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Ann. I. H. Poincaré – AN 27 (2010) 1489–1505

ANNALES DE L'INSTITUT HENRI POINCARÉ
ANALYSE NON LINÉAIRE

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BMO, integrability, Harnack and Caccioppoli inequalities for quasiminimizers

Anders Björn [a], Jana Björn [a], Niko Marola [b,*]

[a] Department of Mathematics, Linköpings universitet, SE-581 83 Linköping, Sweden
[b] Department of Mathematics and Statistics, University of Helsinki, P.O. Box 68, FI-00014 University of Helsinki, Finland

Received 10 November 2009; received in revised form 26 August 2010; accepted 20 September 2010

Available online 6 October 2010

Abstract

In this paper we use quasiminimizing properties of radial power-type functions to deduce counterexamples to certain Caccioppoli-type inequalities and weak Harnack inequalities for quasisuperharmonic functions, both of which are well known to hold for p-superharmonic functions. We also obtain new bounds on the local integrability for quasisuperharmonic functions. Furthermore, we show that the logarithm of a positive quasisuperminimizer has bounded mean oscillation and belongs to a Sobolev type space.
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Résumé

Dans cet article nous utilisons les propriétés des fonctions avec puissance radiale afin d'obtenir des contre-exemples à certaines inéquations de type Caccioppoli et Harnack faible pour les fonctions quasisuperharmoniques, lesquelles sont bien connues être valables pour les fonctions p-superharmoniques. Nous obtenons aussi de nouvelles bornes pour l'intégrabilité locale des fonctions quasisuperharmoniques. De plus nous démontrons que le logarithme d'une fonction positive quasiminimisante est de type BMO, et appartient à un espace de Sobolev.
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MSC: primary 49J20; secondary 30L99, 31C45, 31E05, 35J20, 49J27

Keywords: Bounded mean oscillation; Doubling measure; Metric space; Nonlinear; p-harmonic; Poincaré inequality; Potential theory; Quasiminimizer; Quasisuperharmonic; Quasisuperminimizer; Weak Harnack inequality

1. Introduction

Let $1 < p < \infty$ and let $\Omega \subset \mathbf{R}^n$ be a nonempty open set. A function $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is a *Q-quasiminimizer*, $Q \geqslant 1$, in Ω if

$$\int_{\varphi\neq 0} |\nabla u|^p \, dx \leqslant Q \int_{\varphi\neq 0} \left|\nabla(u+\varphi)\right|^p \, dx \tag{1.1}$$

* Corresponding author.
E-mail addresses: anbjo@mai.liu.se (A. Björn), jabjo@mai.liu.se (J. Björn), niko.marola@helsinki.fi (N. Marola).

0294-1449/$ – see front matter © 2010 Elsevier Masson SAS. All rights reserved.
doi:10.1016/j.anihpc.2010.09.005

for all $\varphi \in W_0^{1,p}(\Omega)$. Quasiminimizers were introduced by Giaquinta and Giusti [19,20] as a tool for a unified treatment of variational integrals, elliptic equations and quasiregular mappings on $\mathbf{R}^n$. They realized that De Giorgi's method could be extended to quasiminimizers, obtaining, in particular, local Hölder continuity. DiBenedetto and Trudinger [15] proved the Harnack inequality for quasiminimizers, as well as weak Harnack inequalities for quasisub- and quasisuperminimizers. We recall that a function $u \in W_{\mathrm{loc}}^{1,p}(\Omega)$ is a *quasisub*(*super*)*minimizer* if (1.1) holds for all nonpositive (nonnegative) $\varphi \in W_0^{1,p}(\Omega)$.

Compared with the theory of p-harmonic functions we have no differential equation for quasiminimizers, only the variational inequality can be used. There is also no comparison principle nor uniqueness for the Dirichlet problem. The following result was recently obtained by Martio [37, Theorem 4.1]. It shows that quasiminimizers are much more flexible under perturbations than solutions of differential equations, which can be useful in applications and in particular shows that results obtained for quasiminimizers are very robust.

Theorem 1.1. *Let u be a Q-quasiminimizer in Ω and $f \in W_{\mathrm{loc}}^{1,p}(\Omega)$ be such that $|\nabla f| \leqslant c|\nabla u|$ a.e. in Ω, where $0 < c < Q^{-1/p}$. Then $u + f$ is a Q'-quasiminimizer in Ω', where $Q' = (1+c)^p/(Q^{-1/p} - c)^p$.*

After the papers by Giaquinta and Giusti [19,20] and DiBenedetto and Trudinger [15], Ziemer [45] gave a Wiener-type criterion sufficient for boundary regularity for quasiminimizers. Tolksdorf [42] obtained a Caccioppoli inequality and a convexity result for quasiminimizers. The results in [15,19,20,45] were extended to metric spaces by Kinnunen and Shanmugalingam [30] and J. Björn [11] in the beginning of this century, see also A. Björn and Marola [9]. Soon afterwards, Kinnunen and Martio [28] showed that quasiminimizers have an interesting potential theory, in particular they introduced quasisuperharmonic functions, which are related to quasisuperminimizers in a similar way as superharmonic functions are related to supersolutions, see Definition 2.4.

The one-dimensional theory was already considered in [19], and has since been further developed by Martio and Sbordone [38], Judin [27], Martio [35] and Uppman [44]. Most aspects of the higher-dimensional theory fit just as well in metric spaces, and this theory, in particular concerning boundary regularity, has recently been developed further in a series of papers by Martio [34–37], A. Björn and Martio [10], A. Björn [1–4] and J. Björn [12].

So far, most of the theory for quasiminimizers has been extending various results known for p-harmonic functions. This paper goes in the opposite direction: we show that some results are not extendable and the class of quasiminimizers behaves in a way that was not expected.

Superminimizers, i.e. 1-quasisuperminimizers, are nothing but supersolutions to the p-Laplace equation

$$\operatorname{div}\bigl(|\nabla u|^{p-2}\nabla u\bigr) = 0.$$

Until now, there have been very few examples of quasi(super)minimizers for which the best quasi(super)minimizer constant is known, apart from a few explicit examples of p-harmonic functions, i.e. with $Q = 1$. In one dimension there are a couple of examples with optimal quasiminimizer constant in Judin [27], Martio [35] and Uppman [44]. As far as we know, the only explicit examples of quasiminimizers with optimal quasiminimizer constant $Q > 1$ in higher dimensions were recently obtained by Björn and Björn [7]. Let $\mathbf{B} = B(0,1)$ denote the unit ball in $\mathbf{R}^n$.

Theorem 1.2. *Let $1 < p < n$, $\alpha \leqslant \beta = (p-n)/(p-1)$ and $u(x) = |x|^\alpha$. Then u is a Q-quasiminimizer in $\mathbf{B} \setminus \{0\}$ and a Q-quasisuperharmonic function in $\mathbf{B}$, where*

$$Q = \left(\frac{\alpha}{\beta}\right)^p \frac{p\beta - p + n}{p\alpha - p + n}$$

is the best quasiminimizer constant in both cases.

Trudinger [43] obtained a sharp weak Harnack inequality for supersolutions ($Q = 1$), whose exponent coincides with the sharp exponent for local integrability of superharmonic functions. As a consequence of Theorem 1.2, we will show that the best exponent in the weak Harnack inequality for Q-quasisuperminimizers must depend on Q, and tends to 0, as $Q \to \infty$. The same is true for the best exponent of local integrability for Q-quasisuperharmonic functions. Theorem 1.2 implies upper bounds for these exponents, which for $Q = 1$ coincide with the known sharp bounds. It is therefore natural to expect that these bounds are sharp also for $Q > 1$.

Similar conclusions can be drawn for Caccioppoli inequalities for quasisuperminimizers: some of the "classical" Caccioppoli type inequalities cannot be true with exponents independent of the quasiminimizing constant Q. There is a gap between the sharp exponents for $Q = 1$ and the known exponents for $Q > 1$, and Theorem 1.2 implies bounds for these exponents, which for $Q = 1$ coincide with the known sharp bound.

Caccioppoli inequalities and the weak Harnack inequality for quasisuperminimizers are essential for extending the Moser iteration technique in full to quasiminimizers. A. Björn and Marola [9] have shown that the scheme applies to a large extent to this setting, but a logarithmic Caccioppoli inequality for quasisuperminimizers still needs to be proved in order to obtain the full result. It is traditionally used to show that the logarithm of a positive superminimizer belongs to BMO. We show that the logarithm of a positive quasisuperminimizer belongs to BMO simply by exploiting a classical tool from harmonic analysis in our setting. Interesting enough, in light of Theorem 1.2, this qualitative result seems to be the best we can hope for.

The outline of the paper is as follows: In Section 2 we introduce the relevant background on metric spaces and quasiminimizers. Those readers only interested in Euclidean spaces may simply replace the minimal upper gradient g_u by the modulus $|\nabla u|$ of the usual gradient and the Newtonian space $N^{1,p}$ by the Sobolev space $W^{1,p}$ throughout the paper.

