The limiting behavior of the value-function for variational problems arising in continuum mechanics

Alexander J. Zaslavski

Department of Mathematics, The Technion-Israel Institute of Technology, 32000 Haifa, Israel

Received 17 May 2009; received in revised form 16 July 2009; accepted 16 July 2009
Available online 5 August 2009

Abstract

In this paper we study the limiting behavior of the value-function for one-dimensional second order variational problems arising in continuum mechanics. The study of this behavior is based on the relation between variational problems on bounded large intervals and a limiting problem on $[0, \infty)$. © 2009 Elsevier Masson SAS. All rights reserved.

MSC: 49J99

Keywords: Good function; Infinite horizon; Minimal long-run average cost growth rate; Variational problem

1. Introduction

The study of properties of solutions of optimal control problems and variational problems defined on infinite domains and on sufficiently large domains has recently been a rapidly growing area of research. See, for example, [3,5,6,15–19,21–24] and the references mentioned therein. These problems arise in engineering [8], in models of economic growth [10,25], in infinite discrete models of solid-state physics related to dislocations in one-dimensional crystals [2,20] and in the theory of thermodynamical equilibrium for materials [7,9,11–14]. In this paper we study the limiting behavior of the value-function for variational problems arising in continuum mechanics which were considered in [7,9,11–14,21–24]. The study of this behavior is based on the relation between variational problems on bounded large intervals and a limiting problem on $[0, \infty)$.

In this paper we consider the variational problems

$$\int_0^T f(w(t), w'(t), w''(t)) \, dt \to \min, \quad w \in W^{2,1}([0, T]),$$

$$(w(0), w'(0)) = x \quad \text{and} \quad (w(T), w'(T)) = y,$$  \hspace{1cm} (P)

E-mail address: ajzasl@tx.technion.ac.il.
where \( T > 0, x, y \in R^2, W^{2,1}([0, T]) \subset C^1([0, T]) \) is the Sobolev space of functions possessing an integrable second derivative \([1]\) and \( f \) belongs to a space of functions to be described below. The interest in variational problems of the form \((P)\) and the related problem on the half line:

\[
\lim_{T \to \infty} T^{-1} \int_0^T f(w(t), w'(t), w''(t)) \, dt \to \min, \quad w \in W^{2,1}_{\text{loc}}([0, \infty))
\]

\((P_{\infty})\)

stems from the theory of thermodynamical equilibrium for second-order materials developed in [7,9,11–14]. Here \( W^{2,1}_{\text{loc}}([0, \infty)) \subset C^1([0, \infty)) \) denotes the Sobolev space of functions possessing a locally integrable second derivative \([1]\) and \( f \) belongs to a space of functions to be described below.

We are interested in properties of the valued-function for the problem \((P)\) which are independent of the length of the interval, for all sufficiently large intervals.

Let \( a = (a_1, a_2, a_3, a_4) \in R^4, a_i > 0, i = 1, 2, 3, 4 \) and let \( \alpha, \beta, \gamma \) be positive numbers such that \( 1 \leq \beta < \alpha, \beta \leq \gamma, \gamma > 1 \). Denote by \( \mathcal{M}(\alpha, \beta, \gamma, a) \) the set of all functions \( f : R^3 \to R^1 \) such that:

\[
f(w, p, r) \geq a_1|w|^\alpha - a_2|p|^\beta + a_3|r|^\gamma - a_4 \quad \text{for all } (w, p, r) \in R^3;
\]

\[
f, \partial f/\partial p \in C^2, \quad \partial f/\partial r \in C^3, \quad \partial^2 f/\partial r^2(w, p, r) > 0 \quad \text{for all } (w, p, r) \in R^3;
\]

there is a monotone increasing function \( M_f : [0, \infty) \to [0, \infty) \) such that for every \( (w, p, r) \in R^3 \)

\[
\max \{f(w, p, r), |\partial f/\partial p(w, p, r)|, |\partial f/\partial r(w, p, r)|\} \leq M_f([w] + |p|)(1 + |r|^\gamma).
\]

Let \( f \in \mathcal{M}(\alpha, \beta, \gamma, a) \). Of special interest is the minimal long-run average cost growth rate

\[
\mu(f) = \inf \left\{ \liminf_{T \to \infty} T^{-1} \int_0^T f(w(t), w'(t), w''(t)) \, dt : w \in A_x \right\},
\]

where

\[
A_x = \left\{ v \in W^{2,1}_{\text{loc}}([0, \infty)) : (v(0), v'(0)) = x \right\}.
\]

It was shown in [9] that \( \mu(f) \in R^1 \) is well defined and is independent of the initial vector \( x \). A function \( w \in W^{2,1}_{\text{loc}}([0, \infty)) \) is called an \((f)\)-good function if the function

\[
\phi_w^f : T \to \int_0^T \left[ f(w(t), w'(t), w''(t)) - \mu(f) \right] \, dt, \quad T \in (0, \infty)
\]

is bounded. For every \( w \in W^{2,1}_{\text{loc}}([0, \infty)) \) the function \( \phi_w^f \) is either bounded or diverges to \( \infty \) as \( T \to \infty \) and moreover, if \( \phi_w^f \) is a bounded function, then

\[
\sup \{ \|w(t), w'(t)\| : t \in [0, \infty) \} < \infty
\]

[22, Proposition 3.5]. Leizarowitz and Mizel [9] established that for every \( f \in \mathcal{M}(\alpha, \beta, \gamma, a) \) satisfying \( \mu(f) < \inf f(w, 0, s) : (w, s) \in R^2 \) there exists a periodic \((f)\)-good function. In [21] it was shown that a periodic \((f)\)-good function exists for every \( f \in \mathcal{M}(\alpha, \beta, \gamma, a) \).

Let \( f \in \mathcal{M}(\alpha, \beta, \gamma, a) \). For each \( T > 0 \) define a function \( U^f_T : R^2 \times R^2 \to R^1 \) by

\[
U^f_T(x, y) = \inf \left\{ \int_0^T f(w(t), w'(t), w''(t)) \, dt : w \in W^{2,1}([0, T]),
\right\}
\]

\[
(w(0), w'(0)) = x \text{ and } (w(T), w'(T)) = y
\]

(1.5)
In [9], analyzing problem \((P_\infty)\) Leizarowitz and Mizel studied the function \(U_f^T : R^2 \times R^2 \to R^1, T > 0\) and established the following representation formula
\[
U_f^T(x, y) = T \mu(f) + \pi_f^T(x) - \pi_f^T(y) + \theta_f^T(x, y), \quad x, y \in R^2, \quad T > 0,
\] (1.6)
where \(\pi_f : R^2 \to R^1\) and \((T, x, y) \to \theta_f^T(x, y)\) and \((T, x, y) \to U_f^T(x, y), x, y \in R^2, T > 0\) are continuous functions,
\[
\pi_f(x) = \inf \left\{ \liminf_{T \to \infty} \int_0^T \left[ f(w(t), w'(t), w''(t)) - \mu(f) \right] dt : \right. \\
\left. w \in W^{2,1}_{loc}(0, \infty) \right\} \quad , \quad x \in R^2,
\] (1.7)
\(\theta_f^T(x, y) \geq 0\) for each \(T > 0\), and each \(x, y \in R^2\), and for every \(T > 0\), and every \(x \in R^2\) there is \(y \in R^2\) satisfying \(\theta_f^T(x, y) = 0\).

