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# Global existence and collisions for symmetric configurations of nearly parallel vortex filaments

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**Abstract**

We consider the Schrödinger system with Newton-type interactions that was derived by R. Klein, A. Majda and K. Damodaran (1995) [17] to modelize the dynamics of $N$ nearly parallel vortex filaments in a 3-dimensional homogeneous incompressible fluid. The known large time existence results are due to C. Kenig, G. Ponce and L. Vega (2003) [16] and concern the interaction of two filaments and particular configurations of three filaments. In this article we prove large time existence results for particular configurations of four nearly parallel filaments and for a class of configurations of $N$ nearly parallel filaments for any $N \geqslant 2$. We also show the existence of travelling wave type dynamics. Finally we describe configurations leading to collision.


## 1. Introduction

In this paper we study the dynamics of $N$ interacting vortex filaments in a 3-dimensional homogeneous incompressible fluid. We focus on filaments that are all nearly parallel to the $z$-axis. They are described by means of complex-valued functions $\Psi_j(t,\sigma) \in \mathbb{C}$, $1 \leqslant j \leqslant N$, where $t \in \mathbb{R}$ is the time, $\sigma \in \mathbb{R}$ parameterizes the $z$-axis, and $\Psi_j(t,\sigma)$ is the position of the $j$-th filament. A simplified model for the dynamics of such nearly parallel filaments has been derived by R. Klein, A. Majda and K. Damodaran [17] in the form of the following 1-dimensional Schrödinger system of equations

$$\begin{cases} i\partial_t \Psi_j + \Gamma_j \partial_\sigma^2 \Psi_j + \displaystyle\sum_{k \neq j} \Gamma_k \frac{\Psi_j - \Psi_k}{|\Psi_j - \Psi_k|^2} = 0, \quad 1 \leqslant j \leqslant N, \\ \Psi_j(0,\sigma) = \Psi_{j,0}(\sigma). \end{cases} \tag{1.1}$$

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Here $\Gamma_j$ is a real number representing the circulation of the $j$-th filament.[2] In the case where $\Psi_j(t,\sigma)=\Psi_j(t)=X_j(t)$ are exactly parallel filaments, system (1.1) reduces to the well-known point vortex system arising in 2-dimensional homogeneous incompressible fluids

$$\begin{cases} i\dfrac{dX_j}{dt}+\displaystyle\sum_{k\neq j}\Gamma_k\frac{X_j-X_k}{|X_j-X_k|^2}=0, \quad 1\leqslant j\leqslant N,\\ X_j(0)=X_{j,0}. \end{cases} \tag{1.2}$$

The system (1.1) combines on the one hand the linearized self-induction approximation for each vortex filament, given by the linear Schrödinger equation, and on the other hand the interaction of the filaments, for any $\sigma$, by the point vortex system. Solutions of the simplified model (1.1) have remarkable mathematical and physical properties, as described in [20]. The main issue in this context is the possibility of collision of at least two of the filaments in finite time at some point $\sigma$.

Before presenting the known results on nearly parallel vortex filaments let us briefly review some classical facts on the point vortex system (1.2). Its dynamics preserves the center of inertia $\sum_j\Gamma_jX_j(t)$, the angular momentum $\sum_j\Gamma_j|X_j(t)|^2$ and the quantities

$$\sum_{j\neq k}\Gamma_j\Gamma_k\ln\bigl|X_j(t)-X_k(t)\bigr|^2,\quad \sum_{j\neq k}\Gamma_j\Gamma_k\bigl|X_j(t)-X_k(t)\bigr|^2.$$

In case of circulations having all the same signs this implies that no collision among the vortices can occur in finite time. Therefore there exists a unique global $\mathcal{C}^1$ solution $(X_j(t))_j$ to (1.2). For $N=2$ global existence still holds independently of the circulation signs since $|X_1(t)-X_2(t)|$ remains constant. When dealing with more than two vortices the single-sign assumption of the circulations really matters – explicit examples of configurations leading to collapse in finite time have been given by self-similar shrinking triangles [1]. For any circulations the equilateral triangle is a rotating or translating configuration, and for identical circulations the ends and the middle of a segment form also a relative equilibrium configuration. For $N\geqslant 4$ and identical circulations $\Gamma_j=\Gamma\ \forall j$, vertices of regular polygons also form relative equilibrium configurations. They rotate around the center of inertia with constant angular velocity $\omega=\Gamma(N-1)/(2R^2)$, where $R$ is the size of the polygon. These polygon configurations are stable if and only if $N\leqslant 7$. The proof of this result, conjectured by Kelvin in 1878, was recently completed by L.G. Kurakin and V.I. Yudovitch in 2002 [18] (see also [22]). Finally, the configuration formed by adding to an $N$-polygon configuration one point of arbitrary circulation $\Gamma_0$ at the center of inertia, is a relative equilibria rotating with constant angular velocity $\omega=[\Gamma(N-1)+2\Gamma_0]/(2R^2)$. A natural observation to be done is that as $N$ increases the dynamics gets more and more sophisticated.

The first result on nearly parallel vortex filaments has been given in [17]. The authors proved that for $N=2$ the linearized system around the exactly parallel filaments solution of (1.2) is stable if the circulations have the same sign and unstable otherwise. Moreover, they made numerical simulations suggesting global existence for (1.1) in the first case and collision in finite time in the second case. Their first conjecture on global existence was proved then by C. Kenig, G. Ponce and L. Vega [16] for filaments $\Psi_j$ obtained as small $H^1$ perturbations of exactly parallel filaments $X_j$,

$$\Psi_j(t,\sigma)=X_j(t)+u_j(t,\sigma),\quad 1\leqslant j\leqslant N. \tag{1.3}$$

More precisely, it has been proved in [16] (see also the survey [23]) that for $u_j(0)$ sufficiently small in $H^1(\mathbb{R})$ – and therefore in $L^\infty(\mathbb{R})$ – global existence and uniqueness of the solution to system (1.1) hold for all vortex solutions $(X_j)_j$ of equal circulations and such that $|X_j(t)-X_k(t)|=d$ for $1\leqslant j\neq k\leqslant N$. The only such possible configurations are $N=2$ with any pair $(X_1,X_2)$, and $N=3$ with $(X_1,X_2,X_3)$ an equilateral triangle. Moreover, local existence and uniqueness hold for any number $N$ of filaments and any circulations $\Gamma_j$ and the solution exists at least up to times of order $|\ln\sum_j\|u_j(0)\|_{H^1}|$.

Finally, let us mention that P.-L. Lions and A. Majda [19] developed an equilibrium statistical theory for nearly parallel filaments using the approximation given by system (1.1).

[2] The free Schrödinger operator derived in [17] is actually $i\partial_t+\alpha_j\Gamma_j\partial_\sigma^2$, where $\alpha_j$ is another vortex core parameter related to the $j$-th filament. For simplicity we assume throughout the paper that $\alpha_j=1$.

The purpose of this article is to study other specific configurations of vortex filaments. In order to obtain large time existence results we will strongly use the symmetry properties of the configuration of the straight filaments $(X_j)_j$ in itself, and those of the perturbation $(u_j)_j$ on the other hand.

In the first part of this paper we focus on the case where $N \geqslant 3$ and $(X_j)_j$ is a regular rotating polygon of radius 1 with $N$ vertices, with or without its center. The index $j = 0$ refers to the center of the polygon and $1 \leqslant j \leqslant N$ to the vertices of the polygon. Since (1.1) is invariant under translations, we can suppose that the center of inertia of the polygon is set at the origin, i.e. $X_0(t) = 0$ for all $t$. We shall impose that the circulations in the vertices have the same value $\Gamma$ and that $\omega$ has the same sign as $\Gamma$. For simplicity we consider

$$\Gamma_j = 1, \quad 1 \leqslant j \leqslant N.$$

In the cases where the center of the polygon is not considered, the angular speed $\omega$ is $(N-1)/2$, hence positive. In the cases when the center of the polygon is considered, the circulation $\Gamma_0$ must be larger than $-(N-1)/2$.

We will consider very specific perturbations of the configuration $(X_j)_j$, assuming that all the perturbations are the same for each of the straight filaments, a dilation combined with a rotation. More precisely we shall focus on solutions having the form

$$\Psi_j(t,\sigma) = X_j(t)\Phi(t,\sigma), \tag{1.4}$$

with $\Psi(t,\sigma)$ close to $X_j(t)$ in some sense as $|\sigma| \to \infty$. Let us notice that this dilation–rotation type of perturbations keeps the symmetry of the polygon for all $(t,\sigma)$. A natural example of such perturbations are the ones with $\Phi - 1$ small in $H^1(\mathbb{R})$. Our result below allows to handle a larger class of perturbations of the regular rotating polygon, including also for example all small constant rotations of the polygon.

**Theorem 1.1.** *Let $N \geqslant 3$ and $(X_j)_j$ be the equilibrium solution given by a regular rotating polygon of radius* 1*, with or without its center, with $\Gamma_j = 1$ for $1 \leqslant j \leqslant N$ and positive angular velocity $\omega$. Assume that*

$$\Psi_{j,0}(\sigma) = X_{j,0}\Phi_0(\sigma),$$

*with $\Phi_0$ such that*

$$\mathcal{E}(\Phi_0) = \frac{1}{2}\int |\partial_\sigma \Phi_0|^2 + \frac{\omega}{2}\int \big(|\Phi_0|^2 - 1 - \ln|\Phi_0|^2\big)$$

*satisfies $\mathcal{E}(\Phi_0) \leqslant \eta_1$, where $\eta_1$ is an absolute constant.*[3] *Then there exists a unique global solution $(\Psi_j)_j$ of* (1.1)*, with this initial datum, such that*

$$\Psi_j(t,\sigma) = X_j(t)\Phi(t,\sigma), \quad t \in \mathbb{R}$$

*with $\Phi - \Phi_0 \in C(\mathbb{R}, H^1(\mathbb{R}))$. Moreover*

$$\frac{3}{4} \leqslant \frac{|\Psi_j(t,\sigma) - \Psi_k(t,\sigma)|}{|X_j(t) - X_k(t)|} \leqslant \frac{5}{4}, \quad t,\sigma \in \mathbb{R}.$$

*In particular, if $\Phi_0(\sigma) \overset{|\sigma|\to\infty}{\longrightarrow} 1$ then $\Psi_j(t,\sigma) \overset{|\sigma|\to\infty}{\longrightarrow} X_j(t)\ \forall t$, and if $\Phi_0 \in 1 + H^1(\mathbb{R})$ then $\Psi_j - X_j \in C(\mathbb{R}, H^1(\mathbb{R}))$.*

**Remark 1.** Theorem 1.1 does not assert that if initially $\|\Phi_0 - 1\|_{H^1}$ is small then $\|\Phi(t) - 1\|_{H^1}$ remains small for all $t$.

