Fading absorption in non-linear elliptic equations

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Abstract

We study the equation

$$-\Delta u + h(x)|u|^{q-1}u = 0,$$

in $\mathbb{R}_+^N = \mathbb{R}^{N-1} \times \mathbb{R}_+$ where $h \in C(\mathbb{R}_+^N)$, $h \geq 0$. Let $(x_1, \ldots, x_N)$ be a coordinate system such that $\mathbb{R}_+^N = [x_N > 0]$ and denote a point $x \in \mathbb{R}^N$ by $(x', x_N)$. Assume that $h(x', x_N) > 0$ when $x' \neq 0$ but $h(x', x_N) \to 0$ as $|x'| \to 0$. For this class of equations we obtain sharp necessary and sufficient conditions in order that singularities on the boundary do not propagate in the interior.

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1. Introduction

In this paper we study solutions of the equation

$$-\Delta u + h(x)|u|^{q-1}u = 0,$$  \hspace{1cm} (1.1)

in $\mathbb{R}_+^N = \mathbb{R}^{N-1} \times \mathbb{R}_+$ where $q > 1$ and $h \in C(\mathbb{R}_+^N)$, $h \geq 0$. (If $x \in \mathbb{R}_+^N$ we write $x = (x', x_N)$ where $x' = (x_1, \ldots, x_{N-1})$.)

If $h > 0$ in $\mathbb{R}_+^N$ then boundary singularities of solutions of (1.1) cannot propagate to the interior. This is due to the presence of the absorption term $h|u|^{q-1}u$ and the Keller–Osserman estimates [3] and [7]. In fact, in this case, (1.1) possesses a maximal solution $U$ in $\mathbb{R}_+^N$ and,

$$\lim_{x_N \to 0} \frac{U(x)}{|x|} = \infty \quad \forall M > 0.$$  \hspace{1cm} (1.2)

A solution satisfying this boundary condition is called a large solution. If, in addition, $h$ is bounded away from zero then the large solution is unique. (See [1] for the case of bounded domains. If $h$ is bounded away from zero, the extension to unbounded domains is standard.)
On the other hand, if \( h \) vanishes on a set \( F \subset \mathbb{R}^N_+ \) which has limit points on \( [x_N = 0] \) then a singularity at these limit points may propagate to the interior. By this we mean that there may exist a sequence \( \{u_n\} \) of solutions of (1.1) in \( \mathbb{R}^N_+ \) which converges in
\[
\Omega = \mathbb{R}^N_+ \setminus F
\]
but tends to infinity at some points of \( F \).

In this paper we shall study the case where \( h \) is positive in \( \mathbb{R}^N_+ \) but may vanish on
\[
F = \{(0, x_N) \in \mathbb{R}^N_+: x_N > 0\}.
\]

Since \( h \) is positive in \( \mathbb{R}^N_+ \setminus F \) a singularity at the origin may propagate only along the set \( F \). Furthermore a weak singularity, such as that of the Poisson kernel, cannot propagate to the interior because any solution of (1.1) is dominated by the harmonic function with the same boundary behavior. Therefore we must consider only strong singularities, i.e. singularities which cannot occur in the case of a harmonic function but may occur with respect to solutions of (1.1).

Suppose that
\[
h(x', x_N) \leq h_0(|x'|),
\]
where
\[
h_0 \in C^1[0, \infty), \quad h_0(s) > 0 \quad \text{for } s > 0, \quad h_0(0) = 0.
\]
It is clear that, the faster \( h_0(s) \) tends to zero as \( s \to 0 \) the greater the chance that a strong boundary singularity at the origin will propagate to the interior.

Our aim is to determine a sharp criterion for the propagation of singularities with respect to solutions of (1.1) with \( h \in C(\mathbb{R}^N_+) \) such that \( h > 0 \) in \( \mathbb{R}^N_+ \setminus F \). It turns out that such a criterion can be expressed in terms of functions of the form
\[
\bar{h}(s) := e^{-\frac{\omega(s)}{s}}. \tag{1.3}
\]
We assume that \( \omega \) satisfies the following conditions:
\begin{enumerate}
\item[(i)] \( \omega \in C(0, \infty) \) is a positive nondecreasing function,
\item[(ii)] \( s \mapsto \mu(s) := \frac{\omega(s)}{s} \) is monotone decreasing on \( \mathbb{R}_+ \), \hspace{1cm} (1.4)
\item[(iii)] \( \lim_{s \to 0} \mu(s) = \infty \)
\end{enumerate}
bounded. We establish the following results.

**Theorem 1.1.** Suppose that
\[
\liminf_{x \to 0} \frac{h(x)}{\bar{h}(|x'|)} > 0 \tag{1.5}
\]
where \( \bar{h} \) is given by (1.3) and that (1.4) holds.

Suppose that \( \omega \) satisfies the Dini condition,
\[
\int_0^1 (\omega(t)/t) \, dt < \infty. \tag{1.6}
\]
If \( \{u_n\} \) is a sequence of positive solutions of (1.1) in \( \mathbb{R}^N_+ \) converging (pointwise) in
\[
\Omega = \mathbb{R}^N_+ \setminus F
\]
then the sequence converges in \( \mathbb{R}^N_+ \) and its limit is a solution of (1.1) in \( \mathbb{R}^N_+ \).

In particular, (1.1) possesses a maximal solution \( U \) in \( \mathbb{R}^N_+ \) and \( U \) is a large solution.
Theorem 1.2. Suppose that there exists a constant $c > 0$ such that
\[ h(x) \leq c \bar{h}(|x'|) \quad \forall x \in \mathbb{R}^N_+ \]  \hspace{1cm} (1.7)
where $\bar{h}$ is given by (1.3). Assume that (1.4) and the following additional conditions hold:
\[ \limsup_{j \to \infty} \frac{\mu(a^{-j+1})}{\mu(a^{-j})} < 1 \quad \text{for some } a > 1 \]  \hspace{1cm} (1.8)
and
\[ \lim_{s \to 0} \frac{\mu(s)}{\ln s} = \infty. \]  \hspace{1cm} (1.9)
Condition (1.9) guarantees that, for every real $k$, (1.1) has a solution $u_{0,k}$ with boundary data $k \delta_0$ (where $\delta_0$ denotes the Dirac measure at the origin).

Under these assumptions, if
\[ \int_0^1 \frac{\omega(t)}{t} \, dt = \infty \]  \hspace{1cm} (1.10)
then
\[ u_{0,\infty} = \lim_{k \to \infty} u_{0,k} \]  \hspace{1cm} (1.11)
is a solution of (1.1) in $\Omega$ but
\[ u_{0,\infty}(x) = \infty \quad \forall x \in F. \]

Corollary 1.1. Suppose that there exists a positive constant $c$ such that
\[ c^{-1} \bar{h}(|x'|) \leq h(x) \leq c \bar{h}(|x'|) \quad \forall x \in \mathbb{R}^N_+ \]  \hspace{1cm} (1.12)
where $\bar{h}$ is given by (1.3) and satisfies conditions (1.4), (1.8) and (1.9). Then the Dini condition (1.6) is necessary and sufficient for the existence of a large solution of (1.1) in $\mathbb{R}^N_+$. It is also necessary and sufficient for the existence of the strongly singular solution $u_{0,\infty}$.

Problems concerning the propagation of singularities for semilinear equations with absorption have been studied in [6,9] (elliptic case) and in [5,8,10] (parabolic case). In the elliptic case it was assumed that the absorption term is positive everywhere in the interior of the domain, fading only on the boundary. Consequently singularities could propagate only along the boundary.

In [5] the authors studied the equation
\[ \partial_t u - \Delta u + e^{-\frac{1}{\tau t^q}} u^q = 0 \quad \text{in } \mathbb{R}^N \times \mathbb{R}_+ \]  \hspace{1cm} (1.13)
and proved that if $u$ is a positive solution with strong singularity at a point on $t = 0$ then $u$ blows up at every point of the initial plane. In [6] the authors studied the corresponding elliptic problem in a domain $D$ where the coefficient of the absorption term is $e^{-\frac{1}{\rho(x)}}$, $\rho(x) = \text{dist}(x, \partial D)$, proving a similar result.

In [8] the authors considered the equation,
\[ \partial_t u - \Delta u + e^{-\frac{\omega(t)}{t}} u^q = 0 \quad \text{in } \mathbb{R}^N \times \mathbb{R}_+ \]  \hspace{1cm} (1.14)
where $\omega$ is a positive, continuous and increasing function on $\mathbb{R}_+$. They proved that if $\sqrt{\omega}$ satisfies the Dini condition then there exist solutions with a strong isolated singularity at a point on $t = 0$. Similar sufficient conditions were obtained in [9] and [10] with respect to an elliptic (respectively parabolic) equation, where the absorption term vanishes at the boundary (respectively along the axis $x = 0, t > 0$). In addition some necessary conditions were presented, but a considerable gap remained between these and the corresponding sufficient conditions. In the recent preprint [11] the authors provide a rough condition for the propagation of singularities for equation (1.14) when the absorption term vanishes along an ascending curve. Conditions for the non-propagation of singularities are not discussed.
In the present paper, the proof of the sufficiency of the Dini condition (Theorem 1.1) is based on a refinement of the energy estimates technique. Given \( R > 0 \) denote by \( x^R \) the point \((x', x_N) = (0, R)\) and let \( D_R = B_R(x^R) \). For \( M > 0 \) denote by \( V_M \) the solution of (1.1) in \( D_R \) such that \( V_M = M \) on \( \partial D_R \). For \( r \in (0, R) \) let \( D'_r := \{(x', x_N) : |x'| < r, |x_N - x_N|^2 < r\} \). By estimating various energy integrals of \( V_M \) over domains \( D'_r \) and using a double iteration scheme we show that, for a specific sequence \( \{M_j\} \) tending to infinity and a related sequence \( \{r_j\} \) decreasing to a positive number \( b \),

\[
\sup_{D'_{r_j}} \int \left( |\nabla V_{M_j}|^2 + h(x)V_{M_j}^{q+1} \right) dx < \infty
\]

for every sufficiently small \( R \). By a standard argument this leads to the conclusion that \( V^R := \lim V_M \) is bounded in a neighborhood of \( F \cap D_R \) and consequently it is a solution of (1.1) in \( D_R \). Let \( w_k \) denote the solution of (1.1) in \( \mathbb{R}^N_+ \) such that \( w_k = k \) on \( x_N = 0 \). Then \( w_k \leq V^R \) in \( D_R \) and consequently \( w = \lim w_k < V^R \) in \( D_R \). This implies that \( w \) is finite everywhere on \( F \) so that \( w \) is a large solution of (1.1) in \( \mathbb{R}^N_+ \). Using this fact we prove the conclusion of Theorem 1.1, first in the case that \( h = \bar{h} \) and then in the general case.

