A local symmetry result for linear elliptic problems with solutions changing sign

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Abstract

We prove that the only domain $\Omega$ such that there exists a solution to the following problem

\[
\begin{array}{ll}
\Delta u + \omega^2 u = -1 & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{array}
\]

with

\[
\frac{1}{|\partial \Omega|} \int_{\partial \Omega} \partial_n u = c,
\]

for a given constant $c$, is the unit ball $B_1$, if we assume that $\Omega$ lies in an appropriate class of Lipschitz domains.

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1. Introduction

Let us consider the following problem: for $\omega \in \mathbb{R}$, is it true that the only domain $\Omega$ such that there exists a solution $u$ to the problem

\[
\begin{cases}
\Delta u + \omega^2 u = -1 & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

with

\[
\frac{1}{|\partial \Omega|} \int_{\partial \Omega} \partial_n u = c
\]

is a ball? Here $\Omega$ is a sufficiently smooth bounded domain in $\mathbb{R}^N$, $N \geq 2$, $\partial_n u$ is the external normal derivative to the boundary $\partial \Omega$, and $c$ is a given constant. By using the Alexandrov method of moving planes J. Serrin [20] has proved that if there exists a solution $u$ to (1.1), (1.2), and if $u$ has a sign in $\Omega$, then $\Omega = B_1$ (for example for $\omega = 0$, by the maximum principle it follows that $u$ is positive in $\Omega$). For the particular case $\omega = 0$ see also the proofs of H. Weinberger [23], based on a Rellich-type identity and on the maximum principle, and M. Choulli, A. Henrot [7], which use the technique of domain derivative. We point out that Serrin in [20] has studied the same type of problem for more general nonlinear elliptic equations. For further references concerning symmetry (and non-symmetry) results for overdetermined elliptic problems, see also [1–4,8–19,21,22]. All these results need hypothesis on the sign of $u$.

In [5] the authors have given a positive answer to the above question by supposing that

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(i) \( \omega^2 \notin \{ \lambda_n \}_{n \geq 1} \) \((\{ \lambda_n \}_{n \geq 1} \) being the sequence, in increasing order, of eigenvalues of \( -\Delta \) in \( B_1 \) with Dirichlet boundary conditions),
(ii) \( \omega \notin \Lambda \), where \( \Lambda \) is an enumerable set of \( \mathbb{R}^+ \), whose limit points are the values \( \lambda_{1m} \), for some integer \( m \geq 1 \), \( \lambda_{1m} \) being the \( m \)-th-zero of the first-order Bessel function \( I_1 \),
(iii) \( \Omega \) is such that the ker\( (\Delta + \omega^2) = \{ 0 \} \) in \( \Omega \),
(iv) the boundary \( \partial \Omega \) is a Lipschitz perturbation of the unit sphere \( \partial B_1 \) of \( \mathbb{R}^N \).

We point out that in [5] no hypothesis are required on the sign of the solution \( u \). We can say that paper [6] can be considered as preparatory of [5] (in the sense that some ideas developed in [6] are used in [5]). In the present paper we give a new proof of the result proved in [5], which let us permit to avoid hypothesis (i)–(iii) above.

We recall that if let us denote by \((\lambda_n)_{n \geq 1}\) the sequence, in increasing order, of eigenvalues of \( -\Delta \) in \( B_1 \) with Dirichlet boundary conditions, we have that the eigenvalue \( \lambda_n \), for some \( n \in \mathbb{N} \), coincides, for some integers \( \ell \geq 0 \) and \( m \geq 1 \), with \( \lambda_{1m} \). Here and in what follows \( \lambda_{1m} \) will denote the \( m \)-th-zero of the so-called \( N \)-dimensional \( \ell \)-order Bessel function of the first kind \( I_\ell \), i.e. \( I_\ell (\lambda_{1m}) = 0 \) (see Section 2). We recall in particular that (see [5, Lemma 3.5])

\[
I_0' = -I_1 \quad \text{in} \quad \mathbb{R}.
\]

From these remarks it follows that the function \( u^{(0)} \) given by

\[
u^{(0)}(x) = \frac{1}{\omega^2} \left( \frac{I_0(\omega |x|)}{I_0(\omega)} - 1 \right) \quad \text{in} \quad B_1,
\]

solves (1.1), (1.2) when \( \Omega = B_1 \). Here \( r = |x|, |\cdot| \) denoting the Euclidean norm in \( \mathbb{R}^N \). We observe that if the constant \( \omega \) is smaller or equal than \( \lambda_{11} \), the solution \( u^{(0)} \) is positive in \( B_1 \), while if \( \omega \) is bigger than \( \lambda_{11} \), then \( u^{(0)} \) changes sign. In the rest of the paper we will assume \( \omega \geq 0 \). The same conclusions hold true for \( \omega < 0 \), since the coefficient \( \omega^2 \) is even in (1.1). We stress out that in order that (1.3) makes sense, in the rest of the paper we will suppose that

\( \omega \notin \{ \lambda_{0m} \}_{m \geq 1} \).

Here and in what follows \( c = \partial_n u^{(0)} \) on \( \partial B_1 \). By (1.3), we obtain that

\[
c = \frac{I_0'(\omega)}{\omega I_0(\omega)} \quad \text{in} \quad B_1.
\]

In the present paper we prove the following

**Theorem 1.1.** For \( \omega \notin \{ \lambda_{0m} \}_{m \geq 1} \), there exists a class \( \mathcal{D} \) of \( C^{2,\alpha} \)-domains such that if \( u \) is a solution to (1.1) verifying

\[
\frac{1}{|\partial \Omega|} \int_{\partial \Omega} \partial_n u = c,
\]

with \( \Omega \in \mathcal{D} \), and \( c \) given by (1.4), then \( \Omega = B_1 \), and \( u = u^{(0)} \).

