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# Multiple homoclinic solutions for singular differential equations ☆

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**Abstract**

The homoclinic bifurcations of ordinary differential equation under singular perturbations are considered. We use exponential dichotomy, Fredholm alternative and scales of Banach spaces to obtain various bifurcation manifolds with finite codimension in an appropriate infinite-dimensional space. When the perturbative term is taken from these bifurcation manifolds, the perturbed system has various coexistence of homoclinic solutions which are linearly independent.


*Keywords:* Degenerate homoclinic solution; Lyapunov–Schmidt reduction; Singular perturbations; Codimension

## 1. Introduction

The homoclinic bifurcations are important topics in dynamical systems since it is related to variously complicated dynamical behaviors, such as chaos. In recent decades, many authors studied this problem [1–16]. In 1980, S.N. Chow, J.K. Hale and J. Mallet-Parret [7] introduced Lyapunov–Schmidt reduction to investigate the persistence of homoclinic orbit for Duffing's equation under damping and excitation in $\mathbb{R}^2$. Under the assumptions that the unperturbed system had an orbit which homoclinic to a hyperbolic equilibrium, and the dimension of the intersection of the stable and unstable manifolds of the equilibrium was one, K.J. Palmer [14] extended this method to $\mathbb{R}^n$.

In 1984, J.K. Hale [12] suggested that this method could be extended to more general case where the perturbative terms were multiple parameters and the dimension of the intersection of the stable and unstable manifolds was greater than one. For regular perturbations, J. Knobloch, U. Schalk and A. Vanderbauwhede [13,15,16] investigated the case where the dimension of the intersection was two. Later, J. Gruendler, F. Battelli and C. Lazzari [1,9,10] gave the general theory for the case where the intersection of the dimension was arbitrary. F. Battelli and C. Lazzari [1] studied the persistence of degenerate heteroclinic orbit for ordinary differential equations with nonautonomous perturbations. J. Gruendler [9,10] considered the persistence of homoclinic solutions under autonomous and nonautonomous perturbations. For arbitrary dimension of intersection, the homoclinic bifurcations were also investigated in [18].

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Meantime, the homoclinic bifurcations under singular perturbations were rising a lot of interest [2–4,8,11]. In [11], J. Gruendler considered the following singular system

$$\epsilon\dot{x} = f_0(x) + \epsilon f_1(x, \epsilon, t)$$

where $x \in \mathbb{R}^n$, $\epsilon \in \mathbb{R}$ and $f_1$ is periodic in $t$. In [3], F. Battelli and K.J. Palmer also investigated this system in $\mathbb{R}^2$ but with the coefficient of $\epsilon^2$ for $f_1$. The general theory for the arbitrary dimension of the intersection was developed in [11]. By using Lyapunov–Schmidt reduction, he obtained a bifurcation function $H : \mathbb{R}^{d-1} \times \mathbb{R}^1 \times \mathbb{R}^1 \to \mathbb{R}^d$, where $d$ was the dimension of the intersection of the stable and unstable manifolds. The zeros of $H(\beta, \epsilon, \alpha) = 0$ were one-to-one correspondence to the existence of transversal homoclinic solutions for the perturbed system. He gave some applicable conditions on the bifurcation function.

Motivated by these works, we will consider the system

$$\epsilon\dot{x}(t) = f\big(x(t)\big) + g\big(x(t), \epsilon, t\big) \tag{1.1}$$

where $\epsilon \in \mathbb{R}$, $x \in \mathbb{R}^n$ and $\|g\|_{C^3}$ is small. Let $t \leftrightarrow t/\epsilon$, Eq. (1.1) is equivalent to

$$\dot{x}(t) = f\big(x(t)\big) + g\big(x(t), \epsilon, \epsilon t\big). \tag{1.2}$$

Different from the regular perturbations, there are some difficulties when one considers the problem in a usual Banach space. Let $X$ and $Y$ be certain Banach spaces. If we convert (1.2) into integral equation $F(x, g, \epsilon) = 0$ where $F : X \times C^3 \times \mathbb{R} \to Y$. There is a difficulty to solve $F(x, g, \epsilon) = 0$ since the map $F(x, g, \epsilon)$ is not differentiable in $\epsilon$. Different choices of Banach spaces also lead to the similar problem. In [3,11], they dealt with this difficulty by using a scale of Banach spaces. This idea was introduced by A. Vanderbauwhede and S.A. Van Gils in [17].

In the present paper, we consider Eq. (1.1) and its equivalent form (1.2). Let $X$ and $Y$ be Banach spaces and $\Omega \in X$ be an open set. Let $C^k(\Omega, Y)$ denote all functions $f : \Omega \to Y$ with continuous derivatives up to order $k$. The space $C^k(\Omega, Y)$ is Banach space with norm $\|f\|_{C^k} = \sup_{x\in\Omega} \sum_{i=0}^{k} |D^i f(x)|$. We make some assumptions.

(H1) $f$ and $g$ are $C^3$ in all their variables.
(H2) $f(0) = 0$ and the eigenvalues of $Df(0)$ lie off the imaginary axis.
(H3) The unperturbed system

$$\dot{x}(t) = f\big(x(t)\big) \tag{1.3}$$

has a homoclinic orbit $\gamma(t)$. That is, there is differentiable function $\gamma(t)$ such that $\dot{\gamma}(t) = f(\gamma(t))$ and $\lim_{t\to\pm\infty} \gamma(t) = 0$.
(H4) $g(0, \epsilon, t) = 0$ and $\|g\|_{C^3}$ is small.

In this paper, both scalar $\epsilon$ and function $g$ are treated as parameters. We will investigate various choices of $g \in C^3(\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}, \mathbb{R}^n)$ and give conditions on $g$ to obtain the various situations of linearly independent multiple homoclinic solutions for system (1.2). These conditions determine some bifurcation submanifolds, containing zero, with finite codimension in $C^3(\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}, \mathbb{R}^n)$. When $g$ is chosen from different submanifold, the system (1.2) has different number linearly independent homoclinic solutions.

## 2. Preliminary and main results

If $h : X_1 \times \cdots \times X_m \to Y$ where $X_1, \ldots, X_m, Y$ are Banach spaces, let $D_i h$ denote the first partial derivative with respect to the $i$-th variable, $D_{ij} h$ denote the second partial derivative with respect to the $i$-th and $j$-th variables. If there is only one variable, we often omit the subscript. From (H2), we know that $x = 0$ is hyperbolic equilibrium of (1.3). Let $W^s$ and $W^u$ be the stable and unstable manifolds of the origin. From (H3) we see that Eq. (1.3) has a homoclinic orbit $\gamma(t)$. It is clear that $\gamma \in W^s \cap W^u$. Let $d = T_{\gamma(0)} W^s \cap T_{\gamma(0)} W^u$.

For $c \in \mathbb{C}$, let $\mathrm{Re}(c)$ denote the real part of $c$. Since the eigenvalue of $Df(0)$ lies off the imaginary axis, we can choose a constant $\alpha > 0$ such that $|\mathrm{Re}(\lambda_i)| > 3\alpha$ where $\lambda_i$ is the eigenvalue of $Df(0)$, $i = 1, \ldots, n$.

The linearly variational equation of (1.3) along $\gamma(t)$ is

$$\dot{u}(t) = Df\big(\gamma(t)\big)u(t). \tag{2.1}$$

The following lemma is Theorem 2 in [10] with some changes of notations.

**Lemma 2.1.** *There are fundamental matrix solution, $U$, for* (2.1)*, constants $K>0$ and projections $P_{ss}$, $P_{su}$, $P_{us}$ and $P_{uu}$ such that $P_{ss}+P_{su}+P_{us}+P_{uu}=I$ and the following hold*:

(a) $|U(t)(P_{ss}+P_{su})U^{-1}(s)| \leqslant Ke^{2\alpha(t-s)}$, *for* $t \leqslant s \leqslant 0$,
(b) $|U(t)(P_{uu}+P_{us})U^{-1}(s)| \leqslant Ke^{2\alpha(s-t)}$, *for* $s \leqslant t \leqslant 0$,
(c) $|U(t)(P_{ss}+P_{us})U^{-1}(s)| \leqslant Ke^{2\alpha(s-t)}$, *for* $0 \leqslant s \leqslant t$,
(d) $|U(t)(P_{uu}+P_{su})U^{-1}(s)| \leqslant Ke^{2\alpha(t-s)}$, *for* $0 \leqslant t \leqslant s$.

*Furthermore,* $\mathrm{Rank}(P_{ss}) = \mathrm{Rank}(P_{uu}) = d$.

Let $u_0 \in \mathbb{R}^n$. We consider the solution of (2.1) with initial condition $u(0)=u_0$. It is that $u(t,u_0)=U(t)U^{-1}(0)u_0$ with $u(0,u_0)=u_0$. From Lemma 2.1, we have the following observations:

$$\begin{cases} \text{For } u_0 \in P_{ss}\mathbb{R}^n, & |u(t,u_0)|e^{2\alpha|t|} \to 0, & \text{as } t \to \pm\infty, \\ \text{For } u_0 \in P_{su}\mathbb{R}^n, & |u(t,u_0)|e^{-2\alpha t} \to 0(\infty), & \text{as } t \to -\infty(\infty), \\ \text{For } u_0 \in P_{us}\mathbb{R}^n, & |u(t,u_0)|e^{2\alpha t} \to \infty(0), & \text{as } t \to -\infty(\infty), \\ \text{For } u_0 \in P_{uu}\mathbb{R}^n, & |u(t,u_0)|e^{-2\alpha|t|} \to \infty, & \text{as } t \to \pm\infty. \end{cases} \tag{2.2}$$

Renumbering if necessary, we can assume that

$$P_{uu} = \begin{pmatrix} I_d & 0 & 0 \\ 0 & 0_d & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad P_{ss} = \begin{pmatrix} 0_d & 0 & 0 \\ 0 & I_d & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where $I_d$ and $0_d$ are $d \times d$ identity and zero matrix respectively.

Let $u_j$ denote the $j$-th column of the fundamental solution $U$ defined in Lemma 2.1. From the observations (2.2), we have

$$\lim_{t\to\pm\infty} |u_i(t)|e^{-2\alpha|t|} = \infty, \quad i=1,\dots,d,$$
$$\lim_{t\to\pm\infty} |u_{d+i}(t)|e^{2\alpha|t|} = 0, \quad i=1,\dots,d.$$

For any $i, i=1,\dots,n$, we define $u_i^\perp$ by $\langle u_i^\perp, u_j\rangle = \delta_{ij}$, $j=1,\dots,n$, where $\delta_{ij}$ is the Kronecker delta. The vector functions $u_i^\perp$ can be obtained as following. Let $U^\perp$ be matrix with $u_i^\perp$ in the $i$-th column. We can get that ${U^\perp}^T U = I$ where $T$ denote the transpose. Through differentiating, we have $\dot{U^\perp}^T U + {U^\perp}^T \dot{U} = 0$. Thus $\dot{U}^\perp = -({U^\perp}^T \dot{U} U^{-1})^T = -Df(\gamma)^T U^\perp$. Thus $U^\perp$ is the fundamental matrix solution for the adjoint equation of (2.1). Obviously, $U^{-1} = {U^\perp}^T$, $\{u_1^\perp,\dots,u_d^\perp,0,\dots,0\} = P_{uu}U^{-1}$ and $\{0,\dots,0,u_{d+1}^\perp,\dots,u_{2d}^\perp,0,\dots,0\} = P_{ss}U^{-1}$. It is clear that

$$\lim_{t\to\pm\infty} |u_i^\perp(t)|e^{2\alpha|t|} = 0, \quad i=1,\dots,d,$$
$$\lim_{t\to\pm\infty} |u_{d+i}^\perp(t)|e^{-2\alpha|t|} = \infty, \quad i=1,\dots,d.$$

Now we introduce some notations. Let

$$\Delta_{ij} := \int_{-\infty}^{\infty} \langle u_i^\perp(t), D^2 f(\gamma(t)) u_{d+j}(t) u_{d+j}(t)\rangle\, dt, \quad i,j=1,\dots,d.$$

We further make an assumption.