In Section 3 we turn to Caccioppoli inequalities for quasisuperminimizers and deduce bounds on the exponents. In Section 4 we obtain bounds on the exponents in weak Harnack inequalities for quasisuperminimizers and for the local integrability of quasisuperharmonic functions. In Section 5 we prove some new Caccioppoli inequalities for quasiminimizers. We also show that the logarithm of a positive quasisuperminimizer belongs to BMO, qualitatively, and to $W^{1,q}_{\mathrm{loc}}$ for every $q < p$.

Finally, in Section 6 we make a digression and discuss the relation between the quasiconvexity constant L and the dilation constant λ in the weak Poincaré inequality; both constants play essential roles in Sections 4 and 5.

2. Preliminaries

The theory of quasiminimizers fits naturally into the analysis on metric spaces, as it uses variational integrals rather than partial differential equations. In this case, the metric space (X, d) is assumed to be complete and equipped with a doubling measure μ, i.e. there exists a doubling constant C_μ such that

$$0 < \mu(2B) \leqslant C_\mu \mu(B) < \infty$$

for every ball $B = B(x, r) = \{y \in X\colon\ d(x, y) < r\}$, where $\lambda B = B(x, \lambda r)$. Moreover we require the measure to support a weak $(1, p)$-Poincaré inequality, see below. Throughout the paper we assume that $1 < p < \infty$.

We follow Heinonen and Koskela [25] in introducing upper gradients as follows (they called them very weak gradients).

Definition 2.1. A nonnegative Borel function g on X is an *upper gradient* of an extended real-valued function f on X if for all curves $\gamma : [0, l_\gamma] \to X$,

$$\bigl|f\bigl(\gamma(0)\bigr) - f\bigl(\gamma(l_\gamma)\bigr)\bigr| \leqslant \int_\gamma g\, ds \tag{2.1}$$

whenever both $f(\gamma(0))$ and $f(\gamma(l_\gamma))$ are finite, and $\int_\gamma g\, ds = \infty$ otherwise.

A nonnegative measurable function g on X is a *p-weak upper gradient* of f if (2.1) holds for p-almost every curve in the sense of Definition 2.1 in Shanmugalingam [40]. It is implicitly assumed that $\int_\gamma g\, ds$ is defined (with a value in $[0, \infty]$) for p-almost every rectifiable curve γ.

The p-weak upper gradients were introduced in Koskela and MacManus [31]. They also showed that if $g \in L^p(X)$ is a p-weak upper gradient of f, then one can find a sequence $\{g_j\}_{j=1}^\infty$ of upper gradients of f such that $g_j \to g$ in $L^p(X)$. If f has an upper gradient in $L^p(X)$, then it has a *minimal p-weak upper gradient* $g_f \in L^p(X)$ in the sense that for every p-weak upper gradient $g \in L^p(X)$ of f, $g_f \leqslant g$ a.e., see Corollary 3.7 in Shanmugalingam [41]. (The reader may also consult Björn and Björn [8] where proofs of all the facts mentioned in this section are given, apart from those about quasiminimizers.)

Next we define a version of Sobolev spaces on the metric space X due to Shanmugalingam in [40]. Cheeger [14] gave an alternative definition which leads to the same space (when $p > 1$) see [40].

Definition 2.2. Whenever $u \in L^p(X)$, let

$$\|u\|_{N^{1,p}(X)} = \left(\int_X |u|^p \, d\mu + \inf_g \int_X g^p \, d\mu \right)^{1/p},$$

where the infimum is taken over all upper gradients of u. The *Newtonian space* on X is the quotient space

$$N^{1,p}(X) = \{u: \ \|u\|_{N^{1,p}(X)} < \infty\}/\sim,$$

where $u \sim v$ if and only if $\|u - v\|_{N^{1,p}(X)} = 0$.

The space $N^{1,p}(X)$ is a Banach space and a lattice, see Shanmugalingam [40].

A function u belongs to the *local Newtonian space* $N^{1,p}_{\mathrm{loc}}(\Omega)$ if $u \in N^{1,p}(V)$ for all bounded open sets V with $\overline{V} \subset \Omega$, the latter space being defined by considering V as a metric space with the metric d and the measure μ restricted to it.

If $u, v \in N^{1,p}_{\mathrm{loc}}(X)$, then $g_u = g_v$ a.e. in $\{x \in X: \ u(x) = v(x)\}$, in particular $g_{\min\{u,c\}} = g_u \chi_{u \neq c}$ for $c \in \mathbf{R}$. For these and other facts on p-weak upper gradients, see, e.g., Björn and Björn [5, Section 3] (which is not included in Björn and Björn [6]).

Definition 2.3. We say that X supports a *weak (q, p)-Poincaré inequality* if there exist constants $C > 0$ and $\lambda \geqslant 1$ such that for all balls $B \subset X$, all integrable functions f on X and all upper gradients g of f,

$$\left(\fint_B |f - f_B|^q \, d\mu \right)^{1/q} \leqslant C \operatorname{diam}(B) \left(\fint_{\lambda B} g^p \, d\mu \right)^{1/p}, \tag{2.2}$$

where $f_B := \fint_B f \, d\mu := \int_B f \, d\mu / \mu(B)$.

In the definition of Poincaré inequality we can equivalently assume that g is a p-weak upper gradient—see the comments above.

For more details see any of the papers on metric spaces in the reference list. Note, in particular, that $\mathbf{R}^n$ with the Lebesgue measure $d\mu = dx$, as well as weighted $\mathbf{R}^n$ with p-admissible weights are special cases of metric spaces satisfying our assumptions.

In metric spaces, g_u is the natural substitute for the scalar $|\nabla u|$. Note that we have no natural counterpart to the vector ∇u. (See however Cheeger [14].) The Newtonian space $N^{1,p}$ replaces the Sobolev space $W^{1,p}$. The definition of quasiminimizers on metric spaces is thus as follows. A function $u \in N^{1,p}_{\mathrm{loc}}(\Omega)$ is a *Q-quasiminimizer*, $Q \geqslant 1$, in Ω if

$$\int_{\varphi \neq 0} g_u^p \, dx \leqslant Q \int_{\varphi \neq 0} g_{u+\varphi}^p \, d\mu \tag{2.3}$$

for all $\varphi \in \mathrm{Lip}_c(\Omega)$. Similarly, a function $u \in N^{1,p}_{\mathrm{loc}}(\Omega)$ is a *Q-quasisub*(*super*)*minimizer* if (2.3) holds for all nonpositive (nonnegative) $\varphi \in \mathrm{Lip}_c(\Omega)$. Note also that a function is a Q-quasiminimizer in Ω if and only if it is both a Q-quasisubminimizer and a Q-quasisuperminimizer in Ω.

Our definition of quasiminimizers (and quasisub- and quasisuperminimizers) is one of several equivalent possibilities, see Proposition 3.2 in A. Björn [1]. In fact it is enough to test (1.1) with (all, nonpositive and nonnegative, respectively) $\varphi \in \mathrm{Lip}_c(\Omega)$, the space of Lipschitz functions with compact support in Ω.

Every quasiminimizer can be modified on a set of measure zero so that it becomes locally Hölder continuous in Ω. This was proved in $\mathbf{R}^n$ by Giaquinta and Giusti [20, Theorem 4.2], and in metric spaces by Kinnunen and Shanmugalingam [30, Proposition 3.8 and Corollary 5.5]. A *Q-quasiharmonic* function is a continuous Q-quasiminimizer.

Kinnunen and Martio [28, Theorem 5.3] showed that if u is a Q-quasisuperminimizer in Ω, then its *lower semicontinuous regularization* $u^*(x) = \operatorname{ess\,lim\,inf}_{y\to x} u(y)$ is also a Q-quasisuperminimizer in Ω belonging to the same equivalence class as u in $N^{1,p}_{\mathrm{loc}}(\Omega)$. Furthermore, u^* is Q-quasisuperharmonic in Ω. For our purposes we make the following definition.

Definition 2.4. A function $u : \Omega \to (-\infty, \infty]$ is *Q-quasisuperharmonic* in Ω if u is not identically ∞ in any component of Ω, u is lower semicontinuously regularized, and $\min\{u, k\}$ is a Q-quasisuperminimizer in Ω for every $k \in \mathbf{R}$.

This definition is equivalent to Definition 7.1 in Kinnunen and Martio [28], see Theorem 7.10 in [28]. (Note that there is a misprint in Definition 7.1 in [28]—the functions v_i are assumed to be Q-quasisuperminimizers.)

A function is *p-harmonic* if it is 1-quasiharmonic, it is *superharmonic* if it is 1-quasisuperharmonic, and it is a *sub*(*super*)*minimizer* if it is a 1-quasisub(super)minimizer.

Unless otherwise stated, the letter C denotes various positive constants whose exact values are unimportant and may vary with each usage.