The idea underlying the proof of Theorem 1.1 is the following. Let \( E \) be the vector space of \( C^{2,\alpha} \) functions defined on the unit sphere \( \partial B_1 \), i.e.

\[
E = \{ k \in C^{2,\alpha}(\partial B_1) \},
\]

\( 0 < \alpha < 1 \). For \( k \in E \), let \( \Omega_k \) be the domain whose boundary \( \partial \Omega_k \) can be written as perturbation of \( \partial B_1 \), i.e.

\[
\partial \Omega_k = \{ x = (1 + k) y, \quad y \in \partial B_1 \}
\]

(in particular for \( k \equiv 0 \) on \( \partial B_1 \), \( \Omega_0 = B_1 \)). We denote by \( \Phi \) the following operator

\[
\Phi : E \mapsto \mathbb{R},
\]

defined by

\[
\Phi(k) = \int_{\partial \Omega_k} \partial_n u_p - c \int_{\partial \Omega_k} \cdot ,
\]
where \( u_p \) is a particular solution to (1.1), when \( \Omega = \Omega_k \) (\( u_p \) will be defined in Section 3 below). We observe that \( \Phi \) has not a sign in a neighborhood of 0 in \( E \) (i.e. \( \Phi \) is neither positive nor negative). In fact \( \Phi(0) = 0 \) (since \( u_p = u(0) \) when \( \Omega = B_1 \)). Moreover since the unit sphere centered at the point \( x_0 \in \mathbb{R}^N \) is parametrized by
\[
\partial B_1(x_0) = \{ x = (1 + k')y, \ y \in \partial B_1 \},
\]
where \( k' \) is given by
\[
k'(y) = x_0 \cdot y - 1 + \sqrt{1 + |x_0 \cdot y|^2 - |x_0|^2},
\]
we have that \( \Phi(k') = 0 \), with
\[
k' \to 0 \quad \text{in} \ E, \quad \text{as} \ x_0 \to 0.
\]
So the best one can expect is that \( \Phi \) is different to 0 in \( \mathcal{O} \setminus \{ k \in E; \ k = k' \} \), for some neighborhood \( \mathcal{O} \) of 0 in \( E \). By studying the behavior of the operator \( \Phi \) at 0, we prove that if \( \omega \not\in \{ \lambda_{\ell m} \}_{\ell \geq 2, m \geq 1} \), with \( \lambda_{\ell m} \neq \lambda_{1m'} \), for all \( m' \geq 1 \), then \( \Phi \) is differentiable at zero in \( E \). On the other hand if \( \omega = \lambda_{\ell m} \), for some \( \ell \geq 2 \), and \( m \geq 1 \) (with \( \lambda_{\ell m} \neq \lambda_{1m'} \), for all \( m' \geq 1 \)), then \( \Phi \) is differentiable at zero in the vector space
\[
\mathcal{E}_\ell = \{ k \in E; \ k_{\ell q} = 0, \ k_{pq'} = 0, \ p, q \in I \}
\]
of functions \( k \in E \) which don’t have either the frequency \( \ell \) or the frequency \( p, I \) being a (eventually empty) finite set of positive integer such that \( I_p(\lambda_{\ell m}) = 0 \) (the cardinality of \( I \) depending on the multiplicity of the eigenvalue \( \lambda_{\ell m}^2 \), see Section 2 for more details). Here and in what follows \( k_{st} = \frac{1}{|\partial B_1|} \int_{\partial B_1} k \Phi_{st} \) is the \( s \)-order (Fourier) coefficient of \( k \), and \( \Phi_{st} \) is the spherical harmonic of degree \( s \), with \( s = 1, \ldots, d_\ell \). More precisely we have that the differential at zero in the direction \( k \) has a sign if \( k_0 \neq 0 \) (see Lemma 3.3), \( k_0 \) being the zeroth-order coefficient of \( k \) (i.e. \( k_0 = \frac{1}{|\partial B_1|} \int_{\partial B_1} k \)). We can show then that there exists a neighborhood \( \mathcal{O} \) of 0 in \( E \) such that \( \Phi \) is positive in \( \mathcal{O} \cap E^+ \), and \( \Phi \) is negative in \( \mathcal{O} \cap E^- \), where \( E^+ \) and \( E^- \) are two circular sectors respectively in the subset \( \{ k \in E; \ k_0 < 0 \} \), and \( \{ k \in E; \ k_0 > 0 \} \). Now, since if there exists a solution \( u \) to (1.1), when \( \Omega = \Omega_k \), verifying \( \frac{1}{|\partial \Omega_k|} \int_{\partial \Omega_k} \partial_n u = c \), one can prove that \( \Phi(c = 0, \text{if we assume that} \ k \in \mathcal{O} \cap (E^+ \cup E^- \cup \{ 0 \}) \). Finally, since the operator \( \Phi \) is invariant up to isometries, we obtain that the class \( \mathcal{D} \) in Theorem 1.1 is defined as
\[
\mathcal{D} = \{ \Omega; \ \Omega = \sigma(\Omega_k) \}
\]
for some \( \sigma \in \Sigma \), and some \( \Omega_k \in \mathcal{G} \), where \( \Sigma \) is the set of isometries of \( \mathbb{R}^N \), and
\[
\mathcal{G} = \{ \Omega_k; \ k \in \mathcal{O} \cap (E^+ \cup E^- \cup \{ 0 \}) \}.
\]
We stress out that \( E \) through the paper is the space of functions of class \( C^{2,\alpha} \) on \( \partial B_1 \) (this means that we consider only regular perturbations of the unit sphere), but, up to obvious changes, the same conclusions hold true in the case where \( E \) is the space of functions of class \( C^{0,1} \) on \( \partial B_1 \), i.e. the boundary \( \partial \Omega_k \) is of Lipschitz class. The paper is organized as follows: in the next section we give some notations used through the paper, in Section 3 we give the first-order approximation of the operator \( \Phi \) in a neighborhood of 0, and in Section 4 we prove Theorem 1.1, and we consider the Lipschitz case. Finally in Section 5 counter-examples to Theorem 1.1 are given.