(H5) $\Delta_{1j} \neq 0$, $j=1,\dots,d$.

For the perturbative term, we define a subset of $C^3(\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}, \mathbb{R}^n)$ by

$$\mathcal{G} = \left\{ g \in C^3\colon \int_{-\infty}^{\infty} \langle u_i^{\perp}(t), g(\gamma(t), 0, 0)\rangle\, dt = 0,\ i = 1, \ldots, d,\ g(0, \epsilon, t) = 0, \text{ and } \|g\|_{C^3} \text{ is small} \right\}.$$

The following is our main result.

**Theorem 2.1.** *Assume that* (H1)–(H5) *hold. Let $d = T_{\gamma(0)}W^s \cap T_{\gamma(0)}W^u$. Then there are $\epsilon_0 > 0$, neighborhood $\mathcal{U} \subset \mathcal{G}$ which contains origin, and $d$ submanifolds $\mathfrak{M}_j \subset \mathcal{G}$, $0 \in \mathfrak{M}_j$, with codimension $dj$, $j = 1, \ldots, d$, such that for every $g \in \mathcal{U} \cap (\mathfrak{M}_k/(\mathfrak{M}_{k+1} \cup \cdots \cup \mathfrak{M}_d))$ and $\epsilon \in (-\epsilon_0, 0) \cup (0, \epsilon_0)$, the system* (1.1) *has $k$ linearly independent homoclinic solutions where $k = 1, \ldots, d$.*

## 3. The homoclinic bifurcations

For each $\beta \in (0, \alpha)$, we define the space

$$\mathcal{Z}_\beta = \left\{ z \in C^0(\mathbb{R}, \mathbb{R}^n) \;\Big|\; \sup_{t \in \mathbb{R}} |z(t)| e^{-\beta|t|} < \infty \right\}.$$

We know that $\mathcal{Z}_\beta$ is a Banach space with the sup norm $\|\cdot\|$. Clearly, for any $z \in \mathcal{Z}_\beta$, the function $|z(t)|$ grows no faster than $e^{\alpha|t|}$ as $t \to \pm\infty$. Thus we can define a subspace of $\mathcal{Z}_\beta$ by

$$\tilde{\mathcal{Z}}_\beta = \left\{ z \in \mathcal{Z}_\beta \;\Big|\; \int_{-\infty}^{\infty} \langle P_{uu} U^{-1}(t), z(t)\rangle\, dt = 0 \right\}.$$

Using $\mathcal{Z}_\beta$, we define another space $\mathcal{Z}_0 = \bigcap_{\beta \in (0,\alpha)} \mathcal{Z}_\beta$.

Let $S$ denote subspace of $\mathcal{Z}_\beta$, such that $\tilde{\mathcal{Z}}_\beta = S \oplus \operatorname{span}\{u_{d+1}, \ldots, u_{2d}\}$. We make transformation

$$x(t) = \gamma(t) + \sum_{i=1}^{d} k_i u_{d+i}(t) + z(t) \tag{3.1}$$

where $k_i \in \mathbb{R}$ and $z \in S$. We take some special forms of (3.1). For any fixed $j$, we choose $k_i = 0$ if $i \neq j$. Then (3.1) is

$$x_j(t) = \gamma(t) + k_j u_{d+j}(t) + z_j(t) \tag{3.2}$$

where $x_j$ and $z_j$ are $x$ and $z$ in (3.1) respectively. Under the transformation (3.2), Eq. (1.2) is

$$\dot{z}_j(t) = Df(\gamma(t)) z_j(t) + h_j(z_j, \beta_j, g, \epsilon)(t) \tag{3.3}$$

where

$$\begin{aligned} h_j(z_j, \beta_j, g, \epsilon)(t) = {} & f(\gamma(t) + k_j u_{d+j}(t) + z_j(t)) - Df(\gamma(t))(k_j u_{d+j}(t) + z_j(t)) \\ & - f(\gamma(t)) + g(\gamma(t) + k_j u_{d+j}(t) + z_j(t), \epsilon, \epsilon t). \end{aligned}$$

Naturally, we wish to convert (3.3) to integral equation $F(z_j, k_j, g, \epsilon) = 0$. By implicit function theorem, we find the solution of $F(z_j, k_j, g, \epsilon) = 0$ for $z_j \in \mathcal{Z}_\beta$. There is a problem that we cannot control the growth of $h_j(z_j, \beta_j, g, \epsilon)$. As in [17], we can introduce a so-called cut-off function. Let $\chi : \mathbb{R}^n \to [0, 1]$ be a $C^\infty$ function, such that

$$\chi(x) = \begin{cases} 1 & \text{if } |x| \leqslant 1, \\ 0 & \text{if } |x| \geqslant 2. \end{cases}$$

For each $\rho > 0$, we define new function $h_{j_\rho} : \mathcal{Z}_\beta \times \mathbb{R} \times C^3 \times \mathbb{R} \to \mathbb{R}^n$ by

$$h_{j_\rho}(z_j, k_j, g, \epsilon) = h_j(z_j, k_j, g, \epsilon)\chi(z_j/\rho). \tag{3.4}$$

For $\rho_i > 0$, $r_i > 0$ and $\sigma_i > 0$, let $B_1(\rho_i) \in \mathcal{Z}_\beta$, $B_2(r_i) \in \mathbb{R}$ and $B_3(\sigma_i) \in C^3$ be balls with radius $\rho_i$, $r_i$ and $\sigma_i$ centered at respective origin.

**Proposition 3.1.** *For any given $K_1 > 0$, there are $\rho_1 > 0, r_1 > 0$ and $\sigma_1 > 0$ such that the function $h_{j_{\rho_1}}(z_j, k_j, g, \epsilon)$ satisfies*

(a) $|D_1 h_{j_{\rho_1}}(z_j, k_j, g, \epsilon)(t)| < K_1$ *for* $(z_j, k_j, g, \epsilon) \in \mathcal{Z}_\beta \times B_2(r_1) \times B_3(\sigma_1) \times \mathbb{R}$,

(b) *for any* $z_j^{(1)}, z_j^{(2)} \in \mathcal{Z}_\beta$,

$$\left|h_{j_{\rho_1}}\big(z_j^{(1)}, k_j, g, \epsilon\big)(t) - h_{j_{\rho_1}}\big(z_j^{(2)}, k_j, g, \epsilon\big)(t)\right| < K_1\left|z_j^{(1)}(t) - z_j^{(2)}(t)\right|$$

*for* $(k_j, g, \epsilon) \in B_2(r_1) \times B_3(\sigma_1) \times \mathbb{R}$.

**Proof.** (a). From (3.4), we have

$$\begin{aligned} D_1 h_{j_\rho}(z_j, k_j, g, \epsilon)(t) = {} & \big[Df\big(\gamma(t) + k_j u_{d+j}(t) + z_j(t)\big) - Df\big(\gamma(t)\big) \\ & + D_1 g\big(\gamma(t) + k_j u_{d+j}(t) + z_j(t), \epsilon, \epsilon t\big)\big] \cdot \chi(z_j/\rho) \\ & + \frac{1}{\rho}\big[f\big(\gamma(t) + k_j u_{d+j}(t) + z_j(t)\big) - Df\big(\gamma(t)\big)\big(k_j u_{d+j}(t) + z_j(t)\big) \\ & - f\big(\gamma(t)\big) + g\big(\gamma(t) + k_j u_{d+j}(t) + z_j(t), \epsilon, \epsilon t\big)\big] \cdot D\chi(z_j/\rho). \end{aligned} \tag{3.5}$$

Let $C_1 = \sup_{t\in\mathbb{R}} |u_{d+j}(t)|$, $C_2 = \sup_{t\in\mathbb{R}} \sup_{|x|\leqslant 2, a\in[-1,1]} |D^2 f(\gamma(t) + a u_{d+j}(t) + x(t))|$, $C_3 = \sup_x |D\chi(x/\rho)|$. For any given $K_1 > 0$, we choose $0 < \rho_1 \leqslant \min\{\frac{1}{2}, \frac{K_1}{16C_2}, \frac{K_1}{64C_2C_3}\}$. Let

$$r_1 = \min\left\{1, \frac{2\rho_1}{C_1}\right\}, \qquad \sigma_1 = \min\left\{\frac{K_1}{4}, \frac{K_1\rho_1}{4C_3}\right\}.$$

Since $h_{j_{\rho_1}} = 0$ for $|z_j(t)| > 2\rho_1$, we see that (a) and (b) hold if $|z_j(t)| > 2\rho_1$. Thus we assume $|z_j(t)| \leqslant 2\rho_1$.

For $(z_j, k_j) \in \mathcal{Z}_\beta \times B_2(r_1)$, define $\psi_1[0,1] \to L(\mathcal{Z}_\beta, \mathcal{Z}_\beta)$ by $\psi_1(s) = Df(\gamma(t) + sk_j u_{d+j}(t) + sz_j(t)) - Df(\gamma(t))$. It is clear that $\psi_1 \in C^1$. Then there is $s_1 \in [0,1]$ such that

$$\begin{aligned} & \big|Df\big(\gamma(t) + k_j u_{d+j}(t) + z_j(t)\big) - Df\big(\gamma(t)\big)\big| \\ & \quad = \big|\psi_1(1) - \psi_1(0)\big| = \big|\psi_1'(s_1)\big| \\ & \quad \leqslant \big|D^2 f\big(\gamma(t) + s_1 k_j u_{d+j}(t) + s_1 z_j(t)\big)\big|\big(|k_j| \cdot \big|u_{d+j}(t)\big| + \big|z_j(t)\big|\big) \\ & \quad \leqslant C_2\left(\frac{2\rho_1}{C_1}C_1 + 2\rho_1\right) \\ & \quad = 4C_2\rho_1 \leqslant \min\left\{\frac{K_1}{4}, \frac{K_1}{16C_3}\right\}. \end{aligned} \tag{3.6}$$

For $(z_j, k_j) \in \mathcal{Z}_\beta \times B_2(r_1)$, define $\psi_2 : [0,1] \to \mathcal{Z}_\beta$ by $\psi_2(s) = f(\gamma(t) + sk_j u_{d+j}(t) + sz_j(t)) - f(\gamma(t)) - Df(\gamma(t))(sk_j u_{d+j}(t) + sz_j(t))$. It is clear that $\psi_2 \in C^1$. Then there is $s_2 \in [0,1]$ such that

$$\begin{aligned} & \big|f\big(\gamma(t) + k_j u_{d+j}(t) + z_j(t)\big) - f\big(\gamma(t)\big) - Df\big(\gamma(t)\big)\big(k_j u_{d+j}(t) + z_j(t)\big)\big| \\ & \quad = \psi_2(1) - \psi_2(0) = \psi_2'(s_2) \\ & \quad \leqslant \big|Df\big(\gamma(t) + s_2 k_j u_{d+j}(t) + s_2 z_j(t)\big) - Df\big(\gamma(t)\big)\big|\big(|k_j| \cdot \big|u_{d+j}(t)\big| + \big|z_j(t)\big|\big) \\ & \quad \leqslant \frac{K_1}{16C_3}\left(\frac{2\rho_1}{C_1}C_1 + 2\rho_1\right) = \frac{K_1\rho_1}{4C_3} \end{aligned} \tag{3.7}$$

where (3.6) is used.