The proof of the necessity of the Dini condition (Theorem 1.2) is based on the analysis of a sequence of boundary value problems whose solutions are dominated by \( u_{0, \infty} \) and blow up on \( F \). The sequence of boundary value problems is of the form:

\[
\begin{align*}
-\Delta u_j + a_j u_j^q &= 0 \quad \text{in} \quad \Omega_j := \left[ |x'| < 2^{-j} \right] \cap [x_N > 0], \\
\ u_j(x) &= 0 \quad \text{on} \quad \left[ |x'| = 2^{-j} \right] \cap [x_N > 0], \\
\ u_j(x', 0) &= \gamma_j(x') \quad \text{for} \quad |x'| \leq 2^{-j}
\end{align*}
\]

where \( a_j = \sup_{|x'| < 2^{-j}} \bar{h} \). The boundary data \( \gamma_j \) is chosen in such a way that, by a transformation akin to the similarity transformation, each problem is reduced to a boundary value problem in \( \left[ |x'| < 1 \right] \cap [x_N > 0] \), which is independent of \( j \). Using this fact and a result of Brada [2] we derive precise upper and lower estimates for \( u_j \). By an iterative technique these estimates lead to the conclusion that \( \{u_j\} \) blows up at \( F \).

The methods of the present paper can be applied to many of the problems with fading absorption mentioned before (parabolic and elliptic). In a subsequent paper we shall consider a parabolic problem involving the equation

\[
\partial_t u - Lu + b(x)|u|^{q-1} u = 0 \tag{1.15}
\]

in a cylindrical domain \( D \times \mathbb{R}_+ \) where \( D \subset \mathbb{R}^N_+ \), \( 0 \in D \) and the absorption term fades along the axis \( x = 0, t > 0 \). Here \( L \) is a linear, second order, uniformly elliptic operator with smooth coefficients which may depend on both space and time variables.

Assuming that \( h = \bar{h}(|x|) \), with \( \bar{h} \) as in (1.3), we shall study the question of propagation of singularities along this axis, using the tools developed in the present paper.

2. Proof of Theorem 1.1

Given \( R > 0 \) let \( x^R = (0, R) \) and denote by \( B_R \) the ball of radius \( R \) centered at \( x^R \). We shall prove the following:

**Theorem 2.1.** Suppose that \( h = \bar{h} \). Then, under the assumptions of Theorem 1.1, for every \( R > 0 \), (1.1) has a solution \( V^R \) in \( B_R \) which blows up on \( \partial B_R \), i.e.,

\[
V^R(x) \to \infty \quad \text{as} \quad x \to \partial B_R.
\]

Before proving this theorem, we show that it implies Theorem 1.1.

Let \( v_k \) denote the (smallest) solution of (1.1) in \( \mathbb{R}^N_+ \) such that \( v_k = k \) on the boundary. This means that

\[
v_k = \lim_{r \to \infty} v_{k,r},
\]

where \( v_{k,r} \) is the solution of (1.1) in \( B_r(0) \cap \mathbb{R}^N_+ \) such that \( v_{k,r} = k \) on \( \partial B_r(0) \cap [x_N = 0] \)
and \( v_{k,r} = 0 \) on \( \partial B_r(0) \cap \{x_N > 0\} \). Note that, for fixed \( k \), \( v_{k,r} \) increases with respect to \( r \). Put \[ V = \lim_{k \to \infty} v_k. \]

Condition (1.5) implies that there exist positive constants \( c_1 \) and \( R_1 \) such that
\[ h(x) \geq c_1 \bar{h}(\{x'\}) \quad \text{for} \ |x| < 2R_1, \]
(2.1)

Without loss of generality we assume that \( c_1 = 1 \). Therefore, if \( V_R \) is as in Theorem 2.1 then, for \( R \in (0, R_1) \), \( V_R \) is a supersolution of (1.1) in \( B_R \). Since \( V_R \) blows up at \( \partial B_R \) we conclude that \( v_k \ll V_R \) in \( B_R \) and consequently
\[ V \leq V_R \quad \forall R \in (0, R_1). \]

Further this implies that \( V \) is locally bounded in the strip \( \mathbb{R}^N_+ \cap \{x_N < R_0'\} \) and therefore, everywhere in \( \mathbb{R}^N_+ \). By its definition \( V \) is the smallest large solution of (1.1) in \( \mathbb{R}^N_+ \).

Now let us assume that \( h = \bar{h} \). In this case we may apply Theorem 2.1 in the half-space \( \{x_N > a\} \), for every real \( a \). In particular we deduce that for every \( a > 0 \) there exists a large solution of (1.1) in \( B_r(a) \) for every \( r \in (0, a) \). The smallest large solution is denoted by \( V'_a \). Let \( \{u_n\} \) be a sequence of positive solutions of (1.1) in \( \mathbb{R}^N_+ \) converging pointwise to \( u \) in \( \mathbb{R}^N_+ \setminus F \). For every \( a > 0 \) and \( r \in (0, a) \), \( u_n < V'_a \) in \( B_r(a) \). Consequently \( \{u_n\} \) is bounded in \( B_r(a) \) for every \( r \) and \( a \) as above. This implies that \( \{u_n\} \) converges pointwise in \( \mathbb{R}^N_+ \) and the limit \( u \) is a solution of (1.1) in \( \mathbb{R}^N_+ \).

In the general case, it remains true that \( \{u_n\} \) is bounded in \( B_r(a) \) for every \( a \in (0, R_1) \) and \( r \in (0, a) \). This implies that \( \{u_n\} \) converges pointwise in the strip \( \mathbb{R}^N_+ \cap \{x_N < R_1\} \) and the limit \( u \) is a solution of (1.1) in this strip. This in turn implies that \( \{u_n\} \) converges to a solution \( u \) in \( \mathbb{R}^N_+ \) which is the first assertion of Theorem 1.1. By a standard argument, if \( U \) is the supremum of all solutions of (1.1) in \( \mathbb{R}^N_+ \) then there exists a sequence of solutions \( \{u_n\} \) that converges to \( U \) in \( \mathbb{R}^N_+ \setminus F \). Therefore, the first assertion implies that \( U \) is a solution of (1.1) in \( \mathbb{R}^N_+ \).

The proof of Theorem 2.1 is based on estimates of certain energy integrals of solutions of (1.1). In a half-space these integrals are infinite. Therefore we shall estimate integrals over a bounded domain for solutions with arbitrary large boundary data.

Condition (1.6) implies that \( \lim_{s \to 0} \omega(s) = 0 \) while (1.4) implies that \( \lim_{s \to 0} \bar{h}(s) = 0 \). We extend both of these functions to \([0, \infty)\) by setting them equal to zero at the origin.

In the course of the proof we denote by \( c, c', c_i \) constants which depend only on \( N, q \). The value of the constant may vary from one formula to another. A notation such as \( C(b) \) denotes a constant depending on the parameter \( b \) as well as on \( N, q \).

2.1. Part 1

Let \( R, b \) be positive numbers such that \( R/8 < b < R/2 \). Denote by \( U_M, M > 0 \), the solution of (1.1) in \( B_R(0) \) such that \( U_M = M \) on \( \partial B_R(0) \).

