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Blow up and grazing collision in viscous fluid solid interaction systems

Matthieu Hillairet [a,1], Takéo Takahashi [b,c,*,2]

[a] IMT, Université Paul Sabatier Toulouse 3, 31062 Toulouse Cedex 9, France
[b] Institut Élie Cartan, UMR 7502, INRIA, Nancy-Université, CNRS, POB 239, Vandœuvre-lès-Nancy 54506, France
[c] Team-project CORIDA, INRIA Nancy – Grand Est, France

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Abstract

We investigate qualitative properties of strong solutions to a classical system describing the fall of a rigid ball under the action of gravity inside a bounded cavity filled with a viscous incompressible fluid. We prove contact between the ball and the boundary of the cavity implies blow up of strong solutions and such a contact has to occur in finite time under symmetry assumptions on the initial data.

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1. Introduction

In this paper, we compute blowing-up solutions for a fluid solid interaction system. We consider a bounded domain $\Omega \subset \mathbb{R}^3$ of class C^2 containing a viscous incompressible fluid and a rigid ball. The equations of motion for the fluid and the rigid body are the classical Navier–Stokes equations coupled with the Newton laws

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = \operatorname{div} \mathbb{T}(\mathbf{u}, p) + \mathbf{f}, & \text{in } \mathcal{F}_t, \\ \operatorname{div} \mathbf{u} = 0, & \text{in } \mathcal{F}_t, \\ \mathbf{u} = \dot{\mathbf{G}} + \boldsymbol{\omega} \times (\mathbf{x} - \mathbf{G}), & \text{on } \partial \mathcal{B}_t, \\ \mathbf{u} = 0, & \text{on } \partial \Omega, \end{cases} \tag{1}$$

* Corresponding author at: Institut Élie Cartan, UMR 7502, INRIA, Nancy-Université, CNRS, POB 239, Vandœuvre-lès-Nancy 54506, France.
E-mail address: takeo8@gmail.com (T. Takahashi).
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$$\begin{cases} -\displaystyle\int_{\partial\mathcal{B}} \mathbb{T}(\mathbf{u},p)\mathbf{n}\,\mathrm{d}\sigma + \int_{\mathcal{B}} \rho_{\mathcal{B}}\mathbf{f}\,\mathrm{d}\mathbf{x} = m\ddot{\mathbf{G}}, \\ -\displaystyle\int_{\partial\mathcal{B}} (\mathbf{x}-\mathbf{G})\times\mathbb{T}(\mathbf{u},p)\mathbf{n}\,\mathrm{d}\sigma + \int_{\mathcal{B}} \rho_{\mathcal{B}}(\mathbf{x}-\mathbf{G})\times\mathbf{f}\,\mathrm{d}\mathbf{x} = J\dot{\boldsymbol{\omega}}. \end{cases} \tag{2}$$

In the above system, the set $\mathcal{B}_t$ stands for the domain of the solid $\mathcal{B}$: it is a ball with center $\mathbf{G}(t)$ and radius 1. Its complement in Ω, $\mathcal{F}_t = \Omega \setminus \overline{\mathcal{B}}_t$, is the domain occupied by the fluid. For simplicity, we have assumed the fluid has a constant density $\rho_{\mathcal{F}} = 1$ and its stress tensor is given by:

$$\mathbb{T}(\mathbf{u},p) = \mu\big(\nabla\mathbf{u} + [\nabla\mathbf{u}]^\top\big) - pI_3 = 2\mu D(\mathbf{u}) - pI_3,$$

where μ is the viscosity of the fluid. For any matrix M, we denote by $M^\top$ the transpose of M. We also assume the solid is homogeneous with constant density $\rho_{\mathcal{B}} > 1$ so that $\mathbf{G}(t)$ is the position of the center of mass of $\mathcal{B}$ at time t and

$$m = \rho_{\mathcal{B}}|\mathcal{B}_t|, \quad J = \bigg[\int_{\mathcal{B}_t} \rho_{\mathcal{B}}\big|\mathbf{x}-\mathbf{G}(t)\big|^2\,\mathrm{d}\mathbf{x}\bigg]I_3, \quad \forall t \geqslant 0.$$

The vector $\boldsymbol{\omega}$ stands for the angular velocity of $\mathcal{B}$. We take into account the action of the fluid in the Newton laws. The whole system evolves under the action of gravity

$$\mathbf{f} = -g\mathbf{e}_3.$$

The main unknown in the system (1)–(2) is $(\mathbf{u},\mathbf{G},\boldsymbol{\omega})$. This system is completed by adding the following initial conditions:

$$\mathbf{u}(0,\cdot) = \mathbf{u}_0, \qquad \mathbf{G}(0) = \mathbf{G}_0, \qquad \dot{\mathbf{G}}(0) = \dot{\mathbf{G}}_0, \qquad \boldsymbol{\omega}(0) = \boldsymbol{\omega}_0. \tag{3}$$

1.1. Previous results

We call Fluid Solid Interaction System ((FSIS) for short) the set of Eqs. (1)–(2). This system, and its many-bodies variant, is relevant on the theoretical level as in applications. It is the motivation for many recent studies. First, some authors obtain existence of weak solutions (in a sense which will be made precise later on) up to collision between two bodies [1,4,7,8,17]. These results "up-to-collision" are extended to global ones by J.A. San Martín, V. Starovoitov and M.Tucsnak in two dimensions [14], and by E. Feireisl in three dimensions [5]. In both papers, the authors prove there exist global weak solutions to (FSIS) regardless collisions. The two-dimensional result is slightly more general than the three-dimensional one. Indeed, in [5], it is imposed in the construction of weak solutions that solids remain "stuck" forever if contact occurs. The Cauchy theory for (FSIS) has also been studied in the frame of strong solutions. In two dimensions, it is proved that, for a given initial data, such a solution exists and is unique up to collision between solid bodies. In the three dimensions case, existence of a unique maximal strong solution before collision has also been proved. However, in this second case, it is not known whether this maximal solution blows up before collision or not.