For $(z_j, k_j, g, \epsilon) \in \mathcal{Z}_\beta \times B_2(r_1) \times B_3(\sigma_1) \times \mathbb{R}$, we can get from (3.5)–(3.7) that

$$\begin{aligned} \big|D_1 h_{j_{\rho_1}}(z_j, k_j, g, \epsilon)(t)\big| \leqslant {} & \big|Df\big(\gamma(t) + k_j u_{d+j}(t) + z_j(t)\big) - Df\big(\gamma(t)\big)\big| \\ & + \big|D_1 g\big(\gamma(t) + k_j u_{d+j}(t) + z_j(t), \epsilon, \epsilon t\big)\big| \\ & + \frac{1}{\rho_1}\big[\big|f\big(\gamma(t) + k_j u_{d+j}(t) + z_j(t)\big) - Df\big(\gamma(t)\big)\big(k_j u_{d+j}(t) + z_j(t)\big) - f\big(\gamma(t)\big)\big| \end{aligned}$$

$$+\big|g\big(\gamma(t)+k_j u_{d+j}(t)+z_j(t),\epsilon,\epsilon t\big)\big|\big]\cdot\big|D\chi(z_j/\rho_1)\big|$$
$$\leqslant \frac{K_1}{4}+\sigma_1+\frac{1}{\rho_1}\left(\frac{K_1\rho_1}{4C_3}+\sigma_1\right)C_3 \leqslant K_1. \tag{3.8}$$

The proof of (a) is completed.

(b). For any $z_j^{(1)}, z_j^{(1)} \in \mathcal{Z}_\beta$, and $(k_j,g,\epsilon)\in B_2(r_1)\times B_3(\sigma_1)\times\mathbb{R}$, define $\psi_3:[0,1]\to\mathcal{Z}_0$ by $\psi_3(s)=h_{j_{\rho_1}}(sz_j^{(1)}+(1-s)z_j^{(2)},k_j,g,\epsilon)(t)$. There is $s_3\in[0,1]$, such that

$$\begin{aligned}
&\big|h_{j_{\rho_1}}\big(z_j^{(1)},k_j,g,\epsilon\big)(t)-h_{j_{\rho_1}}\big(z_j^{(2)},k_j,g,\epsilon\big)(t)\big|\\
&\quad=\big|\psi_3(1)-\psi_3(0)\big|=\big|\psi_3'(s_3)\big|\\
&\quad=\big|D_1h_{j_{\rho_1}}\big(s_3z_j^{(1)}+(1-s_3)z_j^{(2)},k_j,g,\epsilon\big)(t)\big|\big|z_j^{(1)}(t)-z_j^{(2)}(t)\big|\\
&\quad\leqslant K_1\big|z_j^{(1)}(t)-z_j^{(2)}(t)\big|.
\end{aligned}$$

The proof of (b) is completed. □

Let $b:\mathbb{R}\to\mathbb{R}$ be smooth function with $\int_{-\infty}^{\infty}b(t)\,dt=1$. Define a map $P:\mathcal{Z}_\beta\to\mathcal{Z}_\beta$ by

$$(Pw)(t)=b(t)U(t)\int_{-\infty}^{\infty}\big\langle P_{uu}U^{-1}(s),w(s)\big\rangle\,ds.$$

**Lemma 3.1.** *The operator $P$ is a projection and $P(\dot z-Df(\gamma(t))z)=0$ for $z\in\mathcal{Z}_\beta$.*

**Proof.** For $z\in\mathcal{Z}_\beta$, we have

$$\begin{aligned}
\big(P^2z\big)(t)&=b(t)U(t)\int_{-\infty}^{\infty}\big\langle P_{uu}U^{-1}(s),(Pz)(s)\big\rangle\,ds\\
&=b(t)U(t)\int_{-\infty}^{\infty}\left\langle P_{uu}U^{-1}(s),b(s)U(s)\int_{-\infty}^{\infty}\big\langle P_{uu}U^{-1}(\tau),z(\tau)\big\rangle\,d\tau\right\rangle ds\\
&=b(t)U(t)\int_{-\infty}^{\infty}\big\langle P_{uu}U^{-1}(\tau),z(\tau)\big\rangle\,d\tau=(Pz)(t).
\end{aligned}$$

Thus $P$ is projection.

Note that $|u_i^\perp(t)|$, $i=1,\dots,d$, approach zero like $e^{-2\alpha|t|}$ as $t\to\pm\infty$. For $z\in\mathcal{Z}_\beta$, $|z(t)|$ grows no faster than $e^{\alpha|t|}$ as $t\to\pm\infty$. Thus $\langle u_i^\perp(t),z(t)\rangle|_{-\infty}^{\infty}=0$. Then for $z\in\mathcal{Z}_\beta$, we have

$$\begin{aligned}
\int_{-\infty}^{\infty}\big\langle u_i^\perp,\dot z-Df\big(\gamma(t)\big)z\big\rangle\,dt&=\int_{-\infty}^{\infty}\big\langle u_i^\perp,\dot z\big\rangle\,dt-\int_{-\infty}^{\infty}\big\langle u_i^\perp,Df\big(\gamma(t)\big)z\big\rangle\,dt\\
&=-\int_{-\infty}^{\infty}\big\langle u_i^\perp,\dot z\big\rangle\,dt-\int_{-\infty}^{\infty}\big\langle Df\big(\gamma(t)\big)^*(t)u_i^\perp,z\big\rangle\,dt\\
&=\int_{-\infty}^{\infty}\big\langle u_i^\perp,\dot z\big\rangle\,dt+\int_{-\infty}^{\infty}\big\langle \dot u_i^\perp,z\big\rangle\,dt\\
&=\int_{-\infty}^{\infty}\frac{d}{dt}\big\langle u_i^\perp,z\big\rangle\,dt=\big\langle \dot u_i^\perp(t),z(t)\big\rangle\big|_{-\infty}^{\infty}=0.
\end{aligned}$$

Thus we get $P(\dot z-Df(\gamma(t))z)=0$ for $z\in\mathcal{Z}_\beta$. The proof finishes. □

Consider the equation

$$\dot{v}(t) = Df(0)v(t). \tag{3.9}$$

Since $|\mathrm{Re}(\lambda_i)| \geqslant 3\alpha$ where $\lambda_i$ are the eigenvalues of $Df(0)$, Eq. (3.9) has a fundamental matrix solution, $V(t)$, with projections $Q$, $(I-Q)$ and constants $A>0$, $\beta_0 \in (0,\alpha)$, such that

$$\begin{aligned} &\left|V(t)QV^{-1}(s)\right| \leqslant Ae^{2\beta_0(s-t)}, \quad \text{for } s \leqslant t, \\ &\left|V(t)(I-Q)V^{-1}(s)\right| \leqslant Ae^{2\beta_0(t-s)}, \quad \text{for } t \leqslant s. \end{aligned} \tag{3.10}$$

Let $\rho_1$ and $P$ be as in Proposition 3.1 and Lemma 3.1 respectively. We define

$$\eta_{j_{\rho_1}}(z_j,k_j,g,\epsilon)(t) := \big(Df\big(\gamma(t)\big) - Df(0)\big)z_j(t) + (I-P)h_{j_{\rho_1}}(z_j,k_j,g,\epsilon)(t). \tag{3.11}$$

**Proposition 3.2.** *Let $\beta_0$ and $A$ be as in* (3.10). *There are $t_0>0, r_2>0$ and $\sigma_2>0$ such that*

(1) *for $z_j^{(1)}, z_j^{(2)} \in \mathcal{Z}_{\beta_0}$, $(k_j,g) \in B_2(r_2)\times B_3(\sigma_2)$ and $t \in (-\infty,-t_0]\cup[t_0,\infty)$,*

$$\left|\eta_{j_{\rho_1}}\big(z_j^{(1)},k_j,g,\epsilon\big)(t) - \eta_{j_{\rho_1}}\big(z_j^{(2)},k_j,g,\epsilon\big)(t)\right| \leqslant \frac{3\beta_0}{8A}\left|z_j^{(1)}(t) - z_j^{(2)}(t)\right|,$$

(2) *for $(k_j,g) \in B_2(r_2)\times B_3(\sigma_2)$ and $t\in\mathbb{R}$,*

$$\left|\eta_{j_{\rho_1}}(0,k_j,g,\epsilon)(t)\right| \leqslant \frac{3\beta_0\rho_1}{16A}e^{-\beta_0|t|}.$$

**Proof.** Since $u_{d+j}$ is bounded solution of (2.1) and $\gamma$ is homoclinic solution, there is $c_1>0$ such that

$$\left|\gamma(t)\right| \leqslant c_1 e^{-\beta_0|t|}, \qquad \left|u_{d+j}(t)\right| \leqslant c_1 e^{-\beta_0|t|}.$$

Let $c_2 = \sup_{t\in\mathbb{R}}\max_{s\in[-1,1]}|Df(\gamma(t)+su_{d+j}(t)) - Df(\gamma(t))|$ and

$$r_2 = \min\left\{1, r_1, \frac{3\beta_0\rho_1}{32Ac_1c_2}\right\}, \qquad \sigma_2 = \min\left\{\sigma_1, \frac{3\beta_0\rho_1}{64Ac_1}\right\}$$

where $\rho_1$, $r_1$ and $\sigma_1$ are defined in Proposition 3.1. Since $\lim_{t\to\pm\infty}\gamma(t)=0$, there is $t_0>0$ such that

$$\left|Df\big(\gamma(t)\big) - Df(0)\right| \leqslant \frac{3\beta_0}{16A} \quad \text{for } |t| \geqslant t_0. \tag{3.12}$$

(1). We only give the proof for $t\in[t_0,\infty)$ since the similar argument can be used to prove the case of $t\in(-\infty,-t_0]$.

For any $z_j^{(1)}, z_j^{(2)} \in \mathcal{Z}_{\beta_0}$, $(k_j,g)\in B_2(r_2)\times B_3(\sigma_2)$, we can get from (b) in Proposition 3.1 that

$$\left|h_{j_{\rho_1}}\big(z_j^{(1)},k_j,g,\epsilon\big)(t) - h_{j_{\rho_1}}\big(z_j^{(2)},k_j,g,\epsilon\big)(t)\right| \leqslant \frac{3\beta_0}{16A}\left|z_j^{(1)}(t) - z_j^{(2)}(t)\right| \tag{3.13}$$

where we have taken $K_1 = \frac{3\beta_0}{16A}$. Note that $\|(I-P)\| \leqslant 1$ since $(I-P)$ is projection. Then we can get

$$\begin{aligned} &\left|\eta_{j_{\rho_1}}\big(z_j^{(1)},k_j,g,\epsilon\big)(t) - \eta_{j_{\rho_1}}\big(z_j^{(2)},k_j,g,\epsilon\big)(t)\right| \\ &\quad \leqslant \left|Df\big(\gamma(t)\big) - Df(0)\right|\cdot\left|z_j^{(1)}(t) - z_j^{(2)}(t)\right| + \left|h_{j_{\rho_1}}\big(z_j^{(1)},k_j,g,\epsilon\big)(t) - h_{j_{\rho_1}}\big(z_j^{(2)},k_j,g,\epsilon\big)(t)\right| \\ &\quad \leqslant \frac{3\beta_0}{8A}\left|z_j^{(1)}(t) - z_j^{(2)}(t)\right| \end{aligned}$$

for $t\in[t_0,\infty)$, where (3.12) and (3.13) are used. The proof of (1) finishes.