Let
\[ \Omega_b = \{x = (x', x_N) \in \mathbb{R}^N: \ |x'| < b, \ |x_N| < b\}. \]

We start with an elementary estimate of the energy integral:
\[ I_b(M) = \int_{\Omega_b} (|\nabla U_M|^2 + h(x)U_M^{q+1}) \, dx. \]
(2.2)

**Lemma 2.1.** Let \( h \) be as in (1.3) and assume (1.4). Then
\[ I_b(M) \leq C_1(b)M^{q+1}, \quad C_1(b) = cb^N \bar{h}(8b). \]
(2.3)

**Proof.** Let \( v_M := U_M - M \). Multiplying (1.1) (for \( u = U_M \)) by \( v_M \) and integrating by parts we obtain,
\[ \int_{B_R(0)} (|\nabla U_M|^2 + h(x)U_M^{q}v_M) \, dx = 0. \]
Therefore
\[
I_b(M) \leq \int_{B_R(0)} \left( |\nabla U_M|^2 + h(x)U_M^{q+1} \right) dx
\]
\[
= M \int_{B_R(0)} h(x)U_M^q dx \leq c'M^{q+1}\tilde{h}(R)R^N \leq cb^N\tilde{h}(8b)M^{q+1}. \quad \square
\]

**Notation.** Put
\[
\Omega_b(s) := \{ x \in \mathbb{R}^N : s < |x'| < b - s, |x_N| < b - s \} \quad \forall s \in (0, b/2).
\]
If \( v \) is a positive solution of (1.1) in \( B_R(0) \), denote
\[
J_b(s; v) := \int_{\Omega_b(s)} \left( |\nabla_x v|^2 + \tilde{h}(|x'|)v^{q+1} \right) dx.
\]
Finally denote,
\[
\varphi_b(s) := \int_{\partial\Omega_b(s)} h(x)\frac{\sqrt{\pi}}{q-1} d\sigma.
\]

**Proposition 2.1.** There exists a constant \( c \) such that, for every positive solution \( v \) of (1.1) in \( B_R(0) \),
\[
J_b(s; v) \leq c\left( \int_0^s \varphi_b(r)^{-\frac{q-1}{q+1}} dr \right)^{-\frac{q+3}{q+1}} \quad \forall s \in (0, b/2).
\]

**Proof.** Put \( S_b(s) := \partial\Omega_b(s) \) and denote by \( \vec{n} = \vec{n}(x) \) the unit outward normal to \( S_b(s) \) at \( x \).

Multiplying Eq. (1.1) by \( v \) and integrating by parts over \( \Omega_b(s) \) we obtain,
\[
\int_{\Omega_b(s)} \left( |\nabla_x v|^2 + \tilde{h}(|x'|)v^{q+1} \right) dx = \int_{S_b(s)} \frac{\partial v}{\partial \vec{n}} v d\sigma. \quad (2.9)
\]

We estimate the term on the right-hand side using first Hölder’s inequality (for a product of three terms) and secondly Young’s inequality:
\[
\left| \int_{S_b(s)} v \frac{\partial v}{\partial \vec{n}} d\sigma \right| \leq \int_{S_b(s)} |\nabla_x v||v| d\sigma
\]
\[
\leq \left( \int_{S_b(s)} |\nabla_x v|^2 d\sigma \right)^{\frac{1}{2}} \left( \int_{S_b(s)} h(x)|v|^{q+1} d\sigma \right)^{\frac{1}{q+1}} \varphi_b(s)^{-\frac{q-1}{q+1}}
\]
\[
\leq c_1 \left( \int_{S_b(s)} \left( |\nabla_x v|^2 + h(x)v^{q+1} \right) d\sigma \right)^{\frac{q+1}{2(q+1)}} \varphi_b(s)^{-\frac{q-1}{2(q+1)}}. \quad (2.10)
\]

Substituting estimate (2.10) into (2.9) we obtain:
\[
J_b(s; v) \leq c_2 \left( \int_{S_b(s)} \left( |\nabla_x v|^2 + h(x)v^{q+1} \right) d\sigma \right)^{\frac{q+3}{2(q+1)}} \varphi_b(s)^{-\frac{q-1}{2(q+1)}}. \quad (2.11)
\]

Since
\[
-\frac{d}{ds} J_b(s; v) = \int_{S_b(s)} \left( |\nabla_x v|^2 + h(x)v^{q+1} \right) d\sigma,
\]
\[ |x_N| = b - s, \text{ inequality (2.11) is equivalent to} \]
\[ J_b(s; v) \leq c_3 \varphi_b(s) \frac{q-1}{2q+3} \left( \frac{d}{ds} J_b(s; v) \right)^{\frac{q+3}{2q+3}} \forall s \in (0, b/2). \]

Solving this differential inequality, with initial data \( J_b(b/2; v) = 0 \), we obtain (2.8). \( \square \)

In continuation we derive a more explicit estimate for \( h \) as in (1.3). We need the following technical lemma.

**Lemma 2.2.** Let \( A > 0, m \in \mathbb{N}, l \in \mathbb{R}^1 \) and let \( \omega \in C^1(0, \infty) \) be a positive function satisfying condition (1.4). Then there exist \( \tilde{s} \in (0, 1) \), depending on \( A, l \) and \( \omega \) such that the following inequality holds:

\[
\int_0^s t^{m-1} \omega(t)^l \exp(-A \mu(t)) \, dt \geq \frac{s^{m+1} \omega(s)^l - 1}{(m+1) \mu(s)^{-1} + A} \exp(-A \mu(s)) \quad \forall s: 0 < s < \tilde{s}. \tag{2.12}
\]

**Proof.** Due to condition (1.4)(ii) integration by parts yields:

\[
\int_0^s t^{m-1} \omega(t)^l \exp(-A \mu(t)) \, dt = \frac{s^{m+1} \omega(s)^l - 1}{m+1} \exp(-A \mu(s)) - \int_0^s \frac{At^{m-1}}{m+1} \exp(-A \mu(t)) \omega(t)^l \, dt \\
+ \int_0^s \frac{t^{m+1}}{m+1} \exp(-A \mu(t)) \omega'(t) \omega^l - 1 (A \mu(t) - l) \, dt. \tag{2.13}
\]

Again due to (1.4)(ii), there exists \( \tilde{s} > 0 \) such that

\[ A \mu(s) \geq l \quad \forall s \in (0, \tilde{s}). \]

For later estimates it is convenient to choose \( \tilde{s} \) in \( (0, 1) \).

As \( \omega(s) \) is non-decreasing, it follows that, for \( 0 < s \leq \tilde{s} \),

\[
\left( \frac{s + A \omega(s)}{m+1} \right) \int_0^s t^{m-1} \omega(t)^l \exp(-A \mu(t)) \, dt \geq \frac{s^{m+1}}{m+1} \omega(s)^l \exp(-A \mu(s)).
\]

This inequality is equivalent to (2.12). \( \square \)

**Proposition 2.2.** Assume that \( h \) is given by (1.3) and satisfies (1.4). Then there exists a constant \( s^* \in (0, b/2) \), depending on \( N, q \) and the rate of blow-up of \( \mu(s) = \omega(s)/s \) as \( s \to 0 \), such that

\[
J_b(s; v) \leq cb^{N-1} \exp Q(s) \quad \forall s \in (0, s^*),
\]

\[
Q(s) = \frac{2 \mu(s)}{q - 1} + \frac{q + 3}{q - 1} \ln \mu(s) - \frac{q + 3}{q - 1} \ln s,
\tag{2.14}
\]

for every positive solution \( v \) of (1.1) in \( B_R(0) \).

If, in addition, there exists a positive constant \( \beta \) such that

\[
\beta \ln \frac{1}{s} \leq \mu(s) \quad 0 < s \leq s^*,
\tag{2.15}
\]

then

\[
Q(s) \leq Q_0 \mu(s) \quad 0 < s \leq s^*
\tag{2.16}
\]

where

\[
Q_0 := \frac{2}{q - 1} + \frac{q + 3}{(q - 1)} + \frac{q + 3}{\beta (q - 1)}.
\tag{2.17}
\]
Proof. Denote
\[ S_{b,1}(s) = \left\{ x: |x'| = s, |x_N| < b \right\} \cup \left\{ x: |x'| = b - s, |x_N| < b \right\} \]
and
\[ S_{b,2}(s) = \left\{ x: s < |x'| < b - s, |x_N| = b \right\}. \]
Then
\[
\int_{S_{b,1}} \tilde{h}(|x'|)^{-\frac{2}{q+1}} d\sigma = 2\gamma_{N-1}(b - s)(\tilde{h}(s)^{-\frac{2}{q+1}}s^{N-2} + \tilde{h}(b)^{-\frac{2}{q+1}}(b - s)^{N-2})
\leq 4b^{N-1}\gamma_{N-1} \exp \frac{2\mu(s)}{q-1} \quad 0 < s < b/2, \tag{2.18}
\]
where $\gamma_{N-1}$ denotes the area of the unit sphere in $\mathbb{R}^{N-1}$. Further, since $\mu$ is monotone decreasing,
\[
\int_{S_{b,2}} \tilde{h}(|x'|)^{-\frac{2}{q+1}} d\sigma = 2\gamma_{N-1} \int_{s}^{b-s} \exp \frac{2\mu(\rho)}{q-1} \rho^{N-2} d\rho 
\leq 2(N-1)^{-1}b^{N-1}\gamma_{N-1} \exp \frac{2\mu(s)}{q-1}. \tag{2.19}
\]
By (2.18) and (2.19):
\[
\varphi_b(s) = \int_{S_{b}(s)} \tilde{h}(|x'|)^{-\frac{2}{q+1}} d\sigma \leq cb^{N-1} \exp \frac{2\mu(s)}{q-1}, \quad 0 < s < b/2,
\]
where $c = (4 + 2(N-1)^{-1})\gamma_{N-1}$. This implies,
\[
\int_{0}^{s} \varphi_b(r)^{-\frac{q+1}{q+3}} dr \geq c_1 b \left( \frac{(N-1)(q-1)}{q+3} \right) \int_{0}^{s} \exp \left( -\frac{2\mu(r)}{q+3} \right) dr, \quad c_1 = c^{-\frac{q+1}{q+3}}. \tag{2.20}
\]
Let $s^*$ be the largest number in $(0, b/2)$ such that
\[
\bullet \; s^* \leq \tilde{s} \text{ as in Lemma 2.2 for } l = 0, m = 1 \text{ and } A = \frac{2}{q+3}, \\
\bullet \; \mu(s^*) \geq A^{-1} = (q + 3)/2.
\]
Then (2.20) and (2.12) imply
\[
\int_{0}^{s} \varphi_b(r)^{-\frac{q+1}{q+3}} dr \geq c_2 b^2 \left( \frac{(N-1)(q-1)}{q+3} \right) \frac{\exp \left( -\frac{2\mu(s)}{q+3} \right)}{\omega(s)}, \quad c_2 = c_1(q + 3)/6, \tag{2.21}
\]
for all $s \in (0, s^*)$. This inequality and (2.8) imply (2.14). Suppose now that the function $\mu(\cdot)$ given by (1.3) satisfies (2.15). Since $\ln r \leq r$ for $r \geq 1$, conditions (1.4), (2.14) and (2.15) imply (2.16). □