These results show that the existence of collisions is a major issue in (FSIS). Such a question has been already tackled in two ways, to our knowledge. The first method uses the fact that, in these fluid solid interaction systems, the bodies follow characteristics of the extended velocity field:

$$\bar{\mathbf{u}} = \mathbf{1}_{\mathcal{F}_t}\mathbf{u} + \mathbf{1}_{\mathcal{B}_t}\mathbf{u}_{\mathcal{B}_t}[\dot{\mathbf{G}},\boldsymbol{\omega}],$$

where $\mathbf{u}_{\mathcal{B}_t}[\mathbf{V},\boldsymbol{\omega}] = \mathbf{V} + \boldsymbol{\omega}\times(\mathbf{x}-\mathbf{G}(t))$. If this velocity field is sufficiently smooth ($\mathcal{C}^1$ uniformly in time, for example), the Cauchy–Lipschitz theorem implies two particles following the characteristics cannot meet each other in finite time. Hence, collision is impossible. We emphasize such a regularity is unexpectable here because the Newton laws impose a jump in the derivatives of $\bar{\mathbf{u}}$ through $\partial\mathcal{B}$. Even though restricting to the fluid domain, estimates on derivatives of $\bar{\mathbf{u}}$ in a solution to (FSIS) are known to depend drastically on the distance between solids (see [3]). Nevertheless, a criterion based on these ideas is derived by V.N. Starovoitov [16]. It does not enable to prevent solution to (FSIS) from collision between solids, but, it follows from this criterion that a certain class of strong solutions cannot persist through collisions in the two-dimensional as in the three-dimensional case. This argument is developed for our class of strong solutions in Section 2.

In the second method, one takes further advantage of the Newton laws. More precisely, in solutions to the above (FSIS) the least one can expect is the decrease of the total energy of the system:

$$E := \int_{\Omega} \bar{\rho}|\bar{\mathbf{u}}|^2 \,\mathrm{d}\mathbf{x} + \int_{\mathcal{B}} [\rho_{\mathcal{B}} - 1] g \mathbf{e}_3 \,\mathrm{d}\mathbf{x},$$

where $\bar{\rho} := \mathbf{1}_{\mathcal{F}_t} + \mathbf{1}_{\mathcal{B}_t}\rho_{\mathcal{B}}$. In particular, in the toy-model of a ball falling over a horizontal ramp $\mathcal{P}$, this yields that the speed of the ball remains bounded with time. Then integrating the Newton law on the linear momentum with respect to time, one deduces

$$\int_0^T \int_{\partial\mathcal{B}_t} \mathbb{T}(\mathbf{u}, p)\mathbf{n} \,\mathrm{d}\sigma \cdot \mathbf{e}_3 \,\mathrm{d}t < m[C_0 + gT], \tag{4}$$

where $\mathbf{e}_3$ is the vertical direction and C_0 is a constant fixed by initial data. In the slow motion regime, computations due to Cooley and O'Neill [2] imply that

$$\int_{\partial\mathcal{B}_t} \mathbb{T}(\mathbf{u}, p)\mathbf{n} \,\mathrm{d}\sigma \cdot \mathbf{e}_3 \sim -\frac{\kappa \dot{d}(t)}{d^{\alpha}(t)},$$

where $d(t) = \mathrm{dist}(\mathcal{B}_t, \mathcal{P})$. The factor κ depends on the radius of $\mathcal{B}$ and the exponent α depends on the dimension. Cooley and O'Neill computed $\alpha = 1$ in the case of a ball falling over a ramp in the three-dimensional case, while $\alpha = 3/2$, in the case of a disk over a ramp in $\mathbb{R}^2$, see [10]. In both cases, (4) implies $\dot{d}/d \in L^1(0, T)$ so that d cannot go to 0 in finite time. These arguments are adapted rigorously to the full non-linear system in the two-dimensional case in [10] and in the three-dimensional case in [11]. They are also extended in [6] to more singular geometries yielding a threshold for the body-shape regularity under which collision can occur. These results are in contrast with the non-viscous case in which it is proved that collision can occur with non-zero relative velocity [12].

In the articles previously quoted, only frontal collisions are taken into account. The aim of this paper is to show that in the three-dimensional setting, grazing collisions between smooth bodies can occur (see Theorem 2). This collision result is the main step in the proof of our main result:

Theorem 1. *There exists a smooth initial condition such that the maximal strong solution* $(T_{max}, (\mathbf{u}, \mathbf{G}, \dot{\mathbf{G}}, \omega))$ *to* (*FSIS*) *before collision with this initial condition satisfies*

(1) $T_{max} < \infty$,
(2) *one of the two following alternatives holds true*:
   (i) $$\limsup_{t \to T_{max}} \int_{\mathcal{F}_t} \left|\nabla \mathbf{u}(t, \mathbf{x})\right|^2 \mathrm{d}\mathbf{x} = \infty,$$
   (ii) $$\liminf_{t \to T_{max}} \mathrm{dist}(\mathcal{B}_t, \partial\Omega) = 0 \quad \textit{and} \quad \lim_{t \to T_{max}} \int_0^t \int_{\mathcal{F}_t} \left|\nabla^2 \mathbf{u}(s, \mathbf{x})\right| \mathrm{d}\mathbf{x}\,\mathrm{d}t = \infty.$$

We state this result in the frame of "strong solution before collision" because maximal strong solutions are well defined only before collision. We emphasize alternative (ii) prevents strong solutions to persist through collision (see Definition 2.2).

In the three-dimensional context, it is not clear whether collision is the most important responsible for ill-posedness of smooth solutions. Indeed, non-linear convective terms in the Navier–Stokes system could make smoothness of solutions to fail before collision. Hence, we cannot exclude that a maximal strong solution blows up by satisfying (i). However, our result does not depend on the size of initial data. In particular, for small data, the only way for maximal strong solution to blow up in finite time is the second one. This alternative corresponds to a collision between solids

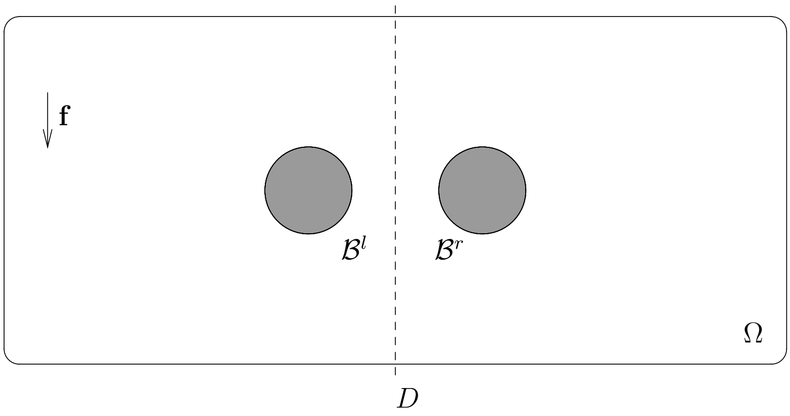

Fig. 1. Geometrical configuration.

in the fluid. In the frame of weak solutions, collision occurrence is also an important phenomenon because it is known that it causes failure of uniqueness [15]. We emphasize this result does not contradict [11] because the geometric configuration under consideration here does not enter in the frame of this former result.