Our analysis is based on the observation that the solution $(\Psi_j)_j$ to system (1.1) satisfies (1.4) if and only if $\Phi$ is solution to the equation

$$i\partial_t\Phi + \partial_\sigma^2\Phi + \omega\frac{\Phi}{|\Phi|^2}\big(1 - |\Phi|^2\big) = 0. \tag{1.5}$$

[3] Introduced in Lemma 2.1 below.

Eq. (1.5) is a hamiltonian equation, which preserves the energy

$$\mathcal{E}(\Phi)=\frac{1}{2}\int|\partial_\sigma\Phi|^2+\frac{\omega}{2}\int\big(|\Phi|^2-1-\ln|\Phi|^2\big). \tag{1.6}$$

Note that in the setting of Theorem 1.1 the solutions satisfy $|\Phi|\simeq 1$, so that Eq. (1.5) is formally similar to the well-known Gross–Pitaevskii equation

$$i\partial_t\Phi+\partial_\sigma^2\Phi+\omega\Phi\big(1-|\Phi|^2\big)=0, \tag{1.7}$$

with energy given by

$$\mathcal{E}_{GP}(\Phi)=\frac{1}{2}\int|\partial_\sigma\Phi|^2+\frac{\omega}{4}\int\big(|\Phi|^2-1\big)^2.$$

In fact we shall see that both functionals $\mathcal{E}(\Phi)$ and $\mathcal{E}_{GP}(\Phi)$ are comparable whenever $|\Phi|\simeq 1$. A key point in the proof is, as in [16], the fact that if $\mathcal{E}(\Phi_0)$ is small then the solution $\Phi$ enjoys the property

$$\sup_{t\in\mathbb{R}}\big\||\Phi(t)|^2-1\big\|_{L^\infty}\leqslant\frac{1}{4}. \tag{1.8}$$

This allows us to establish Theorem 1.1 by using the techniques introduced in [24] by P.E. Zhidkov (see also P. Gérard [11,12]) for solving the Gross–Pitaevskii equation in the energy space.

In the case where $\Phi_0\in 1+H^1(\mathbb{R})$ we mention that the proof in [16] can be adapted here, by showing that some quantities are still conserved even though $|X_j(t)-X_k(t)|$ are not all the same.

As far as we have seen, global existence and uniqueness of the filaments hold for $N=2$ with any $(X_j)_j$ and any small perturbations, for $N=3$ with $(X_j)_j$ the equilateral triangle stable equilibrium and any small perturbations, for any $N\geqslant 2$ with $(X_j)_j$ the regular polygon equilibrium and any small perturbations with strong symmetry conditions. We expect then that global existence might hold for small $N$ and less restrictive conditions on the perturbations.

In the second part of this paper we study the case

$$N=4,\qquad \Gamma_j=1,$$

and we assume that $(X_j)_j=(X_1,X_2,X_3,X_4)$ is a square of radius 1 rotating with constant angular speed. Again, since (1.1) is invariant under translations, we can suppose that the square is centered at the origin. Our main result in this case may be formulated as follows.

**Theorem 1.2.** *Let $N=4$ and $(X_j)_j$ be the equilibrium solution given by a rotating square of radius 1 with $\Gamma_j=1$. Let $(u_{j,0})_j\in H^1(\mathbb{R})^4$ and set $\Psi_{j,0}=X_{j,0}+u_{j,0}$.*

*We introduce the energy*[4]

$$\mathcal{E}_0=\frac{1}{2}\sum_j\int\big|\partial_\sigma\Psi_{j,0}(\sigma)\big|^2\,d\sigma$$
$$+\frac{1}{2}\sum_{j\neq k}\int-\ln\bigg(\frac{|\Psi_{j,0}(\sigma)-\Psi_{k,0}(\sigma)|^2}{|X_{j,0}-X_{k,0}|^2}\bigg)+\bigg(\frac{|\Psi_{j,0}(\sigma)-\Psi_{k,0}(\sigma)|^2}{|X_{j,0}-X_{k,0}|^2}-1\bigg)\,d\sigma.$$

*We also introduce the quantity*

$$\widetilde{\mathcal{E}_0}=\max\bigg\{\mathcal{E}_0;\ \frac{\|u_{1,0}+u_{3,0}\|_{L^2}^2}{2}+\frac{\|u_{2,0}+u_{4,0}\|_{L^2}^2}{2}\bigg\}$$

*and we assume that*

$$\widetilde{\mathcal{E}_0}\leqslant\eta_2$$

[4] Note that $\mathcal{E}_0\geqslant 0$.

*for an absolute small constant $\eta_2 > 0$. Then there exists an absolute constant $C > 0$, and there exists a time $T$, with*

$$T \geqslant C \min\left\{\frac{1}{\widetilde{\mathcal{E}_0}^{1/4} \max_{j,k} \|u_{j,0} - u_{k,0}\|_{L^2}^{1/2}}, \frac{1}{\widetilde{\mathcal{E}_0}^{1/3}}\right\},$$

*such that there exists a unique corresponding solution $(\Psi_j)_j$ to system (1.1) on $[0,T]$, satisfying $\Psi_j = X_j + u_j$, with $u_j \in C([0,T], H^1(\mathbb{R}))$, and such that*

$$\frac{3}{4} \leqslant \frac{|\Psi_j(t,\sigma) - \Psi_k(t,\sigma)|}{|X_j(t) - X_k(t)|} \leqslant \frac{5}{4}, \quad t \in [0,T],\ \sigma \in \mathbb{R}.$$

*Finally, if the initial perturbation is parallelogram-shaped, namely*

$$\|u_{1,0} + u_{3,0}\|_{L^2} = \|u_{2,0} + u_{4,0}\|_{L^2} = 0,$$

*then the solution $(\Psi_j)_j$ is globally defined.*

**Remark 2.** In the proof of Theorem 1.2 we shall actually establish a local existence result for any $N$, any parallel configuration $(X_j)_j$, any set of positive circulations $(\Gamma_j)_j$ and any perturbations with small energy, but not necessarily small in $H^1$. This is a slight improvement of the result in [16], see also the next two remarks.

**Remark 3.** As we shall see, we can infer from the smallness of the energy $\mathcal{E}_0$ and from Sobolev embeddings that the nearly parallel filaments $\Psi_{j,0}$ are not too far from the straight filaments $X_{j,0}$ and that $\mathcal{E}_0 \leqslant C\sum_j \|u_{j,0}\|_{H^1}^2$. Conversely, if we assume that $\sum_j \|u_{j,0}\|_{H^1}$ is sufficiently small then one can show that $\widetilde{\mathcal{E}_0} \leqslant C\sum_j \|u_{j,0}\|_{H^1}^2$ and the assumptions of Theorem 1.2 are satisfied. Therefore the hypothesis on the energy is less restrictive than the one on the $H^1$ norm, see also the next remark.

**Remark 4.** From $0 \leqslant \mathcal{E}_0 \leqslant C\sum_j \|u_{j,0}\|_{H^1}^2$ it follows that $\widetilde{\mathcal{E}_0} \leqslant C\sum_j \|u_{j,0}\|_{H^1}^2$ so the time of existence is a priori larger than in [16]. Moreover, for all $\epsilon > 0$ Theorem 1.2 allows for initial perturbations of the form

$$\Psi_{j,0}^{\epsilon}(\sigma) = e^{i\varphi^{\epsilon}(\sigma)} X_{j,0} + T^{\epsilon}(\sigma),$$

with $\varphi^\epsilon, T^\epsilon$ such that $\|(\varphi^\epsilon, T^\epsilon)\|_{H^1} = O(1)$. This amounts to rotating and translating the square $(X_j)_j$ at each level $\sigma$. By taking oscillating phases of the form $\varphi^\epsilon(\sigma) = \sqrt{\varepsilon}\varphi_0(\varepsilon\sigma)$ with a fixed $\varphi_0 \in H^1$, which implies $\|\varphi^\epsilon\|_{L^2} \geqslant O(1)$, $\|\nabla\varphi^\epsilon\|_{L^2} = O(\epsilon)$ and by choosing $T^\epsilon$ such that $\|T^\epsilon\|_{H^1} = O(\epsilon)$ we compute

$$\widetilde{\mathcal{E}_0} = O(\epsilon^2), \qquad \sum_j \|u_{j,0}\|_{H^1}^2 \geqslant O(1).$$

Therefore Theorem 1.2 provides a unique solution at least up to time of order $1/\sqrt{\varepsilon}$, while the $H^1$ norm of the perturbations is of order one. This suggests that the energy space is more appropriate for the analysis of (1.1) than classical Sobolev spaces.

The proof of Theorem 1.2 follows the one of Theorem 1.1 combined with the one in [16]. In particular, we consider, as in [16], the energy

$$\begin{aligned}\mathcal{E}(t) = {}& \frac{1}{2}\sum_j \int \left|\partial_\sigma \Psi_j(t,\sigma)\right|^2 d\sigma \\ & + \frac{1}{2}\sum_{j\neq k}\int -\ln\left(\frac{|\Psi_j(t,\sigma) - \Psi_k(t,\sigma)|^2}{|X_j(t) - X_k(t)|^2}\right) + \left(\frac{|\Psi_j(t,\sigma) - \Psi_k(t,\sigma)|^2}{|X_j(t) - X_k(t)|^2} - 1\right) d\sigma, \end{aligned} \tag{1.9}$$

and show that the solution can be extended as long as $\mathcal{E}(t)$ remains small. For this purpose we show that $u_j$ can be extended locally from a time $t_0$ by a fixed point argument for small $H^1$ perturbations $w_j$ of the linear evolutions of the initial data, i.e. $u_j(t) = e^{i(t-t_0)\partial_\sigma^2} u_j(t_0) + w_j(t)$. In here we use crucially the fact that the deviation $e^{i(t-t_0)\partial_\sigma^2} u_j(t_0) - u_j(t_0)$ can be upper-bounded in $L^\infty$ in terms of the energy at the initial time $\mathcal{E}(t_0)$. As observed in [16], for any two parallel filaments and for the equilateral triangle configuration the energy is conserved, i.e. $\mathcal{E}(t) = \mathcal{E}(0) = \mathcal{E}_0$, so that global existence follows for small energy perturbations. Unfortunately, under the assumptions of Theorem 1.2 the energy is no longer conserved (unless the perturbation $(u_j)_j$ is parallelogram-shaped). Instead, we estimate its evolution in time, showing that it does not increase too fast, and this control enables us to obtain a large time of existence.