Denote by \(| \cdot |\) the Euclidean norm in \(R^n\). For every \(x \in R^n\) and every nonempty set \(\Omega \subset R^n\) set
\[
d(x, \Omega) = \inf \{|x - y| : y \in \Omega\}.
\]
For each function \(g : X \to R^1 \cup \{\infty\}\), where the set \(X\) is nonempty, put
\[
\inf(g) = \inf \{g(z) : z \in X\}.
\]
Let \(f \in \mathfrak{M}(\alpha, \beta, \gamma, a)\). It is easy to see that
\[
\mu(f) \leq \inf \{f(t, 0, 0) : t \in R^1\}.
\]
If \(\mu(f) = \inf \{f(t, 0, 0) : t \in R^1\}\), then there is an \((f)\)-good function which is a constant function. If \(\mu(f) < \inf \{f(t, 0, 0) : t \in R^1\}\), then there exists a periodic \((f)\)-good function which is not a constant function. It was shown in [14] that in this case the extremals of \((P_\infty)\) have interesting asymptotic properties. In [26] we equipped the space \(\mathfrak{M}(\alpha, \beta, \gamma, a)\) with a natural topology and showed that there exists an open everywhere dense subset \(\mathcal{F}\) of this topological space such that for every \(f \in \mathcal{F}\),
\[
\mu(f) < \inf \{f(t, 0, 0) : t \in R^1\}.
\]
In other words, the inequality above holds for a typical integrand \(f \in \mathfrak{M}(\alpha, \beta, \gamma, a)\).

In the present paper for an integrand \(f \in \mathfrak{M}(\alpha, \beta, \gamma, a)\) satisfying
\[
\mu(f) < \inf \{f(t, 0, 0) : t \in R^1\}
\]
we study the limiting behavior of the value-function \(U_f^T\) as \(T \to \infty\) and establish the following two results.

**Theorem 1.1.** Let \(f \in \mathfrak{M}(\alpha, \beta, \gamma, a)\) satisfy \(\mu(f) < \inf \{f(t, 0, 0) : t \in R^1\}\). Then for each \(x, y \in R^2\) there exists
\[
U_f^T(x, y) := \lim_{T \to \infty} \{U_f^T(x, y) - T \mu(f)\}.
\]
Moreover, \(U_f^T(x, y) - T \mu(f) \to U_f^\infty(x, y)\) as \(T \to \infty\) uniformly on bounded subsets of \(R^2 \times R^2\).

**Theorem 1.2.** Let \(f \in \mathfrak{M}(\alpha, \beta, \gamma, a)\) satisfy \(\mu(f) < \inf \{f(t, 0, 0) : t \in R^1\}\). Then there exists a nonempty compact set \(E_\infty \subset R^2 \times R^2\) such that
\[
E_\infty = \{(x, y) \in R^2 \times R^2 : U_f^T(x, y) = \inf \{U_f^T\}\}.
\]
Moreover, for any \(\epsilon > 0\) there exist \(\delta > 0\) and \(\bar{T} > 0\) such that if \(T \geq \bar{T}\) and if \(x, y \in R^2\) satisfy \(U_f^T(x, y) \leq \inf(U_f^T) + \delta\), then \(d((x, y), E_\infty) \leq \epsilon\).

The paper is organized as follows. Section 2 contains preliminaries. In Section 3 we prove several auxiliary results. Theorems 1.1 and 1.2 are proved in Sections 4 and 5 respectively.
2. Preliminaries

For \( \tau > 0 \) and \( v \in W^{2,1}([0, \tau]) \) we define \( X_v : [0, \tau] \to \mathbb{R}^2 \) as follows:

\[
X_v(t) = (v(t), v'(t)), \quad t \in [0, \tau].
\]

We also use this definition for \( v \in W^{2,1}_{\text{loc}}([0, \infty)) \) and \( v \in W^{2,1}_{\text{loc}}(\mathbb{R}^1) \).

Put

\[
\mathcal{M} = \mathcal{M}(\alpha, \beta, \gamma, a).
\]

We consider functionals of the form

\[
\Gamma^f (T_1, T_2, v) = \int_{T_1}^{T_2} f(v(t), v'(t), v''(t)) dt,
\]

where \(-\infty < T_1 < T_2 < +\infty, v \in W^{2,1}([T_1, T_2]) \) and \( f \in \mathcal{M} \).

If \( v \in W^{2,1}_{\text{loc}}([0, \infty)) \) satisfies

\[
\sup\{ |X_v(t)| : t \in [0, \infty) \} < \infty,
\]

then the set of limiting points of \( X_v(t) \) as \( t \to \infty \) is denoted by \( \Omega(v) \).

For each \( f \in \mathcal{M} \) denote by \( \mathcal{A}(f) \) the set of all \( w \in W^{2,1}_{\text{loc}}([0, \infty)) \) which have the following property:

There is \( T_w > 0 \) such that

\[
w(t + T_w) = w(t) \quad \text{for all} \ t \in [0, \infty) \quad \text{and} \quad \Gamma^f (0, T_w, w) = \mu(f)T_w.
\]

In other words \( \mathcal{A}(f) \) is the set of all periodic \((f)\)-good functions. By a result of [21], \( \mathcal{A}(f) \neq \emptyset \) for all \( f \in \mathcal{M} \).

The following result established in [13, Lemma 3.1] describes the structure of periodic \((f)\)-good functions.

**Proposition 2.1.** Let \( f \in \mathcal{M} \). Assume that \( w \in \mathcal{A}(f) \).

\[
w(0) = \inf\{ w(t) : t \in [0, \infty) \}
\]

and \( w'(t) \neq 0 \) for some \( t \in [0, \infty) \). Then there exist \( \tau_1(w) > 0 \) and \( \tau(w) > \tau_1(w) \) such that the function \( w \) is strictly increasing on \( [0, \tau_1(w)] \), \( w \) is strictly decreasing on \( [\tau_1(w), \tau(w)] \),

\[
w(\tau_1(w)) = \sup\{ w(t) : t \in [0, \infty) \} \quad \text{and} \quad w(t + \tau(w)) = w(t) \quad \text{for all} \ t \in [0, \infty).
\]

In [24, Theorem 3.15] we established the following result.

**Proposition 2.2.** Let \( f \in \mathcal{M} \). Assume that \( w \in \mathcal{A}(f) \) and \( w'(t) \neq 0 \) for some \( t \in [0, \infty) \). Then there exists \( \tau > 0 \) such that

\[
w(t + \tau) = w(t), \quad t \in [0, \infty) \quad \text{and} \quad X_w(T_1) \neq X_w(T_2)
\]

for each \( T_1 \in [0, \infty) \) and each \( T_2 \in (T_1, T_1 + \tau) \).

In the sequel we use the following result of [23, Proposition 5.1].