2. Preliminaries and notations

Let us denote by \( B_1 \) the ball of radius 1 in \( \mathbb{R}^N \) centered at zero. By \( \overline{B}_1 \) we define the Euclidean closure of \( B_1 \). Let us denote by \( I_\ell \) the so-called \( N \)-dimensional \( \ell \)-order Bessel function of the first kind, i.e.
\[
I_\ell(r) = r^{-\nu} J_{\nu+\ell}(r),
\]
where \( \nu = \frac{N}{2} - 1 \), and \( J_{\nu+\ell} \) is the well-known \( \nu + \ell \)-order Bessel function of the first kind (we observe that for \( N = 2 \), \( I_\ell \) coincides with the \( \ell \)-order Bessel function of the first kind \( J_\ell \)). \( I_\ell \) solves the following Bessel equation
\[
I''_\ell + \frac{N-1}{r} I'_\ell + \left( 1 - \frac{\ell(\ell + N - 2)}{r^2} \right) I_\ell = 0 \quad \text{in} \ \mathbb{R}.
\]
Let $\lambda_{\ell m}$ be the $m$th-zero of the $\ell$-order Bessel function $I_{\ell}$. Let $(\lambda_n)_{n \geq 1}$ be the sequence, in increasing order, of eigenvalues of $-\Delta$ in $B_1$ with Dirichlet boundary conditions. An eigenvalue $\lambda_n$, for some $n \in \mathbb{N}$, coincides, for some integer $\ell \geq 0$, and $m \geq 1$, with $\lambda^2_{\ell m}$. The corresponding eigenfunctions can be written as (in polar coordinates)

$$\varphi_1 = I_{\ell}(\lambda_{\ell m} r)Y_{\ell 1}(\theta),$$
$$\vdots$$
$$\varphi_d \ell = I_{\ell}(\lambda_{\ell m} r)Y_{d \ell}(\theta),$$
$$\varphi_{pq} = I_p(\lambda_{\ell m} r)Y_{pq}(\theta),$$

where $p \in I$, and $I$ is a (eventually empty) finite set (by Fredholm theorem) of integer such that $I_p(\lambda_{\ell m}) = 0$, i.e.

$$I = \{ p \in \mathbb{N}, \ p \neq \ell; \ I_p(\lambda_{\ell m}) = 0 \}. \quad (2.1)$$

Here $Y_{st}$ is the spherical harmonic of degree $s$, with $t = 1, \ldots, d_s$, and

$$d_s = \begin{cases} \frac{1}{2s(N-2)(s+N-3)!} & \text{if } s = 0, \\ \frac{(2s+N-2)(s+N-3)!}{s!(N-2)!} & \text{if } s \geq 1. \end{cases}$$

We will use the following convention: we say that a function $f$ has the frequency $s$, if the $s$-order coefficient of $f$, i.e. $f_{st} = \frac{1}{|\partial B_1|} \int_{\partial B_1} f Y_{st}$, is different to zero. And similarly we say that a function $f$ doesn’t have the frequency $s$, if the $s$-order coefficient of $f$ vanishes.

Let $\tilde{k}$ be a $C^2, \alpha$-extension of $k$ into $\overline{B}_1$. Let us call $A$ the Jacobian matrix of change of variable

$$x = (1 + k(y)) y, \quad y \in \overline{B}_1 \quad (2.2)$$

(where we denote $\tilde{k}$ by $k$). The matrix $A$ is given by

$$A_{ij} = \begin{bmatrix} 1 + k + y_1 \partial_1 k & y_1 \partial_2 k & \cdots & y_1 \partial_N k \\ y_2 \partial_1 k & 1 + k + y_2 \partial_2 k & \cdots & y_2 \partial_N k \\ \vdots & \vdots & \ddots & \vdots \\ y_N \partial_1 k & \cdots & \cdots & 1 + k + y_N \partial_N k \end{bmatrix}. $$

Let $G = A^T A$. The matrix $G$ can be written as

$$G = I_N + G^{(1)} + o(\|k\|).$$

where $I_N$ is the $N$-order identity matrix, and the matrix $G^{(1)}$ depends linearly on $k$ and $\nabla k$. Following [5], the matrix $G^{(1)}$ is given by

$$G^{(1)}_{ij} = 2kI_N + \begin{bmatrix} 2x_1 \partial_1 k & x_1 \partial_2 k + x_2 \partial_1 k & \cdots & x_1 \partial_N k + x_N \partial_1 k \\ x_1 \partial_2 k + x_2 \partial_1 k & 2x_2 \partial_2 k & \cdots & x_2 \partial_N k + x_N \partial_2 k \\ \vdots & \vdots & \ddots & \vdots \\ x_1 \partial_N k + x_N \partial_1 k & \cdots & \cdots & 2x_N \partial_N k \end{bmatrix}. \quad (2.3)$$

3. The first-order expansion of the operator $\Phi$

A function $k \in E$ can be written, in Fourier series expansion, as

$$k = k_0 + \sum_{p \geq 1} \sum_{q=1}^{d_p} k_{pq} Y_{pq} \quad \text{on } \partial B_1.$$

We recall that problem (1.1) cannot have solutions or, if a solution exists, it cannot be unique. This happens all times the kernel $\ker(\Delta + \omega^2) \neq \{0\}$ in $\Omega$. More precisely by Fredholm theorem there exists a solution to (1.1) if and only if