We finally mention another collection of dynamics that is governed by the linear Schrödinger equation. For shifted perturbations $\Psi_j = X_j + u$, for any $X_j$ with $\Gamma_j$ the same, we obtain that $u$ is a solution of the linear Schrödinger equation. So if $u$ is regular enough, it has constant $H^1$ norm, so the filaments remain separated for all time. Moreover, due to the dispersive inequality for the linear Schrödinger equation, the perturbations spread in time along the parallel configuration $X_j$. Finally, we get examples of $\mathcal{C}^\infty$ perturbations decaying at infinity that generate a singularity in finite time by considering less regular perturbations than $H^1$ that lead to an $L^\infty$ dispersive blow-up for the linear Schrödinger. The self-similar linear Schrödinger solution constructed from homogeneous data $|x|^{-p}$ with $0 < p < 1$ in [6] leads to solutions blowing up in $L^\infty$ in finite time at one point. Also, the linear Schrödinger evolution of $e^{i|x|^2}/(1+|x|^2)^m$ with $1/2 < m \leqslant 1$ has been proved in [5] to be an $L^2$ solution whose modulus blows up in finite time at one point.

The third part of this work is devoted to travelling waves for system (1.1). Let us recall that in the case of one single filament, a travelling wave dynamics was exhibited by H. Hasimoto [14] and experimentally observed by E.J. Hopfinger and F.K. Browand [15]. Here we construct travelling waves for several filaments via finite energy travelling wave solutions to Eq. (1.5), i.e. solutions of the form $\Phi(t,\sigma) = v(\sigma + ct)$, with $v$ solution of the equation

$$icv' + v'' + \omega \frac{v}{|v|^2}\big(1 - |v|^2\big) = 0 \tag{1.10}$$

and having finite energy,

$$\mathcal{E}(v) = \frac{1}{2}\int |\partial_\sigma v|^2 + \frac{\omega}{2}\int \big(|v|^2 - 1 - \ln|v|^2\big) < \infty. \tag{1.11}$$

As in Theorem 1.1 we assume that $\omega > 0$. In order to avoid having $v$ approaching zero we shall impose that the energy is small.

Existence, stability issues and qualitative behaviour near the speed of sound of travelling waves for Gross–Pitaevskii-type equations and related problems were extensively studied in the past years (see for instance [9,25,4,10,13,3,21,8] and the references therein). For the 1-dimensional Gross–Pitaevskii equation (1.7), finite energy travelling waves (referred to as “grey solitons” in the context of non-linear optics) are known to exist for all $0 < c < \sqrt{2\omega}$, and they have the explicit form (see e.g. [13])

$$v(\sigma) = v_c(\sigma) = \sqrt{1 - \frac{\frac{1}{2\omega}(2\omega - c^2)}{\cosh^2(\frac{\sqrt{2\omega-c^2}}{2}\sigma)}}\, e^{i \arctan \frac{\omega e^{\sqrt{2\omega-c^2}\sigma} + c^2 - \omega}{c\sqrt{2\omega-c^2}} - i \arctan \frac{c}{\sqrt{2\omega-c^2}}}.$$

The modulus $|v_c|$ of such maps is close to 1 when $c$ is close to $\sqrt{2\omega}$, in which case $\mathcal{E}(v_c) \leqslant C\mathcal{E}_{GP}(v_c) \leqslant C(2\omega - c^2)^{3/2}$ (see [13]), so the energy is finite and as small as needed. Note that therefore the maps $v_c$, with $c$ close to $\sqrt{2\omega}$ enter the class of perturbations presented in Theorem 1.1. Our next result in this context is the following.

**Theorem 1.3.** *Let $c$ be such that $0 < 2\omega - c^2 < \eta_3$ for an absolute small constant $\eta_3 > 0$. There exists a travelling wave solution to system* (1.1)

$$\Psi_j(t,\sigma) = e^{it\omega + i\frac{2\pi j}{N}} v(\sigma + ct),$$

*where $v \in C^\infty(\mathbb{R})$ is a solution to Eq.* (1.10)*, with finite energy $\mathcal{E}(v) \leqslant C(2\omega - c^2)^{3/2}$, such that $v$ never vanishes. The modulus $|v|$ is an even function, increasing on $[0,\infty)$ and satisfying on $\mathbb{R}$*

$$0 < 1 - \big|v(\sigma)\big|^2 < \min\bigg\{\frac{3}{2\omega}\big(2\omega - c^2\big),\ C\sqrt{2\omega - c^2}\, e^{-\sqrt{2\omega - c^2}^{\,-}|\sigma|}\bigg\}.$$

*Finally, we have a limit at infinity*

$$v(\sigma) \to \exp(i\theta_\pm), \quad \sigma \to \pm\infty, \text{ with } |\theta_+ - \theta_-| \leqslant C\sqrt{2\omega - c^2}.$$

*Here $C$ denotes an absolute numerical constant.*

It has been noticed in [16] that the Galilean invariance of system (1.1) leads to helix-shaped vortex filaments. In here, on one hand Eq. (1.5) is invariant under Galilean transform, i.e. $\Phi_\nu(t,\sigma) = e^{-it\nu^2 + i\nu\sigma}\,\Phi(t,\sigma - 2t\nu)$ is also a solution $\forall \nu \in \mathbb{R}$. On the other hand $X_j(t) = e^{it\omega + i\frac{2\pi j}{N}}$ for $j \neq 0$, so

$$\Psi_{j,\nu}(t,\sigma) = e^{it(\omega-\nu^2)+i\nu\sigma+i\frac{2\pi j}{N}}\,\Phi(t,\sigma - 2t\nu)$$
$$= e^{it(\omega-\nu^2)+i\nu\sigma+i\frac{2\pi j}{N}}\,v\big(\sigma + t(c - 2\nu)\big).$$

Therefore, choosing $\nu = \sqrt{\omega}$, we obtain a stationary $(\theta_+ - \theta_-)$-twisted $N$-helix filament configuration with some localized perturbation travelling in time on each of its filaments.

Last but not least, in the last part of this paper we describe configurations of nearly parallel filaments that lead to a collision in finite time. They are obtained by the same kind of dilation–rotation perturbations as in Theorem 1.1.

**Theorem 1.4.** *Let $N \geqslant 2$ and $(X_j)_j$ be the stationnary configuration given by a regular $N$-polygon with its center and circulations $\Gamma_j = 1$ for $1 \leqslant j \leqslant N$, $\Gamma_0 = -(N-1)/2$. Then the initial condition*

$$\Psi_{j,0}(\sigma) = X_j(0)\left(1 - \frac{e^{-\frac{\sigma^2}{1-4i}}}{\sqrt{1-4i}}\right)$$

*yields a solution $(\Psi_j)_j$ for system (1.1), with $\Psi_j - X_j \in C(\mathbb{R}, H^1(\mathbb{R}))$, that collide at time $t = 1$ at $\sigma = 0$.*

The remainder of this paper is organized as follows. In Section 2 we derive Eq. (1.5). We then present some preliminary lemmas about its energy, which lead to the proof of Theorem 1.1. Section 3 is devoted to the proof of Theorem 1.2. Section 4 contains the construction of travelling waves for Theorem 1.3. Finally, in Section 5 we construct the collision dynamics in Theorem 1.4. In all the following the notation $C$ denotes an absolute constant which can possibly change from a line to another.

## 2. Proof of Theorem 1.1

We first derive Eq. (1.5). Plugging the ansatz $\Psi_j(t,\sigma) = X_j(t)\Phi(t,\sigma)$ into system (1.1) with $\Gamma_j = 1$ for $1 \leqslant j \leqslant N$ we obtain

$$iX_j\partial_t\Phi + i\partial_t X_j\Phi + X_j\partial_\sigma^2\Phi + \frac{\Phi}{|\Phi|^2}\sum_{k\neq j}\frac{X_j - X_k}{|X_j - X_k|^2} = 0.$$

Next we use (1.2) to get

$$X_j\big(i\partial_t\Phi + \partial_\sigma^2\Phi\big) - i\partial_t X_j\frac{\Phi}{|\Phi|^2}\big(1 - |\Phi|^2\big) = 0.$$

Now if we consider a configuration rotating with speed $\omega$ around its steady center of inertia $X_0 = 0$, for $1 \leqslant j \leqslant N$ we have $X_j(t) = e^{it\omega + i\theta_j}$, so that $-i\partial_t X_j = \omega X_j$ and hence we obtain Eq. (1.5),

$$i\partial_t\Phi + \partial_\sigma^2\Phi_j + \omega\frac{\Phi}{|\Phi|^2}\big(1 - |\Phi|^2\big) = 0.$$

Conversely, assume that $\Phi$ is a solution to Eq. (1.5) and set $\Psi_j = X_j\Phi$. Reversing the previous arguments, we obtain

$$i\partial_t\Psi_j + \partial_\sigma^2\Psi_j + \sum_{k\neq j}\frac{\Psi_j - \Psi_k}{|\Psi_j - \Psi_k|^2} = 0, \quad 1 \leqslant j \leqslant N,$$

while, since $\Psi_0(t,\sigma)=0$ for all $(t,\sigma)$,

$$i\partial_t\Psi_0=\partial_\sigma^2\Psi_0=0 \quad \text{and} \quad \sum_{k=1}^N \frac{\Psi_0-\Psi_k}{|\Psi_0-\Psi_k|^2}=-\frac{\Phi}{|\Phi|^2}\sum_{k=1}^N\frac{X_k}{|X_k|^2}=0$$

and therefore $(\Psi_j)_j$ is a solution to system (1.1).

### 2.1. Some preliminary lemmas

**Lemma 2.1.** *There exists an absolute constant $\eta_1$ and a time $t_1$ depending only on $\eta_1$ such that*:

(i) *If $\mathcal{E}(f)\leqslant\eta_1$ then*

$$\big\||f|^2-1\big\|_{L^\infty}\leqslant\frac{1}{4}.$$

(ii) *If $\|\partial_\sigma f\|_{L^2}\leqslant\eta_1$ then for all $0\leqslant t\leqslant t_1$*

$$\frac{1}{\sqrt{2}}\big\|e^{it\partial_\sigma^2}f-f\big\|_{L^\infty}\leqslant\big\|e^{it\partial_\sigma^2}f-f\big\|_{H^1}\leqslant\frac{1}{4}.$$

**Proof.** (i) The function $a(x)=x-1-\ln x$ is positive and convex, and vanishes only at $x=1$, therefore we can adapt standard arguments already used in the context of Ginzburg–Landau-type functionals (see e.g. [2]). More precisely, we assume by contradiction that $||f(\sigma_0)|^2-1|>1/4$ for some $\sigma_0\in\mathbb{R}$. For example, $|f(\sigma_0)|>\sqrt{5/4}$. Next, since $\|\partial_\sigma f\|_{L^2}^2\leqslant 2\mathcal{E}(f)$ we have by Cauchy–Schwarz inequality

$$\big|f(\sigma)\big|\geqslant\big|f(\sigma_0)\big|-\left|\int_{\sigma_0}^{\sigma}\partial_x f(x)\,dx\right|\geqslant\sqrt{\frac{5}{4}}-\sqrt{2\mathcal{E}(f)|\sigma-\sigma_0|}.$$

It follows that $|f|>\sqrt{9/8}$ on $I=[\sigma_0-1/(500\mathcal{E}(f)),\sigma_0+1/(500\mathcal{E}(f))]$. Therefore

$$\mathcal{E}(f)\geqslant\frac{1}{2}a\left(\frac{9}{8}\right)|I|=\frac{1}{500\mathcal{E}(f)}a\left(\frac{9}{8}\right),$$

a contradiction if $\mathcal{E}(f)\leqslant\eta_1$ is sufficiently small.