(2). It is clear that

$$\begin{aligned} \eta_{j_{\rho_1}}(0,k_j,g,\epsilon)(t) = (I-P)\big[&f\big(\gamma(t)+k_ju_{d+j}(t)\big) - Df\big(\gamma(t)\big)k_ju_{d+j}(t) \\ &- f\big(\gamma(t)\big) + g\big(\gamma(t)+k_ju_{d+j}(t),\epsilon,\epsilon t\big)\big]. \end{aligned}$$

For any $(k_j, g) \in B_2(r_2) \times B_3(\sigma_2)$, define $\varphi : [0,1] \to \mathcal{Z}_{\beta_0}$ by

$$\varphi(\tau) = (I-P)\big[f\big(\gamma(t) + \tau k_j u_{d+j}(t)\big) - Df\big(\gamma(t)\big)\tau k_j u_{d+j}(t) \\ - f\big(\gamma(t)\big) + g\big(\tau\big(\gamma(t) + k_j u_{d+j}(t)\big), \epsilon, \epsilon t\big)\big].$$

Then we can get

$$\begin{aligned}\big|\eta_{j\,\rho_1}(0, k_j, g, \epsilon)(t)\big| = \big|\varphi(1) - \varphi(0)\big| &= \left|\int_0^1 \varphi'(\tau)\,d\tau\right| \\ &\leqslant \int_0^1 \big|Df(\gamma + \tau k_j u_{d+j}) - Df(\gamma)(t)\big|\big|k_j u_{d+j}(t)\big|\,d\tau \\ &\quad + \int_0^1 \big|D_1 g\big(\tau(\gamma + k_j u_{d+j}), \epsilon, \epsilon t\big)\big|\big(\big|\gamma(t)\big| + \big|k_j u_{d+j}(t)\big|\big)\,d\tau \\ &\leqslant \big[c_2 r_2 c_1 + \sigma_2(c_1 + r_2 c_1)\big]e^{-\beta_0|t|} \leqslant \frac{3\beta_0\rho_1}{16A}e^{-\beta_0|t|}.\end{aligned}$$

The proof is finished. □

For each $\rho > 0$, we consider the nonlinear ordinary differential equation

$$\dot{z}_j(t) = Df\big(\gamma(t)\big)z_j(t) + h_{j\,\rho}(z_j, k_j, g, \epsilon)(t) \tag{3.14}$$

where $h_{j\,\rho}(z_j, k_j, g, \epsilon)(t)$ is defined in (3.4). It is clear that if there are some $\rho > 0$ such that $|z_j(t)| \leqslant \rho$, then (3.14) is equivalent to Eq. (3.3).

From Lemma 3.1, we see that $P$ and $(I-P)$ are projections. Thus Eq. (3.14) is equivalent to

$$\dot{z}_j(t) = Df\big(\gamma(t)\big)z_j(t) + (I-P)h_{j\,\rho}(z_j, k_j, g, \epsilon)(t), \tag{3.15}$$

$$0 = Ph_{j\,\rho}(z_j, k_j, g, \epsilon)(t). \tag{3.16}$$

Our strategy is to solve (3.15) for $z_j \in \mathcal{Z}_{\beta_0}$. Then (3.16) becomes the bifurcation equation.

**Theorem 3.1.** *For any fixed $\rho > 0$, there are constants $\delta > 0$, $r_3 > 0$, $\sigma_3 > 0$, such that* (3.15) *has a solution $z_j^*(k_j, g, \epsilon) \in \mathcal{Z}_{\beta_0}$ for $(k_j, g, \epsilon) \in B_2(r_3) \times B_3(\sigma_3) \times \mathbb{R}$ satisfying $z_j^*(0,0,\epsilon) = \partial z_j^*/\partial k_j|_{(0,0,\epsilon)} = 0$ and $\|z_j^*\| \leqslant \delta$.*

**Proof.** Using variation of constants, we define map $\mathcal{K} : \tilde{\mathcal{Z}}_{\beta_0} \to \mathcal{Z}_{\beta_0}$ by

$$\mathcal{K}(w)(t) = \begin{cases} U(t)\Big[-\displaystyle\int_0^\infty \big\langle P_{us}U^{-1}(s), w(s)\big\rangle\,ds + \int_0^t \big\langle (P_{ss} + P_{su})U^{-1}(s), w(s)\big\rangle\,ds \\ \quad + \displaystyle\int_{-\infty}^t \big\langle (P_{us} + P_{uu})U^{-1}(s), w(s)\big\rangle\,ds\Big], \quad t \leqslant 0, \\ U(t)\Big[\displaystyle\int_{-\infty}^0 \big\langle P_{us}U^{-1}(s), w(s)\big\rangle\,ds + \int_0^t \big\langle (P_{ss} + P_{us})U^{-1}(s), w(s)\big\rangle\,ds \\ \quad - \displaystyle\int_t^\infty \big\langle (P_{ss} + P_{us})U^{-1}(s), w(s)\big\rangle\,ds\Big], \quad t \geqslant 0. \end{cases}$$

Define map $F : \mathcal{Z}_{\beta_0} \times \mathbb{R} \times C^3 \times \mathbb{R} \to \mathcal{Z}_{\beta_0}$ by

$$F(z_j, k_j, g, \epsilon) = \mathcal{K}(I - P)h_{j_\rho}(z_j, k_j, g, \epsilon). \tag{3.17}$$

It is clear that the fixed points $z_j \in \mathcal{Z}_{\beta_0}$ of (3.17) are solutions of (3.15). Through direct calculations, we have

$$F(0,0,0,\epsilon) = 0, \qquad D_1F(0,0,0,\epsilon) = 0, \qquad D_2F(0,0,0,\epsilon) = 0.$$

For a large $r > 0$, let $B_1(r) \in \mathcal{Z}_{\beta_0}$, $B_2(r) \in \mathbb{R}$ and $B_3(r) \in C^3$ be balls centered at respective origin. There is $M_0 > 0$ such that

$$\begin{gathered}\|F(\cdot,\cdot,\cdot,\epsilon)\| < M_0, \qquad \|D_1F(\cdot,\cdot,\cdot,\epsilon)\| < M_0, \qquad \|D_2F(\cdot,\cdot,\cdot,\epsilon)\| < M_0,\\ \|D_{11}F(\cdot,\cdot,\cdot,\epsilon)\| < M_0, \qquad \|D_{12}F(\cdot,\cdot,\cdot,\epsilon)\| < M_0, \qquad \|D_{13}F(\cdot,\cdot,\cdot,\epsilon)\| < M_0\end{gathered}$$

for $(z_j, k_j, g) \in B_1(r) \times B_2(r) \times B_3(r)$.

We choose $0 < \delta \leqslant \min\{r, \frac{1}{8M_0}\}$. Let

$$r_3 = \min\left\{1, \delta, \frac{\delta}{4M_0}\right\}, \qquad \sigma_3 = \min\left\{\delta, \frac{\delta}{4M_0}\right\}.$$

For $(z_j, k_j, g) \in B_1(\delta) \times B_2(r_3) \times B_3(\sigma_3)$, define $\xi_1 : [0,1] \to L(\mathcal{Z}_{\beta_0}, \mathcal{Z}_{\beta_0})$ by $\xi_1(s) = D_1F(sz_j, sk_j, sg, \epsilon)$. It is clear that $\xi_1 \in C^1$. There is $s_1 \in [0,1]$ such that

$$\begin{aligned}\|D_1F(z_j, k_j, g, \epsilon)\| &= \|\xi_1(1) - \xi_1(0)\| = \|\xi_1'(s_1)\|\\ &\leqslant \|D_{11}F(s_1z_j, s_1k_j, s_1g, \epsilon)\|\|z_j\| + \|D_{12}F(s_1z_j, s_1k_j, s_1g, \epsilon)\||k_j|\\ &\quad + \|D_{13}F(s_1z_j, s_1k_j, s_1g, \epsilon)\|\|g\|\\ &\leqslant M_0 \cdot \frac{1}{8M_0} + M_0 \cdot \frac{1}{8M_0} + M_0 \cdot \frac{1}{8M_0} = \frac{3}{8}.\end{aligned} \tag{3.18}$$

For $(z_j, k_j, g) \in B_1(\delta) \times B_2(r_3) \times B_3(\sigma_3)$, as before we define $\xi_2 : [0,1] \to \mathcal{Z}_{\beta_0}$ by $\xi_2(s) = F(sz_j, sk_j, sg, \epsilon)$. Then there is $s_2 \in [0,1]$ such that

$$\begin{aligned}\|F(z_j, k_j, g, \epsilon)\| &= \|\xi_2(1) - \xi_2(0)\| = \|\xi_2'(s_1)\|\\ &\leqslant \|D_1F(s_2z_j, s_2k_j, s_2g, \epsilon)\|\|z_j\| + \|D_2F(s_2z_j, s_2k_j, s_2g, \epsilon)\||k_j|\\ &\quad + \|D_3F(s_2z_j, s_2k_j, s_2g, \epsilon)\|\|g\|\\ &\leqslant \frac{3}{8} \cdot \delta + M_0 \cdot \frac{\delta}{4M_0} + M_0 \cdot \frac{\delta}{4M_0} < \delta\end{aligned} \tag{3.19}$$

where (3.18) is used.

For $z_j^{(1)}, z_j^{(2)} \in B_1(\delta)$, and $(k_j, g) \in B_2(r_3) \times B_3(\sigma_3)$, let $\xi_3 : [0,1] \to \mathcal{Z}_{\beta_0}$ by $\xi_3(s) = F(sz_j^{(1)} + (1-s)z_j^{(2)}, k_j, g, \epsilon)$. Then there is $s_3 \in [0,1]$ such that

$$\begin{aligned}\|F(z_j^{(1)}, k_j, g, \epsilon) - F(z_j^{(2)}, k_j, g, \epsilon)\| &= \|\xi_3(1) - \xi_3(0)\| = \|\xi_3'(s_3)\|\\ &\leqslant \|D_1F(s_3z_j^{(1)} + (1-s_3)z_j^{(2)}, k_j, g, \epsilon)\|\|z_j^{(1)} - z_j^{(2)}\|\\ &\leqslant \frac{3}{8}\|z_j^{(1)} - z_j^{(2)}\|\end{aligned} \tag{3.20}$$

where (3.18) is used.

From (3.19) and (3.20), we see that the map $F(\cdot, k_j, g, \epsilon) : B_1(\delta) \to B_1(\delta)$ is uniformly contractive. By contraction mapping theorem, there is a $C^1$ map $z_j^* : B_2(r_3) \times B_3(\sigma_3) \times \mathbb{R} \to B_1(\delta)$, such that $z_j^*(0,0,\epsilon) = 0$ and

$$z_j^*(k_j, g, \epsilon) = F(z_j^*(k_j, g, \epsilon), k_j, g, \epsilon). \tag{3.21}$$

It is clear that $\|z_j^*(k_j, g, \epsilon)\| \leqslant \delta$ for $(k_j, g, \epsilon) \in B_2(r_3) \times B_3(\sigma_3) \times \mathbb{R}$.

Differentiating (3.21) in $k_j$ and evaluating at $(0,0,\epsilon)$, we can get that

$$D_1z_j^*(0,0,\epsilon) = D_1F(0,0,0,\epsilon)D_1z_j^*(0,0,\epsilon) + D_2F(0,0,0,\epsilon) = 0.$$

The proof is completed. $\square$

Let $r_i$ and $\sigma_i$, $i=1,2,3$, be as in Propositions 3.1, 3.2 and Lemma 3.1 respectively. Let

$$r_0=\min\{r_1,r_2,r_3\},\qquad \sigma_0=\min\{\sigma_1,\sigma_2,\sigma_3\}.$$

**Lemma 3.2.** *Let $z_j^*(k_j,g,\epsilon)(t)$ be as in Theorem 3.1. Then there is $\rho_0>0$ such that $\sup_{t\in\mathbb{R}}|z_j^*(k_j,g,\epsilon)(t)|\leqslant\rho_0$ for $(k_j,g,\epsilon)\in B_2(r_0)\times B_3(\sigma_0)\times\mathbb{R}$.*

**Proof.** Let $K_2=\sup_{t\in\mathbb{R}}|(Df(\gamma(t))-Df(0))|$, $\beta_0$ and $A$ be as in (3.10), and $t_0$ be as in Proposition 3.2. We choose $\rho_0>0$ such that $\rho_1$ and $\delta$ which are obtained in Proposition 3.1 and Theorem 3.1 respectively satisfy

$$\rho_1\leqslant\rho_0,\qquad \delta\leqslant\min\left\{\frac{\rho_0e^{-3\beta_0t_0}}{4A},\frac{5\rho_0}{12A},\frac{4\beta_0\rho_0e^{-\beta_0t_0}}{3\beta_0+16AK_2}\right\}.$$

We only give the proof of $t\in[0,\infty)$ since the similar method can be used to prove the case of $t\in(-\infty,0]$.