$$-1 \in \ker(\Delta + \omega^2) \perp \text{ in } \Omega.$$

We can write a solution $u$ as

$$u = u_p + u_h,$$
where $u_p$ is a particular solution to (1.1) such that
\[ u_p \in \ker(\Delta + \omega^2) \] in $\Omega$,
\[(3.1)\]
and $u_h$ solves the corresponding homogeneous problem. We observe that $u_p$ is unique and can be written as
\[ u_p = \sum_{\mu \in I} n \sum_{q=1}^{n_p} \alpha_{pq} \psi_{pq}, \]
where $\alpha_{pq} = \int_{\Omega} \psi_{pq} \mu - \lambda_p$ is the $p$-order Fourier coefficient of $u$. Here $\lambda_p$ and $\psi_{pq}$ are respectively the $p$th-eigenvalue and a corresponding eigenfunction of $-\Delta$ in $\Omega$ (with Dirichlet boundary conditions), and $n_p$ is the dimension of the corresponding eigenspace. $I$ is a finite set of integer (by Fredholm theorem), and $I^C$ is the complementary of $I$. On the other hand if the kernel \( \ker(\Delta + \omega^2) = \{0\} \), then a solution $u$ exists and is unique. For example for $\omega = \lambda_{\ell m}$, for some $\ell, m \geq 1$, then $u_p = \frac{1}{\lambda_{\ell m}} (I_{0}(\lambda_{\ell m} r) - 1)$ is a particular solution to (1.1) when $\Omega = B_1$ (lying in the $\ker(\Delta + \lambda_{\ell m}^2) \perp$ in $B_1$), and $u_h$ has the form (in polar coordinates)
\[ u_h = \sum_{\mu \in I} d \sum_{q=1}^{d_p} \alpha_{\ell q} I_{\ell}(\lambda_{\ell m} r) Y_{\ell q}(\theta) + \sum_{\mu \in I} \sum_{q=1}^{d_p} \alpha_{pq} I_{p}(\lambda_{\ell m} r) Y_{pq}(\theta), \]
where $I$ is defined in (2.1), and $\alpha_{\ell 1}, \ldots, \alpha_{\ell d}, \alpha_{pq} \in \mathbb{R}$. We denote by $\Phi$ the following operator
\[ \Phi : E \mapsto \mathbb{R}, \]
defined by
\[ \Phi(k) := \int_{\partial \Omega_k} \partial_n u_p - c \int_{\partial \Omega_k} \sqrt{g}, \]
where $u_p$ is a particular solution to (1.1), verifying (3.1), when $\Omega = \Omega_k$. The operator $\Phi$ is well-defined, since we suppose that a solution $u$ exists for $k$ lying in some neighborhood of 0 in $E$. Using (2.2), we have that the function $\tilde{u}$ defined by
\[ \tilde{u}(y) = u((1 + k)y) \text{ in } \bar{B}_1, \]
solves
\[ \begin{cases} \text{div}(\sqrt{g} G^{-1} \nabla \tilde{u}) + \omega^2 \sqrt{g} \tilde{u} = -\sqrt{g} & \text{in } B_1, \\ \tilde{u} = 0 & \text{on } \partial B_1, \end{cases} \]
where $g = |\det G|$. Following [5], the external normal derivative of $u$ at the point $x = (1 + k)y \in \partial \Omega_k$ is given by
\[ \partial_n u((1 + k)y) = (G^{-1} y \cdot y)^{-1/2} G^{-1} \nabla \tilde{u} \cdot y. \]
The operator $\Phi$ then becomes
\[ \Phi(k) = \int_{\partial B_1} (G^{-1} y \cdot y)^{-1/2} G^{-1} \nabla \tilde{u}_p \cdot y \sqrt{\tilde{g}} - c \int_{\partial B_1} \sqrt{\tilde{g}}, \]
where $\tilde{u}_p(y) = u_p((1 + k)y)$, and $\sqrt{\tilde{g}}$ is the surface element of the new variable $y$. Let us denote $\tilde{u}_p$ by $u_p$, and $y$ by $x$. We begin by proving the following

**Lemma 3.1.** We have
\[ u_p \to u^{(0)} \text{ as } k \to 0. \]

**Proof of Lemma 3.1.** Let $z = u_p - u^{(0)}$. By writing the matrix $\sqrt{\tilde{g}} G^{-1}$ in (3.2) as
\[ \sqrt{\tilde{g}} G^{-1} = I_N + K, \]
it follows that \( z \) solves
\[
\begin{align*}
\Delta w + \omega^2 w &= (1 - \sqrt{g})(\omega^2 u_p + 1) - \text{div}(K \nabla u_p) \quad \text{in } B_1, \\
w &= 0 \quad \text{on } \partial B_1.
\end{align*}
\] (3.4)

Let assume that the ker\((\Delta + \omega^2)\) = \{0\} in \( B_1 \). The solution \( w \) to (3.4) can be written as
\[
w = \sum_{p=1}^{+\infty} \sum_{q=1}^{n_p} \alpha_{pq} \psi_{pq},
\]
where the \( p \)-order Fourier coefficient
\[
\alpha_{pq} = \frac{\int_{B_1} ((1 - \sqrt{g})(\omega^2 u_p + 1) - \text{div}(K \nabla u_p)) \psi_{pq}}{\omega^2 - \lambda_p}.
\]

Since
\[
\sqrt{g} = 1 + Nk + x \cdot \nabla k + o(\|k\|),
\] (3.5)
we obtain
\[
w \to 0 \quad \text{as } k \to 0.
\]

On the other hand, if the ker\((\Delta + \omega^2)\) \(\neq \{0\}\) in \( B_1 \), i.e. \( \omega^2 = \lambda_n \), for some \( n \geq 2 \) (we recall that \( \lambda_n \notin \{\lambda_{0m}^2 \}_{m \geq 1} \)), then a solution \( w \) to (3.4) can be written as
\[
w = w_p + w_h,
\]
where
\[
w_p = \sum_{p \in I^c} \sum_{q=1}^{n_p} \alpha_{pq} \psi_{pq}.
\]

We claim that \( w_p = z \). We have that the function \( w_p - z \) solves
\[
\begin{align*}
\Delta (w_p - z) + \lambda_n (w_p - z) &= 0 \quad \text{in } B_1, \\
w_p - z &= 0 \quad \text{on } \partial B_1.
\end{align*}
\]

So we obtain
\[
w_p - z = \sum_{p \in I^c} \sum_{q=1}^{n_p} \beta_{pq} \psi_{pq},
\]
i.e.
\[
u_p = u^{(0)} + w_p + \sum_{p \in I^c} \sum_{q=1}^{n_p} \beta_{pq} \psi_{pq},
\]
for all \( \beta_{pq} \in \mathbb{R} \). Since \( u_p \) is a solution to (3.2), it follows that
\[
-\sqrt{g} = \text{div}(\sqrt{g} G^{-1} \nabla u_p) + \lambda_n \sqrt{g} u_p
\]
\[
= \text{div}(\sqrt{g} G^{-1} \nabla (u^{(0)} + w_p)) + \lambda_n \sqrt{g} (u^{(0)} + w_p)
\]
\[
+ \sum_{p \in I^c} \sum_{q=1}^{n_p} \beta_{pq} \text{div}(\sqrt{g} G^{-1} \nabla \psi_{pq}) + \lambda_n \sqrt{g} \sum_{p \in I^c} \sum_{q=1}^{n_p} \beta_{pq} \psi_{pq}
\]
\[
= -\sqrt{g} + \sum_{p \in I^c} \sum_{q=1}^{n_p} \beta_{pq} (\text{div}(\sqrt{g} G^{-1} \nabla \psi_{pq}) + \lambda_n \sqrt{g} \psi_{pq}).
\]
In particular we obtain
\[ \beta_{pq} \left( \text{div} \left( \sqrt{g} G^{-1} \nabla \psi_{pq} \right) + \lambda_n \sqrt{g} \psi_{pq} \right) = 0. \]