Next we estimate energy integrals over domains of the form
\[
\Omega_{\sigma}(\tau, \sigma) := \left\{ x = (x', x_N): |x'| < \sigma, |x_N| < b - \tau \right\} \tag{2.22}
\]
where $0 < \sigma < b/2, 0 \leq \tau < b$. Let $\eta \in C^\infty((0, \infty))$ be a monotone decreasing function such that
\[
\eta(s) = 1 \quad \text{if } s < 1, \quad \eta(s) = 0 \quad \text{if } s > 2, \quad \eta'(s) \leq 2 \tag{2.23}
\]
and denote
\[ \eta_\sigma (s) = \eta (s/\sigma). \]

We shall estimate the integrals,
\[ E_b (\tau, \sigma; v) := \int_{\Omega_b (\tau, 2\sigma)} \left( |\nabla \eta_\sigma (|x'|) v|^2 + h(x) \eta_\sigma (|x'|)^2 v^{q+1} \right) \, dx. \]  

(2.24)

**Proposition 2.3.** Assume condition (1.4). Let \( s^* \in (0, b/2) \) be as in Proposition 2.2. Then the following inequality holds for \( 0 < \sigma \leq s^* \) and \( \sigma \leq \tau < b \):
\[ E_b (\tau, \sigma; v) \leq c\sigma \left( - \frac{dE_b (\tau, \sigma; v)}{d\tau} \right) + C_2 (b) \exp H(\sigma), \]  

(2.25)

where
\[ C_2 (b) := cb^2 (N-1) q + 1, \]

(2.26)

and
\[ c^* := \frac{2(q+3) + 2q^2 - 1 - (N-1)(q-1)^2}{q^2 - 1}. \]

If, in addition, condition (2.15) holds then there exists a constant \( H_0 \) depending only on \( q \) and \( \beta \) such that
\[ H(\sigma) \leq H_0 \mu(\sigma), \]  

(2.27)

where
\[ H_0 = \frac{2}{q-1} + \frac{2(q+3)}{(q-1)(q)+1} \frac{c^*}{\beta}. \]  

(2.28)

**Proof.** Multiplying Eq. (1.1) by \( \eta_\sigma (|x'|)^2 v \) and integrating by parts over \( \Omega_b (\tau, 2\sigma) \) we obtain,
\[ \int_{\Omega_b (\tau, 2\sigma)} \nabla v \cdot \nabla (v \eta_\sigma^2) \, dx + \int_{\Omega_b (\tau, 2\sigma)} h(x) v^{q+1} \eta_\sigma^2 \, dx = \int_{S'(\tau, 2\sigma)} \frac{\partial v}{\partial n} \eta_\sigma^2 \, dx', \]  

(2.29)

where \( S'(\tau, \sigma) = \{ x: |x'| < \sigma, |x_N| < b-\tau \} \).

We estimate the first term on the left hand side:
\[ \int_{\Omega_b (\tau, 2\sigma)} \nabla v \cdot \nabla (v \eta_\sigma^2) \, dx = \int_{\Omega_b (\tau, 2\sigma)} \left| \nabla (v \eta_\sigma) \right|^2 \, dx - \int_{\Omega_b (\tau, 2\sigma)} v^2 |\nabla \eta_\sigma|^2 \, dx \]
\[ \geq \int_{\Omega_b (\tau, 2\sigma)} \left| \nabla (v \eta_\sigma) \right|^2 \, dx - 4\sigma^{-2} \int_{\tilde{\Omega}_b (\tau, \sigma)} v^2 \, dx, \]  

(2.30)

where
\[ \tilde{\Omega}_b (\tau, \sigma) := \{ \sigma < |x'| < 2\sigma, |x_N| < b-\tau \}. \]  

(2.31)

Using Hölder’s inequality, conditions (1.3), (1.4) and estimate (2.14) with \( s = \sigma \), we obtain:
\[
\int_{\tilde{\Omega}_b(\tau, \sigma)} v(x)^2 \, dx \leq \left( \int_{\tilde{\Omega}_b(\tau, \sigma)} v^{q+1} h(x) \, dx \right)^{\frac{2}{q+1}} \left( \int_{\tilde{\Omega}_b(\tau, \sigma)} h(x)^{-\frac{2}{q+1}} \, dx \right)^{\frac{q-1}{q+1}} 
\]

\[
\leq c' \left( b^{N-1} \exp Q(\sigma) \right)^{\frac{2}{q+1}} \tilde{h}(\sigma)^{-\frac{2}{q+1}} \left| \tilde{\Omega}_b(\tau, \sigma) \right|^{\frac{q-1}{q+1}} 
\]

\[
\leq cb \left( \frac{2(Q(\sigma) + \mu(\sigma))}{q+1} \right) \sigma^{\frac{(N-1)(q-1)}{q+1}} 
\]

(2.32)

for \( \sigma < \tau < b \) and \( 0 < \sigma < \min\{s^*, \frac{b}{3}\} \). The application of (2.14) here is justified because, for \( \tau \) and \( \sigma \) as above, \( \tilde{\Omega}_b(\tau, \sigma) \subset \Omega_b(\sigma) \).

Combining (2.29)–(2.32) we obtain,

\[
\int_{\Omega_b(\tau, 2\sigma)} \left| \nabla (v_{\eta\sigma}) \right|^2 \, dx + \int_{\Omega_b(\tau, 2\sigma)} h(x) v^{q+1} \eta_{\sigma}^2 \, dx 
\]

\[
\leq \int_{S_b'(\tau, 2\sigma)} \frac{\partial v}{\partial n} \eta_{\sigma}^2 \, dx' + cb \left( \frac{2(Q(\sigma) + \mu(\sigma))}{q+1} \right) \sigma^{\frac{(N-1)(q-1)}{q+1}} - 2. 
\]

(2.33)

Next, by Hölder’s inequality,

\[
\left| \int_{S_b'(\tau, 2\sigma)} \frac{\partial v}{\partial n} \eta_{\sigma}^2 \, dx' \right| \leq \int_{S_b'(\tau, 2\sigma)} \left| \frac{\partial}{\partial x_N} \left( v_{\eta\sigma}(\cdot | x') \right) \right| \eta_{\sigma} \, dx' 
\]

\[
\leq \left( \int_{S_b'(\tau, 2\sigma)} \left( \frac{\partial}{\partial x_N} (v_{\eta\sigma}) \right)^2 \, dx' \right)^{\frac{1}{2}} \left( \int_{S_b'(\tau, 2\sigma)} (v_{\eta\sigma})^2 \, dx' \right)^{\frac{1}{2}} 
\]

and by Poincaré’s inequality in \( S_b'(\tau, \sigma) \),

\[
\int_{S_b'(\tau, 2\sigma)} (v_{\eta\sigma})^2 \, dx' \leq (c_0 \sigma)^2 \int_{S_b'(\tau, 2\sigma)} \left| \nabla x'(v_{\eta\sigma}) \right|^2 \, dx'. 
\]

Therefore

\[
\left| \int_{S_b'(\tau, 2\sigma)} \frac{\partial v}{\partial n} \eta_{\sigma}^2 \, dx' \right| \leq c\sigma \int_{S_b'(\tau, 2\sigma)} \left| \nabla x(v_{\eta\sigma}) \right|^2 \, dx'. 
\]

(2.34)

Since

\[
\frac{d E_b(\tau, \sigma; v)}{d \tau} = - \int_{S_b'(\tau, 2\sigma)} \left( \left| \nabla (v_{\eta\sigma}) \right|^2 + h(x) v^{q+1} \eta_{\sigma}^2 \right) \, dx' 
\]

inequalities (2.33) and (2.34) imply (2.25).