**Proposition 2.3.** Let \( f \in \mathcal{M} \). Then there exists a number \( S > 0 \) such that for every \((f)\)-good function \( v \),

\[
|X_v(t)| \leq S \quad \text{for all large enough} \ t.
\]

The following result was proved in [13, Lemma 3.2].
Proposition 2.4. Let $f \in \mathcal{M}$ satisfy
\[ \mu(f) < \inf \{ f(t, 0, 0): t \in \mathbb{R}^1 \}. \]
Then no element of $\mathcal{A}(f)$ is a constant and $\sup \{ \tau(w): w \in \mathcal{A}(f) \} < \infty$.

Proposition 2.5. Let $f \in \mathcal{M}$ and let $M_1, M_2, c$ be positive numbers. Then there exists $S > 0$ such that the following assertion holds:
If $T_1 > 0, T_2 > T_1 + c$ and if $v \in W^{2,1}([T_1, T_2])$ satisfies
\[ |X_v(T_1)|, |X_v(T_2)| \leq M_1 \quad \text{and} \quad I^f(T_1, T_2, v) \leq U^f_{T_2-T_1}(X_v(T_1), X_v(T_2)) + M_2, \]
then
\[ |X_v(t)| \leq S \quad \text{for all} \quad t \in [T_1, T_2]. \]

For this result we refer the reader to [9] (see the proof of Proposition 4.4).

The next important ingredient of our proofs is established in [13, Lemma B5] which is an extension of [14, Theorem 1.2].

For this result we refer the reader to [9] (see the proof of Proposition 4.4).

The next useful result was proved in [13, Lemma 2.6].

Proposition 2.7. Let $f \in \mathcal{M}$. Then for every compact set $E \subset \mathbb{R}^2$ there exists a constant $M > 0$ such that for every $T > 1$
\[ U^f_T(x, y) \leq T \mu(f) + M \quad \text{for all} \quad x, y \in E. \]

The next important ingredient of our proofs is established in [13, Lemma B5] which is an extension of [23, Lemma 3.7].

Proposition 2.8. Let $f \in \mathcal{M}$, $w \in \mathcal{A}(f)$ and $\epsilon > 0$. Then there exist $\delta, q > 0$ such that for each $T \geq q$ and each $x, y \in \mathbb{R}^2$ satisfying $d(x, \Omega(w)) \leq \delta, d(y, \Omega(w)) \leq \delta$, there exists $v \in W^{2,1}([0, \tau])$ which satisfies
\[ X_v(0) = x, \quad X_v(\tau) = y, \quad I^f(0, \tau, v) \leq \epsilon. \]

We also need the following auxiliary result of [21, Proposition 2.3].

Proposition 2.9. Let $f \in \mathcal{M}$. Then for every $T > 0$
\[ U^f_T(x, y) \to \infty \quad \text{as} \quad |x| + |y| \to \infty. \]

Proposition 2.10. (See [12, Lemma 3.1].) Let $f \in \mathcal{M}$ and $\delta, \tau$ are positive numbers. Then there exists $M > 0$ such that for every $T \geq \tau$ and every $v \in W^{2,1}([0, T])$ satisfying
\[ I^f(0, T, v) \leq \inf \{ U^f_T(x, y): x, y \in \mathbb{R}^2 \} + \delta \]
the following inequality holds:
\[ |X_v(t)| \leq M \quad \text{for all} \quad t \in [0, T]. \]
3. Auxiliary results

Let \( f \in \mathcal{M} \). By Proposition 2.2 for each \( w \in \mathcal{A}(f) \) which is not a constant there exists \( \tau(w) > 0 \) such that

\[
w(t + \tau(w)) = w(t), \quad t \in [0, \infty), \quad X_w(T_1) \neq X_w(T_2) \quad \text{for each} \ T_1 \in [0, \infty)
\]

and each \( T_2 \in (T_1, T_1 + \tau(w)). \) \hfill (3.1)

By Proposition 2.3 there exists a number \( \bar{M} > 0 \) such that

\[
\sup \{ |X_1(t)| : t \in [0, \infty) \} < \bar{M} \quad \text{for all} \ w \in \mathcal{A}(f).
\] \hfill (3.2)

**Proposition 3.1.** Suppose that \( \mu(f) < \inf \{ f(t, 0, 0) : t \in \mathbb{R}^1 \} \). Then

\[
\inf \{ \tau(w) : w \in \mathcal{A}(f) \} > 0.
\]

**Proof.** Let us assume the contrary. Then there exists a sequence \( \{w_n\}_{n=1}^{\infty} \subset \mathcal{A}(f) \) such that \( \lim_{n \to \infty} \tau(w_n) = 0 \). It follows from (3.2), the definition of \( \tau(w) \), \( w \in \mathcal{A}(f) \) and the equality above that for \( n = 1, 2, \ldots, \)

\[
\sup \{ |w_n(t) - w_s(s)| : t, s \in [0, \infty) \} \leq \bar{M} \tau(w_n) \to 0 \quad \text{as} \ n \to \infty.
\] \hfill (3.3)

Since \( \{w_n\}_{n=1}^{\infty} \subset \mathcal{A}(f) \) it follows from (3.2) and the continuity of the functions \( U^f_T, T > 0 \) that for any natural number \( k \) the sequence \( \{U^f_T(0, k, w_n)\}_{n=1}^{\infty} \) is bounded. Combined with (3.2) and the growth condition (1.1) this implies that for any integer \( k \geq 1 \) the sequence \( \{U^f_T(0, k, w_n(t)|^T d t\}_{k=1}^{\infty} \) is bounded. Since this fact holds for any natural number \( k \) it follows from (3.2) that the sequence \( \{w_n\}_{n=1}^{\infty} \) is bounded in \( W^{2,\gamma}([0, k]) \) for any natural number \( k \) and it possesses a weakly convergent subsequence in this space. By using a diagonal process we obtain that there exist a subsequence \( \{w_{n_i}\}_{i=1}^{\infty} \) of \( \{w_n\}_{n=1}^{\infty} \) and \( w_s \in W^{2,\gamma}_{loc}([0, \infty)) \) such that for each natural number \( k \)

\[
\begin{align*}
(w_{n_i}, w'_n) & \to (w_s, w'_s) \quad \text{as} \ i \to \infty \text{ uniformly on} \ [0, k], \quad (3.4) \\
w''_n & \to w''_s \quad \text{as} \ i \to \infty \text{ weakly in} \ L^\gamma[0, k]. \quad (3.5)
\end{align*}
\]

By (3.4), (3.5) and the lower semicontinuity of integral functionals [4] for each natural number \( k \),

\[
I^f(0, k, w_s) \leq \liminf_{i \to \infty} I^f(0, k, w_{n_i}).
\]

Combined with (3.4) and (2.2), the continuity of \( \pi^f_I \) and the inclusion \( w_n \in \mathcal{A}(f), n = 1, 2, \ldots, \) this inequality implies that for any natural number \( k \)

\[
\Gamma^f(0, k, w_s) \leq \liminf_{i \to \infty} \Gamma^f(0, k, w_{n_i}) = 0.
\]

In view of (3.3) and (3.4), \( w_s \) is a constant function. Together with the relation above and (2.2) this implies that

\[
\mu(f) = f(u_s(0), 0, 0) = \inf \{ f(t, 0, 0) : t \in \mathbb{R}^1 \}.
\]