We claim that
\[ \text{div} \left( \sqrt{g} G^{-1} \nabla \psi_{pq} \right) + \lambda_n \sqrt{g} \psi_{pq} \not\equiv 0 \text{ in } B_1. \]

By contradiction let assume that there exists a \( p \in I \) and a \( q \in \{1, \ldots, n_p\} \) such that
\[ \text{div} \left( \sqrt{g} G^{-1} \nabla \psi_{pq} \right) + \lambda_n \sqrt{g} \psi_{pq} = 0 \text{ in } B_1. \]

By defining by \( y = y(x) \) the inverse of the change of variable (2.2), we obtain that
\[ \tilde{\psi}_{pq}(x) = \psi_{pq}(y(x)), \quad x \in \Omega_k, \]
solves
\[ \Delta \tilde{\psi}_{pq} + \lambda_n \tilde{\psi}_{pq} = 0 \text{ in } \Omega_k, \quad \tilde{\psi}_{pq} = 0 \text{ on } \partial \Omega_k. \]

This implies that \( \lambda_n \) is an eigenvalue of \( -\Delta \) in \( \Omega_k \). Then \( u_p \) doesn’t lie in \( \ker(\Delta + \lambda_n)^{+} \) in \( \Omega_k \), which yields a contradiction. This yields that \( \beta_{pq} = 0 \), for all \( p \in I \), and \( q = 1, \ldots, n_p \), and then \( u_p = u(0) + w_p. \)

By (3.3) it follows that
\[ \sqrt{g} I_N - G = KG = (K^{(1)} + o(\|k\|))(I_N + G^{(1)} + o(\|k\|)), \]
where \( K^{(1)} \) denotes the one-order term of the matrix \( K \) (the matrix \( G^{(1)} \) is given by (2.3)). In particular the matrix
\[ K^{(1)} = g^{(1)} I_N - G^{(1)}, \]
where \( g^{(1)} \), the one-order term of \( \sqrt{g} \), is given by
\[ g^{(1)} = Nk + x \cdot \nabla k. \]

By (3.5) we have
\[ \frac{1}{\sqrt{g}} = 1 - Nk - x \cdot \nabla k + o(\|k\|), \]
and by (3.3), (3.6), and (3.7), we obtain
\[ G^{-1} = \frac{I_N}{\sqrt{g}} + \frac{1}{\sqrt{g}} K^{(1)} + \cdots \]
\[ = \frac{I_N}{\sqrt{g}} + G^{(1)} + o(\|k\|). \]

Lemma 3.2. If \( \omega \notin \{\lambda_{\ell m} \mid \ell \geq 2, m \geq 1\} \), with \( \lambda_{\ell m} \neq \lambda_{1m'} \), for all \( m' \geq 1 \), then \( u_p \) has the form
\[ u_p = u(0) + u(1) + o(\|k\|) \quad \text{in } E, \]
where \( u(1) \) solves
\[ \begin{cases} \Delta u^{(1)} + \omega^2 u^{(1)} = f^{(1)} & \text{in } B_1, \\ u^{(1)} = 0 & \text{on } \partial B_1, \end{cases} \]
and \( f^{(1)} \) is given by
\[ f^{(1)} = -(Nk + x \cdot \nabla k) (1 + \omega^2 u^{(0)} - \text{div}(K^{(1)} \nabla u^{(0)})). \]

If \( \omega = \lambda_{\ell m} \), for some \( \ell \geq 2 \), and \( m \geq 1 \) (with \( \lambda_{\ell m} \neq \lambda_{1m'} \), for all \( m' \geq 1 \)), the same holds true by changing \( E \) with \( E_{\ell} \), where \( E_{\ell} \) is defined in (1.6).
To prove Lemma 3.2, we observe that if the \( \ker(\Delta + \omega^2) = \{0\} \) in \( B_1 \), then \( u_p \) admits a one-order expansion in \( E \). The same holds true if the \( \ker(\Delta + \omega^2) \neq \{0\} \) in \( B_1 \), with \( \omega = \lambda_{m'} \), for some \( m' \geq 1 \). On the other hand, if the \( \ker(\Delta + \omega^2) = \{0\} \) in \( B_1 \), i.e. \( \omega = \lambda_{m'} \), for some \( \ell \geq 2 \), and \( m \geq 1 \), then \( u_p \) admits a one-order expansion in the vector space \( E_\ell \) of functions \( k \in E \) which don’t have either the frequency \( \ell \) or the frequency \( p \), with \( p \in I \), the set \( I \) being defined in (2.1).

**Proof of Lemma 3.2.** Let \( \omega \notin \{\lambda_{\ell m}\}_{\ell \geq 2, m \geq 1} \), with \( \lambda_{\ell m} \neq \lambda_{m'} \), for all \( m' \geq 1 \). Let assume that \( u_p \) can be written as in (3.9). Then \( u_p \) solves

\[
\begin{cases}
\Delta u_p + \text{div}(K\nabla u_p) + \omega^2 \sqrt{\gamma} u_p = -\sqrt{\gamma} & \text{in } B_1, \\
u_p = 0 & \text{on } \partial B_1.
\end{cases}
\]

We have

\[
\text{div}(K\nabla u_p) + \sqrt{\gamma}(\omega^2 u_p + 1) = \text{div}(K^{(1)}(\nabla u_0 + \nabla u^{(1)})) + (1 + Nk + x \cdot \nabla k)(\omega^2 (u_0 + u^{(1)}) + 1) + \cdots.
\]

The one-order terms in (3.12) are given by

\[
(Nk + x \cdot \nabla k)(1 + \omega^2 u_0) + \omega^2 u^{(1)} + \text{div}(K^{(1)}\nabla u_0).
\]

By taking the one-order terms in (3.11), we obtain that \( u^{(1)} \) solves (3.10). By a direct calculation \( u^{(1)} \) has the form

\[
u^{(1)} = \frac{I_0'(\lambda_{m'} r)}{\lambda_{m'} I_0(\lambda_{m'})} r k,
\]

if \( \omega = \lambda_{m'} \), since \( I_0' = -I_1 \). Otherwise, for \( \omega \neq \lambda_{m'} \), then \( u^{(1)} \) has the form

\[
u^{(1)} = \frac{I_0'(\omega r)}{\omega I_0(\omega)} r k + \bar{\nu},
\]

where \( \bar{\nu} \) solves

\[
\begin{cases}
\Delta \bar{\nu} + \omega^2 \sqrt{\gamma} \bar{\nu} = 0 & \text{in } B_1, \\
\bar{\nu} = \frac{I_0'(\omega)}{\omega I_0(\omega)} k & \text{on } \partial B_1.
\end{cases}
\]