3. Caccioppoli inequalities for quasisuperminimizers

In this section we discuss Caccioppoli inequalities for quasi(super)minimizers. These inequalities play an important role, e.g., when proving regularity results for quasiminimizers, see, e.g., DiBenedetto and Trudinger [15], Kinnunen and Shanmugalingam [30] and A. Björn and Marola [9]. We will show how some well-known results for sub- and superminimizers (i.e. with $Q = 1$) do not extend to the case $Q > 1$.

Let $\gamma_0 = \gamma_0(Q, p)$ be the largest number (independent of X and Ω) such that for every $\gamma < \gamma_0$ there is a constant $C_\gamma = C_\gamma(Q, p, X, \Omega)$ such that the *Caccioppoli inequality*

$$\int_\Omega u^{-p+\gamma} g_u^p \eta^p \, d\mu \leqslant C_\gamma \int_\Omega u^\gamma g_\eta^p \, d\mu \tag{3.1}$$

holds for all Q-quasisuperminimizers $u \geqslant 0$ in Ω and all $0 \leqslant \eta \in \mathrm{Lip}_c(\Omega)$.

By Proposition 7.3 in A. Björn and Marola [9], (3.1) holds for all $\gamma < 0$ and thus $\gamma_0(Q, p) \geqslant 0$, both in $\mathbf{R}^n$ and in metric spaces. We also know that $\gamma_0(1, p) \geqslant p - 1$, see Heinonen, Kilpeläinen and Martio [24, Lemma 3.57] for the Euclidean case and Kinnunen and Martio [29, Lemma 3.1] for metric spaces. It is probably well known that $\gamma_0(1, p) = p - 1$, but it also follows from the following more general proposition, with $\alpha = \beta$.

Proposition 3.1. *Let $n > p$ and $\alpha \leqslant (p - n)/(p - 1) = \beta$. Let further*

$$Q = \left(\frac{\alpha}{\beta}\right)^p \frac{p\beta - p + n}{p\alpha - p + n} \quad \text{and} \quad \delta(Q, p) = \frac{p - n}{\alpha}. \tag{3.2}$$

Then $\gamma_0(Q, p) \leqslant \delta(Q, p)$.

Note that δ is really a function of Q and p only: there is a one-to-one correspondence between $Q \geqslant 1$ and $\alpha \leqslant \beta$, and thus $\delta(Q, p)$ is a function of Q, p and n a priori, and is independent of n by Proposition 3.3 below.

In view of this result, and the fact that $\gamma_0(1, p) = p - 1$, it feels natural to make the following conjecture.

Conjecture 3.2. *Let $Q > 1$. We conjecture that $\gamma_0(Q, p) = \delta(Q, p)$.*

Recall though that we do not know that $\gamma_0(Q, p)$ is positive nor that (3.1) holds for $\gamma = 0$, for any $Q > 1$. See however Lemma 5.8 below.

Proposition 3.3. *Let $n > p$. Then $\delta(Q, p)$ is the unique solution in $(0, p - 1]$ of the equation*

$$Q = \frac{(p - 1)^{p-1}}{\delta^{p-1}(p - \delta)}. \tag{3.3}$$

In particular, the expression for $\delta(Q, p)$ in (3.2) is independent of $n > p$.

Proof. Let $0 < \delta \leqslant p-1$ be fixed and $\alpha = (p-n)/\delta \leqslant (p-n)/(p-1) = \beta < 0$. Note that $p\beta - p + n = \beta$. Inserting this into (3.2) gives

$$Q = \left(\frac{p-1}{\delta}\right)^p \frac{(p-n)/(p-1)}{p(p-n)/\delta - p + n} = \frac{(p-1)^{p-1}}{\delta^{p-1}(p-\delta)}.$$

Note that $Q = 1$ for $\delta = p-1$, and $Q \to \infty$, as $\delta \to 0$. Differentiating Q with respect to δ gives

$$\frac{\partial Q}{\partial \delta} = (p-1)^{p-1}\left(\frac{1-p}{\delta^p(p-\delta)} + \frac{1}{\delta^{p-1}(p-\delta)^2}\right) = \frac{p(p-1)^{p-1}(\delta+1-p)}{\delta^p(p-\delta)^2} \leqslant 0$$

with equality only for $\delta = p-1$. Thus, Q is strictly decreasing as a function of δ in the interval $(0, p-1]$. Consequently, for every $Q \geqslant 1$, there exists a unique $\delta \in (0, p-1]$ satisfying (3.3). □

It is easy to see that $\delta(Q,2) = 1 - \sqrt{1-1/Q}$ and $\alpha(Q,2,n) = (2-n)(Q + \sqrt{Q^2 - Q})$. For general p we can use (3.3) to obtain the following asymptotic estimates for $\delta(Q,p)$ and α in terms of Q and p.

Corollary 3.4. *Let $Q > 1$ and $n > p$. Then*

$$\frac{p-1}{(pQ)^{1/(p-1)}} < \delta(Q,p) < \frac{p-1}{Q^{1/(p-1)}}. \tag{3.4}$$

Moreover, if $\beta := (p-n)/(p-1)$, then

$$(pQ)^{1/(p-1)}\beta < \alpha < Q^{1/(p-1)}\beta.$$

Proof. Write $\delta := \delta(Q,p)$. As $\alpha = (p-n)/\delta = (p-1)\beta/\delta$, it suffices to prove (3.4). Since $\delta > 0$, we have

$$Q = \frac{(p-1)^{p-1}}{\delta^{p-1}(p-\delta)} > \frac{(p-1)^{p-1}}{\delta^{p-1}p},$$

proving the first inequality in (3.4). Similarly, as $\delta < p-1$, we see that $Q < (p-1)^{p-1}/\delta^{p-1}$, and the second inequality in (3.4) follows. □

Before giving the proof of Proposition 3.1 we show that it is equivalent to study Caccioppoli inequalities for quasisuperharmonic functions, a fact that we will actually use in the proof of Proposition 3.1.

Proposition 3.5. *Let γ, Q, $\widetilde{C}$, Ω and $0 \leqslant \eta \in \operatorname{Lip}_c(\Omega)$ be fixed. Then the Caccioppoli inequality*

$$\int_\Omega u^{-p+\gamma} g_u^p \eta^p \, d\mu \leqslant \widetilde{C} \int_\Omega u^\gamma g_\eta^p \, d\mu \tag{3.5}$$

holds for all Q-quasisuperminimizers $u \geqslant 0$ in Ω if and only if it holds for all Q-quasisuperharmonic functions $u \geqslant 0$ in Ω.

Quasisuperharmonic (and also superharmonic) functions are in general too large to belong to $N^{1,p}_{\mathrm{loc}}(\Omega)$. However the gradient is naturally defined by $g_u = g_{\min\{u,k\}}$ on $\{x\colon u(x) < k\}$, for all $k = 1, 2, \ldots$, see e.g. p. 150 in Heinonen, Kilpeläinen and Martio [24] for the Euclidean case or Kinnunen and Martio [29] for the metric space case. Here it is important to know that a quasisuperharmonic function is infinite only on a set with zero measure. Kinnunen and Martio [28, Theorem 10.6] showed even more: that it is infinite only on a set of zero capacity.

Proof of Proposition 3.5. Assume first that (3.5) holds for all Q-quasisuperharmonic $u \geqslant 0$ and let u be a nonnegative Q-quasisuperminimizer. Then $u^* \geqslant 0$ is Q-quasisuperharmonic and $u^* = u$ a.e. Thus

$$\int_\Omega u^{-p+\gamma} g_u^p \eta^p \, d\mu = \int_\Omega (u^*)^{-p+\gamma} g_{u^*}^p \eta^p \, d\mu \leqslant \widetilde{C} \int_\Omega (u^*)^\gamma g_\eta^p \, d\mu = \widetilde{C} \int_\Omega u^\gamma g_\eta^p \, d\mu.$$

Conversely, let u be a nonnegative Q-quasisuperharmonic function. Then $u_k := \min\{u, k\}$, $k = 1, 2, \dots$, is a nonnegative Q-quasisuperminimizer. Moreover, $g_{u_k} = \chi_{\{u<k\}} g_u$ a.e. By monotone convergence we see that

$$\int_\Omega u^{-p+\gamma} g_u^p \eta^p \, d\mu = \lim_{k\to\infty} \int_\Omega u_k^{-p+\gamma} g_{u_k}^p \eta^p \, d\mu \leqslant \lim_{k\to\infty} \widetilde{C} \int_\Omega u_k^\gamma g_\eta^p \, d\mu = \widetilde{C} \int_\Omega u^\gamma g_\eta^p \, d\mu,$$

where we use the boundedness of $|u_k^\gamma - u^\gamma|$, g_η and $\operatorname{supp} \eta$ to establish the last equality if $\gamma < 0$. $\square$

Proof of Proposition 3.1. By Theorem 1.2, $u(x) = |x|^\alpha$ is Q-quasisuperharmonic in $\mathbf{B}$. Let $\delta := (p-n)/\alpha$ and $\eta(x) = \min\{(2-3|x|)_+, 1\}$. Then

$$\int_{\mathbf{B}} u^{-p+\delta} |\nabla u|^p \eta^p \, dx \geqslant C \int_0^{1/3} r^{\alpha(-p+\delta)} r^{(\alpha-1)p} r^{n-1} \, dr = C \int_0^{1/3} \frac{dr}{r} = \infty.$$