The contradiction we have reached proves Proposition 3.1. \qed

**Proposition 3.2.** Suppose that

\[
\mu(f) < \inf \{ f(t, 0, 0) : t \in \mathbb{R}^1 \}.
\] \hfill (3.6)

Let \( M, l, \epsilon > 0 \). Then there exist \( \delta > 0 \) and \( L > l \) such that for each \( T \geq L \) and each \( v \in W^{2,\gamma}([0, T]) \) satisfying

\[
|X_v(0)|, |X_v(T)| \leq M, \quad \Gamma^f(0, T, v) \leq \delta,
\] \hfill (3.7)

there exist \( s \in [0, T - l] \) and \( w \in \mathcal{A}(f) \) such that

\[
|X_v(s + t) - X_w(t)| \leq \epsilon, \quad t \in [0, l].
\]
Proof. Assume the contrary. Then there exists a sequence \( v_i \in W^{2,1}([0, T_i]) \), \( i = 1, 2, \ldots \), such that

\[
T_i \geq l, \quad i = 1, 2, \ldots, \\
T_i \to \infty \quad \text{as} \quad i \to \infty, \quad \Gamma^f(0, T_i, v_i) \to 0 \quad \text{as} \quad i \to \infty, \\
\left| X_{v_i}(0) \right|, \left| X_{v_i}(T_i) \right| \leq M, \quad i = 1, 2, \ldots, 
\]

(3.8)

and that for each natural number \( i \) the following property holds:

\[
\sup \left\{ \left| X_{v_i}(s + t) - X_w(t) \right| : t \in [0, l] \right\} > \epsilon \quad \text{for each} \quad s \in [0, T - l] \quad \text{and} \quad w \in \mathcal{A}(f). 
\]

(3.10)

We may assume without loss of generality that

\[
\Gamma^f(0, T_i, v_i) \leq 1, \quad i = 1, 2, \ldots. 
\]

(3.11)

It follows from (2.2), (3.11), (1.6) and (1.5) that for each integer \( i \geq 1 \)

\[
\begin{align*}
I^f(0, T_i, v_i) &= \pi^f(X_{v_i}(0)) - \pi^f(X_{v_i}(T_i)) + T_i \mu(f) + \Gamma^f(0, T_i, v_i) \\
&\leq 1 + \pi^f(X_{v_i}(0)) - \pi^f(X_{v_i}(T_i)) + T_i \mu(f) \\
&\leq 1 + U^f_T(X_{v_i}(0), X_{v_i}(T_i)).
\end{align*}
\]

(3.12)

By (3.12), (3.9), (3.8) and Proposition 2.5 there exists a constant \( M_1 > 0 \) such that

\[
\left| X_{v_i}(t) \right| \leq M_1, \quad t \in [0, T_i], \quad i = 1, 2, \ldots 
\]

(3.13)

By (3.13), (3.12) and the continuity of \( U^f_T \), \( T > 0 \), for each natural number \( n \), the sequence \( \{I^f(0, n, v_i)\}_{i=n}^{i=\infty} \) is bounded, where \( i(n) \) is a natural number such that \( T_i > n \) for all integers \( i \geq i(n) \) (see (3.8)). Together with (3.13) and (1.1) this implies that for any natural number \( n \) the sequence \( \{\int_0^n |v_i''(t)|^\gamma \ dt\}_{i=i(n)}^{i=\infty} \) is bounded. Since this fact holds for any natural number \( n \) it follows from (3.13) that the sequence \( \{v_i\}_{i=i(n)}^{i=\infty} \) is bounded in \( W^{2,\gamma}(0, n) \) for any natural number \( n \) and it possesses a weakly convergent subsequence in this space. By using a diagonal process we obtain that there exist a subsequence \( \{v_{i_k}\}_{k=1}^{\infty} \) of \( \{v_i\}_{i=1}^{\infty} \) and \( u \in W^{2,1}_{loc}((0, \infty)) \) such that for each natural number \( n \)

\[
\begin{align*}
(v_{i_k}, v_{i_k}') &\to (u, u') \quad \text{as} \quad k \to \infty \quad \text{uniformly on} \quad [0, n], \\
v_{i_k}'' &\to u'' \quad \text{as} \quad k \to \infty \quad \text{weakly in} \quad L^\gamma[0, k].
\end{align*}
\]

(3.14)

(3.15)

In view of (3.14) and (3.13),

\[
\left| X_u(t) \right| \leq M_1 \quad \text{for all} \quad t \geq 0. 
\]

(3.16)

It follows from (3.14), (3.15), (3.13) and the lower semicontinuity of integral functionals [4] for each natural number \( n \)

\[
I^f(0, n, u) \leq \liminf_{k \to \infty} I^f(0, n, v_{i_k}).
\]

Combined with (3.14), (3.13), (2.2), (1.6), the continuity of \( \pi^f \) and (3.8) the inequality above implies that for any natural number \( n \)

\[
\Gamma^f(0, n, u) \leq \liminf_{k \to \infty} \Gamma^f(0, n, v_{i_k}) = 0.
\]

Thus

\[
\Gamma^f(0, T, u) = 0 \quad \text{for all} \quad T > 0.
\]

(3.17)

By (3.16), (3.17) and Proposition 2.6 there exists \( w \in \mathcal{A}(f) \) such that \( \Omega(w) = \Omega(u) \) and the following assertion holds:

\(\text{(A1)}\) Let \( T_w \) be a period of \( w \) (not necessarily minimal). Then for each \( \gamma > 0 \) there exists \( \tau(\gamma) > 0 \) such that for each \( \tau \geq \tau(\gamma) \) there is \( s \in [0, T_w) \) such that

\[
\left| X_u(t + \tau) - X_w(s + t) \right| \leq \gamma, \quad t \in [0, T_w].
\]
We may assume without loss of generality that a period $T_w$ of $w$ satisfies $T_w > l$. Assumption (A1) implies that there exist $\tau > 0$ and $\tilde{w} \in \mathcal{A}(f)$ such that
$$|X_u(\tau + t) - X_\tilde{w}(t)| \leq \epsilon/4, \quad t \in [0, l].$$
Combined with (3.14) this implies that for all sufficiently large natural numbers $k$
$$|X_{v_k}(\tau + t) - X_\tilde{w}(t)| \leq \epsilon/2, \quad t \in [0, l].$$
This contradicts (3.10). The contradiction we have reached proves Proposition 3.2. \hfill $\Box$

**Proposition 3.3.** Let $M > 0$ and $\delta > 0$. Then there exists a natural number $n$ such that for each number $T \geq 1$ and each $v \in W^{2,1}([0, T])$ satisfying
$$|X_v(0)|, |X_v(T)| \leq M, \quad I^f(0, T, v) \leq U_f^f(X_v(0), X_v(T)) + 1$$
the following property holds:
There exists a sequence $\{t_i\}_{i=0}^m$ with $m \leq n$ such that
$$0 = t_0 < t_1 < \cdots < t_i < t_{i+1} < \cdots < t_m = T,$$
$$\Gamma^f(t_i, t_{i+1}, v) = \delta \quad \text{for any integer } i \text{ satisfying } 0 \leq i < m - 1, \quad \Gamma^f(t_{m-1}, t_m, v) \leq \delta.$$  \hfill (3.19)