1.2. Description of the geometry and formal arguments

The geometry of the problem is crucial to obtain Theorem 2. For simplicity, we set

$$\Omega = B\big((0,0,0), M\big) \setminus \big(\overline{B\big((2,0,0),1\big) \cup B\big((-2,0,0),1\big)}\big),$$

with M sufficiently large. However, our techniques extend to more general geometries. The same result would also hold true under the following assumptions:

G1 The cavity Ω is symmetric with respect to some line D.
G2 The cavity Ω has exactly two holes $\mathcal{B}^l$ and $\mathcal{B}^r$ symmetric with respect to D. These holes have the shape of balls with radius 1. The distance between the holes is 2.
G3 The gravity $\mathbf{f}$ is parallel to D.
G4 Near $\partial D \cap \partial\Omega$ the boundary $\partial\Omega$ is flat.

An example of such a geometry is represented in Fig. 1.

In the following, we denote by $\mathbf{G}^l = (-2,0,0)$ and $\mathbf{G}^r = (2,0,0)$ the centers of the holes and the corresponding holes by $\mathcal{B}^l = B((-2,0,0),1)$ and $\mathcal{B}^r = B((2,0,0),1)$. We emphasize the distance between the holes is chosen so that $\mathcal{B}$ fills exactly the gap between $\mathcal{B}^l$ and $\mathcal{B}^r$. We introduce $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ the orthonormal basis corresponding to our coordinates for $\mathbb{R}^3$. In particular $\mathbf{e}_1$ is a direction of the line joining the two hole centers and the last unit vector $\mathbf{e}_3$ is a direction of the gravity. We underline D is the line which is parallel to the gravity and passes through the origin of our system of coordinates.

Our computations are motivated by the following formal arguments. If the ball $\mathcal{B}$ moves along the axis D, we have $\mathbf{G}(t) = (0,0,a(t))$. In our coordinates $a(t)$ is the "altitude" of $\mathcal{B}$ at time t. We denote by $d(t)$ the distance between $\mathcal{B}$ and the holes $\mathcal{B}^l$ and $\mathcal{B}^r$ at time t. With these conventions, contact occurs between $\mathcal{B}$ and the holes if d or a vanishes. We do not envisage other kinds of collision between $\mathcal{B}$ and $\partial\Omega$ because they are precluded by former arguments (see [11]).

In a first approximation, when $\mathcal{B}$ comes close to the holes, the action of the fluid on $\mathcal{B}$ can be written as the sum of two forces. One is due to the vicinity of $\mathcal{B}^l$ and the other one to the vicinity of $\mathcal{B}^r$. Concerning $\mathcal{B}^l$ for example, we split the force in a frontal resistance preventing $\mathcal{B}$ from going closer to $\mathcal{B}^l$ and a friction. It stems from computations in the lubrication theory we can neglect the frictions in what follows [13] and the frontal resistance reads [2]:

$$-\frac{1}{\big(|\mathbf{G}^l - \mathbf{G}| - 2\big)} \frac{(\dot{\mathbf{G}}^l - \dot{\mathbf{G}}) \cdot (\mathbf{G}^l - \mathbf{G})}{|\mathbf{G}^l - \mathbf{G}|} \frac{(\mathbf{G}^l - \mathbf{G})}{|\mathbf{G}^l - \mathbf{G}|}.$$

We have an equivalent formula for the second contribution replacing $\mathbf{G}^l$ by $\mathbf{G}^r$. Eventually, the projection along $\mathbf{e}_3$ of Newton law on the linear momentum reads:

$$\ddot{a} = -\frac{2\dot{a}a^2}{(\sqrt{a^2+4}-2)(a^2+4)} - \big(m - |\mathcal{B}|\big)g, \tag{5}$$

where we take into account the Archimede law. Standard Cauchy–Lipschitz arguments imply this equation is locally well-posed when completed by adding initial conditions $a(0) = a_0 \in \mathbb{R} \setminus \{0\}$, and $\dot{a}(0) = \dot{a}_0$. Moreover, if a maximal solution to this system blows up at $T_* < \infty$, then one of the three following equalities holds true:

$$\limsup_{t\to T_*} \big|\dot{a}(t)\big| = \infty, \quad \limsup_{t\to T_*} \big|a(t)\big| = \infty, \quad \liminf_{t\to T_*} \big|a(t)\big| = 0.$$

However, multiplying (5) by $\dot{a}$, we obtain that this simplified model dissipates the total energy $\mathcal{E} = |\dot{a}|^2/2 + (m - |\mathcal{B}|)ga$ of the particle $\mathcal{B}$. This implies the only way a maximal solution to (5) blows up is the third one. Furthermore, we remark $\tilde{a}(t) = a_0$ for all $t \geqslant 0$ is a global supersolution to (5) regardless of the value of $a_0 \neq 0$. In particular, if $a_0 > 0$ and $\dot{a}_0 < 0$, then $a(t) \in [0, a_0]$ until blow up of the solution. So, under this assumption the only way the maximal solution may blow up in finite time T_* is that a vanishes when t goes to T_*. In what follows, we assume $a_0 > 0$ and $\dot{a}_0 < 0$.

Integrating (5) between 0 and t, we obtain

$$\dot{a}(t) - \dot{a}_0 = -\int_0^t \left[\frac{2\dot{a}a^2}{(\sqrt{a^2+4}-2)(a^2+4)} + \big(m - |\mathcal{B}|\big)g\right] \mathrm{d}s = -P(t) - \big(m - |\mathcal{B}|\big)gt, \tag{6}$$

where, after a straightforward change of variable:

$$P(t) = \tilde{P}\big(\big|a(t)\big|^2\big) = \int_{|a_0|^2}^{|a(t)|^2} \frac{\sqrt{r}\,\mathrm{d}r}{(\sqrt{r+4}-2)(r+4)}.$$

Standard computations lead to $|\tilde{P}(z)| \leqslant C_0$ for all $z \in (0, a_0]$. Finally, assuming the function a is defined globally, we combine (6) together with dissipation of total energy to obtain:

$$-C_0 \leqslant P(t) \leqslant K_0 - \big(m - |\mathcal{B}|\big)gt, \quad \forall t \in (0, \infty),$$

with a constant K_0 depending only on initial data. Since we assumed $\rho_{\mathcal{B}} > 1$, we obtain a contradiction for:

$$T_0 := \frac{K_0 + C_0}{m - |\mathcal{B}|} + 1.$$

Hence, a has to vanish before T_0. We emphasize considering a three-dimensional example is critical here. Indeed, in the two-dimensional case we would get a function $\tilde{P}(z)$ which diverges when z goes to 0.