(ii) The property (ii) is a known one used in the Gross–Pitaevskii study (see Lemma 3 in [11]) to which we recall the short proof: the Fourier transform of $e^{it\partial_\sigma^2}f-f$ can be written as $\frac{e^{-it\xi^2}-1}{\xi}\,\xi\hat{f}(\xi)$, so the $L^2$ norm is bounded by $C\sqrt{t}\|\partial_\sigma f\|_{L^2}$ and the $\dot{H}^1$ norm is bounded by $C\|\partial_\sigma f\|_{L^2}$, i.e.

$$\big\|e^{it\partial_\sigma^2}f-f\big\|_{H^1}\leqslant C(1+\sqrt{t})\|\partial_\sigma f\|_{L^2}\leqslant C(1+\sqrt{t})\eta_1.$$

We choose $\eta_1$ small enough and $t_1$ small with respect to $\eta_1$ such that for $0\leqslant t\leqslant t_1$,

$$\big\|e^{it\partial_\sigma^2}f-f\big\|_{H^1}\leqslant\frac{1}{4},$$

and the conclusion of the lemma follows. □

Since $(x-1)^2/4\leqslant x-1-\ln x\leqslant 10(x-1)^2$ on $[3/4,5/4]$ we immediately obtain the second lemma.

**Lemma 2.2.** *If $\||f|^2-1\|_{L^\infty}\leqslant 1/4$ then we can compare the energies*:

$$\mathcal{E}_{GP}(f)\equiv\frac{1}{2}\|\partial_\sigma f\|_{L^2}^2+\frac{\omega}{8}\big\||f|^2-1\big\|_{L^2}^2\leqslant\mathcal{E}(f)\leqslant 5\mathcal{E}_{GP}(f).$$

So, if we consider an initial perturbation such that $\Phi_0 - 1$ is sufficiently small in $H^1$, we infer from Sobolev embedding that $\mathcal{E}_{GP}(\Phi_0) < \infty$ and that $\||\Phi_0|^2 - 1\|_{L^\infty} < 1/4$. Hence Lemma 2.2 ensures that $\Phi_0$ belongs to the energy space associated to Eq. (1.5).

We will also need the following transposition of a standard property of the Gross–Pitaevskii energy (see [7,11,12]).

**Lemma 2.3.** *Let $f$ be such that $\mathcal{E}(f) \leqslant \eta_1$, with $\eta_1$ defined in Lemma 2.1. Let $h \in H^1(\mathbb{R})$ with $\|h\|_{H^1} \leqslant 1/2$. Then the energy $\mathcal{E}(f+h)$ is finite. More precisely we have, for absolute numerical constants $C$, $C'$,*

$$\mathcal{E}(f+h) \leqslant C\mathcal{E}_{GP}(f+h) \leqslant C'\big(1+\mathcal{E}(f)\big)\big(1+\|h\|_{H^1}^2\big).$$

*Moreover,*

$$\big\||f+h|-1\big\|_{L^\infty} \leqslant \frac{2+\sqrt{2}}{4} < 1.$$

**Proof.** We first infer from Lemma 2.1(i) that $\||f|-1\|_{L^\infty} \leqslant 1/4$, and from Lemma 2.2 that $\mathcal{E}_{GP}(f) < \infty$. Next, applying Gagliardo–Nirenberg inequality we get $\|h\|_{L^\infty} \leqslant \sqrt{2}\|h\|_{H^1} \leqslant \sqrt{2}/2$, so that $\||f+h|-1\|_{L^\infty} \leqslant (2+\sqrt{2})/4 < 1$. By Lemma 2.2 it follows that $\mathcal{E}(f+h) \leqslant C\mathcal{E}_{GP}(f+h)$. Using that $\mathcal{E}_{GP}(f) < \infty$ and $h \in H^1$ as well as Sobolev inequalities we conclude that $\mathcal{E}_{GP}(f+h)$ is finite, with the corresponding estimate (see also, e.g., Lemma 2 in [11]). □

*2.2. Proof of Theorem 1.1*

First we will establish local well-posedness for Eq. (1.5) by performing a fixed point argument for the operator

$$A(w)(t) = i\int_0^t e^{i(t-\tau)\partial_\tau^2}\frac{e^{i\tau\partial_\sigma^2}\Phi_0 + w(\tau)}{|e^{i\tau\partial_\sigma^2}\Phi_0 + w(\tau)|^2}\big(1-\big|e^{i\tau\partial_\sigma^2}\Phi_0 + w(\tau)\big|^2\big)\,d\tau$$

on the ball

$$B_T = \left\{w \in C\big([0,T],H^1\big),\ \sup_{0\leqslant t\leqslant T}\big\|w(t)\big\|_{H^1} \leqslant \frac{1}{4}\right\},$$

with $T$ small to be chosen later. Then $\Phi(t) = e^{it\partial_\sigma^2}\Phi_0 + w(t)$ will be a solution for (1.5) on $[0,T]$ with initial data $\Phi_0$. Observe that the proof of Lemma 2.1(ii) yields that $t \mapsto (e^{it\partial_\sigma^2}\Phi_0 - \Phi_0) \in C([0,T],H^1(\mathbb{R}))$. So the map $\Phi$ will belong to the energy space if $\Phi_0$ belongs to the energy space (by Lemma 2.3 applied to $f = \Phi_0$ and $h = e^{it\partial_\sigma^2}\Phi_0 - \Phi_0 + w(t)$ for $T \leqslant t_1$ with $t_1$ from Lemma 2.1), and it will belong to $1 + H^1(\mathbb{R})$ if $\Phi_0$ is in $1 + H^1(\mathbb{R})$.

The hypothesis of Theorem 1.1 is that we start with $\Phi_0$ verifying

$$\mathcal{E} = \mathcal{E}(\Phi_0) = \frac{1}{2}\|\partial_\sigma\Phi_0\|_{L^2}^2 + \frac{\omega}{2}\int\big(-\ln|\Phi_0| + |\Phi_0|^2 - 1\big) \leqslant \eta_1.$$

We first impose $T \leqslant t_1$, with $t_1$ defined in Lemma 2.1. Let $w \in B_T$, and set for $0 \leqslant t \leqslant T$

$$\widetilde{\Phi}(t) = e^{it\partial_\sigma^2}\Phi_0 + w(t) = \Phi_0 + \big(e^{it\partial_\sigma^2}\Phi_0 - \Phi_0 + w(t)\big).$$

By Lemma 2.1(ii) and by choice of $B_T$, we have $\|\widetilde{\Phi}(t) - \Phi_0\|_{H^1} \leqslant 1/2$ on $[0,T]$. Therefore, applying Lemma 2.3 to $f = \Phi_0$ and $h = \widetilde{\Phi}(t) - \Phi_0$ we obtain that $\||\widetilde{\Phi}(t)| - 1\|_{L^\infty} \leqslant (2+\sqrt{2})/4$ on $[0,T]$. In particular, since $C^{-1} \leqslant |\widetilde{\Phi}| \leqslant C$ for $C > 0$ we can estimate the action of the operator as follows

$$\begin{aligned}\big\|A(w)(t)\big\|_{H^1} &\leqslant t\sup_{0\leqslant\tau\leqslant t}\left\|\frac{\widetilde{\Phi}(\tau)}{|\widetilde{\Phi}(\tau)|^2}\big(1-\big|\widetilde{\Phi}(\tau)\big|^2\big)\right\|_{H^1}\\ &\leqslant C\,t\sup_{0\leqslant\tau\leqslant t}\big(\big\|1-\big|\widetilde{\Phi}(\tau)\big|^2\big\|_{L^2} + \big\|\partial_\sigma\widetilde{\Phi}(\tau)\big\|_{L^2}\big)\\ &\leqslant C\,t\sup_{0\leqslant\tau\leqslant t}\sqrt{\mathcal{E}_{GP}\big(\widetilde{\Phi}(\tau)\big)}.\end{aligned}$$

We use again Lemma 2.3 and the bound $\|\widetilde{\Phi}(\tau)-\Phi_0\|_{H^1}\leqslant 1/2$ to obtain

$$\sup_{0\leqslant t\leqslant T}\big\|A(w)(t)\big\|_{H^1}\leqslant C\,T(1+\mathcal{E}).$$

Arguing similarly, we readily check that for $w_1, w_2\in B_T$

$$\sup_{0\leqslant t\leqslant T}\big\|A(w_1)(t)-A(w_2)(t)\big\|_{H^1}\leqslant C\,T(1+\mathcal{E})\sup_{0\leqslant t\leqslant T}\big\|w_1(t)-w_2(t)\big\|_{H^1}.$$

Hence imposing a second smallness condition on $T$ with respect to $\mathcal{E}$ we obtain a fixed point $w$ for $A$ in $B_T$. Therefore local well-posedness holds for Eq. (1.5) on $[0,T]$ with $T$ depending only on $\mathcal{E}$.

Next, since the energy of Eq. (1.5) is conserved

$$\mathcal{E}\big(\Phi(T)\big)=\mathcal{E}\big(\Phi(0)\big)=\mathcal{E},$$

we re-iterate the local in time argument to get the global existence. Finally, Lemma 2.1 insures us that

$$\sup_{t\in\mathbb{R}}\big\|\big|\Phi(t)\big|^2-1\big\|_{L^\infty}\leqslant\frac{1}{4},$$

so the solution satisfies indeed

$$\frac{1}{4}\leqslant\big|\Phi(t,\sigma)\big|\leqslant\frac{5}{4},\quad t,\sigma\in\mathbb{R}.$$

## 3. Proof of Theorem 1.2

### *3.1. Some useful quantities*

From now on we will write $\Psi_{jk}=\Psi_j-\Psi_k$, $X_{jk}=X_j-X_k$ and $u_{jk}=u_j-u_k$.