Finally, if (2.15) holds, (2.27) is obtained in the same way as (2.16). □

2.2. Part 2

Notation. Given \( M > 0 \) and \( v \in (0, 1) \), let \( s_v = s_v(M) \) be defined by,

\[
\exp(Q_0 \mu(s_v(M))) = \tilde{h}(s_v(M))^{-Q_0} = M^v. 
\]

(2.35)

where \( Q_0 \) is given by (2.17).
Lemma 2.3. Put
\[ \gamma := \frac{2(q + 1 + \beta) - (N - 1)(q - 1)}{\beta Q_0(q + 1)}, \] (2.36)
where \( \beta \) is a positive number satisfying (2.15) and
\[ \nu_0 := \begin{cases} 
1 & \text{if } \gamma \leq 0, \\
\frac{q - 1}{\gamma} & \text{if } \gamma > 0. 
\end{cases} \] (2.37)
If
\[ 0 < \nu < \min(\nu_0, 1) \] (2.38)
then,
\[ E_b(0, s_\nu(M'); U_M) \leq 2(I_b(M) + C_3(b) M^2 \nu \gamma) \quad 1 \leq M' \leq M, \] (2.39)
where
\[ C_3(b) := c b^{\frac{2N + q - 1}{q + 1}} h(8b) \frac{2}{q + 1}. \] (2.40)

Proof. Put
\[ I'_b(s, M) := \int_{\Omega_b} U_M^2 |\nabla \eta| \, dx. \]
Then,
\[ E_b(0, s_\nu(M'), U_M) \leq 2 \int_{\Omega_b} (|\nabla(U_M)|^2 \eta_{x_v}^2 + h(x) U_M^{q+1} \eta_{x_v}^2) \, dx + 2 \int_{\Omega_b} U_M^2 |\nabla \eta| \, dx \]
\[ \leq 2(I_b(M) + I'_b(s_\nu, M), \quad s_\nu = s_\nu(M'). \] (2.41)
By (2.23), \( \nabla \eta_{x_v}(|x'|) = 0 \) for \( |x'| < s_\nu \) and for \( |x'| > 2s_\nu \). Therefore, applying Hölder’s inequality and using the monotonicity of \( h \) we obtain
\[ I'_b(s_\nu(M'), M) \leq 4 s_\nu^{-2} \int_{\Omega_b} U_M^2 \, dx \]
\[ \leq 4 s_\nu^{-2} \left( \int_{\Omega_b} U_M^{q+1} \, dx \right)^{\frac{2}{q+1}} \left( \int_{\Omega_b} h(|x'|) \frac{2}{q + 1} \, dx \right)^{\frac{q - 1}{q + 1}} \]
\[ \leq c s_\nu^{-2} (b^N h(8b) M^{q+1}) \frac{2}{q + 1} h(s_\nu)^{-\frac{2}{q+1}} s_\nu^{\frac{N - 1(q - 1)}{q + 1}} b^{\frac{q - 1}{q + 1}} \]
\[ = c (b^N h(8b)) \frac{2}{q + 1} b^{\frac{q - 1}{q + 1}} M^2 \nu \gamma - 2 + \frac{N - 1(q - 1)}{q + 1} \exp 2\mu(s_\nu) q + 1. \]
By (2.15) and (2.35)
\[ s_{-1} \leq \exp(\mu(s) / \beta), \quad M' = M^{1/\nu_0} = h(s_\nu) = \exp(-\mu(s_\nu)). \]
Therefore the previous inequality yields
\[ I'_b(s_\nu(M'), M) \leq c (b^N h(8b)) \frac{2}{q + 1} M^2 M^\nu \gamma \exp 2\mu(s_\nu) q + 1. \]
Hence
\[ I'_b(s_\nu(M'), M) \leq C_3(b) M^2 \nu \gamma \] (2.42)
with \( \gamma \) and \( C_3(b) \) as in (2.36) and (2.40). By (2.38) \( \nu \gamma \leq q - 1 \). Therefore (2.41) and (2.42) imply (2.39). \( \square \)
Notation. For every $M > 0$ and $0 \leq s \leq b/2$ denote,

$$T_b(s, M) = \{ \tau: s \leq \tau < b, \ E_b(\tau, s; U_M) \geq 2C_2(b) \exp(H_0 \mu(s)) \} \tag{2.43}$$

where $C_2(b)$ is the constant in (2.25) and $H_0$ is given by (2.28).

Note that $\tau \mapsto E_b(\tau, s; U_M)$ is continuous and non-increasing in the interval $[s, b]$. Therefore, if $E_b(s, s; U_M) < 2C_2(b) \exp(H_0 \mu(s))$ then $T_b(s, M) = \emptyset$. Put,

$$\tau_b(s, M) = \begin{cases} s & \text{if } T_b(s, M) = \emptyset, \\ \sup T_b(s, M) & \text{otherwise} \end{cases} \tag{2.44}$$

and

$$\tau_{b, v}(M', M) := \tau_b(s_v(M'), M). \tag{2.45}$$

Since $\lim_{\tau \to b} E_b(\tau, s; U_M) \to 0$ it follows that $s_v(M') \leq \tau_{b, v}(M', M) < b. \tag{2.46}$

Furthermore,

$$E_b(\tau_{b, v}(M', M), s_v(M'); U_M) \leq 2C_2(b) \exp(H_0 \mu(s_v(M'))) \tag{2.47}$$

and, if $\tau_{b, v}(M', M) > s_v(M')$ then,

$$E_b(\tau, s_v(M'); U_M) \geq 2C_2(b) \exp(H_0 \mu(s_v(M'))) \tag{2.48}$$

for every $\tau \in (0, \tau_{b, v}(M', M)]$, with equality for $\tau = \tau_{b, v}(M', M)$.

**Proposition 2.4.** (i) Let

$$b'_v(M', M) := b - \tau_{b, v}(M', M).$$

Then

$$\int_{\Omega_{b'_v(M', M)}} (|\nabla_x U_M|^2 + h(x)U_M^{q+1}) \, dx \leq c_0(b^{N-1}M'^v + C_2(b)M'^{vH_0}Q_0). \tag{2.49}$$

(ii) Assume that

$$0 < v \leq \frac{q + 1}{4} \min(1, Q_0/H_0), \tag{2.50}$$

where $H_0$ is given by (2.28) and $Q_0$ is given by (2.17). Let $a \in (1, 2)$ and assume that $M'$ is large enough so that,

$$C_4(b) := c_0(b^{N-1} + C_2(b))/C_1(b) \leq M'^{(q+1)/2a} \tag{2.51}$$

where $C_1(b)$ and $C_2(b)$ are the constants in Lemma 2.1 and Proposition 2.3 respectively while $c_0$ is the constant in (2.49).

Then

$$I_{b'_v(M', M)}(M) = \int_{\Omega_{b'_v(M', M)}} (|\nabla_x U_M|^2 + h(x)U_M^{q+1}) \, dx \leq C_1(b)M'^{\frac{q+1}{a}}. \tag{2.52}$$

**Proof.** By (2.35),

$$M' = \exp\left(\frac{Q_0}{v} \mu(s_v(M'))\right). \tag{2.53}$$

Therefore, by (2.47),

$$E_b(\tau_{b, v}(M', M), s_v(M'); U_M) \leq 2C_2(b)M'^{vH_0}. \tag{2.54}$$
By Proposition 2.2 applied to the estimate of $J_b(s_v(M'), U_M)$,
$$J_b(s_v(M'), U_M) \leq cb^{N-1} \exp(Q_0 \mu(s_v(M'))) = cb^{N-1}M'^v.$$  \hfill (2.55)

Inequality (2.46) implies that $b'_v(M', M) \leq b - s_v(M')$. Therefore
$$\Omega_{b'_v(M', M)} = \Omega_b(\tau_{b,v}(M', M), s_v(M')) \cup \Omega_b(s_v(M'))$$
(see (2.5) for definition of $\Omega_b(s)$). Consequently
$$I_{b'_v(M', M)}(M) \leq E_b(\tau_{b,v}(M', M), s_v(M'); U_M) + J_b(s_v(M'), U_M).$$

This inequality together with (2.54) and (2.55) imply (2.49).

In view of (2.50) we have,
$$b^{N-1}M'^v + C_2(b)M'^{\frac{\nu b_0}{\nu_0}} \leq (b^{N-1} + C_2(b))M'^{(q+1)/2a}.$$ 

If $M'$ satisfies (2.51), this inequality and (2.49) imply (2.52). \hfill \Box

Next we derive an upper bound for $\tau_{b,v}(M', M)$ in terms of $s_v(M')$.

**Lemma 2.4.** Suppose that $0 < v$ satisfies conditions (2.38) and (2.50) and that
$$M \geq \exp\left(\frac{Q_0 \nu \mu(s^*)}{v}\right)$$
where $s^*$ is as in Proposition 2.3. Then
$$\exp\left(\frac{\tau_{b,v}(M', M)}{2cs_v(M')}\right) \leq c_1(I_b(M) + C_3(b)M^2M'^{q-1})C_2(b)^{-1}M'^{\frac{\nu b_0}{\nu_0}}.$$ \hfill (2.57)

**Proof.** Since $v$ satisfies (2.50) and $1 < a < 2$,
$$0 < Q_0(q+1)\left(1 - \frac{1}{2a}\right) \leq Q_0(q+1) - H_0v.$$ 

By (2.39),
$$E_b(\tau, s_v(M'); M) \leq E_b(0, s_v(M'); M) \leq 2(I_b(M) + C_3(b)M^2M'^{q-1}) \quad \forall \tau \in (0, b)$$ \hfill (2.58)

where $1 < M' < M$.

If $\tau_{b,v} \leq s_v$ inequality (2.57) is trivial. Therefore we may assume that
$$\tau_{b,v}(M', M) > s_v(M').$$

Temporarily denote
$$F(\tau) = E_b(\tau, s_v(M'); M).$$

By Proposition 2.3, (2.56) and (2.48),
$$F(\tau) \leq 2cs_v(M')\left(-\frac{dF(\tau)}{d\tau}\right) \forall \tau : s_v(M') < \tau < \tau_{b,v}(M', M).$$ \hfill (2.59)

Solving this differential inequality with initial condition $F(s_v(M'))$ satisfying (2.58) we obtain,
$$E_b(\tau, s_v(M'); M) \leq c_1(I_b(M) + C_3(b)M^2M'^{q-1})\exp\left(-\frac{\tau}{2cs_v(M')}\right)$$ \hfill (2.60)

for every $\tau \in [s_v(M'), \tau_{b,v}(M', M)]$. Combining (2.60) and (2.48) for $\tau = \tau_{b,v}(M', M)$ (in which case (2.48) holds with equality) we obtain,
$$2C_2(b)\exp\left(H_0\mu(s_v(M'))\right) \leq c_1(I_b(M) + C_3(b)M^2M'^{q-1})\exp\left(-\frac{\tau_{b,v}(M', M)}{2cs_v(M')}\right).$$
In view of (2.53) this inequality implies
\[
\exp\left(\tau_{b,\nu}(M', M)\right) \leq c_1 \left(I_b(M) + C_3(b)M^2M'^{-q-1}C_2(b)^{-1}\exp(-H_0\mu(s_{\nu}(M'))\right) \\
= c_1 \left(I_b(M) + C_3(b)M^2M'^{-q-1}C_2(b)^{-1}M'-\frac{vH_0}{2Q_0}\right). \quad (2.61)
\]

2.3. Part 3

In this part of the proof we apply the previous estimates to a specific sequence \(\{M_j\}\) defined below. As before \(R\) is an arbitrary positive number and we require that \(R/4 < b < R/2\).