1.3. Notations

We use the classical Lebesgue and Sobolev spaces $L^q(\mathcal{O})$, $W^{m,q}(\mathcal{O})$, $H^m(\mathcal{O})$, $H_0^m(\mathcal{O})$ for any open set $\mathcal{O} \subset \mathbb{R}^3$. We define

$$\mathcal{H} = \big\{\boldsymbol{\phi} \in L^2(\Omega);\ \operatorname{div}\boldsymbol{\phi} = 0,\ \boldsymbol{\phi}\cdot\mathbf{n} = 0 \text{ on } \partial\Omega\big\}, \qquad \mathcal{V} = \big\{\boldsymbol{\phi} \in H_0^1(\Omega);\ \operatorname{div}\boldsymbol{\phi} = 0\big\}.$$

We recall that $\mathcal{H}$ and $\mathcal{V}$ are closed subspace of $L^2(\Omega)$ and $H_0^1(\Omega)$ respectively. Thus, they form Hilbert spaces with respect to the induced inner products. For an open subset $\mathcal{O} \subset \Omega$, we also consider the following Hilbert spaces:

$$\mathbb{H}(\mathcal{O}) = \big\{\boldsymbol{\phi} \in \mathcal{H};\ D(\boldsymbol{\phi}) = 0 \text{ in } \mathcal{O}\big\}, \qquad \mathbb{V}(\mathcal{O}) = \big\{\boldsymbol{\phi} \in \mathcal{V};\ D(\boldsymbol{\phi}) = 0 \text{ in } \mathcal{O}\big\}.$$

To simplify, if $\mathbf{G} \in \Omega$ we set $\mathcal{B}_{\mathbf{G}} := B(\mathbf{G}, 1)$ and $\mathcal{F}_{\mathbf{G}} := \Omega \setminus \overline{\mathcal{B}}_{\mathbf{G}}$. Moreover, if $\mathcal{B}_{\mathbf{G}} \subset \Omega$, we define $\mathbb{H}(\mathbf{G}) = \mathbb{H}(\mathcal{B}_{\mathbf{G}})$, $\mathbb{V}(\mathbf{G}) = \mathbb{V}(\mathcal{B}_{\mathbf{G}})$. Under the same assumption, we also denote by $\rho_{\mathbf{G}}$ the function

$$\rho_{\mathbf{G}}(\mathbf{x}) = \begin{cases} \rho_{\mathcal{B}}, & \text{if } \mathbf{x} \in \mathcal{B}_{\mathbf{G}}, \\ 1, & \text{if } \mathbf{x} \in \mathcal{F}_{\mathbf{G}}. \end{cases}$$

If $\mathbf{v} \in \mathbb{H}(\mathbf{G})$, from [18, p. 18], there exists a unique pair $(\mathbf{V}[\mathbf{v}], \boldsymbol{\omega}[\mathbf{v}]) \in \mathbb{R}^3 \times \mathbb{R}^3$ such that

$$\mathbf{v}|_{\mathcal{B}_{\mathbf{G}}} = \mathbf{V}[\mathbf{v}] + \boldsymbol{\omega}[\mathbf{v}] \times (\mathbf{x} - \mathbf{G}).$$

In particular, if $(\mathbf{u}, \mathbf{v}) \in \mathbb{H}(\mathbf{G})^2$,

$$\int_{\Omega} \rho_{\mathbf{G}} \mathbf{u} \cdot \mathbf{v} \, d\mathbf{x} = \int_{\Omega \setminus \mathcal{B}_{\mathbf{G}}} \mathbf{u} \cdot \mathbf{v} \, d\mathbf{x} + m\mathbf{V}[\mathbf{u}] \cdot \mathbf{V}[\mathbf{v}] + J\boldsymbol{\omega}[\mathbf{u}] \cdot \boldsymbol{\omega}[\mathbf{v}].$$

2. Cauchy theory and main result

As classical in Navier–Stokes like systems, there exist two families of solutions. First, we have the weak solutions.

Definition 2.1. Given $\mathbf{G}_0 \in \Omega$ such that $\operatorname{dist}(\mathbf{G}_0, \partial\Omega) \geqslant 1$ and $\mathbf{u}_0 \in \mathbb{H}(\mathbf{G}_0)$, a pair $(\mathbf{u}, \mathbf{G})$ is called weak solution to (FSIS) on $(0, T)$ with initial data $(\mathbf{u}_0, \mathbf{G}_0)$ if

$$\mathbf{G} \in W^{1,\infty}(0, T; \Omega), \quad \text{with } \mathbf{G}(0) = \mathbf{G}_0, \tag{7}$$

$$\operatorname{dist}\big(\mathbf{G}(t), \partial\Omega\big) \geqslant 1, \quad \text{for all } t \in (0, T), \tag{8}$$

$$\mathbf{u} \in L^\infty(0, T; \mathcal{H}) \cap L^2(0, T; \mathcal{V}), \tag{9}$$

$$\mathbf{u} = \mathbf{V} + \boldsymbol{\omega} \times (\mathbf{x} - \mathbf{G}) \quad \text{in } \mathcal{B}_{\mathbf{G}}, \text{ with } \mathbf{V} = \dot{\mathbf{G}}; \tag{10}$$

if for all $\mathbf{v} \in \mathcal{C}([0, T]; H^1_0(\Omega)) \cap H^1(0, T; L^2(\Omega))$ with compact support in $(0, T) \times \Omega$ and such that $\mathbf{v} \in \mathbb{V}(\mathbf{G}(t))$ for all $t \in [0, T]$,

$$-\int_0^T \int_{\Omega} \rho_{\mathbf{G}} \mathbf{u} \cdot \partial_t \mathbf{v} \, d\mathbf{y} \, dt + 2\mu \int_0^T \int_{\Omega} D(\mathbf{u}) : D(\mathbf{v}) \, d\mathbf{y} \, dt - \int_0^T \int_{\Omega} \mathbf{u} \otimes \mathbf{u} : D(\mathbf{v}) \, d\mathbf{y} \, dt = \int_0^T \int_{\Omega} \rho_{\mathbf{G}} \mathbf{f} \cdot \mathbf{v} \, d\mathbf{y} \, dt; \tag{11}$$

if for all $\mathbf{v} \in \mathcal{C}([0, T]; L^2(\Omega))$ with compact support in $[0, T) \times \Omega$ and such that $\mathbf{v} \in \mathbb{H}(\mathbf{G}(t))$ for all $t \in [0, T]$ we have

$$W : t \mapsto \int_{\Omega} \rho_{\mathbf{G}} \mathbf{u} \cdot \mathbf{v} \, d\mathbf{x} \in \mathcal{C}\big([0, T]\big) \quad \text{with } W(0) = \int_{\Omega} \rho_{\mathbf{G}_0} \mathbf{u}_0 \cdot \mathbf{v} \, d\mathbf{x}; \tag{12}$$

and if the energy estimate holds true:

$$\left[\frac{1}{2} \int_{\Omega} \rho_{\mathbf{G}} |\mathbf{u}|^2(t, \mathbf{x}) \, d\mathbf{x} + g\big(m - |\mathcal{B}|\big)\mathbf{G}(t) \cdot \mathbf{e}_3\right] + 2\mu \int_0^t \int_{\Omega} \big|D(\mathbf{u})\big|^2 \, d\mathbf{x} \, ds$$
$$\leqslant \left[\frac{1}{2} \int_{\Omega} \rho_{\mathbf{G}_0} |\mathbf{u}_0|^2(\mathbf{x}) \, d\mathbf{x} + g\big(m - |\mathcal{B}|\big)\mathbf{G}_0 \cdot \mathbf{e}_3\right] \quad \text{for a.a. } t \in (0, T). \tag{13}$$