We first introduce some useful quantities. In the general case where $N\geqslant 1$ and $\Gamma_j\in\mathbb{R}$, the dynamics of system (1.1) preserves the following quantities:

The energy

$$\frac{1}{2}\sum_j\Gamma_j^2\int\big|\partial_\sigma\Psi_j(t,\sigma)\big|^2\,d\sigma-\frac{1}{2}\sum_{j\neq k}\Gamma_j\Gamma_k\int\ln\big|\Psi_{jk}(t,\sigma)\big|^2\,d\sigma,$$

the angular momentum

$$\sum_j\Gamma_j\int\big|\Psi_j(t,\sigma)\big|^2\,d\sigma,$$

and

$$\sum_{j\neq k}\Gamma_j\Gamma_k\int\big|\Psi_{jk}(t,\sigma)\big|^2\,d\sigma.$$

However the previous quantities are not well defined in the framework of Theorem 1.2, not even formally, since $\Psi_j(t,\sigma)$ and $\Psi_{jk}(t,\sigma)$ do not tend to zero at infinity. As in [16], we modify them in order to get well-defined quantities, introducing

$$\mathcal{H}=\frac{1}{2}\sum_j\Gamma_j^2\int\big|\partial_\sigma\Psi_j(t,\sigma)\big|^2\,d\sigma-\frac{1}{2}\sum_{j\neq k}\Gamma_j\Gamma_k\int\ln\left(\frac{|\Psi_{jk}(t,\sigma)|^2}{|X_{jk}(t)|^2}\right)d\sigma,$$

$$\mathcal{A}=\sum_j\Gamma_j\int\big(\big|\Psi_j(t,\sigma)\big|^2-\big|X_j(t)\big|^2\big)\,d\sigma,$$

$$\mathcal{T}=\sum_{j\neq k}\Gamma_j\Gamma_k\int\big(\big|\Psi_{jk}(t,\sigma)\big|^2-\big|X_{jk}(t)\big|^2\big)\,d\sigma.$$

Note that, in view of the properties of the point vortex system (1.2) mentioned in the introduction, the renormalized quantities $\mathcal{H}$, $\mathcal{A}$ and $\mathcal{T}$ are still formally preserved in time.

Finally, we also introduce the time-dependent quantity

$$\mathcal{I}(t)=\frac{1}{2}\sum_{j\neq k}\Gamma_j\Gamma_k\int\left(\frac{|\Psi_{jk}(t)|^2}{|X_{jk}(t)|^2}-1\right)d\sigma,$$

and we consider the energy

$$\mathcal{E}(t)=\mathcal{H}+\mathcal{I}(t), \tag{3.1}$$

which have been already introduced in (1.9) in the introduction.

As noticed in [16], a useful consequence of the convexity estimate $(x-1)^2/4\leqslant x-1-\ln x\leqslant 10(x-1)^2$ on $[3/4,5/4]$ is the inequality

$$\frac{1}{2}\sum_j\Gamma_j^2\int\left|\partial_\sigma\Psi_j(t,\sigma)\right|^2d\sigma+\frac{1}{8}\sum_{j\neq k}\Gamma_j\Gamma_k\int\left(\frac{|\Psi_{jk}(t,\sigma)|^2}{|X_{jk}(t)|^2}-1\right)^2d\sigma\leqslant\mathcal{E}(t), \tag{3.2}$$

which holds as long as the filaments satisfy $3/4\leqslant|\Psi_{jk}(t)|^2/|X_{jk}(t)|^2\leqslant 5/4$.

### *3.2. The approach*

In this subsection we briefly sketch how to combine elements from [16] and from Section 2 to prove local existence and uniqueness of a solution to system (1.1) in the general case of $N$ filaments, with $N\geqslant 2$, and the way to extend this solution as long as the energy $\mathcal{E}(t)$ remains sufficiently small. Here we take positive circulations

$$\Gamma_j>0,\quad 1\leqslant j\leqslant N.$$

Therefore there exists a unique global solution $(X_j)_j$ to system (1.2). We denote by $d>0$ the minimal distance between the point vortices for all time. Here we shall make the extra-assumption that

$$u_{j,0}=\Psi_{j,0}-X_{j,0}\in H^1(\mathbb{R}).$$

We look for a solution $u=(u_j)_j\in C([0,T],H^1(\mathbb{R}))^N$ to the system

$$\begin{cases} i\partial_t u_j+\Gamma_j\partial_\sigma^2u_j+\displaystyle\sum_{k\neq j}\Gamma_k\left(\frac{X_{jk}+u_{jk}}{|X_{jk}+u_{jk}|^2}-\frac{X_{jk}}{|X_{jk}|^2}\right)=0,\\ u_j(0)=u_{j,0},\quad 1\leqslant j\leqslant N.\end{cases} \tag{3.3}$$

By similar arguments as in Section 2, our purpose is to find a fixed point in the Banach space

$$B_T=\left\{w=(w_1,\dots,w_N)\in C\big([0,T],H^1\big)^N,\ \sup_{0\leqslant t\leqslant T}\big\|w(t)\big\|_{H^1}\leqslant\frac{d}{4}\right\}$$

for the operator $A(w)=(A_j(w))_j$ defined by

$$A_j(w)(t)=i\int_0^t\sum_{k\neq j}\Gamma_k\left(\frac{X_{jk}(\tau)+e^{i\tau\Gamma_j\partial_\sigma^2}u_{j,0}+w_j(\tau)-e^{i\tau\Gamma_k\partial_\sigma^2}u_{k,0}-w_k(\tau)}{|X_{jk}(\tau)+e^{i\tau\Gamma_j\partial_\sigma^2}u_{j,0}+w_j(\tau)-e^{i\tau\Gamma_k\partial_\sigma^2}u_{k,0}-w_k(\tau)|^2}-\frac{X_{jk}(\tau)}{|X_{jk}(\tau)|^2}\right)d\tau,$$

and for $T$ sufficiently small with respect to $\eta_2$, $\sum_j\|u_{j,0}\|_{H^1}$, $(\Gamma_j)_j$ and $d$.

Then as in Section 2 the solution will be given by

$$u_j(t)=e^{it\Gamma_j\partial_\sigma^2}u_{j,0}+w_j(t).$$

By transposing the arguments of Section 2 we obtain the following local well-posedness result.

**Lemma 3.1.** *Let $(u_{j,0})_j \in H^1(\mathbb{R})^N$ be such that $\mathcal{E}_0 < 10\eta_2$, with $\mathcal{E}_0$ defined in Theorem 1.2 and $\eta_2 = \eta_2(d)$ a small constant depending only on $d$. There exists $T > 0$, depending only on $\eta_2$, $\sum_j \|u_{j,0}\|_{H^1}$, $(\Gamma_j)_j$ and $d$, and there exists a unique solution $(u_j)_j \in C([0,T], H^1(\mathbb{R}))^N$ to system (3.3) satisfying*

$$\sup_{0\leqslant t\leqslant T} \|u_j(t)\|_{H^1} \leqslant \|u_{j,0}\|_{H^1} + \frac{d}{4}, \quad 1 \leqslant j \leqslant N.$$

*Moreover we can choose $T$ such that*

$$T\Bigl(1 + \eta_2 + \sum_j \|u_{j,0}\|_{H^1}\Bigr) \geqslant C\bigl(d, (\Gamma_j)_j\bigr)$$

*for some constant $C(d, (\Gamma_j)_j)$ depending only on $d$ and $(\Gamma_j)_j$.*

**Remark 5.** As a byproduct of Lemma 3.1 we realize that the solution $(u_j)_j$ to (3.3) exists as long as the energy $\mathcal{E}(t)$ remains bounded by $10\eta_2$. Indeed note that the norm $\sum_j \|u_j(t)\|_{H^1}$ can grow exponentially, but it cannot blow up as long as the energy is sufficiently small.

**Proof.** Let $0 < \Gamma \leqslant 1$ such that $0 < \Gamma \leqslant \min_j \Gamma_j$. Since all the $(\Gamma_j)$'s are positive, we have

$$\max_{j\neq k} \mathcal{E}\left(\frac{\Psi_{jk,0}}{X_{jk,0}}\right) \leqslant \frac{1}{\Gamma^2}\mathcal{E}_0,$$

where we recall that $\mathcal{E}$ is defined by (1.6) (taking $\omega = 1$).

In particular, if $\eta_2$ is such that $10\eta_2/\Gamma^2 \leqslant \eta_1$, with $\eta_1$ defined in Lemma 2.1, then $3/4 \leqslant |\Psi_{jk,0}|/|X_{jk,0}| \leqslant 5/4$ for all $j \neq k$. Then we have for $w \in B_T$

$$\begin{aligned}
&\bigl|X_{jk}(\tau) + e^{i\tau\Gamma_j\partial_\sigma^2}u_{j,0} + w_j(\tau) - e^{i\tau\Gamma_k\partial_\sigma^2}u_{k,0} - w_k(\tau)\bigr| \\
&\quad = \bigl|\Psi_{jk,0} + \bigl(X_{jk}(\tau) - X_{jk,0}\bigr) + \bigl(e^{i\tau\Gamma_j\partial_\sigma^2}u_{j,0} - u_{j,0}\bigr) - \bigl(e^{i\tau\Gamma_k\partial_\sigma^2}u_{k,0} - u_{k,0}\bigr) + w_{jk}(\tau)\bigr| \\
&\quad \geqslant |\Psi_{jk,0}| - \bigl|X_{jk}(\tau) - X_{jk,0}\bigr| - \sqrt{2}\bigl\|\bigl(e^{i\tau\Gamma_j\partial_\sigma^2}u_{j,0} - u_{j,0}\bigr) - \bigl(e^{i\tau\Gamma_k\partial_\sigma^2}u_{k,0} - u_{k,0}\bigr) + w_{jk}(\tau)\bigr\|_{H^1} \\
&\quad \geqslant \frac{3d}{4} - \frac{2(\sum_j \Gamma_j)}{d}T - C(1+T)\eta_2 - \frac{\sqrt{2}d}{4} \geqslant \frac{d}{4}
\end{aligned}$$

provided that $\eta_2$ is small with respect to $d$, and that $T$ is small in terms of $\eta_2, d, (\Gamma_j)_j$. In the last inequality we have used the proof of Lemma 2.1(ii) together with the mean-value theorem for $X_{jk}$. Now, since $X_{jk}(\tau) + e^{i\tau\Gamma_j\partial_\sigma^2}u_{j,0} + w_j(\tau) - e^{i\tau\Gamma_k\partial_\sigma^2}u_{k,0} - w_k(\tau)$ is bounded from below, direct estimates show that $A$ is a contraction on $B_T$ as long as

$$T\Bigl(1 + \eta_2 + \sum_j \|u_{j,0}\|_{H^1}\Bigr) \leqslant C\bigl(d, (\Gamma_j)_j\bigr)$$

and the conclusion of Lemma 3.1 follows. □

*3.3. Proof of Theorem 1.2*

We present now the proof of Theorem 1.2. By Remark 5, there exists a unique solution as long as $\mathcal{E}(t)$ remains sufficiently small. In the cases considered in [16] where the $|X_{jk}(t)|$ are all the same and constant equal to $d$, $\mathcal{I}(t) = \mathcal{T}/(2d^2)$ so $\mathcal{E}(t)$ is conserved. Also under the hypothesis of Theorem 1.1, we have

$$\mathcal{I}(t) = \frac{1}{2}\sum_{j\neq k}\Gamma_j\Gamma_k\int\left(\frac{|\Psi_{jk}(t)|^2}{|X_{jk}|^2} - 1\right)d\sigma = \frac{1}{2}\sum_{j\neq k}\Gamma_j\Gamma_k\int\bigl(\bigl|\Phi(t,\sigma)\bigr|^2 - 1\bigr)\,d\sigma = \omega\mathcal{A},$$

so, although $|X_{jk}|$ are not all equal, $\mathcal{I}(t)$ and $\mathcal{E}(t)$ are still formally preserved. In fact, under the assumptions of Theorem 1.1 we have $\mathcal{E}(t) = N\mathcal{E}(\Phi(t))$ so we retrieve the fact that it is constant. Under the general hypothesis of Theorem 1.2 $\mathcal{E}(t)$ is no longer constant, but it will still be a useful quantity for which we can achieve some control.