**Proof.** By Proposition 2.7 there exists a constant $M_1 > 0$ such that
$$U_f^f(x, y) \leq T\mu(f) + M_1 \quad \text{for each } T \geq 1 \quad \text{and each } x, y \in \mathbb{R}^2 \text{ satisfying } |x|, |y| \leq M.$$  \hfill (3.20)
Together with (2.2) and (3.20) this implies that if $T \geq 1$ and if $v \in W^{2,1}([0, T])$ satisfies (3.18), then
$$\Gamma^f(0, T, v) \leq U_f^f(X_v(0), X_v(T)) + 1 - T\mu(f), \quad -\pi^f(X_v(0)) + \pi^f(X_v(T)) \leq M_1 + 1 + 2M_2,$$  \hfill (3.21)
where
$$M_2 = \sup \left\{ \left| \pi^f(z) \right| : z \in \mathbb{R}^2 \text{ and } |z| \leq M \right\}. \hfill (3.22)$$
Choose a natural number $n > 4$ such that
$$(n - 2)\delta > 2(M_2 + M_1 + 1). \hfill (3.23)$$
Assume now that $T \geq 1$ and that $v \in W^{2,1}([0, T])$ satisfies (3.18). Then by (3.21) and (3.22),
$$\Gamma^f(0, T, v) \leq M_1 + 1 + 2M_2.$$  \hfill (3.24)
Clearly for each $\tau \in [0, T)$, $\lim_{\tau \to T+} \Gamma^f(\tau, s, v) = 0$ and one of the following cases holds:
$$\Gamma^f(\tau, T, v) \leq \delta; \text{ there exists } \tilde{\tau} \in (\tau, T) \text{ such that } \Gamma^f(\tau, \tilde{\tau}, v) = \delta.$$  
This implies that there exist a natural number $m$ and a sequence $\{t_i\}_{i=0}^m$ such that (3.19) is true. In order to complete the proof of the proposition it is sufficient to show that $m \leq n$. By (3.24), (3.19) and (3.23),
$$2M_2 + 1 + M_1 \geq \Gamma^f(0, T, v) \geq (m - 1)\delta$$
and
$$m \leq 1 + \delta^{-1}(2M_2 + 1 + M_1) < n.$$  
Proposition 3.3 is proved. \hfill $\Box$

The following proposition is a result on the uniform equicontinuity of the family $(U_f^f)_{T \geq \tau}$ on bounded sets.

**Proposition 3.4.** Let $M > 0$ and $\tau > 0$. Then for each $\epsilon > 0$ there exists $\delta > 0$ such that for each $T \geq \tau$ and each $x, y, \tilde{x}, \tilde{y} \in \mathbb{R}^2$ satisfying
$$|x|, |y|, |\tilde{x}|, |\tilde{y}| \leq M, \quad |x - \tilde{x}|, |y - \tilde{y}| \leq \delta$$  \hfill (3.25)
the following inequality holds:
$$|U_f^f(x, y) - U_f^f(\tilde{x}, \tilde{y})| \leq \epsilon.$$  \hfill (3.26)
Proof. Let \( \epsilon > 0 \). By Proposition 2.5 there exists a constant \( M_1 > M \) such that for each \( T \geq \tau \) and each \( v \in W^{2,1}([0, T]) \) satisfying
\[
|X_v(0)|, |X_v(T)| \leq M, \quad I^f(0, T, v) \leq U^f_T(X_v(0), X_v(T)) + 1 \tag{3.27}
\]
the following inequality holds:
\[
|X_v(t)| \leq M_1, \quad t \in [0, T]. \tag{3.28}
\]
Since the function \( U_T^f \) is continuous, it is uniformly continuous on compact subsets of \( R^2 \times R^2 \) and there exists \( \delta > 0 \) such that
\[
|U_T^f(x, y) - U_T^f(\bar{x}, \bar{y})| \leq \epsilon/4 \tag{3.29}
\]
for each \( x, y, \bar{x}, \bar{y} \in R^2 \) satisfying
\[
|x|, |y|, |\bar{x}|, |\bar{y}| \leq M_1, \quad |x - \bar{x}|, |y - \bar{y}| \leq \delta. \tag{3.30}
\]
Assume that \( x, y, \bar{x}, \bar{y} \in R^2 \) satisfy (3.25) and that \( T \geq \tau \). In order to prove the proposition it is sufficient to show that
\[
U_T^f(\bar{x}, \bar{y}) \leq U_T^f(x, y) + \epsilon.
\]
There exists \( v \in W^{2,1}([0, T]) \) such that
\[
X_v(0) = x, \quad X_v(T) = y, \quad I^f(0, T, v) = U_T^f(x, y). \tag{3.31}
\]
By (3.31), (3.25) and the choice of \( M_1 \), (3.28) is valid. There exists \( u \in W^{2,1}([0, T]) \) such that
\[
X_u(0) = \bar{x}, \quad X_u(\tau/4) = X_v(\tau/4), \quad I^f(0, \tau/4, u) = U^f_{\tau/4}(\bar{x}, X_v(\tau/4)),
\]
\[
u(t) = v(t), \quad t \in [\tau/4, T - \tau/4],
\]
\[
X_u(T - \tau/4) = X_v(T - \tau/4), \quad X_u(T) = \bar{y}, \quad I^f(T - \tau/4, T, u) = U^f_{\tau/4}(X_v(T - \tau/4), \bar{y}). \tag{3.32}
\]
It follows from (3.25) and (3.28) and the choice of \( \delta \) (see (3.29) and (3.30)) that
\[
|U^f_{\tau/4}(\bar{x}, X_v(\tau/4)) - U^f_{\tau/4}(x, X_v(\tau/4))| \leq \epsilon/4
\]
\[
|U^f_{\tau/4}(X_v(T - \tau/4), \bar{y}) - U^f_{\tau/4}(X_v(T - \tau/4), y)| \leq \epsilon/4.
\]
It follows from the inequalities above, (3.32) and (3.31) that
\[
U_T^f(\bar{x}, \bar{y}) \leq I^f(0, T, u) = I^f(0, \tau/4, u) + I^f(\tau/4, T - \tau/4, u) + I^f(T - \tau/4, T, u)
\]
\[
= U^f_{\tau/4}(\bar{x}, X_v(\tau/4)) + I^f(\tau/4, T - \tau/4, u) + U^f_{\tau/4}(X_v(T - \tau/4), \bar{y})
\]
\[
\leq U^f_{\tau/4}(x, X_v(\tau/4)) + \epsilon/4 + I^f(\tau/4, T - \tau/4, u) + U^f_{\tau/4}(X_v(T - \tau/4), y) + \epsilon/4
\]
\[
= I^f(0, T, v) + \epsilon/2 = U_T^f(x, y) + \epsilon/2.
\]
Proposition 3.4 is proved. \( \square \)

**Proposition 3.5.** Suppose that
\[
\mu(f) < \inf\{ f(t, 0, 0): t \in R^1 \}.
\]
Let \( \epsilon > 0 \). Then there exist \( q > 0 \) and \( \delta > 0 \) such that the following assertion holds:
\[
\text{Let } T \geq q, \ w \in A(f), \quad x, y \in R^2, \quad d(x, \Omega(w)), d(y, \Omega(w)) \leq \delta. \tag{3.33}
\]
Then there exists \( v \in W^{2,1}([0, T]) \) which satisfies
\[
X_v(0) = x, \quad X_v(T) = y, \quad I^f(0, \tau, v) \leq \epsilon. \tag{3.34}
\]
Proof. By Proposition 2.8 for each \( w \in \mathcal{A}(f) \) there exist \( \delta(w), q(w) > 0 \) such that the following property holds:

(P1) If \( T \geq q(w) \) and if \( x, y \in \mathbb{R}^2 \) satisfy \( d(x, \Omega(w)), d(y, \Omega(w)) \leq \delta(w) \), then there exists \( v \in W^{2,1}([0, T]) \) which satisfies (3.34).