There exist slightly different definitions of weak solutions. Here we use the same one we used in [11]. For instance, the link between our weak solutions and the one constructed in [5] is the following. First, as we work with a constant-density fluid, we introduce the position of the center of mass $\mathbf{G}$ as unknown instead of the density $\bar{\rho}$ and isometry η. The density and the position of the center of mass fix one another *via* the identity:

$$\bar{\rho}(t, x) = \mathbf{1}_{\mathcal{F}_t}(x) + \rho_{\mathcal{B}} \mathbf{1}_{\mathcal{B}_t}(x), \quad \forall (t, x) \in (0, T) \times \Omega.$$

The isometry η is computed as the composition of the translation associated to $\mathbf{G}(t) - \mathbf{G}_0$ with some rotation associated to $\boldsymbol{\omega}$. We emphasize that we actually do not need any information on this rotation in our case because $\mathcal{B}$ is a ball.

Concerning energy estimate, we have the above particular form because, in [5, (1.16)], we replace the source term $\mathbf{f}$ by the gravity with direction $\mathbf{e}_3$. Hence, after integration by parts, we get

$$\int_{\Omega} \rho(t,\mathbf{x})\mathbf{f}(t,\mathbf{x}) \cdot \mathbf{u}(t,\mathbf{x})\,\mathrm{d}\mathbf{x} = -\int_{\mathcal{B}_t} (\rho_{\mathcal{B}} - 1) g\mathbf{u}(t,\mathbf{x}) \cdot \mathbf{e}_3\,\mathrm{d}\mathbf{x} = -g\big(m - |\mathcal{B}|\big)\dot{\mathbf{G}}(t) \cdot \mathbf{e}_3.$$

Finally, the existence result in Ref. [5] applies to our case, so that weak solutions to (FSIS), in the sense of Definition 2.1 exist globally regardless of the initial position of $\mathcal{B}$ with $\operatorname{dist}(\mathbf{G}_0, \partial\Omega) > 1$ and the value of initial data.

The second family of solutions are the strong solutions.

Definition 2.2. Given $\mathbf{G}_0 \in \Omega$ such that $\operatorname{dist}(\mathbf{G}_0, \partial\Omega) > 1$ and $\mathbf{u}_0 \in \mathbb{V}(\mathbf{G}_0)$, a pair $(\mathbf{u}, \mathbf{G})$ is a strong solution to (FSIS) on $(0, T)$ if it is a weak solution to (FSIS) with the additional regularity:

$$\mathbf{u} \in \mathcal{C}(0,T;\mathcal{V}), \quad \text{and} \quad \mathbf{u}(t,\cdot) \in H^2(\mathcal{F}_t), \qquad p(t,\cdot) \in H^1(\mathcal{F}_t), \quad \text{for a.a. } t \in (0,T), \tag{14}$$

$$\sup_{(0,t)} \big\|\nabla\mathbf{u}(t,\cdot)\big\|^2_{L^2(\Omega)} + \int_0^t \int_{\mathcal{F}_t} \big|\nabla^2\mathbf{u}(t,\mathbf{x})\big|^2 + \big|\nabla p(t,\mathbf{x})\big|^2\,\mathrm{d}\mathbf{x}\,\mathrm{d}t < \infty, \quad \text{in } (0,T). \tag{15}$$

This is the family of solutions studied in [17] but this reformulation allows us to deal with collision. Contrary to [17], we measure here the regularity of the velocity-field after restriction to the fluid domain. We emphasize that as long as no collision occurs, our definition is equivalent to the one in [17]. Hence, the precollisional existence and uniqueness result of [17] still holds for strong solutions in the sense of Definition 2.2. Moreover, in three dimensions, global existence of strong solutions without collision is also known only for small data. Indeed, for a fixed ball $\mathcal{B}$, (FSIS) becomes a particular case of the Navier–Stokes system. Consequently, (FSIS) contains the complexity of Navier–Stokes system. Moreover, as pointed out in [16], (FSIS) is more singular in the sense that collision is a second way for strong solutions to blow up.

2.1. Collision result

We prove here whenever a strong solution does not blow up satisfying alternative (i) in Theorem 1, the second way (alternative (ii) in the same theorem) has to occur. To this end, we obtain the following fundamental result on collisions:

Theorem 2. *Given* $(\mathbf{G}_0, \mathbf{u}_0)$ *such that* $\operatorname{dist}(\mathbf{G}_0, \partial\Omega) > 0$ *and* $\mathbf{G}_0 = (0,0,a_0)$ *with* $a_0 > 0$, *and* $\mathbf{u}_0 \in \mathbb{H}(\mathbf{G}_0)$, *there exists* $T_* < \infty$ *for which the following property holds true*:

Given a weak solution $(\mathbf{u}, \mathbf{G})$ *on* $(0, T_*)$ *with initial data* $(\mathbf{u}_0, \mathbf{G}_0)$ *such that* $\mathbf{G}(t) = (0,0,a(t))$ *for all* $t \in (0, T_*)$, *then the function* a *vanishes before* T_*.

Before going to the proof of Theorem 2 in further details, we explain how it implies Theorem 1. Given any velocity-field $\mathbf{v}$ defined on Ω, we denote by $S_D[\mathbf{v}]$ the symmetric velocity-field:

$$S_D[\mathbf{v}](\mathbf{x}) = (-v_1, -v_2, v_3)(-x_1, -x_2, x_3), \quad \forall \mathbf{x} \in \Omega.$$

As initial data, we choose a smooth $(\mathbf{G}_0, \mathbf{u}_0)$ satisfying

$$S_D[\mathbf{u}_0] = \mathbf{u}_0, \qquad \mathbf{G}_0 = (0,0,a_0), \quad \text{with } a_0 > 0.$$

Applying Theorem 2, we construct a time T_* with the given property for this initial data. Then we denote by $(T_{max}, (\mathbf{u}, \mathbf{G}))$ the maximal strong solution to (FSIS) with this initial data and we assume $T_{max} > T_*$. One can check that $(\tilde{\mathbf{G}}, \tilde{\mathbf{u}})$ as defined by

$$\tilde{\mathbf{G}}(t) = (-G_1, -G_2, G_3)(t), \qquad \tilde{\mathbf{u}}(t,\cdot) = S_D\big[\mathbf{u}(t,\cdot)\big], \quad \forall t \geqslant 0$$

is also a strong solution to (FSIS) with the same initial data. Hence $\tilde{\mathbf{G}} = \mathbf{G}$ and $\tilde{\mathbf{u}} = \mathbf{u}$ so that $(\mathbf{G}, \mathbf{u})$ is symmetric with respect to D before contact. In particular, it is a weak solution on $(0, T_*)$ such that $\mathbf{G}(t) = (0,0,a(t))$ with $a(t) > 0$

before collision. Applying Theorem 2, there exists $t_0 < T_*$ for which $a(t_0) = 0$, or, equivalently, $\mathrm{dist}(\mathcal{B}_{t_0}, \partial\Omega) = 0$. We obtain a contradiction and $T_{max} \leqslant T_*$.