**Proposition 2.5.** Let
\[
M_j = \exp(a^j), \quad s_j := s_{\nu}(M_j) \quad (2.62)
\]
where \(s_{\nu}(\cdot)\) is defined as in (2.35) and
\[
1 < a < \min\left(1 + \frac{vH_0}{2Q_0}, 2\right). \quad (2.63)
\]

Put \(u_j = U_{M_j}\). Then there exists \(j_0 \in \mathbb{N}\) such that
\[
\int_{\Omega_{b/2}} \left(|\nabla_x u_j|^2 + h(x)u_j^{q+1}\right) dx \leq C_1(b)M'^{q+1}_{j_0} \quad \forall j > j_0 \quad (2.64)
\]
where \(C_1(b) = cb^N\tilde{h}(8b)\).

**Proof.** By (2.62) and (2.35),
\[
a^j v/Q_0 = \mu(s_j). \quad (2.65)
\]
Let \(j_0\) be a positive integer to be determined later on. For each integer \(j \geq j_0\) we define the set of pairs
\[
\{b_{i,j}, \tau^{i,j} : i = j_0, \ldots, j\}
\]
by induction as follows:
\[
\tau^{i,j} = \tau_{b,\nu}(M_j, M_j), \quad (2.66)
\]
\[
b_{j,j} = b - \tau^{j,j},
\]
\[
\tau^{i,j} = \tau_{b_{i+1,j},\nu}(M_i, M_j), \quad (2.67)
\]
\[
b_{i,j} = b_{i+1,j} - \tau^{i,j}, \quad j_0 \leq i < j.
\]
Thus
\[
b_{i,j} = b - \sum_{k=i}^{j} \tau^{k,j}, \quad j_0 \leq i < j.
\]

We show below that if \(j_0\) is sufficiently large then
\[
\sum_{i=j_0}^{j} \tau^{i,j} < b/2 \quad \forall j > j_0, \quad (2.68)
\]
which implies,
\[
b/2 < b_{i,j}.
\]
Specifically we choose \(j_0\) so that,
(i) \[ C_4(b/2) \leq M_j^{(q+1)/2a}, \]

(ii) \[ \exp\left( \frac{Q_0}{v} \mu(s^*) \right) \leq M_{j_0}, \]

(iii) \[ C_5(b) := c_1 \frac{C_1(b) + C_2(b)}{C_2(b)} \leq M_j^{q+1} \]

with \( c_1 \) as in (2.57). For the definition of \( C_1(b), \ldots, C_4(b) \) see (2.3), (2.25), (2.40) and (2.51).

We observe that \( C_4(b) \) decreases as \( b \) increases. Therefore (assuming (2.66)) condition (i) implies,

\[ C_4(b_{i,j}) \leq M_{j_0}^{(q+1)/2a}, \quad j_0 \leq i \leq j, \quad j_0 \leq j. \]  \hfill (2.68)

The left hand side in condition (2.67)(iii) increases as \( b \) increases. Therefore

\[ C_5(b_{i,j}) \leq (q+1) \ln M_i, \quad j_0 \leq i \leq j, \quad j_0 \leq j. \]  \hfill (2.69)

Put \( u_j = U_{M_j} \). Assuming that (2.66) holds, we apply Proposition 2.4 to the case where \( b \) is replaced by \( b_{j_0+1,j} \) and \( M' = M_{j_0+1}, M = M_j \); we obtain,

\[ \int_{b_{j_0,j}} \left( |\nabla x u_j|^2 + h(x) u_j^{q+1} \right) dx \leq C_1(b) M_{j_0}^{q+1} \]  \hfill (2.70)

which implies (2.64).

It remains to verify (2.66). To this end we prove the following estimate:

\[ \tau^{i,j} \leq \tilde{c} Q_0 (q + 1) \frac{\omega(s_j)}{v}, \quad j_0 \leq i \leq j \]  \hfill (2.71)

where \( \tilde{c} = 4c \) (\( c \) as in (2.57)).

The proof is by induction. We apply Lemma 2.4 in the case where

\[ b \text{ is replaced by } b_{i+1,j}, \quad M' = M_i, \quad M = M_j, \quad j_0 \leq i \leq j. \]

For \( i = j \) we put \( b_{j+1,j} := b \). Note that, for \( M \geq M_{j_0} \), condition (2.67)(ii) yields (2.56).

Applying Lemma 2.4 and Lemma 2.1 to the case \( i = j \) we obtain

\[ \exp \frac{\tau^{j,j}}{2cs_j} \leq C_5(b) M_{j_0}^{q+1 - v \frac{H_0}{Q_0}}. \]

Consequently, using (2.62) and condition (2.67)(iii)

\[ \frac{\tau^{j,j}}{2cs_j} \leq \ln C_5(b) + \left( q + 1 - v \frac{H_0}{Q_0} \right) \ln M_j \]

\[ \leq 2(q + 1) \frac{Q_0 \mu(s_j)}{v}, \]  \hfill (2.72)

For the last inequality recall that \( s_j = s_v(M_j) \), which implies,

\[ \ln M_j = \frac{Q_0 \mu(s_j)}{v}. \]

Inequality (2.72) implies (2.71) for \( i = j \).

Observe that \( s_j \downarrow 0 \) as \( j \uparrow \infty \) and consequently, \( \omega(s_j) \downarrow 0 \). Therefore if \( j_0 \) is sufficiently large we have \( \tau^{i,j} < b/2 \) and \( b_{j,j} > b/2 \). By Proposition 2.4,

\[ I_{b_{j,j}}(M_j) \leq C_1(b_{j,j}) M_{j_0}^{(q+1)/a} \leq C_1(b) M_{j_0}^{q+1}. \]  \hfill (2.73)

Here we use condition (2.67)(i) and the fact that \( b_{j,j} = b - \tau_{b,v}(M_j, M_j) \).

Now we apply Lemma 2.4 for \( i = j - 1 \), i.e., when \( b \) is replaced by \( b_{j,j} \) and \( M' = M_{j-1}, M = M_j \). This lemma, combined with (2.73), yields
\[
\exp \frac{\tau^{j-1,j}}{2c s_{j-1}} \leq c_1 (I_{b_{j,j}} (M_j) + C_3 (b_{j,j}) M_j^{q_j} M_{j-1}^{q_j-1}) C_2 (b_{j,j})^{-1} M_j^{-\nu \frac{H_0}{\nu_0}} \\
\leq c_1 (C_1 (b_{j,j}) M_j^{q_j+1} + C_3 (b_{j,j}) M_j^{q_j} M_{j-1}^{q_j-1}) C_2 (b_{j,j})^{-1} M_j^{-\nu \frac{H_0}{\nu_0}}.
\]

By (2.63),
\[
M_j M_{j-1}^{-\nu \frac{H_0}{\nu_0}} \leq M_j^2.
\]

Therefore, similarly to (2.72), we obtain
\[
\tau^{j-1,j} \leq \ln C_5 (b_{j,j}) + (q + 1) \ln M_j
\]
\[
\leq 2(q + 1) Q_0 \mu (s_{j-1}) / \nu,
\]
which, in turn, implies (2.71) for \( i = j - 1 \).

This process can be repeated inductively for \( i = j - 2, j - 3, \ldots, j_0 \) provided that \( b_{i+1,j} \geq b/2 \). For each value of \( i \) in this range we first apply Proposition 2.4 to obtain,
\[
I_{b_{i+1,j}} (M_j) \leq C_1 (b_{i+1,j}) M_{i+1}^{q_{i+1}/a} \leq C_1 (b) M_{i+1}^{q_{i+1}}.
\]

After that we apply Lemma 2.4 combined with (2.76) to obtain (2.71) for the respective value of \( i \), always with the same constant \( \bar{c} \).

To complete the proof, it remains to be shown that there exists \( j_0 \) such that:

If \( j > j_0 \), \( j_0 \leq k < j \) and \( \tau^{i,j} \) satisfies (2.71) for \( k \leq i \leq j \) then,
\[
\sum_{i=k}^{j} \tau^{i,j} < b/2.
\]

By (2.65) and (1.4)
\[
s_i \leq (Q_0 / \nu) a^{-i} \omega (s_i) \leq \ell a^{-i}, \quad \ell := Q_0 \omega (s_0) / \nu.
\]

Since, by assumption, (2.71) holds for \( k \leq i \leq j \),
\[
\sum_{i=k}^{j} \tau^{i,j} \leq C(N, q, \nu) \sum_{i=k}^{j} \omega (s_i) \leq C(N, q, \nu) \sum_{i=k}^{j} \omega (\ell a^{-i}).
\]

Further, using the monotonicity of \( \omega \),
\[
\sum_{i=k}^{j} \omega (\ell a^{-i}) \leq \int_{k}^{j} \omega (\ell a^{-s}) ds < (\ln a)^{-1} \int_{0}^{\beta_k} \omega (r) dr
\]
where \( \beta_k = \ell a^{-k} \). Because of the Dini condition, the last integral tends to zero when \( \beta_k \to 0 \). Therefore, if \( j_0 \) is sufficiently large (depending only on \( N, q, \nu \) and \( a \)) (2.77) holds for all \( k \geq j_0 \).