By Propositions 2.4 and 3.1,

\[ \bar{T} := \sup \{ \tau(w): w \in \mathcal{A}(f) \} < \infty, \quad \inf \{ \tau(w): w \in \mathcal{A}(f) \} > 0. \tag{3.35} \tag{3.36} \]

Define

\[ E = \bigcup \{ \Omega(w) \times \Omega(w): w \in \mathcal{A}(f) \}. \tag{3.37} \]

We will show that \( E \) is compact. In view of (3.2) it is sufficient to show that \( E \) is closed.

Let

\[ \{(x_i, y_i)\}_{i=1}^{\infty} \subset E, \quad \lim_{i \to \infty} (x_i, y_i) = (x, y). \tag{3.38} \]

We show that \((x, y) \in E\). For each natural number \( i \) there exist \( w_i \in \mathcal{A}(f), s_i, t_i \in [0, \infty) \) such that

\[ x_i = (w_i(t_i), w_i'(t_i)), \quad y_i = (w_i(s_i), w_i'(s_i)). \tag{3.39} \]

In view of (3.35) we may assume that

\[ t_i, s_i \in [0, \bar{T}], \quad i = 1, 2, \ldots. \tag{3.40} \]

By (3.2) and the continuity of \( U_{\bar{T}}^f \), the sequence \( \{ I_{\bar{T}}^f (0, \bar{T}, w_i) \}_{i=1}^{\infty} \) is bounded. Combined with (3.2) and (1.1) this implies that the sequence \( \{ J_{\bar{T}}^f (0, \bar{T}, w_i') \}_{i=1}^{\infty} \) is bounded. Extracting a subsequence and re-indexing if necessary we may assume without loss of generality that there exist

\[ t_s = \lim_{i \to \infty} t_i, \quad s_s = \lim_{i \to \infty} s_i, \quad \tau_s = \lim_{i \to \infty} \tau(w_i) \tag{3.41} \]

and there exists \( u \in W^{2,\gamma}([0, \bar{T}]) \) such that

\[ w_i \to u \quad \text{as} \quad i \to \infty \quad \text{weakly in} \quad W^{2,\gamma}([0, \bar{T}]), \]

\[ (w_i, w_i') \to (u, u') \quad \text{as} \quad i \to \infty \quad \text{uniformly on} \quad [0, \bar{T}]. \tag{3.42} \]

By (3.42), (3.2), the continuity of \( \pi^f \), and the lower semicontinuity of integral functionals [4],

\[ \Gamma^f(0, \bar{T}, u) \leq \liminf_{i \to \infty} \Gamma^f(0, \bar{T}, w_i) = 0 \]

and \( \Gamma^f(0, \bar{T}, u) = 0 \).

It follows from (3.38), (3.39), (3.40), (3.42) and (3.41) that

\[ x = \lim_{i \to \infty} x_i = \lim_{i \to \infty} (w_i(t_i), w_i'(t_i)) = \lim_{i \to \infty} (u(t_i), u'(t_i)) = (u(t_s), u'(t_s)), \tag{3.43} \]

\[ y = \lim_{i \to \infty} y_i = \lim_{i \to \infty} (w_i(s_i), w_i'(s_i)) = \lim_{i \to \infty} (u(s_i), u'(s_i)) = (u(s_s), u'(s_s)). \tag{3.44} \]

By (3.42), the inclusion \( w_i \in \mathcal{A}(f), i = 1, 2, \ldots, (3.35) \) and (3.41),

\[ X_u(0) = \lim_{i \to \infty} X_{w_i}(0) = \lim_{i \to \infty} X_{w_i}(\tau(w_i)) = \lim_{i \to \infty} X_u(\tau(w_i)) = X_u(\tau_s). \]

In view of (3.41), (3.40) and (3.36),

\[ 0 < \tau_s \leq \bar{T}. \]

We have shown that

\[ X_u(0) = X_u(\tau_s), \quad 0 \leq \Gamma^f(0, \tau_s, u) \leq \Gamma^f(0, \bar{T}, u) = 0. \]
This implies that $u$ can be extended on the infinite interval $[0, \infty)$ as a periodic $(f)$-good function with the period $\tau^*$. Thus we have that $u \in A(f)$ and in view of (3.43), (3.44) and (3.37)
\[(x, y) \in \Omega(u) \times \Omega(u) \subset E.\]
Therefore $E$ is compact. For each $w \in A(f)$ define an open set $U(w) \subset \mathbb{R}^4$ by
\[U(w) = \{(x, y) \in \mathbb{R}^4: d(x, \Omega(w)) < \delta(w)/4, \ d(y, \Omega(w)) < \delta(w)/4\}. \] (3.45)
Then $U(w), w \in A(f)$ is an open covering of the compact $E$ and there exists a finite set $\{w_1, \ldots, w_n\} \in A(f)$ such that
\[E \subset \bigcup_{i=1}^n U(w_i). \] (3.46)
Set
\[q = \max\{q(w_i): i = 1, \ldots, n\}, \ \ \ \delta = \min\{\delta(w_i)/4: i = 1, \ldots, n\}. \] (3.47)
Let $T \geq q, w \in A(f)$ and let $x, y \in \mathbb{R}^2$ satisfy (3.33). There exist
\[\tilde{x}, \tilde{y} \in \Omega(w) \] (3.48)
such that
\[|x - \tilde{x}|, |y - \tilde{y}| \leq \delta. \] (3.49)
In view of (3.37), (3.46) and (3.48), $(\tilde{x}, \tilde{y}) \in E$ and there is $j \in \{1, \ldots, n\}$ such that
\[(\tilde{x}, \tilde{y}) \in U(w_j). \] (3.50)
Relations (3.50) and (3.45) imply that there exist
\[\tilde{x}, \tilde{y} \in \Omega(w_j) \] (3.51)
such that
\[|\tilde{x} - \bar{x}|, |\tilde{y} - \bar{y}| < \delta(w_j)/4. \] (3.52)
By (3.49), (3.52) and (3.47)
\[|x - \bar{x}|, |y - \bar{y}| < \delta + \delta(w_j)/4 \leq \delta(w_j)/2. \]
It follows from this inequalities, (3.51), property (P1) with $w = w_j$, (3.47) and the inequality $T \geq q$ that there exists $v \in W^{2,1}([0, T])$ satisfying (3.34). Proposition 3.5 is proved. \qed

4. Proof of Theorem 1.1

By Proposition 3.4 in order to prove the theorem it is sufficient to show that for each $x, y \in \mathbb{R}^2$ there exists
\[\lim_{T \to \infty} [U^f_T(x, y) - T \mu(f)]. \]
Let $x, y \in \mathbb{R}^2$ and fix $\epsilon > 0$. We will show that there exist $\bar{T} > 0$ and $q > 0$ such that
\[U^f_S(x, y) - S\mu(f) \leq U^f_T(x, y) - T \mu(f) + \epsilon \] (4.1)
for each $T \geq \bar{T}$ and each $S \geq T + q$.