On the other hand,

$$\int_{\mathbf{B}} u^\delta |\nabla \eta|^p \, dx = C \int_{1/3}^{2/3} r^{\alpha\delta} r^{n-1} \, dr < \infty.$$

Thus the Caccioppoli inequality (3.5) does not hold for all Q-quasisuperharmonic functions with $\gamma = \delta$. In view of Proposition 3.5, this shows that $\gamma_0(Q, p) \leqslant \delta$. $\square$

Remark 3.6. If $p = n$, then by Theorem 7.4 in Björn and Björn [7], $u(x) = (-\log|x|)^\alpha$ is quasisuperharmonic in $\mathbf{B}$ for all $\alpha \geqslant 1$ and $Q = \alpha^n/(n\alpha - n + 1)$ is the best quasisuperminimizer constant. Arguments similar to those in the proofs of Propositions 3.1 and 3.3 then show that $\gamma_0(Q, n) \leqslant (n-1)/\alpha = \delta(Q, n)$. As in the proof of Corollary 3.4, one then obtains the estimates $Q^{1/(n-1)} < \alpha < (nQ)^{1/(n-1)}$ for $Q > 1$.

4. Weak Harnack inequalities and local integrability for quasisuperharmonic functions

(Weak) Harnack inequalities for quasi(super)minimizers in $\mathbf{R}^n$ were obtained by DiBenedetto and Trudinger [15, Corollaries 1–3]. In metric spaces they were given by Kinnunen and Shanmugalingam [30, Theorem 7.1 and Corollary 7.3]. Some necessary modifications of the proofs and results in [30] were provided in Section 10 in A. Björn and Marola [9].

Example 10.1 in [9] or Example 8.19 in Björn and Björn [8] also shows that (weak) Harnack inequalities (both for quasi(super)minimizers and for (super)minimizers) in metric spaces can only hold on balls B such that a sufficiently large blow-up of B (depending on X) lies in Ω. This is usually formulated as $c\lambda B \subset \Omega$, where λ is the dilation constant in the weak Poincaré inequality and c is an absolute constant, such as 5, 20 or 50, depending on the proof. In (weighted) $\mathbf{R}^n$ one can of course take $\lambda = 1$.

In this and the next section we will see how one can improve upon the blow-up constant in various Harnack and Caccioppoli inequalities. The price one has to pay is however that the conditions involve the quasiconvexity constant L, see below, instead of λ. It is therefore relevant to discuss the relationship between L and λ. We postpone this discussion to Section 6.

A metric space Y is *L-quasiconvex* if for all $x, y \in Y$, there is a curve $\gamma : [0, l_\gamma] \to Y$ with $\gamma(0) = x$ and $\gamma(l_\gamma) = y$, parameterized by arc length, such that $l_\gamma \leqslant L d(x, y)$. Under our assumptions, X is quasiconvex, see Section 6 for more details.

In this section, we formulate the (weak) Harnack inequality for quasi(super)minimizers as follows. The proof below also shows that the constant $2L$ can be replaced by $(1+\varepsilon)L$ for any $\varepsilon > 0$. By Kinnunen and Shanmugalingam [30] and A. Björn and Marola [9], similar inequalities hold with L replaced by λ and the requirement that a larger blow-up of the balls lies in Ω.

Proposition 4.1. *Assume that X is L-quasiconvex, $\Omega \subset X$ and $Q \geqslant 1$. Then there exist constants C_* and $s > 0$, depending only on Q, p, L, C_μ and the constants in the weak $(1, p)$-Poincaré inequality, such that for all balls B with $2LB \subset \Omega$,*

(a) *whenever $u>0$ is a Q-quasiminimizer in Ω, the following Harnack inequality holds*:

$$\operatorname*{ess\,sup}_B u \leqslant C_* \operatorname*{ess\,inf}_B u; \tag{4.1}$$

(b) *whenever $v>0$ is a Q-quasisuperminimizer in Ω, the following weak Harnack inequality holds*:

$$\biggl(\fint_B v^s\,d\mu\biggr)^{1/s} \leqslant C_* \operatorname*{ess\,inf}_B v. \tag{4.2}$$

As in Proposition 3.5, it can be shown that (4.2) holds also for quasisuperharmonic functions, in which case ess inf may be replaced by inf as quasisuperharmonic functions are lower semicontinuously regularized.

Proof of Proposition 4.1. By changing u and v on a set of capacity zero, if necessary, we can assume that u is continuous and v is lower semicontinuously regularized in Ω. It is thus possible to replace ess sup and ess inf by sup and inf. Let $B=B(x_0,r)$ and assume that $2LB\subset\Omega$. By Theorem 4.32 in Björn and Björn [8], X supports a weak $(1,p)$-Poincaré inequality with dilation constant $\lambda=L$. Thus, (4.1) and (4.2) hold on all balls B^* such that $cLB^*\subset\Omega$ for some fixed $c>1$. See e.g. Kinnunen and Shanmugalingam [30] and A. Björn and Marola [9].

(a) Let $\varepsilon>0$ be arbitrary and find $x\in B$ such that $u(x)\leqslant \inf_B u+\varepsilon$. By the L-quasiconvexity of X, there exists a curve $\gamma:[0,l_\gamma]\to X$, parameterized by arc length, such that $l_\gamma\leqslant Ld(x,x_0)<Lr$, $x_0=\gamma(0)$ and $x=\gamma(l_\gamma)$.

Let $x_j=\gamma(jr/c)$ and cover γ by the balls $B_j=B(x_j,r/c)$, $j=0,1,\dots,N$, with N being the largest integer such that $Nr/c\leqslant l_\gamma$. Then $B_j\cap B_{j+1}$ is nonempty for $j=0,1,\dots,N-1$. Note that $N<cL$ and $cLB_j\subset 2LB\subset\Omega$. Hence, the Harnack inequality for quasiminimizers holds on each B_j and we have for every $j=0,1,\dots,N-1$,

$$\inf_{B_j} u \leqslant \inf_{B_j\cap B_{j+1}} u \leqslant \sup_{B_{j+1}} u \leqslant C_0 \inf_{B_{j+1}} u.$$

Iterating this estimate, we obtain

$$\inf_{B_0} u \leqslant C_0^N \inf_{B_N} u \leqslant C_0^N u(x) \leqslant C_0^N\Bigl(\inf_B u+\varepsilon\Bigr)$$

and letting $\varepsilon\to 0$ yields

$$u(x_0)\leqslant C_0\inf_{B_0} u \leqslant C_0^{N+1}\inf_B u. \tag{4.3}$$

Similarly, choosing $y\in B$ such that $u(y)\geqslant \sup_B u-\varepsilon$ gives

$$u(x_0)\geqslant C_0^{-N-1}\sup_B u,$$

which together with (4.3) proves (4.1).

(b) First, there exist balls $B_1',\dots,B_m'$ with centres in B and radii $r/3c$ such that $B\subset\bigcup_{j=1}^m B_j'$ and the balls $\frac12 B_j'$ are pairwise disjoint. It follows from the doubling property of μ that the number m of these balls does not exceed a constant depending only on c and C_μ. In particular, the bound for m is independent of x_0 and r. Moreover, for all $j=1,2,\dots,m$, we have $3cLB_j'\subset 2LB\subset\Omega$ and $\mu(B)/C\leqslant\mu(B_j')\leqslant C\mu(B)$.

Let $B'=B(x',r/3c)$ be one of these balls and connect x' to x_0 by a curve of length at most $Ld(x',x_0)$. As in (a), choose $z\in B$ such that $u(z)\leqslant\inf_B v+\varepsilon$ and connect x_0 to z by a curve of length at most $Ld(x_0,z)$. Adding these two curves gives a connecting curve γ from x' to z of length less than $2Lr$ and such that $\gamma\subset LB$. Cover γ by balls B_j, $j=0,1,\dots,N\leqslant 6cL$, with radii $r/3c$ as in (a) and note that $3cLB_j\subset 2LB\subset\Omega$ and $B_{j+1}\subset 3B_j$ for each $j=0,1,\dots,N$. Hence, the weak Harnack inequality for quasisuperminimizers holds on each $3B_j$, implying that

$$\inf_{B_j} v \leqslant \biggl(\fint_{B_j} v^s\,d\mu\biggr)^{1/s} \leqslant C_\mu^2\biggl(\fint_{3B_j} v^s\,d\mu\biggr)^{1/s} \leqslant C_0\inf_{3B_j} v \leqslant C_0\inf_{B_{j+1}} v.$$

Iterating this estimate, we obtain, as $B'=B_0$,

$$\inf_{B'} v = \inf_{B_0} v \leqslant C_0^N \inf_{B_N} v \leqslant C_0^N v(z) \leqslant C_0^N \Big(\inf_B v + \varepsilon \Big)$$

and letting $\varepsilon \to 0$ yields

$$\int_{B'} v^s \, d\mu \leqslant \Big(C_0 \inf_{B'} v \Big)^s \mu(B') \leqslant \Big(C_0^{N+1} \inf_B v \Big)^s \mu(B').$$