We proceed with the second part of Theorem 1. As local existence and uniqueness of strong solutions is known for arbitrary initial data $(\mathbf{u}_0, \mathbf{G}_0)$ where $\mathbf{u}_0 \in H^1_0(\Omega)$, and $\mathrm{dist}(\mathbf{G}_0, \partial\Omega) > 1$, classical arguments imply blow-up of a maximal strong solution can occur for $T_{max} < \infty$ only if

$$\limsup_{t\to T_{max}} \|\mathbf{u}(t,\cdot)\|_{H^1_0(\Omega)} = \infty, \quad \text{or} \quad \liminf_{t\to T_{max}} \mathrm{dist}\big(\mathbf{G}(t), \partial\Omega\big) = 1.$$

In the first alternative, as $\|\mathbf{u}(t,\cdot)\|_{L^2(\Omega)}$ remains bounded, blow-up of the H^1-norm of $\mathbf{u}(t,\cdot)$ means blow-up of the L^2-norm of its gradient and we obtain (i). In the second case, as $\mathbf{G}(t) = (0, 0, a(t))$ for all $t < T_{max}$, collision between $\mathcal{B}_t$ and $\partial\Omega$ can only hold on the top boundary or with the holes. As collision between $\mathcal{B}_t$ and the top boundary are ruled out by former arguments (see [11]), we obtain $\liminf_{t\to T_{max}} a(t) = 0$. In particular the distance $d(t)$ between $\mathcal{B}_t$ and the holds satisfies $d(t) > 0$ for $t \in [0, T_{max})$ and $\liminf_{t\to T_{max}} d(t) = 0$. Moreover, we have (see Appendix B recognizing $\mathbf{V}\cdot\tilde{\mathbf{e}}_3 = \dot{d}$):

$$\big|\dot{d}(t)\big| \leqslant C\big|d(t)\big|^{\frac{3}{2}} \big\|\nabla^2\mathbf{u}(t,\cdot)\big\|_{L^2(\mathcal{F}_t)} \tag{16}$$

for some universal constant C. Integrating this inequality between 0 and $t < T_{max}$ yields:

$$\int_0^t \big\|\nabla^2\mathbf{u}\big\|_{L^2(\mathcal{F}_t)}\,\mathrm{d}t \geqslant C\left[\frac{1}{\sqrt{d(t)}} - \frac{1}{\sqrt{d(0)}}\right]. \tag{17}$$

Letting t go to T_{max}, we obtain (ii).

The remainder of this paper is devoted to the proof of Theorem 2.

2.2. *Sketch of the proof of Theorem 2*

The proof of Theorem 2 is based on the same arguments as in [6,11]. In the remainder of this section $(\mathbf{u}, \mathbf{G})$ is a given weak solution such that $\mathbf{G}(t) = (0, 0, a(t))$ for all t. In particular, it has initial data $(\mathbf{u}(0,\cdot), \mathbf{G}(0))$ where $\mathbf{G}(0) = (0, 0, a_0)$ with $a_0 > 0$. Following similar arguments as in [11], collisions between $\mathcal{B}$ and $\partial\Omega \setminus (\partial\mathcal{B}^l \cup \partial\mathcal{B}^r)$ are ruled out. On the contrary, we prove that simultaneous contact between $\mathcal{B}$ and $(\mathcal{B}^l, \mathcal{B}^r)$ occurs in finite time. So, we denote by $d(t)$ the common distance between $\mathcal{B}$ and the holes $\mathcal{B}^l$ and $\mathcal{B}^r$ for all t. Combining that $\mathbf{G}(t) = (0, 0, a(t))$ with $a(t) > 0$ and $\mathrm{dist}(\mathcal{B}_t, \mathcal{B}^l) = \mathrm{dist}(\mathcal{B}_t, \mathcal{B}^r) = d(t)$ we obtain that, before contact

$$\mathbf{G}(t) = \mathbf{G}_{d(t)} = \big(0, 0, \sqrt{d(t)^2 + 4d(t)}\big). \tag{18}$$

We restrict ourselves to the case $a(t) > 0$, because we assume initially that the solid is "above" the holes. We prove by contradiction that $a(t)$ is bound to vanish before a finite time depending only on initial data.

We emphasize that, as collisions between $\mathcal{B}$ and $\partial\Omega \setminus (\partial\mathcal{B}^l \cup \partial\mathcal{B}^r)$ are impossible, there exists $d_{max} > 0$ such that $d(t) \in (0, d_{max}]$ before collision and

$$\mathrm{dist}\big(\mathcal{B}_d, \partial\Omega \setminus \big(\partial\mathcal{B}^l \cup \partial\mathcal{B}^r\big)\big) \geqslant \delta_0 > 0, \quad \forall d \in [0, d_{max}],$$

where $\mathcal{B}_d = \mathcal{B}_{\mathbf{G}_d}$ with the convention (18). In what follows, we replace $\mathbf{G}$ by d in notation, assuming $\mathbf{G} = \mathbf{G}_d$ implicitly. The distances d_{max} and δ_0 are fixed in the remainder of this paper.

In the next section, we construct a suitable family of "stationary" test functions $(\mathbf{w}[d])_{d>0}$ for the weak formulation (11). In particular, we build them so that $\mathbf{w}[d] \in \mathbb{H}(\mathbf{G}_d)$. They also satisfy the following properties:

Proposition 1. *Given* $d_{min} > 0$, *there holds*:

(1) *for any* $d \in [d_{min}, d_{max}]$, $\mathbf{w}[d] \in \mathcal{C}(\overline{\Omega})$ *with*

$$\mathbf{w}[d] = \mathbf{e}_3 \quad \text{on } \mathcal{B}_d, \qquad \mathbf{w}[d] = 0 \quad \text{on } \partial\Omega,$$

(2) *assume* $\mathcal{Q}_d := \{(d, \mathbf{x}),\ d \in (d_{min}, d_{max}),\ \mathbf{x} \in \mathcal{F}_d\}$, *and*

$$\begin{aligned}\tilde{\mathbf{w}} : (0, d_{max}) \times \Omega &\longrightarrow \mathbb{R}^3,\\ (d, \mathbf{x}) &\longmapsto \mathbf{w}[d](\mathbf{x}),\end{aligned}$$

then $\tilde{\mathbf{w}} \in \mathcal{C}^\infty(\overline{\mathcal{Q}}_d)$.