**Case 1.** If $t\in[t_0,\infty)$.

For $\rho=\rho_1$, we rewrite Eq. (3.15) as

$$\dot z_j(t)=Df(0)z_j(t)+\eta_{j\rho_1}(z_j,k_j,g,\epsilon)(t)\tag{3.22}$$

where $\eta_{j\rho_1}(z_j,k_j,g,\epsilon)(t)$ is defined in (3.11).

We know from Theorem 3.1 that $z_j^*\in\mathcal{Z}_{\beta_0}$ is solution of (3.22) with $\|z_j^*\|\leqslant\delta$, i.e., $|z_j^*(t)|\leqslant\delta e^{\beta_0t}$ for $t\in\mathbb{R}^+$. Thus we have

$$\begin{aligned}
z_j^*(t)&=V(t)V^{-1}(t_0)z_j^*(t_0)+V(t)\int_{t_0}^{t}\big\langle V^{-1}(s),\eta_{j\rho_1}\big(z_j^*,k_j,g,\epsilon\big)(s)\big\rangle\,ds\\
&=V(t)QV^{-1}(t_0)z_j^*(t_0)+V(t)\int_{t_0}^{t}\big\langle QV^{-1}(s),\eta_{j\rho_1}\big(z_j^*,k_j,g,\epsilon\big)(s)\big\rangle\,ds\\
&\quad-V(t)\int_{t}^{\infty}\big\langle (I-Q)V^{-1}(s),\eta_{j\rho_1}\big(z_j^*,k_j,g,\epsilon\big)(s)\big\rangle\,ds\\
&\quad+V(t)(I-Q)\left[V^{-1}(t_0)z_j^*(t_0)+\int_{t_0}^{\infty}\big\langle V^{-1}(s),\eta_{j\rho_1}\big(z_j^*,k_j,g,\epsilon\big)(s)\big\rangle\,ds\right].
\end{aligned}$$

In the last equation, we know that the last term grows like $e^{2\beta_0t}$ as $t\to\infty$ and other terms grow no more than $e^{\beta_0t}$ as $t\to\infty$. Thus the last term must vanish and the solution can be expressed as

$$\begin{aligned}
z_j^*(t)&=V(t)QV^{-1}(t_0)z_j^*(t_0)+V(t)\int_{t_0}^{t}\big\langle QV^{-1}(s),\eta_{j\rho_1}\big(z_j^*,k_j,g,\epsilon\big)(s)\big\rangle\,ds\\
&\quad-V(t)\int_{t}^{\infty}\big\langle (I-Q)V^{-1}(s),\eta_{j\rho_1}\big(z_j^*,k_j,g,\epsilon\big)(s)\big\rangle\,ds.
\end{aligned}\tag{3.23}$$

We define the space

$$X=\left\{x\in C^0\big([t_0,\infty),\mathbb{R}^n\big)\;\Big|\;\sup_{t\geqslant t_0}\big|x(t)\big|e^{\beta_0t}<\infty\right\}.$$

Then $X$ is Banach space with sup norm $\|\cdot\|_X$. Define $\mathcal{F}:X\times\mathbb{R}\times C^3\times\mathbb{R}\to X$ by

$$\mathcal{F}(\phi,k_j,g,\epsilon)(t)=V(t)QV^{-1}(t_0)z_j^*(t_0)+V(t)\int_{t_0}^{t}\left\langle QV^{-1}(s),\eta_{j_{\rho_1}}(\phi,k_j,g,\epsilon)(s)\right\rangle ds$$
$$-V(t)\int_{t}^{\infty}\left\langle (I-Q)V^{-1}(s),\eta_{j_{\rho_1}}(\phi,k_j,g,\epsilon)(s)\right\rangle ds. \tag{3.24}$$

Let $B(\rho_0)\subset X$ be ball centered at origin with radius $\rho_0$. For any $\phi_1,\phi_2\in B(\rho_0)$ and $(k_j,g,\epsilon)\in B_2(r_0)\times B_3(\sigma_0)\times\mathbb{R}$, we have

$$\begin{aligned}
&\left|\mathcal{F}(\phi_1,k_j,g,\epsilon)(t)-\mathcal{F}(\phi_2,k_j,g,\epsilon)(t)\right|\\
&\quad\leqslant\int_{t_0}^{t}\left|V(t)QV^{-1}(s)\right|\left|\eta_{j_{\rho_1}}(\phi_1,k_j,g,\epsilon)(s)-\eta_{j_{\rho_1}}(\phi_2,k_j,g,\epsilon)(s)\right|ds\\
&\qquad+\int_{t}^{\infty}\left|V(t)(I-Q)V^{-1}(s)\right|\left|\eta_{j_{\rho_1}}(\phi_1,k_j,g,\epsilon)(s)-\eta_{j_{\rho_1}}(\phi_2,k_j,g,\epsilon)(s)\right|ds\\
&\quad\leqslant\frac{3\beta_0}{8}\int_{t_0}^{t}e^{2\beta_0(s-t)}\left|\phi_1(s)-\phi_2(s)\right|ds+\frac{3\beta_0}{8}\int_{t}^{\infty}e^{2\beta_0(t-s)}\left|\phi_1(s)-\phi_2(s)\right|ds\\
&\quad\leqslant\frac{3\beta_0}{8}e^{-2\beta_0 t}\int_{t_0}^{t}e^{\beta_0 s}\left(\left|\phi_1(s)-\phi_2(s)\right|e^{\beta_0 s}\right)ds+\frac{3\beta_0}{8}e^{2\beta_0 t}\int_{t}^{\infty}e^{-3\beta_0 s}\left(\left|\phi_1(s)-\phi_2(s)\right|e^{\beta_0 s}\right)ds\\
&\quad\leqslant\left[\frac{3}{8}\left(1-e^{\beta_0(t_0-t)}\right)+\frac{1}{8}\right]\cdot\|\phi_1-\phi_2\|_X\cdot e^{-\beta_0 t}
\end{aligned}$$

where (1) of Proposition 3.2 is used. Thus we can get that

$$\left\|\mathcal{F}(\phi_1,k_j,g,\epsilon)-\mathcal{F}(\phi_2,k_j,g,\epsilon)\right\|_X\leqslant\frac{1}{2}\|\phi_1-\phi_2\|_X. \tag{3.25}$$

Moreover, from (3.24) we have

$$\begin{aligned}
\left|\mathcal{F}(0,k_j,g,\epsilon)(t)\right|&\leqslant\left|V(t)QV^{-1}(t_0)z_j^*(t_0)\right|+\int_{t_0}^{t}\left|V(t)QV^{-1}(s)\right|\left|\eta_{j_{\rho_1}}(0,k_j,g,\epsilon)(s)\right|ds\\
&\quad+\int_{t}^{\infty}\left|V(t)(I-Q)V^{-1}(s)\right|\left|\eta_{j_{\rho_1}}(0,k_j,g,\epsilon)(s)\right|ds\\
&\leqslant Ae^{2\beta_0(t_0-t)}\left|z_j^*(t_0)\right|+\frac{3\beta_0\rho_1}{16}\left(\int_{t_0}^{t}e^{\beta_0(s-2t)}\,ds+\int_{t}^{\infty}e^{\beta_0(2t-3s)}\,ds\right)\\
&\leqslant\left[A\delta e^{\beta_0(3t_0-2t)}+\frac{3\rho_1}{16}\left(1-e^{\beta_0(t_0-t)}\right)+\frac{\rho_1}{16}\right]e^{-\beta_0 t}
\end{aligned}$$

where (2) of Proposition 3.2 is used. Thus we can get that

$$\begin{aligned}
\left\|\mathcal{F}(0,k_j,g,\epsilon)\right\|_X&\leqslant Ae^{3\beta_0 t_0}\delta+\frac{3\rho_0}{16}+\frac{\rho_0}{16}\\
&\leqslant Ae^{3\beta_0 t_0}\frac{\rho_0 e^{-3\beta_0 t_0}}{4A}+\frac{\rho_0}{4}=\frac{\rho_0}{2}.
\end{aligned} \tag{3.26}$$

For any $\phi\in B(\rho_0)$, we can get from (3.25) and (3.26) that

$$\begin{aligned}\big\|\mathcal{F}(\phi,k_j,g,\epsilon)\big\|_X &\leqslant \big\|\mathcal{F}(\phi,k_j,g,\epsilon)-\mathcal{F}(0,k_j,g,\epsilon)\big\|_X+\big\|\mathcal{F}(0,k_j,g,\epsilon)\big\|_X\\ &\leqslant \frac{1}{2}\|\phi\|_X+\frac{\rho_0}{2}\leqslant\rho_0.\end{aligned}\tag{3.27}$$

From (3.25) and (3.27), we see that the map $\mathcal{F}(\cdot,k_j,g,\epsilon): B(\rho_0)\to B(\rho_0)$ is uniformly contractive. The contraction mapping theorem implies that the map $\mathcal{F}(\cdot,k_j,g,\epsilon)$ has unique fixed point $\phi^*\in X$. It is that

$$\begin{aligned}\phi^*(t)&=V(t)QV^{-1}(s)z_j^*(t_0)+V(t)\int_{t_0}^{t}\big\langle QV^{-1}(s),\eta_{j_{\rho_1}}\big(\phi^*,k_j,g,\epsilon\big)(s)\big\rangle\,ds\\ &\quad-V(t)\int_t^\infty\big\langle (I-Q)V^{-1}(s),\eta_{j_{\rho_1}}\big(\phi^*,k_j,g,\epsilon\big)(s)\big\rangle\,ds,\quad \text{for } t\geqslant t_0.\end{aligned}\tag{3.28}$$

Let $\tilde M=\sup_{t\geqslant t_0}|\phi^*(t)-z_j^*(t)|e^{-\beta_0 t}$. Since $z_j^*\in\mathcal{Z}_{\beta_0}$ and $\phi^*\in X$, we know that $0\leqslant\tilde M<\infty$. From (3.23) and (3.28), we see that

$$\begin{aligned}\big|\phi^*(t)-z_j^*(t)\big| &\leqslant \int_{t_0}^{t}\big|V(t)QV^{-1}(s)\big|\big|\eta_{j_{\rho_1}}\big(\phi^*,k_j,g,\epsilon\big)(s)-\eta_{j_{\rho_1}}\big(z_j^*,k_j,g,\epsilon\big)(s)\big|\,ds\\ &\quad+\int_t^\infty\big|V(t)(I-Q)V^{-1}(s)\big|\big|\eta_{j_{\rho_1}}\big(\phi^*,k_j,g,\epsilon\big)(s)-\eta_{j_{\rho_1}}\big(z_j^*,k_j,g,\epsilon\big)(s)\big|\,ds\\ &\leqslant \frac{3\beta_0}{8}e^{-2\beta_0 t}\int_{t_0}^{t}e^{3\beta_0 s}\cdot\big(\big|\phi^*(s)-z_j^*(s)\big|e^{-\beta_0 s}\big)\,ds\\ &\quad+\frac{3\beta_0}{8}e^{2\beta_0 t}\int_t^\infty e^{-\beta_0 s}\cdot\big(\big|\phi^*(s)-z_j^*(s)\big|e^{-\beta_0 s}\big)\,ds\\ &\leqslant\left[\frac{1}{8}\big(1-e^{3\beta_0(t_0-t)}\big)+\frac{3}{8}\right]\cdot\tilde M\cdot e^{\beta_0 t}\\ &\leqslant\frac{1}{2}\tilde M\cdot e^{\beta_0 t}\end{aligned}$$

where (1) of Proposition 3.2 is used. Then we have $\tilde M\leqslant\frac{1}{2}\tilde M$ which implies that $\tilde M=0$. Hence $\phi^*(t)=z_j^*(t)\in X$ for $t\in[t_0,\infty)$. We know that $z_j^*(t)$ approaches zero like $e^{-\beta_0 t}$ as $t\to\infty$. Thus we can take larger $t_0$ if necessary such that $|z_j^*(t)|\leqslant\rho_0$ for $t\in[t_0,\infty)$.