The solution \( \bar{\nu} \) (in polar coordinates) can be written as

\[
\bar{\nu}(r, \theta) = -c \left( k_0 I_0(\omega r)/I_0(\omega) + \sum_{p \geq 1} \sum_{q=1}^{d} k_{pq} I_p(\omega r)/I_p(\omega) Y_{pq}(\theta) \right).
\]

Now obviously (3.13) is well-defined for all \( \omega \notin \{\lambda_{\ell m}\}_{\ell \geq 2, m \geq 1} \). Let us define by

\[
w = u_p - u^{(0)} - u^{(1)}.
\]

The function \( w \) solves

\[
\begin{cases}
\Delta w + \omega^2 w = (1 - \sqrt{\gamma})(\omega^2 u_p + 1) - \text{div}(K\nabla u_p) - f^{(1)} & \text{in } B_1, \\
w = 0 & \text{on } \partial B_1.
\end{cases}
\]

By writing \( u_p \) as

\[
u_p = u^{(0)} + f,
\]

with \( f(k) = o(1) \) as \( k \to 0 \) in \( E \), we obtain

\[
(1 - \sqrt{\gamma})(\omega^2 u_p + 1) - \text{div}(K\nabla u_p) - f^{(1)} = o(\|k\|).
\]

By standard \( C^{2,\alpha} \)-estimates we obtain

\[
\|w\|_{C^{2,\alpha}(B_1)} = o(\|k\|).
\]

Now if \( \omega = \lambda_{\ell m} \), for some \( \ell \geq 2 \), and \( m \geq 1 \), then (3.13) makes sense if and only if \( k \in E_\ell \), and the same above conclusions hold true, by substituting \( E \) with \( E_\ell \). \( \square \)
Lemma 3.3. If \( \omega \notin \{ \lambda_{\ell m} \}_{\ell \geq 2, m \geq 1} \), with \( \lambda_{\ell m} \neq \lambda_{1m'} \), for all \( m' \geq 1 \), then the operator \( \Phi \) is differentiable at 0 in \( E \), and

\[
\langle d\Phi(0) \mid k \rangle = -k_0 \left( I_1'(\omega) I_0(\omega) + \frac{I_0'(\omega)^2}{I_0(\omega)^2} \right) \partial B_1.
\]

Otherwise if \( \omega = \lambda_{\ell m} \), for some \( \ell \geq 2 \), and \( m \geq 1 \), the same holds true by changing \( E \) with \( E_\ell \).

The previous lemma means that if \( \omega = \lambda_{\ell m} \), for some \( \ell \geq 2 \), and \( m \geq 1 \), then \( \Phi \) is not differentiable at 0 in \( k \), with \( k \) having the form

\[
k = \sum_{m=1}^{d_\ell} k_{\ell m} Y_{\ell m}(\theta) + \sum_{p \in I} \sum_{q=1}^{d_p} k_{pq} Y_{pq}(\theta).
\]

**Proof of Lemma 3.3.** By (2.3), (3.8), and (3.9), we obtain

\[
\Phi(k) = \int_{\partial B_1} (G^{-1}x \cdot x)^{-1/2} \nabla G^{-1}u_p \cdot x \sqrt{g} - c \int_{\partial B_1} \sqrt{g} - c \int_{\partial B_1} \int_{\partial B_1} \int_{\partial B_1} (G^{-1}x \cdot x)^{-1/2} \nabla G^{-1}u_p \cdot x \sqrt{g} + \ldots
\]

\[
= c \int_{\partial B_1} (1 - 2k - 2\partial_k k)^{1/2} \sqrt{g} - c \int_{\partial B_1} \sqrt{g} + \int_{\partial B_1} (1 - 2k - 2\partial_k k)^{-1/2} (\partial_k u^{(1)}(\omega) - G^{(1)} \nabla u^{(1)}(\omega) \cdot x \sqrt{g} + \ldots
\]

Since the surface element \( \sqrt{g} \) can be written as

\[
\sqrt{g} = 1 + o(\|k\|),
\]

by taking the one-order terms in (3.15), we obtain

\[
\langle d\Phi(0) \mid k \rangle = -c \int_{\partial B_1} (k + \partial_k k) + \int_{\partial B_1} \partial_k u^{(1)}.
\]

Since

\[
\partial_k u^{(1)} = \left( \frac{I_0''(\omega)}{I_0(\omega)} + c \right) k + c \partial_k k + \partial_k \bar{u},
\]

and

\[
\partial_k \bar{u} = -c\omega \left( k_0 I_0'(\omega)/I_0(\omega) + \sum_{p \geq 1} \sum_{q=1}^{d_p} k_{pq} I_p'(\omega)/I_p(\omega) Y_{pq}(\theta) \right).
\]

we obtain

\[
\langle d\Phi(0) \mid k \rangle = -c \int_{\partial B_1} (k + \partial_k k) + \left( c - \frac{I_1'(\omega)}{I_0(\omega)} \right) \int_{\partial B_1} k + c \int_{\partial B_1} \partial_k k + \int_{\partial B_1} \partial_k \bar{u}
\]

\[
= -\frac{I_1'(\omega)}{I_0(\omega)} \int_{\partial B_1} k - c\omega \frac{I_1'(\omega)}{I_0(\omega)} k_0 |\partial B_1|
\]

\[
= -c \left( \frac{I_1'(\omega)}{I_0(\omega)} + \frac{I_0'(\omega)^2}{I_0(\omega)^2} \right) |\partial B_1|,
\]

being \( c = \frac{I_1'(\omega)}{I_0(\omega)} \).
Lemma 3.4. The number
\[ \frac{I'_1(\omega)}{I_0(\omega)} + \frac{I'_0(\omega)^2}{I_0(\omega)^2} > 0. \] (3.16)