Summing up over all balls $B' = B'_j$ covering B finishes the proof. □

A sharp version of the weak Harnack inequality, due to Trudinger [43], is as follows. Assume that $u \geqslant 0$ is a superminimizer in an open set $\Omega \subset \mathbf{R}^n$. If $0 < s < \varkappa(p-1)$, then for every ball B with $6B \subset \Omega$ we have

$$\left(\fint_B u^s \, dx \right)^{1/s} \leqslant C_s \operatorname*{ess\,inf}_{3B} u, \tag{4.4}$$

where $\varkappa = n/(n-p)$ if $1 < p < n$, and $\varkappa = \infty$ if $p \geqslant n$ in unweighted $\mathbf{R}^n$. In the metric space case, this is due to Kinnunen and Martio [29] and $\varkappa > 1$ is chosen so that X supports a weak $(\varkappa p, p)$-Poincaré inequality. The proof is strongly based on the fact that the Caccioppoli inequality (3.1) holds for all $\gamma < p - 1$ when $Q = 1$. The initial requirement on B in metric spaces is that $60\lambda B \subset \Omega$, see A. Björn and Marola [9, Section 10], or Björn and Björn [8] (the latter gives the condition $150\lambda B \subset \Omega$), but as in Proposition 4.1, it can be shown that if X is L-quasiconvex, then (4.4) holds for all balls B with $6LB \subset \Omega$ (or even $(3+\varepsilon)LB \subset \Omega$ for every fixed $\varepsilon > 0$). Example 10.1 in [9] or Example 8.19 in [8] shows that the dilation constant λ or the quasiconvexity constant L are needed in the weak Harnack inequality.

Arguing as in the proof of Proposition 3.5 it is easy to see that all nonnegative Q-quasisuperminimizers satisfy (4.4) if and only if all nonnegative Q-quasisuperharmonic functions satisfy the inequality with the same positive constants s and C_s. (As quasisuperharmonic functions are lower semicontinuously regularized it is also equivalent to replace ess inf by inf in the quasisuperharmonic case.)

Let $\zeta_0 = \zeta_0(Q, p)$ be the largest number (independent of X and Ω) such that for every positive $s < \varkappa\zeta_0$ there is a constant $C_s = C_s(Q, p, X, \Omega)$ such that (4.4) holds for all Q-quasisuperharmonic functions $u \geqslant 0$. We then have the following consequence of Theorem 1.2.

Proposition 4.2. *$\zeta_0(Q, p) \leqslant \delta(Q, p)$, where $\delta(Q, p)$ is as in* (3.2).

By (4.4) we know that $\zeta_0(1, p) = \delta(1, p) = p - 1$, and that this is valid also in metric spaces.

If we fix a metric space X (e.g. $\mathbf{R}^n$) and Q and p, then, by Proposition 4.1, there is some $s > 0$ such that the weak Harnack inequality (4.4) holds. By the Hölder inequality, it holds for all $0 < s' \leqslant s$ and hence $\zeta_0(Q, p, X) \geqslant s > 0$. The proof of the weak Harnack inequality shows that the exponent s only depends on p, Q, C_μ and the constants in the weak $(1, p)$-Poincaré inequality. Proposition 4.2 however suggests that the upper bound for s only depends on p and Q, and that the only dependence on X is through $\varkappa$. It therefore feels natural to make the following conjecture.

Conjecture 4.3. *Assume that X supports a weak $(\varkappa p, p)$-Poincaré inequality. Let $Q > 1$ and $0 < s < \varkappa\delta(Q, p)$. Then there is a constant C_s such that the weak Harnack inequality* (4.4) *holds for every Q-quasisuperharmonic function $u \geqslant 0$ in Ω and every ball B such that $6LB \subset \Omega$ or $c\lambda B \subset \Omega$.*

It is worth observing that X supports a weak $(\varkappa p, p)$-Poincaré inequality if and only if it supports a weak $(1, p)$-Poincaré inequality and there is a constant C such that for all balls $B = B(x, r) \subset X$ and $B' = B(x', r')$, with $x' \in B$ and $r' \leqslant r$, the estimate

$$\frac{\mu(B')}{\mu(B)} \geqslant C \left(\frac{r'}{r} \right)^\sigma$$

holds with $\sigma = \varkappa p/(\varkappa - 1)$, see Hajłasz and Koskela [22] and Franchi, Gutiérrez and Wheeden [17], or Björn and Björn [8].

Proof of Proposition 4.2. Let $s \geqslant \varkappa\delta(Q,p)$ and recall that $\varkappa = n/(n-p)$ in unweighted $\mathbf{R}^n$, $n > p$. By Theorem 1.2, $u(x) = |x|^\alpha$, with $\alpha = (p-n)/\delta(Q,p)$, is Q-quasisuperharmonic in $\mathbf{B} \subset \mathbf{R}^n$. Then, with $B = B(0, \frac{1}{6})$,

$$\int_B u^s\,dx = C\int_0^{1/6} r^{\alpha s} r^{n-1}\,dr = \infty,$$

as

$$\alpha s + n - 1 \leqslant \alpha\varkappa\delta(Q,p) + n - 1 = -1.$$

Thus, the left-hand side in (4.4) is infinite while the right-hand side is finite, showing that (4.4) does not hold for s. □

It is well known that the sharp weak Harnack inequality implies sharp local integrability results for superharmonic functions: if u is superharmonic in Ω, then $u \in L^s_{\mathrm{loc}}(\Omega)$ for $0 < s < \varkappa(p-1)$, where the range for s is sharp.

We immediately get that if u is Q-quasisuperharmonic in $\Omega \subset X$ then $u \in L^s_{\mathrm{loc}}(\Omega)$ for $0 < s < \varkappa\zeta_0(Q,p,X)$. Moreover, by the proof of Proposition 4.2, we see that if $X = \mathbf{R}^n$ and $n > p$, then there is a Q-quasisuperharmonic $u \notin L^{\varkappa\delta(Q,p)}_{\mathrm{loc}}(\Omega)$.

Conjecture 4.4. *Assume that X supports a weak $(\varkappa p, p)$-Poincaré inequality. Let $Q > 1$ and $0 < s < \varkappa\delta(Q,p)$. Then every Q-quasisuperharmonic function in Ω belongs to $L^s_{\mathrm{loc}}(\Omega)$.*

This conjecture follows directly from Conjecture 4.3. In fact Conjecture 4.3 follows from Conjecture 3.2; to show this one essentially needs to repeat the arguments in Kinnunen and Martio [29].

5. Logarithmic Caccioppoli inequality and BMO

The following proposition is the logarithmic Caccioppoli inequality for superminimizers which plays a crucial role in the proof of the (weak) Harnack inequality using the Moser method. In metric spaces, it was originally proved in Kinnunen and Martio [29] and follows easily from (3.1) with $\gamma = 0$ and a suitable choice of test function.

Proposition 5.1. *Assume that $u > 0$ is a superminimizer in Ω which is locally bounded away from* 0. *Then for every ball B with $2B \subset \Omega$ we have*

$$\fint_B g^p_{\log u}\,d\mu \leqslant \frac{C}{\operatorname{diam}(B)^p}. \tag{5.1}$$

Proposition 5.1 implies, together with a Poincaré inequality, that the logarithm of a positive superminimizer has bounded mean oscillation. This is needed in the Moser iteration to show the weak Harnack inequality for superminimizers.

We have not been able to prove the inequality (5.1) for quasisuperminimizers, and it is therefore not clear whether the Moser iteration runs for quasiminimizers. See however Lemma 5.8 below. In view of Proposition 3.1, the possible proof of (5.1) for quasisuperminimizers should be based on some other method than the proof for superminimizers. In any case, the constant C in the logarithmic Caccioppoli inequality for quasisuperminimizers would have to depend on Q and grow at least as $Q^{p/(p-1)}$, see Example 5.7 below. We know, however, that the logarithm of a positive quasisuperminimizer belongs both to BMO and to $N^{1,q}_{\mathrm{loc}}$ for all $q < p$, qualitatively, see Theorems 5.5 and 5.9 below.

It is also rather interesting to observe that if $u > 0$ is a quasiminimizer then (5.1) follows from the Caccioppoli inequality (3.1) for $\gamma < 0$ (recall that by Proposition 7.3 in A. Björn and Marola [9], it is true for all $\gamma < 0$) and from the Harnack inequality.

Proposition 5.2. *Assume that $u > 0$ is a Q-quasiminimizer in Ω. Then inequality* (5.1) *holds true for every ball $B = B(x,r)$ with $2B \subset \Omega$.*

The proof below shows that the factor 2 in the condition $2B \subset \Omega$ can be replaced by $1+\varepsilon$ for any $\varepsilon > 0$.