We recall that $\mathcal{E}_0 \leqslant \eta_2$. From now on we consider $T > 0$ and the unique solution to system (3.3) on $[0, T]$, with $\mathcal{E}(t) < 10\widetilde{\mathcal{E}_0} \leqslant 10\eta_2$, given by Lemma 3.1. We take $T$ maximal in the sense that $\mathcal{E}(T) = 10\widetilde{\mathcal{E}_0}$ (but $T$ is not necessarily the largest time of existence). We thus have $3/4 < |\Psi_{jk}(t,\sigma)| < 5/2$ on $[0, T] \times \mathbb{R}$ for all $j \neq k$.

**Proposition 3.2.** *We have for $t \in [0, T]$*

$$\mathcal{E}(t) = \mathcal{H} + \frac{1}{2}\mathcal{T} - \mathcal{A} + \frac{\|(u_1+u_3)(t)\|^2 + \|(u_2+u_4)(t)\|^2}{2}.$$

**Proof.** Since $(X_1, X_2, X_3, X_4)$ is a square of radius 1 we have

$$\left|X_{jk}(t)\right|^2 = 2 \quad \text{if } |j-k| = 1, \qquad \left|X_{jk}(t)\right|^2 = 4 \quad \text{if } |j-k| = 2.$$

It follows that

$$\begin{aligned}\sum_{j\neq k}\left(\frac{|\Psi_{jk}|^2}{|X_{jk}|^2} - 1\right) &= \sum_{j\neq k}\frac{|\Psi_{jk}|^2 - |X_{jk}|^2}{|X_{jk}|^2} \\ &= \frac{1}{2}\sum_{j\neq k}\left(|\Psi_{jk}|^2 - |X_{jk}|^2\right) + 2\left(\frac{1}{4} - \frac{1}{2}\right)\left(|\Psi_{13}|^2 - |X_{13}|^2 + |\Psi_{24}|^2 - |X_{24}|^2\right).\end{aligned}$$

On the other hand, we compute

$$\begin{aligned}|\Psi_{13}|^2 + |\Psi_{24}|^2 - |X_{13}|^2 - |X_{24}|^2 &= 2\sum_{j=1}^{4}|\Psi_j|^2 - |\Psi_1+\Psi_3|^2 - |\Psi_2+\Psi_4|^2 - 8 \\ &= 2\sum_{j=1}^{4}\left(|\Psi_j|^2 - |X_j|^2\right) - \left(|\Psi_1+\Psi_3|^2 + |\Psi_2+\Psi_4|^2\right),\end{aligned}$$

so integrating with respect to $\sigma$ and using that $\Psi_1 + \Psi_3 = u_1 + u_3$ and $\Psi_2 + \Psi_4 = u_2 + u_4$ we are led to the conclusion. □

**Corollary 3.3.** *In the case of the parallelogram $\|(u_1+u_3)(0)\|_{L^2}^2 = \|(u_2+u_4)(0)\|_{L^2}^2 = 0$, so it follows that $\|(u_1 + u_3)(t)\|_{L^2}^2 = \|(u_2+u_4)(t)\|_{L^2}^2 = 0$ for all times, using the fact that if $(\Psi_1, \Psi_2, \Psi_3, \Psi_4)$ is a solution of (1.1) then $(-\Psi_3, -\Psi_4, -\Psi_1, -\Psi_2)$ is also a solution. Then $\mathcal{I}$ is conserved in time and global existence follows.*

**Remark 6.** One can do similar computations in others particular cases, for instance for ends and the middle of the segment,

$$\mathcal{E}(t) = -\mathcal{H} + \mathcal{I} - \frac{3}{2}\mathcal{A} + \frac{3}{4}\left(\left\|u_1(t)\right\|_{L^2}^2 + \left\|(u_2+u_3)(t)\right\|_{L^2}^2\right),$$

or for hexagon,

$$\mathcal{E}(t) = -\mathcal{H} + \mathcal{I} - \frac{7}{2}\mathcal{A} + \frac{2}{3}\sum_{j=1}^{2}\left\|(u_j + u_{j+2} + u_{j+4})(t)\right\|_{L^2}^2 + \frac{3}{4}\sum_{j=1}^{3}\left\|(u_j + u_{j+3})(t)\right\|_{L^2}^2.$$

But these quantities have no reason to be conserved, unless the perturbations have the same shape as the shape of $(X_j)$, which enters the framework of the first part of this article. Moreover, when trying to control the growth of $\|u_1(t)\|_{L^2}$ for instance in the first example, the time of control is not satisfactory due to the presence of linear terms in the equation of $u_1$, that cannot be resorbed.

In order to control the evolution of the energy we have to control the quantity $\|(u_1+u_3)(t)\|_{L^2}^2 + \|(u_2+u_4)(t)\|_{L^2}^2$. We are led to introduce the new unknowns

$$v = u_1 + u_3, \qquad w = u_2 + u_4.$$

**Proposition 3.4.** *We have for $t \in [0,T]$, with $v = u_1 + u_3$ and $w = u_2 + u_4$,*

$$\|v(t)\|_{L^2} + \|w(t)\|_{L^2} \leqslant \|v(0)\|_{L^2} + \|w(0)\|_{L^2} + Ct \sup_{s\in[0,T]} \max_{j\neq k} \|u_{jk}(s)\|_{L^2}^{1/2} \mathcal{E}(s)^{1/4} \big(\|v(s)\|_{L^2} + \|w(s)\|_{L^2} + \mathcal{E}(s)^{1/2}\big).$$

**Proof.** In view of system (1.1) and system (1.2), we have

$$\begin{aligned} i\partial_t v + \partial_\sigma^2 v &= -\sum_{k\neq 1,3} \left\{ \left( \frac{\Psi_{1k}}{|\Psi_{1k}|^2} - \frac{X_{1k}}{|X_{1k}|^2} \right) + \left( \frac{\Psi_{3k}}{|\Psi_{3k}|^2} - \frac{X_{3k}}{|X_{3k}|^2} \right) \right\} \\ &= -\sum_{k\neq 1,3} \left\{ X_{1k} \left( \frac{1}{|\Psi_{1k}|^2} - \frac{1}{|X_{1k}|^2} \right) + X_{3k} \left( \frac{1}{|\Psi_{3k}|^2} - \frac{1}{|X_{3k}|^2} \right) \right\} \\ &\quad - \sum_{k\neq 1,3} \left\{ u_{1k} \left( \frac{1}{|\Psi_{1k}|^2} - \frac{1}{|X_{1k}|^2} \right) + u_{3k} \left( \frac{1}{|\Psi_{3k}|^2} - \frac{1}{|X_{3k}|^2} \right) \right\} \\ &\quad - \sum_{k\neq 1,3} \left\{ \frac{u_{1k}}{|X_{1k}|^2} + \frac{u_{3k}}{|X_{3k}|^2} \right\}. \end{aligned}$$

We infer that

$$i\partial_t v + \partial_\sigma^2 v = \mathcal{L}_v(u) + \mathcal{R}_v(u),$$

where $\mathcal{L}_v$ denotes the linear part,

$$\mathcal{L}_v(u) = 2\sum_{k\neq 1,3} \left\{ X_{1k} \frac{\Re e(\overline{u_{1k}} X_{1k})}{|X_{1k}|^4} + X_{3k} \frac{\Re e(\overline{u_{3k}} X_{3k})}{|X_{3k}|^4} \right\} - \sum_{k\neq 1,3} \left\{ \frac{u_{1k}}{|X_{1k}|^2} + \frac{u_{3k}}{|X_{3k}|^2} \right\}$$

and where the remainder $\mathcal{R}_v$ is quadratic in $u$,

$$\begin{aligned} \mathcal{R}_v(u) &= \sum_{k\neq 1,3} \left\{ \frac{X_{1k}}{|X_{1k}|^4} |u_{1k}|^2 + \frac{X_{3k}}{|X_{3k}|^4} |u_{3k}|^2 \right\} \\ &\quad - \sum_{k\neq 1,3} \left\{ X_{1k} \left( \frac{|X_{1k}|^2 - |\Psi_{1k}|^2}{|X_{1k}|^2} \right) \left( \frac{1}{|\Psi_{1k}|^2} - \frac{1}{|X_{1k}|^2} \right) + X_{3k} \left( \frac{|X_{3k}|^2 - |\Psi_{3k}|^2}{|X_{3k}|^2} \right) \left( \frac{1}{|\Psi_{3k}|^2} - \frac{1}{|X_{3k}|^2} \right) \right\} \\ &\quad - \sum_{k\neq 1,3} \left\{ u_{1k} \left( \frac{1}{|\Psi_{1k}|^2} - \frac{1}{|X_{1k}|^2} \right) + u_{3k} \left( \frac{1}{|\Psi_{3k}|^2} - \frac{1}{|X_{3k}|^2} \right) \right\} \\ &= \mathcal{R}_v^1(u) + \mathcal{R}_v^2(u) + \mathcal{R}_v^3(u). \end{aligned}$$

We claim that $\mathcal{L}_v(u) = 0$. Indeed, using that $|X_{1k}|^2 = |X_{3k}|^2 = 2$ for $k \neq 1, 3$,

$$\begin{aligned} \mathcal{L}_v(u) &= \frac{1}{2} \sum_{k\neq 1,3} \big( X_{1k} \Re e(\overline{u_{1k}} X_{1k}) + X_{3k} \Re e(\overline{u_{3k}} X_{3k}) \big) - \frac{1}{2} \sum_{k\neq 1,3} (v - 2u_k) \\ &= \frac{1}{2} \sum_{k\neq 1,3} \big( X_{1k} \Re e(\overline{u_{1k}} X_{1k}) + X_{3k} \Re e(\overline{u_{3k}} X_{3k}) \big) - v + w. \end{aligned}$$