**Case 2.** For $t\in[0,t_0]$.

As in the proof of Proposition 3.2, we take $K_1=\frac{3\beta_0}{16A}$. From (3.11), we have

$$\begin{aligned}\big|\eta_{j_{\rho_1}}\big(z_j^*,k_j,g,\epsilon\big)(t)\big| &\leqslant\big|\eta_{j_{\rho_1}}\big(z_j^*,k_j,g,\epsilon\big)(t)-\eta_{j_{\rho_1}}(0,k_j,g,\epsilon)(t)\big|+\big|\eta_{j_{\rho_1}}(0,k_j,g,\epsilon)(t)\big|\\ &\leqslant\big|h_{\rho_1}\big(z_j^*,k_j,g,\epsilon\big)(t)-h_{\rho_1}(0,k_j,g,\epsilon)(t)\big|\\ &\quad+\big|\big(Df\big(\gamma(t)\big)-Df(0)\big)z_j^*(t)\big|+\big|\eta_{j_{\rho_1}}(0,k_j,g,\epsilon)(t)\big|\\ &\leqslant\frac{3\beta_0}{16A}\big|z_j^*(t)\big|+K_2\big|z_j^*(t)\big|+\frac{3\beta_0\rho_1}{16A}e^{-\beta_0 t}\\ &\leqslant\left(\frac{3\beta_0}{16A}+K_2\right)\delta e^{\beta_0 t}+\frac{3\beta_0\rho_1}{16A}e^{-\beta_0 t}\end{aligned}\tag{3.29}$$

where (b) of Proposition 3.1 and (2) of Proposition 3.2 are used.

Substituting (3.29) in (3.23), we have

$$
\begin{aligned}
\big|z_j^*(t)\big| &\leqslant \big|V(t)QV^{-1}(0)z_j^*(0)\big| + \int_0^t \big|V(t)QV^{-1}(s)\big|\big|\eta_{j_{\rho_1}}\big(z_j^*, k_j, g, \epsilon\big)(s)\big|\,ds \\
&\quad + \int_t^\infty \big|V(t)(I-Q)V^{-1}(s)\big|\big|\eta_{j_{\rho_1}}\big(z_j^*, k_j, g, \epsilon\big)(s)\big|\,ds \\
&\leqslant Ae^{-2\beta_0 t}\big|z_j^*(0)\big| + \int_0^t Ae^{2\beta_0(s-t)}\bigg[\bigg(\frac{3\beta_0}{16A}+K_2\bigg)\delta e^{\beta_0 s} + \frac{3\beta_0\rho_1}{16A}e^{-\beta_0 s}\bigg]\,ds \\
&\quad + \int_t^\infty Ae^{2\beta_0(t-s)}\bigg[\bigg(\frac{3\beta_0}{16A}+K_2\bigg)\delta e^{\beta_0 s} + \frac{3\beta_0\rho_1}{16A}e^{-\beta_0 s}\bigg]\,ds \\
&\leqslant A\delta + \frac{3\beta_0 + 16AK_2}{12\beta_0}\delta e^{\beta_0 t_0} + \frac{\rho_1}{4} \\
&\leqslant \frac{5\rho_0}{12} + \frac{\rho_0}{3} + \frac{\rho_0}{4} = \rho_0.
\end{aligned}
\tag{3.30}
$$

From Case 1 and Case 2, we see that $|z_j^*(t)| \leqslant \rho_0$ for $t \in [0,\infty)$. Using the same method, we can get that $|z_j^*(t)| \leqslant \rho_0$ for $t \in (-\infty, 0]$. Thus we can get that $|z_j^*(t)| \leqslant \rho_0$ for $t \in \mathbb{R}$. The proof is completed. □

Let $\rho = \rho_0$ be as in Lemma 3.2. From Theorem 3.1 and Lemma 3.2, we know that $\chi(z_j^*(k_j, g, \epsilon)(t)/\rho_0) = 1$. Then the bifurcation function is

$$
\begin{aligned}
B_j(k_j, g, \epsilon) &:= Ph_{j_{\rho_0}}\big(z_j^*(k_j, g, \epsilon), k_j, g, \epsilon\big)(t) \\
&= Ph_j\big(z_j^*(k_j, g, \epsilon), k_j, g, \epsilon\big)(t) \\
&= b(t)U(t)\int_{-\infty}^{\infty} \big\langle P_{uu}U^{-1}(s), h_j\big(z_j^*(k_j, g, \epsilon), k_j, g, \epsilon\big)(s)\big\rangle\,ds \\
&= b(t)\sum_{i=1}^{d} u_i(t)\int_{-\infty}^{\infty} \big\langle u_i^\perp(s), h_j\big(z_j^*(k_j, g, \epsilon), k_j, g, \epsilon\big)(s)\big\rangle\,ds = 0.
\end{aligned}
$$

By the independence of $u_1, \ldots, u_d$, the bifurcation function is equivalent to

$$
\begin{aligned}
\tilde{H}_{ij}(k_j, g, \epsilon) &:= \int_{-\infty}^{\infty} \big\langle u_i^\perp(s), h_j\big(z_j^*(k_j, g, \epsilon), k_j, g, \epsilon\big)(s)\big\rangle\,ds \\
&= \int_{-\infty}^{\infty} \big\langle u_i^\perp(s), f\big(\gamma(s) + k_j u_{d+j}(s) + z_j^*(s)\big) - Df\big(\gamma(s)\big)\big(k_j u_{d+j}(s) + z_j^*(s)\big) \\
&\qquad - f\big(\gamma(s)\big) + g\big(\gamma(s) + k_j u_{d+j}(s) + z_j^*(s), \epsilon, \epsilon s\big)\big\rangle\,ds
\end{aligned}
$$

where $i = 1, \ldots, d$. It is clear that if $\tilde{H}_{ij}(k_j, g, \epsilon) = 0$ can be solved for $k_j = k_j^*(g, \epsilon)$, then system (1.2) has a homoclinic orbit which is given by

$$
x_j^*(t) = \gamma(t) + k_j^* u_{d+j} + z_j^*(t)
$$

where $z_j^*$ is derived in Theorem 3.1.

We introduce a function

$$\xi_j(k_j,g,\epsilon)=\begin{cases} z_j^*(k_j,g,\epsilon)/k_j, & \text{if } k_j\neq 0,\\ \dfrac{\partial}{\partial k_j}z_j^*(0,g,\epsilon), & \text{if } k_j=0.\end{cases}$$

From Theorem 3.1, we see that $\xi_j(0,0,\epsilon)=\frac{\partial}{\partial k_j}z_j^*(0,0,\epsilon)=0$. If $k_j\neq 0$, the bifurcation function can be written as

$$\begin{aligned}\tilde{H}_{ij}(k_j,g,\epsilon)={}&\int_{-\infty}^{\infty}\big\langle u_i^{\perp}(s),\, f\big(\gamma(s)+k_ju_{d+j}(s)+k_j\xi_j(s)\big)-f\big(\gamma(s)\big)\\&-Df\big(\gamma(s)\big)\big(k_ju_{d+j}(s)+k_j\xi_j(s)\big)\big\rangle\,ds\\&+\int_{-\infty}^{\infty}\big\langle u_i^{\perp}(s),\, g\big(\gamma(s)+k_ju_{d+j}(s)+k_j\xi_j(s),\epsilon,\epsilon s\big)\big\rangle\,ds\\={}&\int_{-\infty}^{\infty}\big\langle u_i^{\perp}(s),\, f\big(\gamma(s)+k_ju_{d+j}(s)+k_j\xi_j(s)\big)-f\big(\gamma(s)\big)\\&-Df\big(\gamma(s)\big)\big(k_ju_{d+j}(s)+k_j\xi_j(s)\big)\big\rangle\,ds\\&+\int_{-\infty}^{\infty}\big\langle u_i^{\perp}(s),\, g\big(\gamma(s),0,0\big)+D_1g\big(\gamma(s),0,0\big)\big(k_ju_{d+j}(s)+k_j\xi_j(s)\big)\\&+\epsilon\big[D_2g\big(\gamma(s),0,0\big)+sD_3g\big(\gamma(s),0,0\big)\big]\big\rangle\,ds+R(k_j,\epsilon)\end{aligned}\tag{3.31}$$

for $i=1,\dots,d$, where $R(k_j,\epsilon)$ is order $O(|k_ju_{d+j}+k_j\xi_j|^2+|\epsilon|^2)$.

In (3.31), the Taylor's expansion of $g(x(t),\epsilon,\epsilon t)$ along $(\gamma(t),0)$ is considered. Let

$$\begin{aligned}g\big(\gamma(t)+z(t),\epsilon,\epsilon t\big)={}&g\big(\gamma(t),0,0\big)+D_1g\big(\gamma(t),0,0\big)z(t)\\&+\epsilon\big(D_2g\big(\gamma(t),0,0\big)+tD_3g\big(\gamma(t),0,0\big)\big)+\tilde{g}+\epsilon^2\end{aligned}\tag{3.32}$$

where $\tilde{g}$ is remainder and $\|\tilde{g}\|$ is order $O(\|z\|^2)$. Let

$$G=\big\{\tilde{g}\colon\ \tilde{g}\text{ is derived in (3.32)},\ g\in\mathcal{G}\big\}.$$

Note that $\|g\|_{C^3}$ is small and $|u_i^{\perp}(t)|$ approaches zero like $e^{-2\beta_0|t|}$ as $t\to\pm\infty$, $i=1,\dots,d$. We have

$$\begin{aligned}&\left|\int_{-\infty}^{\infty}\big\langle u_i^{\perp}(s),D_2g\big(\gamma(s),0,0\big)+sD_3g\big(\gamma(s),0,0\big)\big\rangle\,ds\right|\\&\quad\leqslant\int_{-\infty}^{\infty}\big|u_i^{\perp}(s)\big|\cdot\big(1+|s|\big)\|g\|_{C^3}\,ds\\&\quad=\|g\|_{C^3}\int_{-\infty}^{\infty}\big|u_i^{\perp}(s)\big|\cdot\big(1+|s|\big)\,ds<\infty,\quad i=1,\dots,d.\end{aligned}$$

For $g\in\mathcal{G}$, Eq. (3.31) is

$$\tilde{H}_{ij}(k_j,\tilde{g},\epsilon)=\int_{-\infty}^{\infty}\big\langle u_i^{\perp}(s),\, f\big(\gamma(s)+k_ju_{d+j}(s)+k_j\xi_j(s)\big)-f\big(\gamma(s)\big)$$

$$- Df\big(\gamma(s)\big)\big(k_j u_{d+j}(s) + k_j\xi_j(s)\big)\big\rangle ds$$
$$+ \int_{-\infty}^{\infty} \big\langle u_i^{\perp}(s), D_1 g\big(\gamma(s),0,0\big)\big(k_j u_{d+j}(s) + k_j\xi_j(s)\big)\big\rangle ds$$
$$+ O\big(\|\tilde{g}\|\big) + O(\epsilon), \quad i = 1,\dots,d. \tag{3.33}$$