Proof of Proposition 5.2. Let $c > 1$ be such that the Harnack inequality for quasiminimizers holds on all balls B^* with $c\lambda B^* \subset \Omega$, where λ is the dilation constant from the weak $(1, p)$-Poincaré inequality, see the discussion at the beginning of Section 4.

As in the proof of Proposition 4.1, cover B by balls $B_1, \dots, B_m$ with centres in B and radii $r/c\lambda$, so that the number m of these balls does not exceed a constant depending only on c, λ and C_μ. Moreover, $\mu(B)/C \leqslant \mu(B_j) \leqslant C\mu(B)$ for $j = 1, 2, \dots, m$.

By construction, $c\lambda B_j \subset \Omega$, and hence, by the Harnack inequality for quasiminimizers, there is a constant C_* depending only on Q, C_μ and the constants in the weak $(1, p)$-Poincaré inequality, such that for all $j = 1, 2, \dots, m$,

$$\sup_{B_j} u \leqslant C_* \inf_{B_j} u.$$

Fix $j \in \{1, \dots, m\}$ for the moment. It follows that u has to be bounded away from 0 in B_j. Let $\eta \in \mathrm{Lip}_c(2B_j)$ so that $0 \leqslant \eta \leqslant 1$, $\eta = 1$ on B_j, and $g_\eta \leqslant 4/\mathrm{diam}(B_j)$. Let also $\gamma < 0$ and recall that (3.1) holds for γ. By the Harnack inequality, and the doubling property of μ, we have

$$\begin{aligned}\int_{B_j} g_{\log u}^p \, d\mu &= \int_{B_j} g_u^p u^{\gamma-p} \eta^p u^{-\gamma} \, d\mu \leqslant \Big(\sup_{B_j} u\Big)^{-\gamma} \int_{B_j} g_u^p u^{\gamma-p} \eta^p \, d\mu \\ &\leqslant C_\gamma \Big(C_* \inf_{B_j} u\Big)^{-\gamma} \int_{2B_j} g_\eta^p u^\gamma \, d\mu \leqslant C \int_{2B_j} g_\eta^p \, d\mu = \frac{C\mu(B_j)}{(r/c\lambda)^p}.\end{aligned}$$

Summing up over all B_j and using the fact that $\mu(B_j)$ is comparable to $\mu(B)$ for all $j = 1, 2, \dots, m$, gives the desired logarithmic Caccioppoli inequality on B. $\square$

We want to remark that in the same way, it can be shown that if $u > 0$ is a quasiminimizer, then u satisfies the Caccioppoli inequality (3.1) with arbitrary exponent γ.

Proposition 5.3. *Assume that X is L-quasiconvex and let $u > 0$ be a Q-quasiminimizer in Ω. Then the Caccioppoli inequality* (3.1) *holds for all $\gamma \in \mathbf{R}$ and all $\eta \in \mathrm{Lip}_c(\Omega)$ such that* $\operatorname{supp} \eta \subset B$ *for some ball $B \subset 2LB \subset \Omega$, with a constant independent of u, η and B.*

This follows easily from Proposition 4.1 and the Caccioppoli inequality for $\gamma < 0$, as in the proof of Proposition 5.2.

The blow-up constant $2L$ can be replaced by $c\lambda$, where λ is the dilation constant in the weak Poincaré inequality and $c > 1$ is such that the Harnack inequality (4.1) for quasiminimizers holds on all balls B^* with $c\lambda B^* \subset \Omega$, see the discussion at the beginning of Section 4. In (weighted) $\mathbf{R}^n$ one can clearly take $L = \lambda = 1$.

For $\gamma \geqslant p$, Proposition 5.3 also follows from the Caccioppoli inequality for quasisubminimizers established in A. Björn and Marola [9, Proposition 7.2] in the metric space setting. (For $\gamma = p$ this was earlier obtained by Tolksdorf [42, Theorem 1.4] in unweighted $\mathbf{R}^n$ and by A. Björn [2, Theorem 4.1] for metric spaces.) For subminimizers, i.e. when $Q = 1$, this also follows for $\gamma > p - 1$, by the Caccioppoli inequality in Marola [33, Lemma 4.1]. In all these cases one can allow for $\operatorname{supp} \eta \subset \Omega$, not only $\operatorname{supp} \eta \subset B$. We do not know whether this is possible in Proposition 5.3.

Before stating the main result of this section, we recall that a locally integrable function $f : \Omega \to \mathbf{R}$ belongs to $\mathrm{BMO}(\Omega)$ or $\mathrm{BMO}_{\tau\text{-loc}}(\Omega)$, $\tau \geqslant 1$, if there is a constant C' such that

$$\fint_B |f - f_B| \, d\mu \leqslant C' \tag{5.2}$$

for all balls $B \subset \Omega$ or $B \subset \tau B \subset \Omega$, respectively. The smallest bound C' for which this inequality is satisfied is said to be the BMO-norm (resp. $\mathrm{BMO}_{\tau\text{-loc}}$-norm) of f, and is denoted by $\|f\|_{\mathrm{BMO}(\Omega)}$ (resp. $\|f\|_{\mathrm{BMO}_{\tau\text{-loc}}(\Omega)}$).

A locally integrable function $w > 0$ is an A_1-*weight* in Ω, if there is a constant A such that

$$\fint_B w \, dx \leqslant A \operatorname*{ess\,inf}_B w$$

for all balls $B \subset \Omega$. The least bound A is called the A_1-*constant* of w. An essential feature of A_1-weights is that the average oscillation of its magnitude on every ball is uniformly controlled. A precise version of this is the following theorem which can be found in García-Cuerva and Rubio de Francia [18, Theorem 3.3, p. 157] for unweighted $\mathbf{R}^n$. The proof therein holds true also in the metric setting. We refer also to [18] and Duoandikoetxea [16] for more properties of A_1-weights and BMO.

Theorem 5.4. *If w is an A_1-weight in Ω, then $\log w \in \mathrm{BMO}(\Omega)$ with a norm depending only on the A_1-constant for w.*

Theorem 5.5. *Assume that X is L-quasiconvex. Let $u > 0$ be a Q-quasisuperminimizer or a Q-quasisuperharmonic function in Ω. Then $\log u \in \mathrm{BMO}_{\tau\text{-loc}}(\Omega)$ with $\tau = 4L$. Moreover*

$$\|\log u\|_{\mathrm{BMO}_{\tau\text{-loc}}(\Omega)} < C',$$

where C' only depends on Q, p, C_μ and the constants in the weak $(1, p)$-Poincaré inequality.

The blow-up constant $\tau = 4L$ can be replaced by $\tau = 2c\lambda$, where λ is the dilation constant in the weak Poincaré inequality and $c > 1$ is such that the weak Harnack inequality (4.2) for quasisuperminimizers holds on all balls B^* with $c\lambda B^* \subset \Omega$, see the discussion at the beginning of Section 4.

Proof of Theorem 5.5. Let $B \subset \Omega$ be a ball in Ω such that $4LB \subset \Omega$. By Proposition 4.1 (or its modification for quasisuperharmonic functions), there is $s > 0$ such that

$$\fint_{B'} u^s\,dx \leqslant C_* \operatorname*{ess\,inf}_{B'} u^s \tag{5.3}$$

for all balls $B' \subset 2LB' \subset \Omega$, and in particular for all subballs B' of B. (Observe that if $B(x', r') \subset B(x, r) \neq X$, then it can happen that $r < r' \leqslant 2r$ but it is impossible to have $r' > 2r$.) Hence, u^s is an A_1-weight in B with A_1-constant C_*.

Thus, Theorem 5.4 implies that $s \log u = \log u^s \in \mathrm{BMO}(B)$, with BMO-constant only depending on C_*. As this holds, with the same constant, for all balls B such that $4LB \subset \Omega$, we find that $\log u \in \mathrm{BMO}_{\tau\text{-loc}}(\Omega)$, with the $\mathrm{BMO}_{\tau\text{-loc}}$-norm only depending on C_* and s, which in turn only depend on Q, p, L, C_μ and the constants in the weak $(1, p)$-Poincaré inequality. □

Note that in (weighted) $\mathbf{R}^n$, $\mathrm{BMO}(\Omega) = \mathrm{BMO}_{\tau\text{-loc}}(\Omega)$ (with comparable norms), so that we can take $\tau = 1$ in this case. This follows from the proof of Hilfssatz 2 in Reimann and Rychener [39, pp. 4, 13–17] for unweighted $\mathbf{R}^n$. This is also true for length metric spaces X (i.e. with $L = 1 + \varepsilon$ for every $\varepsilon > 1$), see Maasalo [32, Theorem 2.2]. For general metric spaces, however, the following example shows that τ is essential in Theorem 5.5, that for $L \geqslant 18$ one cannot take $\tau \leqslant \frac{1}{9}L$, and that $\tau(L) = 4L$ in Theorem 5.5 has the right growth as $L \to \infty$. Moreover, this is so even for $Q = 1$.