By Proposition 3.5 there exist $q > 0, \delta_0 > 0$ such that for the following property holds:

(P2) For each $T \geq q$, each $w \in A(f)$ and each $x, y \in \mathbb{R}^2$ satisfying
\[d(x, \Omega(w)), d(y, \Omega(w)) \leq \delta_0 \] (4.2)
there exists $v \in W^{2,1}([0, T])$ such that
\[X_v(0) = x, \ \ \ \ X_v(T) = y, \ \ \ \ \Gamma^f(0, T, v) \leq \epsilon. \] (4.3)
In view of Proposition 2.4 there exists a real number
\[ l > \sup \{ \tau(w) : w \in \mathcal{A}(f) \}. \] (4.4)

Choose
\[ M_0 > |x| + |y| + 2. \] (4.5)

By Proposition 2.5 there exists \( M_1 > M_0 \) such that for each \( T \geq 1 \) and each \( v \in W^{2,1}([0, T]) \) satisfying
\[ |X_v(0)|, |X_v(T)| \leq M_0, \quad I^f(0, T, v) \leq U_T^f(X_v(0), X_v(T)) + 1 \] (4.6)
the following inequality holds:
\[ |X_v(T)| \leq M_1, \quad t \in [0, T]. \] (4.7)

By Proposition 3.2 there exist \( \delta_1 > 0, L_1 \) such that for each \( T \geq L_1 \) and each \( v \in W^{2,1}([0, T]) \) satisfying
\[ |X_v(0)|, |X_v(T)| \leq M_1, \quad \Gamma^f(0, T, v) \leq \delta_1 \] (4.8)
there exist \( \sigma \in [0, T - l] \) and \( w \in \mathcal{A}(f) \) such that
\[ |X_v(\sigma + t) - X_w(t)| \leq \delta_0, \quad t \in [0, l]. \] (4.9)

By Proposition 3.3 there exists a natural number \( n \) such that for each \( T \geq 1 \) and each \( v \in W^{2,1}([0, T]) \) satisfying
\[ |X_v(0)|, |X_v(T)| \leq M_1, \quad I^f(0, T, v) \leq U_T^f(X_v(0), X_v(T)) + 1 \] (4.10)
there exists a sequence \( \{t_i\}_{i=0}^m \subset [0, T] \) with \( m \leq n \) such that
\[ 0 = t_0 < \cdots < t_i < t_{i+1} < \cdots < t_m = T, \] (4.11)
\[ \Gamma^f(t_i, t_{i+1}, v) = \delta_1 \] for all integers \( i \) satisfying \( 0 \leq i < m - 1, \) \[ \Gamma^f(t_{m-1}, t_m, v) \leq \delta_1. \] (4.12)

Choose a number
\[ \bar{T} > 1 + n L_1. \] (4.13)

Let
\[ T \geq \bar{T}, \quad S \geq T + q. \] (4.14)

There exists \( v \in W^{2,1}([0, T]) \) such that
\[ X_v(0) = x, \quad X_v(T) = y, \quad I^f(0, T, v) = U_T^f(x, y). \] (4.15)

By (4.5), (4.13), (4.14), the choice of \( M_1 \) and (4.15), the inequality (4.7) holds. In view of (4.15), the choice of \( n \) (see (4.10)–(4.12)), (4.14), (4.13) and (4.5) there exists a sequence \( \{t_i\}_{i=0}^m \subset [0, T] \) with \( m \leq n \) such that (4.11) and (4.12) hold. It follows from (4.14), (4.13) and (4.11) that
\[ \max\{t_i+1 - t_i : 0, \ldots, m - 1\} \geq T/m \geq \bar{T}/n > L_1. \]

Thus there exists \( j \in \{0, \ldots, m - 1\} \) such that
\[ t_{j+1} - t_j > L_1. \] (4.16)

By (4.16), (4.7), (4.12) and the choice of \( \delta_1, L_1 \) (see (4.8), (4.9)) there exist
\[ \sigma \in [t_j, t_{j+1} - l], \quad w \in \mathcal{A}(f) \] (4.17)
such that (4.9) holds.

In particular
\[ d(X_v(\sigma), \Omega(w)) \leq \delta_0. \] (4.18)
It follows from (4.14), (4.17), the property (P2) and (4.18) that there exists
\[ h \in W^{2,1}([\sigma, \sigma + S - T]) \]
such that
\[
X_h(\sigma) = X_v(\sigma), \quad X_h(\sigma + S - T) = X_v(\sigma), \\
\Gamma^f (\sigma, \sigma + S - T, h) \leq \epsilon.
\] (4.19)

It is easy to see that there exist \( u \in W^{2,1}([0, S]) \) such that
\[
u(t) = v(t), \quad t \in [0, \sigma], \\
u(t) = h(t), \quad t \in [\sigma, \sigma + S - T],
\]
\[
u(\sigma + S - T + t) = v(\sigma + t), \quad t \in [0, T - \sigma].
\] (4.20)

By (4.20) and (4.15),
\[
X_u(0) = x, \quad X_u(S) = y.
\] (4.21)

By (4.21), (2.2), (4.15), (4.20) and (4.19),
\[
U_T^f(x, y) - T\mu(f) \leq \inf \{ U_S^f(x, y) - S\mu(f): S \in [T_0, \infty) \} + d_{*} - \epsilon.
\]
Thus we have shown that (4.1) holds for each \( T \geq \bar{T} \) and each \( S \geq T + q \). By Proposition 2.7
\[
\sup \{ U_T^f(x, y) - T\mu(f): T \in [1, \infty) \} < \infty.
\]

On the other hand by (1.6) for each \( T \geq 1 \)
\[
U_T^f(x, y) - T\mu(f) \geq \pi^f(x) - \pi^f(y).
\]
Hence the set \( \{ U_T^f(x, y): T \in [1, \infty) \} \) is bounded. Put
\[
d_{*} = \lim_{T \to \infty} \inf \{ U_S^f(x, y) - S\mu(f): S \in [T, \infty) \}.
\] (4.22)

We show that
\[
d_{*} = \lim_{T \to \infty} \inf \{ U_T^f(x, y) - T\mu(f) \}.
\]

Let \( \epsilon > 0 \). We have shown that there exist \( \bar{T} > 0, q > 0 \) such that (4.1) holds for each \( T \geq \bar{T} \) and each \( S \geq T + q \). By (4.22) there exists \( T_0 \geq \bar{T} \) such that
\[
d_{*} \geq \inf \{ U_S^f(x, y) - S\mu(f): S \in [T_0, \infty) \} \geq d_{*} - \epsilon.
\] (4.23)