Now we compute, using that $X_{12} = -X_{34}$ and $X_{23} = X_{14}$,

$$\begin{aligned} &\sum_{k\neq 1,3} \big( X_{1k} \Re e(\overline{u_{1k}} X_{1k}) + X_{3k} \Re e(\overline{u_{3k}} X_{3k}) \big) \\ &\quad = X_{12} \Re e(\overline{u_{12}} X_{12}) + X_{32} \Re e(\overline{u_{32}} X_{32}) + X_{14} \Re e\,(\overline{u_{14}} X_{14}) + X_{34} \Re e(\overline{u_{34}} X_{34}) \\ &\quad = X_{12} \Re e(\overline{u_{12}} X_{12}) + X_{12} \Re e(\overline{u_{34}} X_{12}) + X_{32} \Re e(\overline{u_{32}} X_{32}) + + X_{32} \Re e(\overline{u_{14}} X_{32}) \\ &\quad = X_{12} \Re e\big((\overline{u_{12}} + \overline{u_{34}}) X_{12}\big) + X_{32} \Re e\big((\overline{u_{32}} + \overline{u_{14}}) X_{32}\big). \end{aligned}$$

We observe that

$$u_{12} + u_{34} = u_{32} + u_{14} = u_1 + u_3 - (u_2 + u_4) = v - w.$$

Therefore, inserting that $iX_{12} = X_{23}$ and that $|X_{12}|^2 = 2$ in the previous formula we find

$$\sum_{k\neq 1,3} \big(X_{1k}\Re\mathrm{e}(\overline{u_{1k}}X_{1k}) + X_{3k}\Re\mathrm{e}(\overline{u_{3k}}X_{3k})\big) = X_{12}\Re\mathrm{e}\big((\bar{v}-\bar{w})X_{12}\big) - iX_{12}\Im\mathrm{m}\big((\bar{v}-\bar{w})X_{12}\big)$$
$$= 2(v-w),$$

and finally $\mathcal{L}_v(u) = 0$.

We next estimate the remainder terms. Since $3/4 < |\Psi_{jk}| < 5/2$ we have $||X_{jk}|^2 - |\Psi_{jk}|^2| \leqslant C|u_{jk}|$ on $[0,T]$ and therefore

$$\big|\mathcal{R}_v^2(u) + \mathcal{R}_v^3(u)\big| \leqslant C\max_{j\neq k}|u_{jk}|\left|\frac{|\Psi_{jk}|^2}{|X_{jk}|^2} - 1\right|. \tag{3.4}$$

Expanding the first term $\mathcal{R}_v^1(u)$ and using the symmetries of $(X_1, X_2, X_3, X_4)$, we then have

$$\begin{aligned}\mathcal{R}_v^1(u) &= \frac{1}{4}\sum_{k\neq 1,3}\big\{X_{1k}|u_{1k}|^2 + X_{3k}|u_{3k}|^2\big\}\\ &= \frac{1}{4}\big\{X_{12}\big(|u_{12}|^2 - |u_{34}|^2\big) + X_{14}\big(|u_{14}|^2 - |u_{32}|^2\big)\big\}\\ &= \frac{1}{2}\big\{X_{12}\Re\mathrm{e}\big(\overline{u_{12}-u_{34}}(v-w)\big) + X_{14}\Re\mathrm{e}\big(\overline{u_{14}-u_{32}}(v-w)\big)\big\},\end{aligned}$$

so that

$$\big|\mathcal{R}_v^1(u)\big| \leqslant C\max_{j,k}|u_{jk}||v-w|. \tag{3.5}$$

We perform similar computations for $w$ and from (3.4)–(3.5) we infer the estimate

$$\begin{aligned}\|v(t)\|_{L^2} + \|w(t)\|_{L^2} &\leqslant \|v(0)\|_{L^2} + \|w(0)\|_{L^2} + \int_0^t \big(\|\mathcal{R}_v(u)(s)\|_{L^2} + \|\mathcal{R}_w(u)(s)\|_{L^2}\big)\,ds\\ &\leqslant \|v(0)\|_{L^2} + \|w(0)\|_{L^2}\\ &\quad + t\sup_{s\in[0,t]}\max_{j\neq k}\|u_{jk}(s)\|_{L^\infty}\left(\left\|\frac{|\Psi_{jk}(s)|^2}{|X_{jk}(s)|^2} - 1\right\|_{L^2} + \|v(s)\|_{L^2} + \|w(s)\|_{L^2}\right).\end{aligned}$$

Finally we apply Gagliardo–Nirenberg inequality and (3.2) to obtain the conclusion. □

**Proposition 3.5.** *We have for $t \in [0,T]$*

$$\sum_{j\neq k}\|u_{jk}(t)\|_{L^2} \leqslant C\sum_{j\neq k}\|u_{jk}(0)\|_{L^2} + Ct\sup_{s\in[0,t]}\mathcal{E}(s)^{1/2}.$$

**Proof.** By (3.3),

$$i\partial_t u_{jk} + \partial_\sigma^2 u_{jk} = -\sum_{l\neq j}\frac{u_{jl}}{|\Psi_{jl}|^2} + \sum_{l\neq k}\frac{u_{kl}}{|\Psi_{kl}|^2} - \sum_{l\neq j}X_{jl}\left(\frac{1}{|\Psi_{jl}|^2} - \frac{1}{|X_{jl}|^2}\right) + \sum_{l\neq k}X_{kl}\left(\frac{1}{|\Psi_{kl}|^2} - \frac{1}{|X_{kl}|^2}\right).$$

We multiply the equation by $\overline{u_{jk}}$, take the imaginary part and perform the sum over $j$ and $k$, cancelling the first two terms in the right-hand side. Indeed,

$$\sum_{j,k}\sum_{l\neq j}\frac{\Im\mathrm{m}(u_{jk}\overline{u_{jl}})}{|\Psi_{jl}|^2}=\sum_{j,k}\sum_{l\neq j}\frac{\Im\mathrm{m}((u_{jl}+u_{lk})\overline{u_{jl}})}{|\Psi_{jl}|^2}$$
$$=\sum_{j,k}\sum_{l\neq j}\frac{\Im\mathrm{m}(u_{lk}\overline{u_{jl}})}{|\Psi_{jl}|^2}$$
$$=-\sum_{j,k}\sum_{l\neq j}\frac{\Im\mathrm{m}(u_{jk}\overline{u_{jl}})}{|\Psi_{jl}|^2},$$

by exchanging $j$ and $l$ in the last equality. Therefore the latter sum vanishes. By the same arguments we also have

$$\sum_{j,k}\sum_{l\neq k}\frac{\Im\mathrm{m}(u_{jk}\overline{u_{kl}})}{|\Psi_{kl}|^2}=0.$$

It follows that

$$\frac{d}{dt}\sum_{j\neq k}\|u_{jk}\|_{L^2}^2\leqslant C\sum_{j\neq k}\sum_{l\neq j}\int|u_{jk}||X_{jl}|\frac{1}{|\Psi_{jl}|^2}\left|\frac{|\Psi_{jl}|^2}{|X_{jl}|^2}-1\right|d\sigma$$
$$\leqslant C\left(\sum_{j\neq k}\|u_{jk}\|_{L^2}^2\right)^{1/2}\max_{j\neq k}\left\|\frac{|\Psi_{jk}|^2}{|X_{jk}|^2}-1\right\|_{L^2},$$

and we finally obtain by (3.2)

$$\left|\frac{d}{dt}\left(\sum_{j,k}\|u_{jk}(t)\|_{L^2}^2\right)^{1/2}\right|\leqslant C\mathcal{E}(t)^{1/2}.$$

The conclusion follows. □

We are now able to control the evolution of $\mathcal{E}(t)$ and to complete the proof of Theorem 1.2. First we recall that by Proposition 3.2,

$$\frac{1}{2}\left(\|v(t)\|_{L^2}^2+\|w(t)\|_{L^2}^2\right)-\widetilde{\mathcal{E}_0}\leqslant\mathcal{E}(t)\leqslant\widetilde{\mathcal{E}_0}+\frac{1}{2}\left(\|v(t)\|_{L^2}^2+\|w(t)\|_{L^2}^2\right)$$

so in particular

$$\mathcal{E}(t)+\|v(t)\|_{L^2}^2+\|w(t)\|_{L^2}^2\leqslant C\widetilde{\mathcal{E}_0}\quad\text{on }[0,T].$$

Next, in view of Proposition 3.4 we have

$$\mathcal{E}(t)\leqslant\widetilde{\mathcal{E}_0}+\left(\|v(t)\|_{L^2}+\|w(t)\|_{L^2}\right)^2\leqslant\widetilde{\mathcal{E}_0}+2\left(\|v(0)\|_{L^2}+\|w(0)\|_{L^2}\right)^2$$
$$+Ct^2\sup_{s\in[0,t]}\max_{j,k}\|u_{jk}(s)\|_{L^2}\mathcal{E}(s)^{1/2}\left(\mathcal{E}(s)^{1/2}+\|v(s)\|_{L^2}+\|w(s)\|_{L^2}\right)^2$$
$$\leqslant 9\widetilde{\mathcal{E}_0}+Ct^2\sup_{s\in[0,t]}\max_{j,k}\|u_{jk}(s)\|_{L^2}\widetilde{\mathcal{E}_0}^{3/2}$$

and finally by Proposition 3.5

$$\mathcal{E}(t)\leqslant 9\widetilde{\mathcal{E}_0}+Ct^2\max_{j,k}\|u_{jk,0}\|_{L^2}\widetilde{\mathcal{E}_0}^{3/2}+Ct^3\widetilde{\mathcal{E}_0}^2.$$

Setting $t=T$ in the above inequality and recalling that $\mathcal{E}(T)=10\widetilde{\mathcal{E}_0}$, we infer that

$$1\leqslant Ct^2\max_{j,k}\|u_{jk,0}\|_{L^2}\widetilde{\mathcal{E}_0}^{1/2}+Ct^3\widetilde{\mathcal{E}_0}.$$

We conclude that $T$ is larger than

$$C\min\left\{\frac{1}{\widetilde{\mathcal{E}_0}^{1/4}\max_{j,k}\|u_{jk,0}\|_{L^2}^{1/2}},\frac{1}{\widetilde{\mathcal{E}_0}^{1/3}}\right\},$$

as we wanted. This concludes the proof of Theorem 1.2.