Example 5.6. Let $0 < \theta \leqslant \frac{1}{6}$ and

$$X = \mathbf{R}^2 \setminus \{(x, y)\colon\ 0 < x < 1 \text{ and } 0 < y < \theta x\}.$$

By J. Björn and Shanmugalingam [13, Theorem 4.4], X supports a weak $(1, 1)$-Poincaré inequality. Also, X is L-quasiconvex with $L = (1 + \sqrt{1 + \theta^2})/\theta < 3/\theta$.

The balls $B_j = B((2^{-j}, 0), 2^{-j}/3)$, $j = 1, 2, \ldots,$ are pairwise disjoint. Let now

$$G = (0, 1) \times (-1, 0], \qquad \Omega = G \cup \bigcup_{j=1}^{\infty} B_j, \qquad u(x, y) = \begin{cases} 1, & \text{in } G, \\ j, & \text{in } B_j \setminus G,\ j = 1, 2, \ldots. \end{cases}$$

As u is constant in every component of Ω it is a quasiminimizer. Assume that $\tau < 1/3\theta$ and let $B'_j = (1/\tau)B_j$, $j = 1, 2, \ldots,$ and $v = \log u$. Then

$$\fint_{B'_j} |v - v_{B'_j}|\, d\mu \to \infty, \quad \text{as } j \to \infty,$$

showing that $v \notin \mathrm{BMO}_{\tau\text{-loc}}(\Omega)$ for $\tau < 1/3\theta$.

Moreover, we have the following example.

Example 5.7. Let $p < n$ and $\alpha \leqslant (p-n)/(p-1) = \beta$. Let $k > 0$ and $u(x) = \min\{|x|^\beta, k\}$. By Theorem 1.2, the function $|x|^\alpha$ is Q-quasisuperharmonic in $\mathbf{B} \subset \mathbf{R}^n$ with the best quasisuperminimizer constant Q depending on α. For every $k > 0$, the function $v = u^{\alpha/\beta} = \min\{|x|^\alpha, k^{\alpha/\beta}\}$ belongs to $W^{1,p}(\mathbf{B})$ and is thus a Q-quasisuperminimizer in $\mathbf{B}$. We then have

$$\log v = \begin{cases} \frac{\alpha}{\beta}\log k, & \text{if } |x| \leqslant k^{1/\beta}, \\ \alpha \log|x|, & \text{otherwise,} \end{cases} \quad \text{and} \quad |\nabla \log v| = \begin{cases} 0, & \text{if } |x| \leqslant k^{1/\beta}, \\ \alpha/|x|, & \text{otherwise.} \end{cases}$$

It follows that for $r > 2k^{1/\beta}$,

$$\begin{aligned} \fint_{B(0,r)} |\nabla \log v|^p\, dx &= \frac{C}{r^n} \int_{k^{1/\beta}}^{r} \left(\frac{|\alpha|}{\rho}\right)^p \rho^{n-1}\, d\rho \\ &= \frac{C|\alpha|^p}{r^n}\left(r^{n-p} - k^{(n-p)/\beta}\right) \geqslant \frac{C|\alpha|^p}{r^p}, \end{aligned}$$

where the constant C depends only on n and p. Corollary 3.4 shows that $|\alpha|$ is comparable to $Q^{1/(p-1)}$. Hence, the constant in the logarithmic Caccioppoli inequality for quasisuperminimizers, if it holds true, must depend on Q and grow at least as $Q^{p/(p-1)}$.

At the same time, u is a superminimizer in $\mathbf{B}$ and thus $\log u \in \mathrm{BMO}(\mathbf{B})$. Note that $\log v = (\alpha/\beta)\log u$ and hence $\|\log v\|_{\mathrm{BMO}(\mathbf{B})} = (\alpha/\beta)\|\log u\|_{\mathrm{BMO}(\mathbf{B})}$. As $|\alpha|$ is comparable to $Q^{1/(p-1)}$, by Corollary 3.4, this shows that the constant C' in Theorem 5.5 must depend on Q and grow at least as $Q^{1/(p-1)}$.

We finish this section by showing that the logarithm of a positive quasisuperminimizer belongs to $N^{1,q}_{\mathrm{loc}}$ for every $q < p$. For superminimizers, this is known even for $q = p$, and follows directly from Proposition 5.1. We start by proving a weak version of Proposition 5.1 for functions satisfying a Caccioppoli inequality.

Lemma 5.8. *Let $B = B(x,r)$ and assume that $u \geqslant 0$ satisfies the Caccioppoli inequality*

$$\int_B u^{-p+\gamma} g_u^p\, d\mu \leqslant \frac{C_\gamma}{r^p} \int_{2B} u^\gamma\, d\mu \tag{5.4}$$

for all $\gamma < 0$. Assume moreover that for some $\sigma > 0$,

$$\fint_{2B} u^\sigma\, d\mu \fint_{2B} u^{-\sigma}\, d\mu \leqslant C_0. \tag{5.5}$$

Then for all $q < p$,

$$\fint_B \frac{g_u^q}{u^q}\, d\mu \leqslant \frac{C_\mu C_0^{1-q/p} C_\gamma^{q/p}}{r^q},$$

where $\gamma = -\sigma(p-q)/q$ and C_μ is the doubling constant of μ.

Proof. We can assume that $q > p/2$. By the Hölder inequality and the doubling property of μ, we have for all $\varepsilon > 0$,

$$\fint_B \frac{g_u^q}{u^q}\, d\mu = \fint_B u^{-q-\varepsilon} g_u^q u^\varepsilon\, d\mu \leqslant \left(\fint_B u^{(-q-\varepsilon)p/q} g_u^p\, d\mu\right)^{q/p} \left(C_\mu \fint_{2B} u^{\varepsilon p/(p-q)}\, d\mu\right)^{1-q/p}. \tag{5.6}$$

The Caccioppoli inequality (5.4) with $\gamma = -\varepsilon p/q$ and the Hölder inequality show that the first integral on the right-hand side can be estimated by

$$\left(\frac{C_\gamma C_\mu}{r^p} \fint_{2B} u^{-\varepsilon p/q}\,d\mu\right)^{q/p} \leqslant \frac{(C_\gamma C_\mu)^{q/p}}{r^q}\left(\fint_{2B} u^{-\varepsilon p/(p-q)}\,d\mu\right)^{1-q/p}. \tag{5.7}$$

Here we have used the assumption that $q > p/2$, i.e. $q > p - q$. Choosing $\varepsilon = \sigma(p-q)/p$ and applying (5.5) to (5.6) and (5.7) yields

$$\begin{aligned}\fint_B \frac{g_u^q}{u^q}\,d\mu &\leqslant \frac{(C_\gamma C_\mu)^{q/p}}{r^q}\left(\fint_{2B} u^{-\varepsilon p/(p-q)}\,d\mu\right)^{1-q/p}\left(C_\mu \fint_{2B} u^{\varepsilon p/(p-q)}\,d\mu\right)^{1-q/p}\\ &\leqslant \frac{C_\mu C_0^{1-q/p} C_\gamma^{q/p}}{r^q},\end{aligned}$$

where $\gamma = -\varepsilon p/q = -\sigma(p-q)/q$. □

Theorem 5.9. *Let $u > 0$ be a Q-quasisuperminimizer in Ω. Then $\log u \in N^{1,q}_{\mathrm{loc}}(\Omega)$ for every $q < p$.*

Proof. By Theorem 5.5, $v := \log u \in \mathrm{BMO}_{\tau\text{-loc}}(\Omega)$ with $\tau = 4L$, where L is the quasiconvexity constant of X. Moreover, the $\mathrm{BMO}_{\tau\text{-loc}}(\Omega)$-norm of v depends only on Q, p, L, C_μ and the constants in the weak $(1, p)$-Poincaré inequality. In particular, for every ball B with $8LB \subset \Omega$, $\|v\|_{\mathrm{BMO}(2B)} \leqslant C'$, where C' is independent of B and v. Let $\sigma := 1/6C_\mu C'$. Theorem 9.1 in Björn and Marola [9] then implies that

$$\fint_B e^{\sigma|v-v_B|}\,d\mu \leqslant 16.$$

It follows that

$$\begin{aligned}\fint_B u^{-\sigma}\,d\mu \fint_B u^{\sigma}\,d\mu &= \fint_B e^{-\sigma v}\,d\mu \fint_B e^{\sigma v}\,d\mu = \fint_B e^{\sigma(v_B - v)}\,d\mu \fint_B e^{\sigma(v-v_B)}\,d\mu\\ &\leqslant \left(\fint_B e^{\sigma|v-v_B|}\,d\mu\right)^2 \leqslant 256.\end{aligned} \tag{5.8}$$

Choosing $0 \leqslant \eta \in \mathrm{Lip}_c(B)$ in (3.1) with $\eta = 1$ on $\frac{1}{2}B$ and $g_\eta \leqslant 4/\mathrm{diam}(B)$ yields

$$\int_{\frac{1}{2}B} u^{-p+\gamma} g_u^p\,d\mu \leqslant \frac{C_\gamma}{\mathrm{diam}(B)^p}\int_B u^\gamma\,d\mu$$

for all $\gamma < 0$. Thus, (5.8) and Lemma 5.8 then imply that for all $q < p$,

$$\int_{\frac{1}{2}B} g_{\log u}^q\,d\mu = \int_{\frac{1}{2}B} \frac{g_u^q}{u^q}\,d\mu \leqslant \frac{C\mu(B)}{\mathrm{diam}(B)^q}.$$

Finally, as $\operatorname{ess\,inf}_{\frac{1}{2}B} u > 0$, by the weak Harnack inequality, and $u \in L^p(\frac{1}{2}B)$, we easily obtain that $\log u \in L^q(\frac{1}{2}B)$ for all $q < p$.