Completion of proof of Theorem 2.1. Since \( U_M \) increases as \( M \) increases
\[
U^R := \lim_{M \to \infty} U_M = \lim_{j \to \infty} u_j.
\]

The function \( V_M \) defined by
\[
V_M (x) = U_M (x', x_N + R)
\]
is a solution of (1.1) in the ball \( B_R(x^R) \), where \( x^R = (0, R) \). Put
\[
V^R := \lim_{M \to \infty} V_M \quad \text{in} \quad B_R(x^R).
\]

We show that \( V^R \) is bounded in a neighborhood of the point \((0, R)\).
By interior elliptic estimates, (2.64) implies that
\[
\sup_{j_0 \leq j} \int_{\Omega_{b/3}} |u_j|^2 \, dx < \infty. \tag{2.78}
\]
Since \( h(x) \geq 0 \), \( u_j \) is subharmonic in \( \Omega_b \). Therefore (2.78) implies
\[
\sup \{ u_j(x) : j_0 \leq j, x \in \Omega_{b/4} \} < \infty. \tag{2.79}
\]
Thus \( UR \) is bounded in a neighborhood of the origin which means that \( VR \) is bounded in a neighborhood of \((0, R)\). For every \( r \in (0, R) \), \( VR < V_r \) in \( Br(x_r) \). (Recall that \( x_r \) denotes the point \( (x', x_N) = (0, r) \).) As \( V_r \) is bounded in a neighborhood of \((0, r)\) we conclude that \( VR \) is locally bounded in \( BR \cap [R < x_N] \).

3. Proof of Theorem 1.2

Put
\[
r_j := 2^{-j}, \quad \Omega_j = \{ (x', x_N) : |x'| < r_j, \ 0 < x_N \}, \ j = 1, 2, \ldots.
\]
Further denote,
\[
a_j := \exp(-\mu(r_j)), \quad A_j = (a_j r_j^2)^{1/q}
\]
and, for \( x' \in \mathbb{R}^{N-1} \),
\[
\gamma_j(x') = \begin{cases} A_j^{-1} \phi_1(x'/r_j+1) & \text{if } |x'| < r_j+1, \\ 0 & \text{if } |x'| \geq r_j+1 \end{cases}
\]
where \( \phi_1 \) is the first eigenfunction of the Dirichlet problem to \( -\Delta y \) in \( B_1^{N-1} \) normalized by \( \phi_1(0) = 1 \). Recall that \( \mu(s) = \omega(s)/s \).

We consider the boundary value problems
\[
-\Delta u_j + a_j u_j^q = 0 \quad \text{in } \Omega_j,
\]
\[
u_j(x) = 0 \quad \text{on } \{ x \in \partial \Omega_j : x_N > 0 \},
\]
\[
u_j(x', 0) = \gamma_j(x') \quad \text{for } |x'| \leq r_j. \tag{3.3}
\]
In view of (1.4), \( \{a_j\} \) is a decreasing sequence converging to zero and
\[
a_j = \sup_{s \in (0, r_j)} \exp(-\mu(s)).
\]
Therefore, for every \( x_N > 0 \), \( \{u_j(0, x_N)\} \) is an increasing sequence and \( u_j \) is a subsolution of the problem
\[
-\Delta w + h(x) w^q = 0 \quad \text{in } \Omega_j,
\]
\[
w(x) = 0 \quad \text{on } \{ x \in \partial \Omega_j : x_N > 0 \},
\]
\[
w(x', 0) = \gamma_j(x') \quad \text{for } |x'| \leq r_j. \tag{3.4}
\]

The proof of Theorem 1.2 is based on the following:

**Proposition 3.1.** For every \( x_N > 0 \),
\[
\lim_{j \to \infty} u_j(0, x_N) = \infty.
\]

In the next lemma we collect several results of Brada [2] that are used in the proof of this proposition.
Lemma 3.1. Let \( a \) be a positive number, let \( q > 1 \) and let \( f \) be a positive function in \( L^\infty(B_1^{N-1}) \), where \( B_1^{N-1} \) denotes the unit ball in \( \mathbb{R}^{N-1} \) centered at the origin.

Consider the problem

\[
-\Delta u + bu^q = 0 \quad \text{in } D_0, \\
u(y) = 0 \quad \text{for } y \in \partial D_0: 0 < y_N, \\
u(y', 0) = f(y') \quad \text{for } |y'| \leq 1,
\]

where

\[
D_0 = \{ y = (y', y_N) \in \mathbb{R}^{N}: |y'| < 1, \ 0 < y_N \}.
\]

If \( u \) is the solution of this problem then there exists a number \( \alpha > 0 \) such that

\[
\lim_{y_N \to \infty} \exp(\sqrt{\lambda_1 y_N}) u(y) = \alpha \phi_1(y')
\]

uniformly in \( B_1^{N-1} \). Here \( \lambda_1 \) is the first eigenvalue and \( \phi_1 \) the corresponding eigenfunction of \( -\Delta y' \) in \( B_1^{N-1} \) normalized by \( \phi_1(0) = 1 \).

The limit \( \alpha \) satisfies

\[
\alpha \leq c b^{-\frac{1}{q-1}} \sup f.
\]

**Proof.** By [2, Theorem 4], (3.6) holds for some \( \alpha \in \mathbb{R} \). Under our assumptions \( u \) is positive so that \( \alpha \geq 0 \). By the remark in [2, p. 357], if \( \alpha = 0 \) then there exists \( k > 1 \) such that

\[
\lim_{y_N \to \infty} \exp(\sqrt{\lambda_k y_N}) u(y) = \phi_k(y')
\]

where \( \phi_k \) an eigenfunction of \( -\Delta y' \) in \( B_1^{N-1} \) corresponding to the \( k \)-th eigenvalue. However this is impossible because \( \phi_k \) changes signs. Thus \( \alpha > 0 \).

Inequality (3.7) is a consequence of [2, Proposition 1]. \( \square \)

3.1. An estimate of \( u_j \)

We start by rescaling problem (3.3). Put

\[
y = x/r_j, \quad \tilde{u}_j(y) = A_j u_j(r_j y),
\]

where \( A_j \) is given by (3.1). Then \( v := \tilde{u}_j \) is the solution of the problem

\[
-\Delta v + v^q = 0 \quad \text{in } D_0, \\
v(y) = 0 \quad \text{for } y \in \partial D_0: 0 < y_N, \\
v(y', 0) = \tilde{v}(y') \quad \text{for } |y'| \leq 1,
\]

where

\[
\tilde{v}(y') := \begin{cases} 
\phi_1(2y') & \text{if } |y'| < \frac{1}{2}, \\
0 & \text{otherwise}.
\end{cases}
\]

Applying Lemma 3.1 to the solution \( v \) of (3.9) we obtain,

\[
\lim_{y_N \to \infty} \exp(\sqrt{\lambda_1 y_N}) v(y', y_N) = \alpha \phi_1(y')
\]

where \( \alpha \) is a positive number depending only on \( q, N \). Consequently there exists \( \beta > 0 \) such that

\[
\frac{1}{2} \alpha \phi_1(y') \exp(-\sqrt{\lambda_1 y_N}) \leq A_j u_j(r_j y) \\
\leq 2 \alpha \phi_1(y') \exp(-\sqrt{\lambda_1 y_N}) \quad \forall y_N \geq \beta, \ |y'| \leq 1.
\]
This inequality is equivalent to
\[ \frac{\alpha}{2A_j} \phi_1 \left( \frac{x'}{r_j} \right) \exp \left( -\sqrt{\lambda_1 x_N/r_j} \right) \leq u_j(x) \leq \frac{2\alpha}{A_j} \phi_1 \left( \frac{x'}{r_j} \right) \exp \left( -\sqrt{\lambda_1 x_N/r_j} \right) \forall x_N \geq \beta r_j, \ |x'| \leq r_j. \] (3.12)

3.2. Comparison of \( u_j \) and \( u_{j-1} \)

Let \( \tau_j \) be the number determined by the equation,
\[ \frac{\alpha}{2} \exp \left( -\sqrt{\lambda_1 \tau_j/r_j} \right) = \left( \frac{a_j}{a_{j-1}} \right)^{\frac{1}{q-1}} 2^{-\frac{2}{q-1}} \]
\[ = 2^{-\frac{2}{q-1}} \exp \left( -\frac{\mu(r_j) + \mu(r_{j-1})}{q-1} \right). \] (3.13)

By (3.1) and (3.2), this is equivalent to
\[ \frac{\alpha}{2A_j} \phi_1 \left( \frac{x'}{r_j} \right) \exp \left( -\sqrt{\lambda_1 \tau_j/r_j} \right) = \gamma_{j-1}(x'). \] (3.14)

Without loss of generality we may assume that (1.8) holds for \( a = 2 \). Therefore there exists \( \kappa \in (0, 1) \) such that
\[ \mu(r_j) - \mu(r_{j-1}) \geq \kappa \mu(r_j). \] (3.15)

By (3.13),
\[ \frac{\alpha}{2} \exp \left( -\sqrt{\lambda_1 \tau_j/r_j} \right) = \frac{\mu(r_j) - \mu(r_{j-1})}{q-1} + c(N, q). \]

Therefore, by (3.15) and (1.4), there exist positive numbers \( c_0, c_1 \) and \( j_0 \) (depending only on \( \kappa, N, q \)) such that
\[ \beta r_j < c_0 \omega(r_j) \leq \tau_j \leq c_1 \omega(r_j) \] (3.16)
for every \( j \geq j_0 \) (\( \beta \) as in (3.12)).