For any $m$, $m = 1,\dots,d$, our goal is find $m$ homoclinic solutions of (1.2). Our strategy is to find some special subclass $\mathfrak{M}_m$ with finite codimension of $\mathcal{G}$ to realize the goal. Let

$$\tilde{M}(i,j) = (a_{kl})_{d\times d}$$

where $a_{kl} = \delta_{ik}\delta_{jl}$, $i,j,k,l = 1,\dots,d$. We choose some special $g \in \mathcal{G}$ such that

$$D_1 g(\gamma,0,0) = \sum_{i,j=1}^{d} \beta_{ij} M_{ij} P_{ss} \tag{3.34}$$

where $M_{ij} : \operatorname{span}\{u_{d+1},\dots,u_{2d}\} \to \operatorname{span}\{u_1,\dots,u_d\}$ is defined by

$$M_{ij}(u_{d+1},\dots,u_{2d}) = (u_1,\dots,u_d)\tilde{M}(i,j).$$

With the choice of (3.34), Eq. (3.33) is

$$\begin{aligned}
\tilde{H}_{ij}(k_j,\beta,\tilde{g},\epsilon) &= \int_{-\infty}^{\infty} \big\langle u_i^{\perp}(s), f\big(\gamma(s) + k_j u_{d+j}(s) + k_j\xi_j(s)\big) - f\big(\gamma(s)\big) \\
&\quad - Df\big(\gamma(s)\big)\big(k_j u_{d+j}(s) + k_j\xi_j(s)\big)\big\rangle ds \\
&\quad + \int_{-\infty}^{\infty} \left\langle u_i^{\perp}(s), \sum_{i,j=1}^{d} k_j\beta_{ij} M_{ij} P_{ss}\big(u_{d+j}(s) + \xi_j(s)\big)\right\rangle ds \\
&\quad + O\big(\|\tilde{g}\|\big) + O(\epsilon) \\
&= \int_{-\infty}^{\infty} \big\langle u_i^{\perp}(s), D^2 f\big(\gamma(s)\big)\big(k_j u_{d+j}(s) + k_j\xi_j(s)\big)^2\big\rangle ds \\
&\quad + \int_{-\infty}^{\infty} \left\langle u_i^{\perp}(s), \sum_{i,j=1}^{d} k_j\beta_{ij} M_{ij} P_{ss}\big(u_{d+j}(s) + \xi_j(s)\big)\right\rangle ds \\
&\quad + O\big(\|k_j u_{d+j} + k_j\xi_j\|^3\big) + O\big(\|\tilde{g}\|\big) + O(\epsilon)
\end{aligned} \tag{3.35}$$

where $\beta = (\beta_{11},\dots,\beta_{dd})$.

Let

$$H_{ij}(k_j,\beta,\tilde{g},\epsilon) = \begin{cases} \tilde{H}_{ij}(k_j,\beta,\tilde{g},\epsilon)/k_j, & \text{if } k_j \neq 0, \\ \frac{\partial \tilde{H}_{ij}}{\partial k_j}(0,\beta,\tilde{g},\epsilon), & \text{if } k_j = 0. \end{cases}$$

If $k_j \neq 0$, we know that $\tilde{H}_{ij}(k_j,\beta,\tilde{g},\epsilon) = 0$ if and only if $H_{ij}(k_j,\beta,\tilde{g},\epsilon) = 0$. Through direct calculation, we have

$$\begin{aligned}
\frac{\partial \tilde{H}_{ij}}{\partial k_j}(0,\beta,\tilde{g},\epsilon) &= \int_{-\infty}^{\infty} \left\langle u_i^{\perp}(s), \sum_{i,j=1}^{d} \beta_{ij} M_{ij} P_{ss}\big(u_{d+j}(s) + \xi_j(s)\big)\right\rangle ds + O\big(\|\tilde{g}\|\big) + O(\epsilon) \\
&= \beta_{ij} + \int_{-\infty}^{\infty} \left\langle u_i^{\perp}(s), \sum_{i,j=1}^{d} \beta_{ij} M_{ij} P_{ss}\xi_j(s)\right\rangle ds + O\big(\|\tilde{g}\|\big) + O(\epsilon).
\end{aligned}$$

Then we have

$$H_{ij}(0,0,0,0)=0$$

and

$$\frac{\partial H_{ij}}{\partial \beta_{kl}}\bigg|_{(0,0,0,0)}=\delta_{ik}\delta_{jl}+o(1) \tag{3.36}$$

as $\|g\|$ goes to zero. Moreover,

$$\begin{aligned}
\frac{H_{ij}}{\partial k_j}&=\frac{\partial}{\partial k_j}\left(\frac{\tilde{H}_{ij}}{k_j}\right)\\
&=\int_{-\infty}^{\infty}\big\langle u_i^{\perp}(s),D^2f\big(\gamma(s)\big)\big(u_{d+j}(s)+\xi_j(s)\big)^2\big\rangle\,ds+O\big(|k_j|\big)+O\big(\|\tilde{g}\|\big)+O(\epsilon)\\
&=\int_{-\infty}^{\infty}\big\langle u_i^{\perp}(s),D^2f\big(\gamma(s)\big)u_{d+j}(s)u_{d+j}(s)\big\rangle\,ds\\
&\quad+\int_{-\infty}^{\infty}\big\langle u_i^{\perp}(s),2D^2f\big(\gamma(s)\big)u_{d+j}(s)\xi_j(s)+D^2f\big(\gamma(s)\big)\xi_j(s)\xi_j(s)\big\rangle\,ds\\
&\quad+O\big(|k_j|\big)+O\big(\|\tilde{g}\|\big)+O(\epsilon).
\end{aligned}$$

Since $\xi_j(0,0,\epsilon)=0$, we have

$$\int_{-\infty}^{\infty}\big\langle u_i^{\perp}(s),2D^2f\big(\gamma(s)\big)u_{d+j}(s)\xi_j(s)+D^2f\big(\gamma(s)\big)\xi_j(s)\xi_j(s)\big\rangle\,ds=o(1)$$

as $\|g\|$ goes to zero. Then we can get that

$$\frac{H_{ij}}{\partial k_j}\bigg|_{(0,0,0,0)}=\Delta_{ij}+o(1) \tag{3.37}$$

as $\|g\|$ goes to zero. For convenience, let

$$H_l=(H_{1l},\dots,H_{dl}),\qquad \eta_l=(k_l,\beta_{2l},\dots,\beta_{dl}),\qquad l=1,\dots,d.$$

From (3.36) and (3.37), we have

$$M_j:=\frac{\partial H_j}{\partial \eta_j}\bigg|_{(0,0,0,0)}=\begin{bmatrix}\Delta_{1j}+o(1) & o(1) & \dots & o(1)\\ \Delta_{2j}+o(1) & 1+o(1) & \dots & o(1)\\ \dots & \dots & \dots & \dots\\ \Delta_{dj}+o(1) & o(1) & \dots & 1+o(1)\end{bmatrix}_{d\times d} \tag{3.38}$$

as $\|g\|$ goes to zero. It is clear that $M_j$ is nonsingular for small $\|g\|$ since $\Delta_{1j}\neq 0$. With the same argument, we have

$$M_l:=\frac{\partial H_l}{\partial \eta_l}\bigg|_{(0,0,0,0)}=\begin{bmatrix}\Delta_{1l}+o(1) & o(1) & \dots & o(1)\\ \Delta_{2l}+o(1) & 1+o(1) & \dots & o(1)\\ \dots & \dots & \dots & \dots\\ \Delta_{dl}+o(1) & o(1) & \dots & 1+o(1)\end{bmatrix}_{d\times d} \tag{3.39}$$

where $l=1,\dots,d$ and $M_l$ is nonsingular.

**Theorem 3.2.** *Assume that* (H1)–(H5) *hold. For any $m$, $m=1,\dots,d$, there are $\epsilon_m>0$ and submanifold $\mathfrak{M}_m\subset\mathcal{G}$ with codimension $dm$, $0\in\mathfrak{M}_m$, such that for $\epsilon\in(-\epsilon_m,0)\cup(0,\epsilon_0)$ and each small $g\in\mathfrak{M}_m$, the system* (1.2) *has $m$ linearly independent homoclinic solutions.*

**Proof.** Let

$$\hat{H}(k,\beta,\tilde{g},\epsilon) = \big(H_1(k,\beta,\tilde{g},\epsilon),\dots,H_m(k,\beta,\tilde{g},\epsilon)\big)$$

where $k=(k_1,\dots,k_m)$. We rearrange variables.

$$H(\eta,\bar{\beta},\tilde{\beta},\tilde{g},\epsilon) := \hat{H}(k,\beta,\tilde{g},\epsilon)$$

where $\eta=(\eta_1,\dots,\eta_m)\in\mathbb{R}^{md}$, $\bar{\beta}=(\beta_{11},\dots,\beta_{1m})\in\mathbb{R}^m$, $\tilde{\beta}=(\beta_{1m+1},\dots,\beta_{2d})\in\mathbb{R}^{d(d-m)}$. It is clear that the system (1.2) has $m$ homoclinic solutions if $H(k,\beta,\tilde{g},\epsilon)=0$. Notice that

$$H(0,0,0,0)=0.$$

From (3.39), we have

$$\frac{\partial H}{\partial(\eta_1,\dots,\eta_m)}\bigg|_{(0,0,0,0)} = \begin{bmatrix} M_1 & \mathcal{O} & \dots & \mathcal{O} \\ \mathcal{O} & M_2 & \dots & \mathcal{O} \\ & \dots & \dots & \\ \mathcal{O} & \mathcal{O} & \dots & M_m \end{bmatrix}_{md\times md}$$

as $\|g\|$ goes to zero, where

$$\mathcal{O} = \begin{bmatrix} 0 & o(1) & \dots & o(1) \\ 0 & o(1) & \dots & o(1) \\ & \dots & \dots & \\ 0 & o(1) & \dots & o(1) \end{bmatrix}_{d\times d}.$$

It is clear that $\frac{\partial H}{\partial(\eta_1,\dots,\eta_m)}|_{(0,0,0,0)}$ is nonsingular matrix. Then the implicit function theorem implies that there are neighborhoods $U_1\in\mathbb{R}^m$, $U_2\in\mathbb{R}^{d(d-m)}$, $U_3\in G$, $\epsilon_m>0$ and $C^1$ functions

$$\begin{cases} k_1 = k_1^*(\bar{\beta},\tilde{\beta},\tilde{g},\epsilon),\dots,k_m = k_m^*(\bar{\beta},\tilde{\beta},\tilde{g},\epsilon) \\ \beta_{21} = \beta_{21}^*(\bar{\beta},\tilde{\beta},\tilde{g},\epsilon),\dots,\beta_{d1} = \beta_{d1}^*(\bar{\beta},\tilde{\beta},\tilde{g},\epsilon) \\ \dots \qquad\qquad \dots \\ \beta_{2m} = \beta_{2m}^*(\bar{\beta},\tilde{\beta},\tilde{g},\epsilon),\dots,\beta_{dm} = \beta_{dm}^*(\bar{\beta},\tilde{\beta},\tilde{g},\epsilon) \end{cases} \tag{3.40}$$

for $(\bar{\beta},\tilde{\beta},\tilde{g},\epsilon)\in U_1\times U_2\times U_3\times(-\epsilon_m,0)\cup(0,\epsilon_m)$ such that

$$k_i^*(0,0,0,0)=0,\quad \beta_{jk}^*(0,0,0,0)=0,\quad \text{for } i,k=1,\dots,m,\ j=2,\dots,m$$

and

$$H\big(k^*(\bar{\beta},\tilde{\beta},\tilde{g},\epsilon),\beta^*(\bar{\beta},\tilde{\beta},\tilde{g},\epsilon),\bar{\beta},\tilde{\beta},\tilde{g},\epsilon\big)=0 \tag{3.41}$$

for $(\bar{\beta},\tilde{\beta},\tilde{g},\epsilon)\in U_1\times U_2\times U_3\times(-\epsilon_m,0)\cup(0,\epsilon_0)$, where $k^* := (k_1^*,\dots,k_m^*)$ and $\beta^* := (\beta_{21}^*,\dots,\beta_{dm}^*)$. From the transformation (3.2), the $m$ homoclinic solutions are given by

$$\begin{cases} x_1(t) = \gamma(t) + k_1^* u_{d+1}(t) + z_1^*(t) \\ \dots \qquad \dots \\ x_m(t) = \gamma(t) + k_m^* u_{d+m}(t) + z_m^*(t) \end{cases} \tag{3.42}$$

where $k_j^*$ and $z_j^*$ are as in (3.40) and Theorem 3.1 respectively, $j=1,\dots,m$.