As every p-weak upper gradient is also a q-weak upper gradient for $q < p$, it follows that $\log u \in N^{1,q}_{\mathrm{loc}}(\Omega)$. (See Björn and Björn [8, Section 2.9] for more on the relation between p-weak and q-weak upper gradients.) □

6. Quasiconvexity and the blow-up in the Poincaré inequality

In Sections 4 and 5 we saw how one can improve upon the blow-up constant in various Harnack and Caccioppoli inequalities by replacing λ with the quasiconvexity constant L.

Under our assumptions, X is L-quasiconvex. This was proved by Semmes, see Cheeger [14, Theorem 17.1]. It follows that X supports a weak $(1, p)$-Poincaré inequality with dilation constant $\lambda = L$, see e.g. Björn and Björn [8]. For geodesic spaces ($L = 1$), the validity of a strong $(1, p)$-Poincaré inequality with dilation constant $\lambda = 1$ was proved already in Hajłasz and Koskela [21] (see also Heinonen [23, pp. 30–31]). Even before that, in the setting of vector fields on $\mathbf{R}^n$, Jerison [26] showed that a weak Poincaré inequality (with $\lambda = 2$) self-improves to a strong Poincaré inequality (with $\lambda = 1$).

The quasiconvexity constant L has a clear advantage of being very geometrical. At the same time, it is not well understood when a space supports a Poincaré inequality, and it is not easy to determine the optimal dilation constant λ, nor even to determine when one can have $\lambda < L$.

Let us show that λ can both be much smaller than L in some situations, but can also be quite close to L even for arbitrarily large L in other situations.

We will need the *inner metric* $d'(x, y)$ on X which is defined as the length of the shortest curve in X connecting x and y. Let also diam$'$ denote diameters taken with respect to the inner metric d'.

Example 6.1. Let $0 < \alpha < \frac{1}{2}\pi$ and let X consist of two rays with opening α, i.e.

$$X = [0, \infty) \cup \{te^{i\alpha}: t \geqslant 0\} \subset \mathbf{C} = \mathbf{R}^2,$$

equipped with the induced distance from $\mathbf{R}^2$ and the one-dimensional Lebesgue measure μ, which is doubling on X.

We want to show that X supports a weak $(1, 1)$-Poincaré inequality. Let $B = B(x, r)$ be arbitrary, where we may assume that $x \in \mathbf{R}$, without loss of generality. We will use that $\mathbf{R}$ supports a strong $(1, 1)$-Poincaré inequality, i.e. with dilation $\lambda = 1$. For X, let $\lambda = 1/\sin\alpha$. Let further f be integrable and g be an upper gradient of f on X.

If $r \leqslant x \sin\alpha$, then $B \subset \mathbf{R}$, and we get that

$$\fint_B |f - f_B| \, d\mu \leqslant C \operatorname{diam}(B) \fint_B g \, d\mu \leqslant C \operatorname{diam}(B) \fint_{\lambda B} g \, d\mu,$$

by the strong Poincaré inequality on $\mathbf{R}$ and the doubling property of μ.

If $x \sin\alpha < r \leqslant x$, then B is not connected, showing that we cannot have a strong Poincaré inequality on X. However λB is connected, and using that λB, equipped with the inner metric d', is isomorphic to an interval on $\mathbf{R}$ we find that

$$\begin{aligned}\fint_B |f - f_B| \, d\mu &\leqslant \fint_B |f - f_{\lambda B}| \, d\mu \leqslant C \fint_{\lambda B} |f - f_{\lambda B}| \, d\mu \\ &\leqslant C \operatorname{diam}'(\lambda B) \fint_{\lambda B} g \, d\mu \leqslant C \operatorname{diam}(B) \fint_{\lambda B} g \, d\mu,\end{aligned}$$

by the strong Poincaré inequality on $\mathbf{R}$ and the doubling property of μ. The constant $\lambda = 1/\sin\alpha$ is the smallest possible always making λB connected, showing that we cannot have a Poincaré inequality with any dilation $\lambda' < \lambda$: If $1 \leqslant \lambda' < \lambda$, then we can find B such that B and $\overline{\lambda' B}$ are both disconnected. Let then f be a Lipschitz function such that $f|_{\lambda B} = \chi_{\mathbf{R}}|_{\lambda B}$, yielding

$$\fint_B |f - f_B| \, d\mu > 0 = \fint_{\lambda' B} g \, d\mu.$$

Thus we cannot have a Poincaré inequality with dilation λ'.

Finally if $r > x$, then B is connected and we get, using that B, equipped with the inner metric d', is isomorphic to an interval on $\mathbf{R}$, that

$$\fint_B |f - f_B| \, d\mu \leqslant C \operatorname{diam}'(B) \fint_B g \, d\mu \leqslant C \operatorname{diam}(B) \fint_{\lambda B} g \, d\mu.$$

A straightforward calculation, or a symmetry argument, shows that

$$L = \frac{1}{\sin(\alpha/2)} \leqslant \frac{2}{\sin\alpha} \leqslant 2\lambda.$$

This shows that L can be strictly larger than λ, but also that it is possible to have arbitrarily large λ, while $\lambda < L \leqslant 2\lambda$.

Example 6.2. In this example we consider the von Koch snowflake curve, which is a famous example of a curve of infinite length containing no rectifiable curves, and thus not supporting a Poincaré inequality. For our discussion, it is not the von Koch snowflake curve itself that is useful, but the sets generating it.

Let $K_0 \subset \mathbf{R}^2$, the 0th generation, be an equilateral triangle with side length 1. For each of the three sides split it into three intervals of equal lengths and replace the middle one I by two sides I' and I'' of an equilateral triangle (with sides I, I' and I'') outside K_0. We have thus produced the 1st generation K_1 of the von Koch snowflake curve consisting of 12 pieces of length $\frac{1}{3}$ each.

Continuing in this way we obtain the nth generation K_n consisting of $3 \cdot 4^n$ pieces, each of length 3^{-n}. Let also E_n be the set of the end points of the pieces forming K_n.

Now let $X = K_n$ for some fixed integer n, equipped with the induced distance from $\mathbf{R}^2$ and the one-dimensional Lebesgue measure μ, which is doubling on X.

As in the previous example we will use that $\mathbf{R}$ supports a strong Poincaré inequality. Let f be integrable and g be an upper gradient of f on X. Let further $B = B(x,r)$ and find j such that $3^{-j-1} < r \leqslant 3^{-j}$.

Assume first that $1 \leqslant j-1 \leqslant n$. Then we can find $y \in E_j \setminus E_{j-1}$ such that $d(x,y) \leqslant 3^{-j}$. Let I be the piece containing y in the $(j-1)$th generation, and let I' and I'' be its two neighbors in the $(j-1)$th generation. Let further E be the union of all pieces in K_n stemming from any of these three pieces. (Hence E is the union of $3 \cdot 4^{n-j+1}$ pieces.) Then it is relatively easy to see that

$$B = B(x,r) \subset B\big(y, 2 \cdot 3^{-j}\big) \subset E \subset B\big(y, 5 \cdot 3^{-j}\big) \subset B(y, 15r) \subset B(x, 18r).$$

Let thus $\lambda = 18$.

As E is connected, and isomorphic to an interval on $\mathbf{R}$, we see that

$$\begin{aligned} \fint_B |f - f_B| \, d\mu &\leqslant \fint_B |f - f_E| \, d\mu \leqslant C \fint_E |f - f_E| \, d\mu \\ &\leqslant C \operatorname{diam}'(E) \fint_E g \, d\mu \leqslant C \operatorname{diam}(B) \fint_{\lambda B} g \, d\mu. \end{aligned}$$

In the cases when $j-1 < 1$ and when $j-1 > n$ this is easier to obtain, and thus we have shown that X supports a weak $(1,1)$-Poincaré inequality with $\lambda = 18$.

Observe that λ is independent of n. However, $C \to \infty$, as $n \to \infty$. It is also easy to see that $L \to \infty$ as $n \to \infty$, thus showing that λ can be much much smaller than L.

Acknowledgements

We would like to thank Juha Kinnunen for valuable comments and suggestions.

The first two authors were supported by the Swedish Research Council, and belong to the European Science Foundation Networking Programme *Harmonic and Complex Analysis and Applications* and to the Scandinavian Research Network *Analysis and Application*.

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