By (3.12), (3.14) and (3.16)
\[ \gamma_{j-1}(x') \leq u_j(x', \tau_j), \ |x'| \leq r_j, \ j \geq j_0. \] (3.17)

By the maximum principle, (3.3), (3.17) and the fact that \( a_{j-1} > a_j \) imply
\[ u_{j-1}(x', x_N) \leq u_j(x', x_N + \tau_j) \forall j \geq j_0, \ x \in \Omega_j. \] (3.18)

3.3. Proof of Proposition 3.1

Let \( j_0 \leq k < m \). Iterating inequality (3.18) for \( j = k + 1, \ldots, m \) we obtain,
\[ u_k(x', x_N) \leq u_m \left( x', x_N + \sum_{j=k+1}^m \tau_j \right) \forall x \in \Omega_m. \] (3.19)

Combining this inequality (for \( x' = x_N = 0 \)) with (3.12) yields
\[ \frac{1}{2} \alpha (a_k r_k^2)^{-\frac{1}{q-1}} = \frac{\alpha}{2A_k} \leq u_k(0) \leq u_m \left( 0, \sum_{j=k+1}^m \tau_j \right) \] (3.20)
for every \( m, k \) such that \( j_0 \leq k < m \). By (1.10),
\[ \sum_{j=k}^{\infty} \omega(r_j) = \infty. \]
Therefore, by (3.16)
\[ \sum_{j=k}^\infty \tau_j = \infty. \]  
(3.21)

Consequently,
\[ s_{m,k} := \sum_{j=k+1}^m \tau_j \implies \lim_{m \to \infty} s_{m,k} = \infty. \]  
(3.22)

Note that \( a_k r_k^2 \to 0 \); therefore, by (3.20), for every \( M > 0 \) there exists \( j_M \) such that
\[ M < u_m(0, s_{m,k}) \quad j_M \leq k < m. \]  
(3.23)

We claim that
\[ \sup_{x_N > 0} u_j(0, x_N) = \infty \]  
(3.24)

By negation, assume that
\[ \exists s > 0: \sup_{x_N > 0} u_j(0, s) = K < \infty. \]  
(3.25)

By (3.12)
\[ \frac{u_j(x', s)}{u_j(0, s)} \leq 4\alpha \quad |x'| \leq r_j. \]

Here we use the fact that \( 1 = \phi(0) = \max \phi \). It follows that, for every \( j \) such that \( 2^j > \beta / s \),
\[ \sup_{x_N > 0} u_j(x', s) \leq 4\alpha K, \quad |x'| \leq r_j. \]

Therefore, by the maximum principle, for every \( j \) as above,
\[ u_j(x', x_N) \leq 4\alpha K \quad \forall x \in \Omega_j \cap [x_N \geq s]. \]

In view of (3.22), this contradicts (3.23). \( \square \)

3.4. Proof of Theorem 1.2

Let \( P_0(x, y) = c_N x_N |x - y|^{-N} \) be the Poisson kernel for \(-\Delta\) in \( \mathbb{R}_+^N \). Condition (1.9) implies that, for any positive constants \( a, R \)
\[ \sup_{|x'| < R} |x'|^{-a} h(x) < \infty. \]  
(3.25)

For every \( q > 1 \) choose \( a > 0 \) such that \( q < (N + 1 + a)/(N - 1) \). Then for every \( R > 0 \),
\[ \int_{[|x| < R, 0 < x_N]} h(x) P_0^q(x, 0) x_N \, dx < C_a \int_{[|x| < R, 0 < x_N]} \ |x'|^a P_0^q(x, 0) x_N \, dx < \infty. \]

Consequently, for every \( k > 0 \), the problem
\[ -\Delta v + h(x) v^q = 0 \quad \text{in} \quad D_0, \]
\[ v = 0 \quad \text{on} \quad \partial \Omega D_0 := \{ |x'| = 1, \ x_N > 0 \}, \]
\[ v = k \delta \quad \text{on} \quad [x_N = 0] \]
possesses a unique solution dominated by the supersolution \( k P_0 \) (see [4]).

The function
\[ v_{0,\infty} := \lim_{k \to \infty} v_{0,k} \quad \text{in} \quad D_0 \]  
(3.26)

is a solution of (1.1) in \( D_0 \cap [x' > 0] \) but it may blow up as \( |x'| \to 0. \)
Put
\[ f(x_N) = \int_{|x'|<1} v_{0,\infty}(x',\bar{x}_N) \, dx' \quad \forall x_N > 0. \]

If \( f(a) < \infty \) for some \( a > 0 \) then \( v_{0,\infty} \) is finite in \( D_0 \cap [x_N > a] \) so that \( f(x_N) < \infty \) for every \( x_N > a \). Thus
\[ f(a) < \infty \quad \text{for some } a > 0 \quad \implies \quad f(x_N) < \infty \quad \forall x_N \geq a. \] (3.27)

Let
\[ b = \inf \{ x_N > 0 : f(x_N) < \infty \}. \] (3.28)

By (3.27)
\[ f(x_N) = \infty \quad \forall x_N \in (0, b), \quad f(x_N) < \infty \quad \forall x_N \in (b, \infty). \] (3.29)

We have to show that \( b = \infty \). By negation assume that \( b < \infty \). First consider the case \( 0 < b \). Let \( a \in (0, b) \) and put
\[ \eta(x') = v_{0,\infty}(x', \bar{x}_N + a). \]

Then
\[ \int_{|x'|<1} \phi \eta \, dx' = \infty \quad \forall \phi \in C([|x'| \leq 1]) \text{ such that } \phi(0) > 0. \]

Thus the measure \( \mu_{\eta} = \eta \, dx' \) is larger than \( k\delta_0 \) for every \( k > 0 \). The function \( V \) given by
\[ V(x) = v_{0,\infty}(x', x_N + a) \]
satisfies
\[ -\Delta V + h(x)V = 0 \quad \text{in } D_0, \]
\[ V = 0 \quad \text{on } \partial\Omega_0 := [|x'| = 1, \ x_N > 0], \]
\[ V = \eta \quad \text{on } [x_N = 0]. \]

Therefore \( V \geq v_{0,\infty} \), i.e.,
\[ v_{0,\infty}(x', x_N + a) \geq v_{0,\infty}(x', x_N). \]

But this implies
\[ f(x_N + a) = \infty \quad \forall x_N \in (0, a + b) \]
which contradicts (3.28).

Next assume that \( b = 0 \). In this case,
\[ v_{0,\infty}(0, x_N) < \infty \quad \forall x_N > 0 \] (3.30)

and consequently \( v_{0,\infty} \) is a solution of (1.1) in \( D_0 \). Let \( w_j \) be the unique solution of the boundary value problem:
\[ -\Delta w_j + a_j w_j^q = 0 \quad \text{in } \Omega_j, \]
\[ w_j = 0 \quad \text{on } \partial \Omega_j \cap [x_N > 0], \]
\[ w_j = \infty \delta_0 \quad \text{on } [x_N = 0], \] (3.31)

where \( a_j = h(r_j) \). As usual, this means that \( w_j = \lim_{k \to \infty} w_{j,k} \) where \( w_{j,k} \) is the solution of the modified problem where the boundary data on \( x_N = 0 \) is \( w_{j,k}(x', 0) = k\delta_0 \). Since \( a_j \geq h(x) \) in \( \Omega_j \) it follows that
\[ w_j \leq v_{0,\infty} \quad \text{in } \Omega_j. \] (3.32)

The function \( w_j^* \) given by \( w_j^*(x) := A_j w_j(r_jx) \) for \( x \in D_0 \) is a solution of the problem:
\[ -\Delta w + w^q = 0 \quad \text{in } D_0, \]
\[ w = 0 \quad \text{on } \partial \Omega_0, \]
\[ w(x', 0) = \infty \delta_0 \quad \text{on } [x_N = 0]. \] (3.33)

The solution of this problem is unique; consequently \( w_j^* \) is independent of \( j \) and we denote it by \( w^* \).
Let $C := \sup_{|x'| < 1/2} w^*(x', 1)$. Then $w_j(y) = A_j^{-1} w^*(y/r_j)$ satisfies

$$w_j(y', r_j) \geq c A_j^{-1}, \quad |y'| < r_{j+1}.$$  

As $\gamma_j(x') = 0$ for $|x'| > r_{j+1}$ it follows that

$$w_j(y', r_j) \geq c \gamma_j(x'), \quad |x'| < r_j.$$  

Hence

$$w_j(x', x_N + r_j) \geq u_j(x) \quad \text{in } \Omega_j.$$  

Therefore, by Proposition 3.1,

$$\lim_{j \to \infty} w_j(0, x_N) = \infty \quad \forall x_N > 0.$$  

Hence, by (3.32),

$$v_{0, \infty}(0, x_N) = \infty \quad \forall x_N > 0$$  

in contradiction to (3.30). \hfill \Box

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