Proof of Lemma 3.4. We have
\[ \Phi(k_0) = \int_{\partial B_{1+k_0}} \partial_n u - c \int_{\partial B_{1+k_0}} = \left( \frac{I'_0((1+k_0)\omega)}{I_0((1+k_0)\omega)} - \frac{I'_0(\omega)}{I_0(\omega)} \right) \frac{|\partial B_{1+k_0}|}{\omega}. \]

Now since the function
\[ \frac{I'_0(\omega)}{I_0(\omega)} \]
is decreasing in \( \omega \), it follows that for \( k_0 > 0 \) sufficiently small, the function
\[ \frac{I'_0((1+k_0)\omega)}{I_0((1+k_0)\omega)} - \frac{I'_0(\omega)}{I_0(\omega)} < 0. \]
So \( \Phi \) is decreasing in the direction \( tk_0 \), for some \( t \in I \), and then
\[ \langle d\Phi(0) | k_0 \rangle < 0, \]
which yields (3.16). \( \square \)

4. Proof of Theorem 1.1

Before proceeding with the proof of Theorem 1.1, we need the following

Lemma 4.1. There exists a neighborhood \( \mathcal{O} \) of the origin in \( E \), such that if \( k \in \mathcal{O} \cap E^C_1 \), then the mass center \( \bar{x} \) of \( \Omega_k \) is different to zero.

Here \( E_1 \) is the vector space
\[ E_1 = \{ k \in E ; k_{1q} = 0 \}, \]
of functions \( k \in E \) which don’t have the frequency 1, and
\[ E^C_1 = \{ k \in E ; k_{1q} \neq 0 \text{ for some } q = 1, \ldots, N \}, \]
the complementary of \( E_1 \), is the set of functions \( k \) which have the frequency 1. We recall that the mass center of a domain \( \Omega \) is the point \( \bar{x} \) of coordinates
\[ \bar{x}_i = \frac{1}{|\Omega|} \int_{\Omega} x_i, \quad i = 1, \ldots, N. \]

Proof of Lemma 4.1. For \( i = 1, \ldots, N \), let us denote by \( F_i \) the following operator
\[ F_i : E \to \mathbb{R}, \]
defined by
\[ F_i(k) = \frac{1}{|\Omega_k|} \int_{\Omega_k} x_i, \]
i.e. the operator \( F_i \) associates to \( k \) the \( i \)th component of the mass center \( \bar{x} \) of the domain \( \Omega_k \). By the change of variable (2.2), we obtain
\[ F_i(k) = \frac{1}{|\Omega_k|} \int_{\partial B_1} x_i = \frac{1}{\mu_1} \int_{B_1} (1 + k)x_i \sqrt{g} \]

\[ = \int_{B_1} (1 - Nk - x \cdot \nabla k + \cdots) \int_{B_1} \left( (x_i + (N + 1)k x_i + x \cdot \nabla k x_i + \cdots) \right) \]

By taking the one-order terms, we have that the differential of \( F_i \) at zero in \( k \) is given by

\[ \langle dF_i(0) \mid k \rangle = (N + 1) \sum_{p \geq 1} \sum_{q = 1}^{d_p} k_{pq} \int_{\partial B_1} Y_{pq} Y_{1i} + \sum_{p \geq 1} \sum_{q = 1}^{d_p} p k_{pq} \int_{\partial B_1} r^{p+N-1} \int_{\partial B_1} Y_{pq} Y_{1i} \]

\[ = (N + 1) k_{11} \int_{\partial B_1} r^{N+1} + k_{1i} \int_{\partial B_1} r^N \]

\[ = \left( 1 + \frac{1}{(N + 2)(N + 1)} \right) k_{11}. \]

Let \( k \in E_1^C \). Then there exists at least a \( q \in \{1, \ldots, N\} \) such that \( k_{1q} \neq 0 \). So there exists a neighborhood \( \mathcal{O} \) of the origin in \( E \) such that \( F_q \) is increasing (or decreasing) in \( \mathcal{O} \cap E_1^C \). Now, since \( F_i(0) = 0 \), we obtain that \( \bar{x}_q \neq 0. \]

The previous lemma implies in particular that if the mass center of \( \Omega_k \) is at the point zero, then \( k \) doesn’t have the frequency 1, i.e. \( k_{1q} = 0 \) for all \( q = 1, \ldots, N \). This means that a domain \( \Omega_k \), with \( k \in \mathcal{O} \cap E_1 \) is either a domain with mass center at 0, or \( \Omega_k = \sigma(\Omega_k) \), for some \( \sigma \in \Sigma \), and some domain \( \overline{\Omega_k} \), where \( \Sigma \) is the set of isometries of \( \mathbb{R}^N \), and \( \Omega_k \) has mass center at zero. Now since the operator \( \Phi \) is invariant up to isometries, we obtain that \( \Phi \) has a sign in a neighborhood \( \mathcal{O} \) of 0 in \( E \), if \( \Phi \) has a sign in \( \mathcal{O} \cap E_1 \). For this reason in what follows we will concentrate our attention on the space \( E_1 \). We observe for example that the function

\[ k' = x_0 \cdot y - 1 + \sqrt{1 + |x_0 \cdot y|^2 - |x_0|^2}, \]

which parametrizes the sphere \( \partial B_1(x_0) \) centered at \( x_0 \), has the frequency 1, which is equal to \( x_0 \), i.e. \( k' \in E_1^C \). In fact the function

\[ h(y) = \sqrt{1 + |x_0 \cdot y|^2 - |x_0|^2} \]

is even in the variable \( y \), and then the function \( hY_{1m} \) is odd, which implies that \( \int_{\partial B_1} hY_{1m} = 0 \), for all \( m = 1, \ldots, N \).