In order to prove that the $m$ solutions given in (3.42) are linear independence, we need to prove $k_j^*\neq 0$. From system (3.41), we choose $m$ equations and get

$$\bar{H}\big(k^*,\beta^*,\bar{\beta},\tilde{\beta},\tilde{g},\epsilon\big) := \big(H_{11}\big(k^*,\beta^*,\bar{\beta},\tilde{\beta},\tilde{g},\epsilon\big),\dots,H_{1m}\big(k^*,\beta^*,\bar{\beta},\tilde{\beta},\tilde{g},\epsilon\big)\big)=0. \tag{3.43}$$

Differentiating (3.43) in $\bar{\beta}$, we have

$$D_1\bar{H}\cdot\frac{\partial k^*}{\partial\bar{\beta}} + D_2\bar{H}\cdot\frac{\partial\beta^*}{\partial\bar{\beta}} + \frac{\partial\bar{H}}{\partial\bar{\beta}} = 0. \tag{3.44}$$

From (3.37), we have

$$D_1\bar{H}|_{(0,0,0,0)} = \begin{bmatrix} \Delta_{11}+o(1) & o(1) & \dots & o(1) \\ o(1) & \Delta_{12}+o(1) & \dots & o(1) \\ & \dots & \dots & \\ o(1) & o(1) & \dots & \Delta_{1m}+o(1) \end{bmatrix}_{m\times m}.$$

Moreover, from (3.36) we get

$$\frac{\partial \bar{H}}{\partial \bar{\beta}}\bigg|_{(0,0,0,0)} = \begin{bmatrix} 1+o(1) & o(1) & \dots & o(1) \\ o(1) & 1+o(1) & \dots & o(1) \\ & \dots & \dots & \\ o(1) & o(1) & \dots & 1+o(1) \end{bmatrix}_{m\times m}$$

and $|D_2\bar{H}|_{(0,0,0,0)} = o(1)$. Since $\beta^* \in C^1$, we have that $|D_2\bar{H}\cdot \frac{\partial \beta^*}{\partial \bar{\beta}}|_{(0,0,0,0)} = o(1)$. Thus we can get from (3.44) that

$$\frac{\partial k^*}{\partial \bar{\beta}}\bigg|_{(0,0,0,0)} = -\left(D_1\bar{H}|_{(0,0,0,0)}\right)^{-1}\frac{\partial \bar{H}}{\partial \bar{\beta}}\bigg|_{(0,0,0,0)} + o(1)$$

as $\|g\|$ goes to zero. It is clear that $\frac{\partial k^*}{\partial \bar{\beta}}|_{(0,0,0,0)}$ is nonsingular. Thus there exists appropriate $\bar{\beta}^* = (\beta_1^*, \dots, \beta_m^*)$, such that $k_j^*(\beta^*, 0, 0, 0) \neq 0$. From the continuity of $k_j^*$, we can shrink $U_2$, $U_3$ and $\epsilon_m > 0$ if necessary, such that

$$k_j^*\big(\bar{\beta}^*, \tilde{\beta}, \tilde{g}, \epsilon\big) \neq 0, \quad j = 1, \dots, m \tag{3.45}$$

for $(\tilde{\beta}, \tilde{g}, \epsilon) \in U_2 \times U_3 \times (-\epsilon_m, 0) \cup (0, \epsilon_m)$.

From (3.42), to prove the linear independence of the $m$ homoclinic solutions is sufficient to prove the linear independence of the functions

$$k_1^* u_{d+1} + z_1^*, \dots, k_m^* u_{d+m} + z_m^*.$$

If there are some $a_j \in \mathbb{R}$, such that

$$\sum_{j=1}^m a_j\big(k_j^* u_{d+j} + z_j^*\big) = \sum_{j=1}^m a_j k_j^* u_{d+j} + \sum_{j=1}^m a_j z_j^* = 0.$$

From (3.2), we see that $\sum_{j=1}^m a_j z_j^* \in S$ and $\sum_{j=1}^m a_j k_j^* u_{d+j} \in \operatorname{span}\{u_{d+1}, \dots, u_{2d}\}$. Thus we have

$$\sum_{j=1}^m a_j k_j^* u_{d+j} = 0.$$

By the independence of $u_{d+1}, \dots, u_{d+m}$, we have $a_j k_j^* = 0$ which implies from (3.45) that $a_j = 0$. Thus the $m$ homoclinic solution are linearly independent.

We now establish the codimension of the bifurcation manifold. Let $\mathcal{G}_0$ be the subclass of $\mathcal{G}$ such that $D_1 g(\gamma, 0, 0)$ has the form of (3.34) for each $g \in \mathcal{G}_0$. Let

$$\mathfrak{M}_m = \big\{g \in \mathcal{G}_0\colon\ (\beta_{21}, \dots, \beta_{dm}) = \big(\beta_{21}^*\big(\bar{\beta}^*, \tilde{\beta}, \tilde{g}, \epsilon\big), \dots, \beta_{dm}^*\big(\bar{\beta}^*, \tilde{\beta}, \tilde{g}, \epsilon\big)\big)\big\}$$

for $(\tilde{\beta}, \tilde{g}, \epsilon) \in U_2 \times U_3 \times (-\epsilon_m, 0) \cup (0, \epsilon_m)$. In $\mathfrak{M}_m$, the $dm$ parameters, $(\beta_{11}, \dots, \beta_{d1}, \dots, \beta_{md})$, are restricted. Thus $\mathfrak{M}_m$ defines a submanifold with codimension $md$ in $\mathcal{G}_0$ and hence in $\mathcal{G}$. The proof is completed. □

Let $\epsilon_0 = \min\{\epsilon_1, \dots, \epsilon_d\}$. From Theorem 3.2, we see that there are $d$ submanifolds $\mathfrak{M}_k \subset \mathcal{G}$ with codimension $dk$, $0 \in \mathfrak{M}_k$, $k = 1, \dots, d$ and neighborhood $\mathcal{U} \subset \mathcal{G}$, $0 \in \mathcal{U}$ such that for any $k$, $k = 1, \dots, d$, the system (1.2) has $k$ linearly independent homoclinic solutions for every $g \in \mathcal{U} \cap (\mathfrak{M}_k/(\mathfrak{M}_{k+1} \cup \dots \cup \mathfrak{M}_d))$ and $\epsilon \in (-\epsilon_0, 0) \cup (0, \epsilon_0)$. The proof of Theorem 2.1 is finished.

We give an example to illustrate the theory. As in [11], we consider the system

$$\dot{x}(t) = f\big(x(t)\big) + g\big(x(t), \epsilon, \epsilon t\big) \tag{3.46}$$

where $x=(x_1,x_2,x_3,x_4)$, $f(x)=(x_2,x_1-x_1^3-x_1x_3^2,x_3,x_3-\frac{4}{3}x_3^3-\frac{2}{3}x_1^3)$ and $g:\mathbb{R}^4\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}^4$. Eq. (3.46) is the form of (1.2). The corresponding unperturbed system is

$$\begin{cases}\dot{x}_1(t)=x_2,\\ \dot{x}_1(t)=x_1-x_1^3-x_1x_3^2,\\ \dot{x}_3(t)=x_4,\\ \dot{x}_4(t)=x_3-\dfrac{4}{3}x_3^3-\dfrac{2}{3}x_1^3.\end{cases}\tag{3.47}$$

Let $r(t)=\operatorname{sech}(t)$. One can check that Eq. (3.47) has a homoclinic solution $\gamma(t)=(r(t),\dot{r}(t),r(t),\dot{r}(t))$. It is clear that 0 is a fixed point and $\lim_{t\to\pm\infty}\gamma(t)=0$. Moreover, the matrix $Df(0)$ has eigenvalues $\{-1,-1,1,1\}$ which lie off the imaginary axis. Thus (H1)–(H4) are satisfied. The variation equation of (3.47) along $\gamma(t)$ is

$$\dot{u}(t)=Df\bigl(\gamma(t)\bigr)u(t)\tag{3.48}$$

where

$$Df\bigl(\gamma(t)\bigr)=\begin{bmatrix}0&1&0&0\\ 1-4r^2(t)&0&-2r^2(t)&0\\ 0&0&0&1\\ -2r^2(t)&0&1-4r^2(t)&0\end{bmatrix}.$$

As in [11], let $P(t)$ and $Q(t)$ be differential functions satisfying $\dot{P}\dot{r}^2=1$ and $\dot{Q}r^2=1$. Then (3.48) has fundamental solutions

$$\begin{aligned}&u_1=\bigl(Qr,(Qr)^{\cdot},-Qr,-(Qr)^{\cdot}\bigr), && u_2=\bigl(P\dot{r},(Q\dot{r})^{\cdot},Q\dot{r},(Q\dot{r})^{\cdot}\bigr),\\ &u_3=(r,\dot{r},-r,-\dot{r}), && u_4=(r+\dot{r},\dot{r}+\ddot{r},-r+\dot{r},-\dot{r}+\ddot{r}).\end{aligned}$$

It is clear that $u_1$ and $u_2$ are unbounded solutions of (3.48), $u_3$ and $u_4$ are bounded ones. Thus $d=2$. The bounded solutions of the adjoint equation of (3.48) are

$$u_1^{\perp}=\frac{1}{2}(-\dot{r},r,\dot{r},-r),\qquad u_2^{\perp}=\frac{1}{2}(-\ddot{r},\dot{r},-\ddot{r},\dot{r}).$$

As required in Theorem 2.1, we need to calculate $\Delta_{1j}$, $j=1,2$.

$$\begin{aligned}\Delta_{11}&=\int_{-\infty}^{\infty}\bigl\langle u_1^{\perp}(t),D^2f\bigl(\gamma(t)\bigr)u_3(t)u_3(t)\bigr\rangle\,dt\\ &=\int_{-\infty}^{\infty}\bigl\langle u_1^{\perp}(t),\operatorname{col}\bigl(0,-4r^3(t),0,-12r^3(t)\bigr)\bigr\rangle\,dt\\ &=\int_{-\infty}^{\infty}4r^3(t)\,dt=\frac{16}{3},\\ \Delta_{12}&=\int_{-\infty}^{\infty}\bigl\langle u_1^{\perp}(t),D^2f\bigl(\gamma(t)\bigr)u_4(t)u_4(t)\bigr\rangle\,dt\\ &=\int_{-\infty}^{\infty}\bigl\langle u_1^{\perp}(t),\operatorname{col}\bigl(0,-4r^3(t)-12r(t)\dot{r}^2(t),0,-12r^3(t)-12r(t)\dot{r}^2(t)\bigr)\bigr\rangle\,dt\\ &=\int_{-\infty}^{\infty}4r^3(t)\,dt=\frac{16}{3}.\end{aligned}$$

Thus the assumption (H5) holds. As well as (H1)–(H4) hold, Theorem 2.1 can be applied to system (3.46). There are $\epsilon_0>0$ and two submanifolds $\mathfrak{M}_m\subset\mathcal{G}$ with codimension $2m$, $0\in\mathfrak{M}_m$, $m=1,2$, such that for $\epsilon\in(-\epsilon_0,0)\cup(0,\epsilon_0)$ and every small $g\in\mathfrak{M}_1/\mathfrak{M}_2$ (respectively $g\in\mathfrak{M}_2$) the system (3.46) has one (respectively two) homoclinic solutions.

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