## 4. Proof of Theorem 1.3

Before proving Theorem 1.3 we start with some preliminary computations. We mainly follow the Appendix of [13]. Assume that $v$ is a $\mathcal{C}^\infty$ small energy solution to Eq. (1.10) such that $v'$ vanishes at infinity. We set

$$\eta = 1 - |v|^2,$$

then $\eta$ vanishes at infinity. We decompose $v$ into its real and imaginary parts, $v = v_1 + iv_2$. Eq. (1.10) gives then the system

$$\begin{cases} -cv_2' + v_1'' + \omega \dfrac{v_1}{v_1^2 + v_2^2} - \omega v_1 = 0, \\ cv_1' + v_2'' + \omega \dfrac{v_2}{v_1^2 + v_2^2} - \omega v_2 = 0. \end{cases}$$

By subtracting the first equation multiplied by $v_2$ from the second one multiplied by $v_1$

$$\left( v_1 v_2' - v_1' v_2 - \frac{c}{2}\eta \right)' = 0,$$

so since $v$ has finite energy we can integrate from infinity and get

$$v_1 v_2' - v_1' v_2 = \frac{c}{2}\eta. \tag{4.1}$$

Next we add the first equation multiplied by $v_1'$ to the second one multiplied by $v_2'$,

$$\left( v_1'^2 + v_2'^2 + \omega \ln\left(v_1^2 + v_2^2\right) - \omega\left(v_1^2 + v_2^2\right) \right)' = 0,$$

so

$$\left|v'\right|^2 = -\omega \ln(1-\eta) - \omega\eta. \tag{4.2}$$

Finally, in view of (4.1) and (4.2) we can compute

$$\begin{aligned} \eta'' &= -2\left|v'\right|^2 - 2\left(v_1 v_1'' + v_2 v_2''\right) \\ &= -2\left|v'\right|^2 - 2v_1\left( cv_2' - \omega \frac{v_1}{v_1^2 + v_2^2} + \omega v_1 \right) - 2v_2\left( -cv_1' - \omega \frac{v_2}{v_1^2 + v_2^2} + \omega v_2 \right) \\ &= -2\left|v'\right|^2 - 2c\left(v_1 v_2' - v_1' v_2\right) + 2\omega - 2\omega\left(v_1^2 + v_2^2\right) \\ &= 2\omega \ln(1-\eta) + 4\omega\eta - c^2\eta. \end{aligned}$$

So we find

$$\eta'' - 2\omega \ln(1-\eta) + \left(c^2 - 4\omega\right)\eta = 0. \tag{4.3}$$

Multiplying by $\eta'$ and integrating we obtain

$$\left(\eta'\right)^2 + \left(c^2 - 4\omega\right)\eta^2 - 4\omega\left((\eta - 1)\ln(1-\eta) - \eta\right) = 0,$$

which is satisfied if $\eta$ verifies

$$\eta' = \alpha\left( -\left(c^2 - 4\omega\right)\eta^2 + 4\omega\left((\eta-1)\ln(1-\eta) - \eta\right) \right)^{1/2}, \quad \alpha = \alpha(\sigma) = \pm 1. \tag{4.4}$$

We now turn to the proof of Theorem 1.3. From now on we look for solutions such that $\eta$ is sufficiently small on the whole of $\mathbb{R}$ and for which the right-hand side in (4.4) makes sense. We introduce

$$a(\eta) = -\left(c^2 - 4\omega\right)\eta^2 + 4\omega\left((\eta - 1)\ln(1-\eta) - \eta\right).$$

For $0 < \eta < 1$, we perform a Taylor expansion for $a$,

$$a(\eta) = \left(2\omega - c^2\right)\eta^2 - 2\omega\frac{\eta^3}{3} - 4\omega \sum_{k \geqslant 4} \frac{\eta^k}{k(k-1)}$$

therefore

$$b(\eta) \equiv \frac{a(\eta)}{\eta^2} = 2\omega - c^2 - 2\omega\frac{\eta}{3} + r(\eta)$$

with $r(\eta) = o(\eta) \leqslant 0$ such that $r'(\eta) = O(\eta)$. Let us set

$$\sigma_0 = \frac{2\omega - c^2}{\frac{2\omega}{3}} > 0,$$

then $b(\sigma_0) \leqslant 0$. Since on the other hand $b(0) > 0$, there exists $\sigma_1 \in (0, \sigma_0]$ such that $b(\sigma_1) = 0$. Moreover, since for $\eta \in [0, \sigma_0]$ we have $b'(\eta) = -\frac{2\omega}{3} + r'(\eta) \leqslant -\frac{2\omega}{3} + C(2\omega - c^2) < 0$ for $2\omega - c^2$ sufficiently small, we infer that $b$ is strictly decreasing on $[0, \sigma_0]$ and therefore $\sigma_1$ is the unique zero of $a$ on $]0, \sigma_0]$.

Next, we fix a small parameter $\varepsilon > 0$ and we consider the ODE

$$\begin{cases} y_\varepsilon'(\sigma) = -\sqrt{a\big(y_\varepsilon(\sigma)\big)}, \\ y_\varepsilon(0) = \sigma_1 - \varepsilon. \end{cases}$$

Since $\sqrt{a}$ is Lipschitz on $[0, x_1 - \varepsilon/2)$ we can find a unique maximal solution on some interval $I$ containing the origin. We claim that $\sup I = +\infty$. We show first that $0 < y_\varepsilon < \sigma_1 - \varepsilon$ on $I \cap [0, \infty)$. Indeed, $y_\varepsilon$ is strictly decreasing on $I \cap [0, \infty)$. Assume by contradiction that there exists $\overline{\sigma}$ such that $y_\varepsilon(\overline{\sigma}) = 0$ and $y_\varepsilon > 0$ on $[0, \overline{\sigma})$. We recall that $b(y) \sim 2\omega - c^2$ when $y \to 0$. Therefore

$$y_\varepsilon'(\sigma) \geqslant -2\sqrt{2\omega - c^2}\, y_\varepsilon(\sigma) \quad \text{for } \sigma \in [\overline{\sigma} - \delta, \overline{\sigma}]$$

with $\delta$ small. Integrating the differential inequality above yields

$$y_\varepsilon(\sigma) \geqslant y_\varepsilon(\overline{\sigma} - \delta)\exp\big(-2\sqrt{2\omega - c^2}(\sigma - \overline{\sigma} + \delta)\big) \quad \text{on } [\overline{\sigma} - \delta, \overline{\sigma}],$$

which contradicts the fact that $y_\varepsilon(\overline{\sigma}) = 0$. Next, since $y \mapsto \sqrt{a(y)}$ is Lipschitz and bounded on $[0, \sigma_1 - \varepsilon]$ the maximal solution $y_\varepsilon$ exists on $[0, \infty)$ which proves the claim.

We next let $\varepsilon \to 0$. Noting that $y_\varepsilon$ and $y_\varepsilon'$ are uniformly bounded on $[0, \infty)$ we can pass to the limit to find a solution[5] $y$ to the ODE

$$\begin{cases} y' = -\sqrt{a(y)}, \quad \sigma \geqslant 0, \\ y(0) = \sigma_1. \end{cases}$$

We finally set

$$\eta(\sigma) = y(\sigma) \quad \text{for } \sigma \in [0, +\infty) \quad \text{and} \quad \eta(-\sigma) = \eta(\sigma) = y(\sigma) \quad \text{for } \sigma \in (-\infty, 0].$$

Thanks to $\eta(0) = \sigma_1$ and $a(\sigma_1) = 0$ we check that $\eta \in C^\infty(\mathbb{R})$ is a solution of the ODE (4.3). Moreover, by the same kind of arguments as before we have $\eta \to 0$, hence $\eta'(\sigma) \sim -\sqrt{2\omega - c^2}\eta(\sigma)$ as $\sigma \to \infty$, which yields the exponential decay $\eta(\sigma) \leqslant C_\delta \eta(0)\exp(-(\sqrt{2\omega - c^2} - \delta)|\sigma|)$ for all $0 < \delta < \sqrt{2\omega - c^2}$.

We complete the proof of Theorem 1.3 by looking for a solution of the form

$$v = \sqrt{1 - \eta}\exp(i\theta). \tag{4.5}$$

Then according to (4.1) we must have

$$(1 - \eta)\theta' = \frac{c\eta}{2} \tag{4.6}$$

(note that in particular $\theta$ is an increasing function on $\mathbb{R}$). Therefore for $\theta(\sigma) = \theta_0 + \int_0^\sigma \frac{c\eta}{2(1-\eta)}\, d\tau$ where $\theta_0 \in \mathbb{R}$, then

$$\big|\theta(+\infty) - \theta(-\infty)\big| \leqslant \frac{C\eta(0)}{\sqrt{2\omega - c^2}} \leqslant C\sqrt{2\omega - c^2}.$$

[5] We do not claim that such a solution is unique or maximal.

Also, the map defined by (4.5) is a solution to (1.10). It only remains to show that $v$ has finite energy. This clearly holds in view of the exponential decay of $\eta$, of $\eta'$ (by (4.4)) and of $\theta'$ (by (4.6)) at infinity. Moreover in view of (4.6) we obtain

$$\mathcal{E}(v) \leqslant C\|\eta\|_{H^1}^2 \leqslant C\left(2\omega - c^2\right)^{3/2}$$

and the conclusion of Theorem 1.3 follows.

## 5. Proof of Theorem 1.4

Under the hypothesis of Theorem 1.4, the angular speed of the configuration $(X_j)_j$ is $\omega = 0$ so if we set

$$\Psi_j(t,\sigma) = X_j(t)\Phi(t,\sigma)$$

a solution of system (1.1), we have shown in Section 2 that $\Phi$ has to solve the linear Schrödinger equation,

$$i\partial_t\Phi + \partial_\sigma^2\Phi = 0.$$

Since the linear evolution of a Gaussian $G_0(\sigma) = e^{-\sigma^2}$ is

$$e^{it\partial_\sigma^2}G_0(t,\sigma) = \frac{e^{-\frac{\sigma^2}{1+4it}}}{\sqrt{1+4it}},$$

it follows that the linear evolution of

$$\Phi_0(\sigma) = 1 - \frac{e^{-\frac{\sigma^2}{1-4i}}}{\sqrt{1-4i}}$$

is precisely

$$\Phi(t,\sigma) = 1 - \frac{e^{-\frac{\sigma^2}{1-4i(1-t)}}}{\sqrt{1-4i(1-t)}}.$$

We notice that $\Phi(t,\sigma) \overset{|\sigma|\to\infty}{\longrightarrow} 1$ for $t \in [0,1]$, and for $t \in [0,1[$

$$\left|\Phi(t,\sigma)\right| > 1 - \frac{1}{\sqrt{1+16(1-t)^2}} > 0.$$

On the other hand we have

$$\Phi(1,\sigma) = 1 - e^{-\sigma^2},$$

so $\sigma = 0$ is a vanishing point at $t = 1$ and Theorem 1.4 follows.

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