There exists \( T_1 \geq T_0 \) such that
\[
|U_{T_1}^f(x, y) - T_1\mu(f) - \inf \{ U_S^f(x, y) - S\mu(f): S \in [T_0, \infty) \}| \leq \epsilon.
\] (4.24)

Let \( T \geq T_1 + q \). Then in view of (4.23)
\[
U_T^f(x, y) - T\mu(f) \geq \inf \{ U_S^f(x, y) - S\mu(f): S \in [T_0, \infty) \} \geq d_{*} - \epsilon.
\]

On the other hand by the relation \( T \geq T_1 + q \geq T_0 + q \geq \bar{T} + q \), (4.1) (which holds with \( T = T_1, S = T \)) and (4.23)
\[
U_T^f(x, y) - T\mu(f) \leq U_{T_1}^f(x, y) - T_1\mu(f) + \epsilon
\]
\[
\leq \inf \{ U_S^f(x, y) - S\mu(f): S \in [T_0, \infty) \} + 2\epsilon \leq d_{*} + 2\epsilon.
\]
Therefore
\[ |U_T^f(x, y) - T \mu(f) - d_\epsilon| \leq 2\epsilon \quad \text{for all } T \geq T_1 + q. \]

Since \( \epsilon \) is an arbitrary positive number we conclude that
\[ d_\epsilon = \lim_{T \to \infty} [U_T^f(x, y) - T \mu(f)]. \]

Theorem 1.1 is proved.

5. Proof of Theorem 1.2

Consider the function \( U_T^f : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^1 \) defined in Theorem 1.1:
\[ U_T^f(x, y) = \lim_{T \to \infty} [U_T^f(x, y) - T \mu(f)], \quad x, y \in \mathbb{R}^2. \] (5.1)

By Proposition 2.10 there exists \( M > 0 \) such that for each \( T \geq 1 \) and each \( v \in W^{2,1}([0, T]) \) satisfying
\[ I^f(0, T, v) \leq \inf \{ U_T^f(x, y) : x, y \in \mathbb{R}^2 \} + 1 \] (5.2)
the following inequality holds:
\[ |X_v(t)| \leq M, \quad t \in [0, T]. \] (5.3)

Let \( x, y \in \mathbb{R}^2 \) satisfy \( \max \{|x|, |y|\} > T \geq 1 \). Then by the choice of \( M \),
\[ U_T^f(x, y) > \inf \{ U_T^f(z_1, z_2) : z_1, z_2 \in \mathbb{R}^2 \} + 1. \]

This implies that for each \( T \geq 1 \)
\[ \inf \{ U_T^f(x, y) : x, y \in \mathbb{R}^2 \text{ and } \max \{|x|, |y|\} > M \} \geq \inf \{ U_T^f(x, y) : x, y \in \mathbb{R}^2 \} + 1. \] (5.4)

Put
\[ E_1 = \{(x, y) \in \mathbb{R}^2 \times \mathbb{R}^2 : \max \{|x|, |y|\} > M\}, \quad E_2 = (\mathbb{R}^2 \times \mathbb{R}^2) \setminus E_1. \] (5.5)

In view of (5.5) and (5.4) for any \( T \geq 1 \)
\[ \inf \{ U_T^f(x, y) - T \mu(f) : (x, y) \in E_1 \} \geq \inf \{ U_T^f(x, y) - T \mu(f) : (x, y) \in E_2 \} + 1. \] (5.6)

By Theorem 1.1
\[ U_T^f(x, y) - T \mu(f) \to U_\infty^f(x, y) \quad \text{as } T \to \infty \] (5.7)
uniformly on \( E_2 \). This implies that
\[ \lim_{T \to \infty} \inf \{ U_T^f(x, y) - T \mu(f) : (x, y) \in E_2 \} = \inf \{ U_\infty^f(x, y) : (x, y) \in E_2 \}. \] (5.8)

Let \( (z, \bar{z}) \in E_1 \). Then by (5.1), (5.6) and (5.8)
\[ U_\infty^f(z, \bar{z}) = \lim_{T \to \infty} [U_T^f(z_1, \bar{z}) - T \mu(f)] \geq \lim_{T \to \infty} [\inf \{ U_T^f(x, y) - T \mu(f) : (x, y) \in E_2 \} + 1] \]
\[ = \inf \{ U_\infty^f(x, y) : (x, y) \in E_2 \} + 1. \] (5.9)

Since the function \( U_\infty^f \) is continuous the set
\[ E_\infty := \{(x, y) \in E_2 : U_\infty^f(x, y) = \inf \{ U_\infty^f(z) : z \in E_2 \}\} \] (5.10)
is nonempty and compact. In view of (5.9) and (5.10)
\[ U_\infty^f(z) \geq U_\infty^f(y) + 1 \quad \text{for each } z \in E_1 \text{ and each } y \in E_\infty. \] (5.11)
Let $\epsilon > 0$. Using standard arguments and compactness of $E_2$ we can show that there exists $\delta \in (0, 8^{-1})$ such that

if $z \in R^4$ satisfies $U^f_\infty(z) \leq \inf\{U^f_\infty(y): y \in R^4\} + 4\delta$, then $d(z, E_\infty) \leq \epsilon$.  \hspace{1cm} (5.12)

By Theorem 1.1 there exists $\bar{T} > 1$ such that

$$|U^f_\infty(x, y) - T\mu(f) - U^f_\infty(x, y)| \leq \delta$$

for any $T \geq \bar{T}$ and any $(x, y) \in E_2$.  \hspace{1cm} (5.13)

Assume that $T > \bar{T}$, $(x, y) \in R^2 \times R^2$, $U^f_\infty(x, y) \leq \inf\{U^f_\infty(z): z \in R^4\} + \delta$.  \hspace{1cm} (5.14)

In view of (5.14), (5.5) and (5.6),

$(x, y) \in E_2$.  \hspace{1cm} (5.15)

By (5.15), (5.14) and (5.13),

$$|U^f_\infty(x, y) - \mu(f)T - U^f_\infty(x, y)| \leq \delta.$$  \hspace{1cm} (5.16)

By (5.14), (5.6), (5.9) and (5.13),

$$|\inf\{U^f_\infty(z) - T\mu(f): z \in R^4\} - \inf\{U^f_\infty(z): z \in R^4\}|$$

$$= |\inf\{U^f_\infty(z) - T\mu(f): z \in E_2\} - \inf\{U^f_\infty(z): z \in E_2\}| \leq \delta.$$  \hspace{1cm} (5.17)

Combined with (5.16) and (5.14) this implies that

$$U^f_\infty(x, y) \leq U^f_\infty(x, y) - \mu(f)T + \delta \leq \inf\{U^f_\infty(z) - T\mu(f): z \in R^4\} + 2\delta$$

$$\leq \inf\{U^f_\infty(z): z \in R^4\} + 3\delta.$$  \hspace{1cm} (5.18)

By the relation above and (5.12), $d((x, y), E_\infty) \leq \epsilon$. Theorem 1.2 is proved.

References