Assuming at first the function d does not vanish on $(0, T)$ (where T is arbitrary) there exists $d_{min} > 0$ such that $d(t) > d_{min}$ for any $t \in [0, T]$. Hence, for any $\chi \in \mathcal{D}(0, T)$ we can use the following test function in (11):

$$\begin{aligned}\mathbf{w} : (0, T) \times \Omega &\longrightarrow \mathbb{R}^3,\\ (t, \mathbf{x}) &\longmapsto \chi(t)\mathbf{w}\big[d(t)\big](\mathbf{x}).\end{aligned}$$

This yields

$$\int_0^T \int_\Omega \rho_d \mathbf{u} \cdot \partial_t \mathbf{w} \,\mathrm{d}\mathbf{y}\,\mathrm{d}t + \int_0^T \int_\Omega \rho_d \mathbf{f} \cdot \mathbf{w} \,\mathrm{d}\mathbf{y}\,\mathrm{d}t = -\int_0^T \int_\Omega \mathbf{u} \otimes \mathbf{u} : D(\mathbf{w}) \,\mathrm{d}\mathbf{y}\,\mathrm{d}t + 2\mu \int_0^T \int_\Omega D(\mathbf{u}) : D(\mathbf{w}) \,\mathrm{d}\mathbf{y}\,\mathrm{d}t. \tag{19}$$

We split this identity in $I_1^l + I_2^l = I_1^r + I_2^r$ where, after straightforward computations:

$$I_1^l = \int_0^T \dot{\chi} \int_\Omega \rho_d \mathbf{u} \cdot \mathbf{w}[d] \,\mathrm{d}\mathbf{y}\,\mathrm{d}t + \int_0^T \dot{d}\chi \int_\Omega \rho_d \mathbf{u} \cdot \partial_d \mathbf{w}[d] \,\mathrm{d}\mathbf{y}\,\mathrm{d}t, \qquad I_2^l = -\big(m - |B|\big) g \int_0^T \chi(s) \,\mathrm{d}s.$$

In Section 4, we prove:

Proposition 2. *There exists a positive constant C depending only on $\rho_{\mathcal{B}}$ and d_{max} such that, for all $d < d_{max}$ and $\mathbf{v} \in \mathbb{H}(\mathbf{G}_d)$, there hold*:

$$\left| \int_\Omega \rho_d \mathbf{v} \cdot \mathbf{w}[d] \,\mathrm{d}\mathbf{y} \right| \leqslant C \|\mathbf{v}\|_{L^2(\Omega)}, \tag{20}$$

$$\left| \int_\Omega \rho_d \mathbf{v} \cdot \partial_d \mathbf{w}[d] \,\mathrm{d}\mathbf{y} \right| \leqslant \frac{C \|\nabla \mathbf{v}\|_{L^2(\Omega)}}{\sqrt{d}}, \tag{21}$$

$$\left| \int_\Omega \mathbf{v} \otimes \mathbf{v} : D\big(\mathbf{w}[d]\big) \,\mathrm{d}\mathbf{y} \right| \leqslant C \|\nabla \mathbf{v}\|_{L^2(\Omega)}^2. \tag{22}$$

Moreover, if $\mathbf{v} = \ell \mathbf{e}_3$ on $\mathcal{B}_d$ then

$$\int_\Omega D(\mathbf{v}) : D\big(\mathbf{w}[d]\big) \,\mathrm{d}\mathbf{y} = \ell b(d) + R \tag{23}$$

with $|R| \leqslant C \|\nabla \mathbf{v}\|_{L^2(\Omega)}$ and with $0 \leqslant b(d) \leqslant C|\ln(d)|$.

The content of this proposition is twofold. First, inequalities (20)–(22) enable to dominate remainder terms in (19). Indeed, combining these inequalities and energy estimate (13), this yields

$$\big|I_1^l\big| \leqslant \int_0^T \big[C|\dot{\chi}| \big\|\mathbf{u}(t, \cdot)\big\|_{L^2(\Omega)} + C\big|\chi(t)\big|\big|\dot{d}(t)\big|\big|d(t)\big|^{-\frac{1}{2}} \big\|\nabla \mathbf{u}(t, \cdot)\big\|_{L^2(\Omega)}\big] \,\mathrm{d}t,$$

where, applying [16, Theorem 3.1], there exists a universal constant for which

$$\big|\dot{d}(t)\big|\big|d(t)\big|^{-\frac{1}{2}} \leqslant C \big\|\nabla \mathbf{u}(t, \cdot)\big\|_{L^2(\Omega)}.$$

Consequently

$$\left|I_1^l\right| \leqslant C_0\left(\|\dot{\chi}\|_{L^1(0,T)} + \|\chi\|_{L^\infty(0,T)}\right)$$

with a constant C_0 depending only on initial data and on the size of Ω. We emphasize that here (13) implies decrease of the total energy of the system. As Ω is bounded this implies a T-independent control on the solution $\mathbf{u}$ in $L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$. This uniform estimate would not persist if $\mathbf{f}$ were not deriving from such a potential. Similarly

$$\left|I_1^r\right| \leqslant \int_0^T C\|\chi\|_{L^\infty(0,T)}\left\|\nabla\mathbf{u}(t,\cdot)\right\|_{L^2(\Omega)}^2 \,\mathrm{d}t \leqslant C_0\|\chi\|_{L^\infty(0,T)}.$$

Second, inequality (23) computes the drag acted on a body in a fluid flow $\mathbf{v}$ with a precision $O(|\nabla\mathbf{v}|_{L^2(\Omega)})$. In the frame of our weak solution, this leads to

$$\left|I_2^r - 2\mu\int_0^T \chi(t)\dot{a}(t)b\big(d(t)\big)\,\mathrm{d}t\right| \leqslant C_0\|\chi\|_{L^\infty(0,T)}\sqrt{T},$$

where $\dot{a}(t) = \dot{d}(d+2)/\sqrt{d^2+4d}$ (see (18)). Eventually (19) reduces to:

$$\int_0^T \chi(t)\left[2\mu\frac{\dot{d}(t)(d(t)+2)b(d(t))}{\sqrt{d(t)^2+4d(t)}} + \big(m-|\mathcal{B}|\big)g\right]\mathrm{d}t \leqslant C_0(1+\sqrt{T})\big[\|\chi\|_{L^\infty(0,T)} + \|\dot{\chi}\|_{L^1(0,T)}\big].$$

Using a family of functions χ increasing toward the characteristic function of $(0,T)$, we obtain

$$\int_0^T 2\mu\frac{\dot{d}(t)(d(t)+2)}{\sqrt{d(t)^2+4d(t)}}b\big(d(t)\big)\,\mathrm{d}t \leqslant -\big(m-|\mathcal{B}|\big)gT + C_0(1+\sqrt{T}). \tag{24}$$

On the other hand, the above control on b implies

$$d \mapsto \int_{d_0}^d b(s)\frac{s+2}{\sqrt{s^2+4s}}\,\mathrm{d}s$$

is bounded continuous when d goes to 0. Hence, (24) leads to a contradiction for a time T_0 depending only on initial conditions, as in our toy-model. This completes the proof of Theorem 2.

