int_I \frac{D^2 f(s)}{Df(s)}\, ds$. By Proposition 3.9 and ν-Hölder continuity of $\frac{D^2 f}{Df}$ we have

$$(\mathrm{V}) = \ell\left((\alpha, \star, n)\right) \cdot \int_I \frac{D^2 f(s)}{Df(s)}\, ds + O\left(\lambda^{n/k}\right).$$

This finishes the proof. □

Before proving Theorem 3 we need the following lemma whose proof is left to the reader.

Lemma 4.8. *Let $a, b \in \mathbb{R}$. Then for every $C > 0$ there is $C_{25} > 0$ such that if $|a|, |b| \leqslant C$ then*

$$|M_a - M_b|_{C^2} \leqslant C_{25} \cdot |a - b|,$$

where M_a and M_b are defined in (2).

Proof of Theorem 3. By assumption $\int_I \frac{D^2 f(s)}{Df(s)}\, ds = 0$, so by Theorem 2 we have

$$\left|\int_{I_\alpha^n} \frac{D^2 f_n(s)}{Df_n(s)}\, ds\right| \leqslant C_{26} \cdot \lambda_4^{\sqrt{\frac{n}{2}}}.$$

Therefore by Lemma 4.8

$$\begin{aligned} |M_{\int_{I_\alpha^n} \frac{D^2 f_n(s)}{Df_n(s)}\, ds} - \mathrm{Id}|_{C^2} &\leqslant C_{25} \cdot \left|\int_{I_\alpha^n} \frac{D^2 f_n(s)}{Df_n(s)}\, ds - 0\right| \\ &\leqslant C_{25} \cdot C_{26} \cdot \lambda_4^{\sqrt{\frac{n}{2}}}. \end{aligned} \tag{26}$$

Theorem 1 together with (26) gives us that

$$\left|\mathcal{Z}_{I_\alpha^n}\left(R^n(f)\right) - \mathrm{Id}\right|_{C^2} \leqslant C_{27} \cdot \lambda_4^{\sqrt{\frac{n}{2}}}. \qquad \square$$

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