**Proof of Theorem 1.1.** Step 1. Let assume that \( \omega \notin \{ \lambda_{\ell m} \}_{\ell \geq 1, m \geq 1} \), with \( \lambda_{\ell m} \neq \lambda_{1m'} \), for all \( m' \geq 1 \). Let us define by

\[ E^+_{\varepsilon} = \{ k \in E_1; \|k\| = 1, \ k_0 \leq -\varepsilon \}, \]

and by

\[ E^-_{\varepsilon} = \{ k \in E_1; \|k\| = 1, \ k_0 \geq \varepsilon \}, \]

for some positive constant \( \varepsilon < 1 \). We have

\[ \langle d\Phi(0) \mid k \rangle \geq \varepsilon C|\partial B_1| \quad \text{for all} \ k \in E^+_{\varepsilon}, \]

and

\[ \langle d\Phi(0) \mid k \rangle \leq -\varepsilon C|\partial B_1| \quad \text{for all} \ k \in E^-_{\varepsilon}. \]
where $C = \frac{i'(\omega)}{\bar{\omega}i(\omega)} + \frac{i'(\omega)^2}{\bar{\omega}i(\omega)}$. So there exists a sufficiently small interval $I$ of $0$ in $\mathbb{R}^+$ such that $\Phi$ is positive in

$$E^+ = \{tk; \ t \in I, \ k \in E^+_1\},$$

and $\Phi$ is negative in

$$E^- = \{tk; \ t \in I, \ k \in E^-_1\}.$$  

Let $\mathcal{O}$ be a neighborhood of $0$ in $E$ such that $\mathcal{O} \cap E^+ \cup \{0\}$ is contained in $E^+ \cup \{0\}$, and $\mathcal{O} \cap E^- \cup \{0\}$ is contained in $E^- \cup \{0\}$. Now if $\omega = \lambda \ell m$, for some $\ell \geq 2$, and $m \geq 1$, the same above conclusions hold true by changing $E_1$ with the subspace

$$E_\ell = \{k \in E_1; \ k_\ell q = 0, \ k_p q' = 0, \ p \in I\}$$

of $E_1$. Now since for example $\Phi$ is positive in $E^+ \cap E_\ell$ and is continuous in $E^+$, and $E_\ell$ is finite dimensional, it follows that $\Phi$ is positive in $E^+$.

**Step 2.** Let $D$ be the class of $C^{2,\alpha}$-domains defined as

$$D = \{\Omega; \ \Omega = \sigma(\Omega_k)\},$$

for some $\sigma \in \Sigma$, and some $\Omega_k \in \mathcal{G}$, where $\Sigma$ is the set of isometries of $\mathbb{R}^N$, and $\mathcal{G} = \{\Omega_k; \ k \in \mathcal{O} \cap (E^+ \cup E^- \cup \{0\})\}$.

Let assume that there exists a $\Omega \in D$ such that $\frac{1}{|\Omega_k|}\int_{\partial \Omega} \partial_n u = c$. Since the problem is invariant up to isometries we have that $\frac{1}{|\Omega_k|}\int_{\partial \Omega_k} \partial_n u = c$, for some $k \in \mathcal{O} \cap (E^+ \cup E^- \cup \{0\})$.

**Step 3.** Let assume that the kernel $\text{ker}(\Delta + \omega^2) = \{0\}$ in $\Omega_k$. Then $u$ coincides with $u_p$, and

$$\Phi(k) = 0.$$  

Let assume that $k \in \mathcal{O} \cap E^+ \cup \{0\}$. This yields that $k = 0$, since $\Phi$ is positive in $\mathcal{O} \cap E^+$. Now if the kernel $\text{ker}(\Delta + \omega^2) \neq \{0\}$ in $\Omega_k$, then $u$ can be written as

$$u = u_p + u_h \text{ in } \Omega_k.$$  

Since by Fredholm theorem $-1 \in \text{ker}(\Delta + \omega^2)^\perp$, by divergence theorem we obtain

$$0 = \int_{\Omega_k} u_h = -\frac{1}{\omega^2} \int_{\Omega_k} \Delta u_h = -\frac{1}{\omega^2} \int_{\partial \Omega_k} \partial_n u_h.$$  

Then we have

$$\Phi(k) = \int_{\partial \Omega_k} \partial_n u_p - c \int_{\partial \Omega_k} \partial_n u = \int_{\partial \Omega_k} \partial_n u - c \int_{\partial \Omega_k} = 0. \ \square$$

We conclude this section by examining briefly the Lipschitz case. Let us define by

$$E = \{k \in C^{0,1}(\partial B_1)\}.$$  

Let $u \in H^1(\Omega_k)$ be a weak solution to (1.1), when $\Omega = \Omega_k$, and $k \in E$. Then $u$ solves

$$\int_{\Omega_k} \nabla u \cdot \nabla \phi - \omega^2 \int_{\Omega_k} u \phi = \int_{\Omega_k} \phi,$$

for all $\phi \in C^\infty(\Omega_k)$. Since, by regularity results, $u \in C^{0,1}(\Omega_k)$, the operator $\Phi$ is well-defined in $E$. By repeating the same arguments as in the regular case, one can prove the following
Theorem 4.2. For $\omega \notin \{\lambda_{\text{dom}}\}_{m \geq 1}$, there exists a class $D$ of Lipschitz domains, such that if $u \in H^1(\Omega)$ is a weak solution to (1.1) verifying
\[
\frac{1}{|\partial \Omega|} \int_{\partial \Omega} \partial_n u = c,
\]
with $\Omega \in D$, and $c$ given by (1.4), then $\Omega = B_1$, and $u = u^{(0)}$.

5. Concluding remark

We recall that by the proof of Theorem 1.1 it follows that $\Phi$ is positive in the circular sector $E^+$ in $\{k \in \mathbb{E}; k_0 < 0\}$, and is negative in the circular sector $E^-$ in $\{k \in \mathbb{E}; k_0 > 0\}$. So the operator $\Phi$ must vanish somewhere. In fact let $\epsilon > 0$ be fixed. Let $k \in E^-$. Then $\Phi(k)$ is negative. Now the domain $\tilde{\Omega}_k$, whose boundary is given by
\[
\partial \tilde{\Omega}_k = \{x = (1 + (a + k))y, \ y \in \partial B_1\},
\]
with $-1 < a < 0$, is a contraction of the domain $\Omega_k$. We can find then a value $a$ such that $a + k \in E^+$. But $\Phi(a + k)$ is positive. Then there exists a $\tilde{k}$ such that $\Phi(\tilde{k}) = 0$. By repeating the same argument for all $\epsilon > 0$, and for all $k \in E^-$, we can find a variety $M$ in $E_1$ (whose tangent space at 0 is contained or coincides with $E_0 = \{k; k_0 = 0\}$), such that $\Phi$ vanishes identically on $M$. In particular we obtain that all domains $\Omega$ lying in the class
\[
D = \{\Omega; \Omega = \sigma(\Omega_k)\},
\]
for some $\sigma \in \Sigma$, and some $k \in M$, are counter-examples to Theorem 1.1.

References