3. Constructing the test functions

In this section we construct the test functions we have used to prove Theorem 2. We compute an analytic formula in the half space $\mathcal{P}^l := \{(x_1,x_2,x_3) \in \mathbb{R}^3;\ x_1 \leqslant 0\}$, the test functions are extended to the half space $\mathcal{P}^r := \{(x_1,x_2,x_3) \in \mathbb{R}^3;\ x_1 \geqslant 0\}$ by applying the symmetry operator S_D. In each half space, it is more convenient to work in a local orthonormal frame attached to the moving ball $\mathcal{B}$. The origin of this local frame is $\mathbf{G}$ and the associated direct orthonormal basis is $(\tilde{\mathbf{e}}_1,\tilde{\mathbf{e}}_2,\tilde{\mathbf{e}}_3)$. We consider $\tilde{\mathbf{e}}_2 = \mathbf{e}_2$ and $\tilde{\mathbf{e}}_3$ is such that $(\mathbf{G}-\mathbf{G}^l) = (2+d)\tilde{\mathbf{e}}_3$. For any $\mathbf{x} \in \mathbb{R}^3$ we denote by $\tilde{\mathbf{x}} = (\tilde{x}_1,\tilde{x}_2,\tilde{x}_3)$ its coordinates in this new frame. In particular, the following holds:

$$\tilde{\mathbf{x}} = Q_\alpha(\mathbf{x}-\mathbf{G}) \quad \text{or} \quad \mathbf{x} = \mathbf{G} + Q_\alpha^{-1}\tilde{\mathbf{x}}$$

with Q_α the rotation with axis $\mathbb{R}\mathbf{e}_2$ and angle α (see Fig. 2 for the definition of α). We also introduce (r,θ,z) the cylindrical coordinates:

$$\tilde{x}_1 = r\cos(\theta), \qquad \tilde{x}_2 = r\sin(\theta), \qquad \tilde{x}_3 = z.$$

In what follows, we keep this convention for sets. Namely, if not precisely mentioned, for any set $\mathcal{S} \subset \mathbb{R}^3$ the following holds:

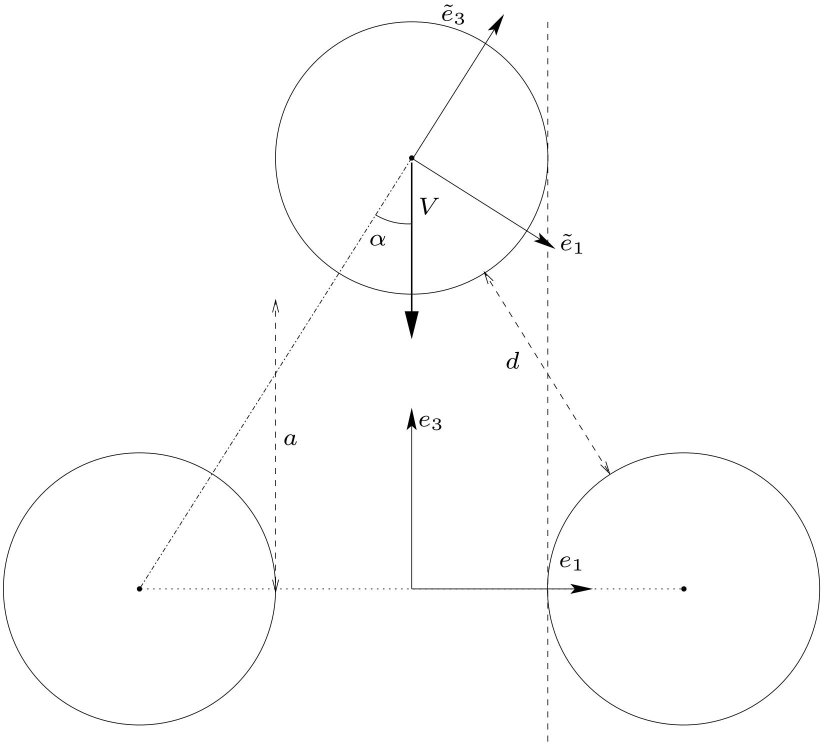

Fig. 2. Detailed description of the geometry.

$$\tilde{\mathcal{S}} = Q_\alpha(\mathcal{S} - \mathbf{G}) \quad \text{or} \quad \mathcal{S} = \mathbf{G} + Q_\alpha^{-1}\tilde{\mathcal{S}}.$$

Actually, we shall only make one exception. Indeed, in this new frame the ball $\mathcal{B}$ is fixed and centered in $\mathbf{0}$ whereas the center $\mathbf{G}^l$ of $\mathcal{B}^l$ has moving coordinates $(0, 0, -2 - d)$. Consequently, we prefer to use $\tilde{\mathcal{B}}_*$ for the image of $\mathcal{B}$ (which is fixed) and $\tilde{\mathcal{B}}_d$ for the image of $\mathcal{B}^l$. Hence, $\tilde{\mathcal{B}}_*$ and $\tilde{\mathcal{B}}_d$ are the unit balls in $\mathbb{R}^3$ having the origin and $(0, 0, -2 - d)$ for respective centers.

When $d = 0$, the fluid domain $\tilde{\mathcal{F}}_0$ has a cusp where $\tilde{\mathcal{B}}_*$ is in contact with $\tilde{\mathcal{B}}_0$. In order to surround this singularity we introduce a family of neighborhoods of the points realizing the distance between $\tilde{\mathcal{B}}_d$ and $\tilde{\mathcal{B}}_*$. Given $d \in (0, d_{max})$ and $l \in (0, 1)$, we denote by $\tilde{\Omega}_{d,l}$ the cylindric domain between $\tilde{\mathcal{B}}_*$ and $\tilde{\mathcal{B}}_d$ with radius l:

$$\tilde{\Omega}_{d,l} := \big\{(r, \theta, z) \in \tilde{\mathcal{F}}_d \text{ such that } r \in [0, l),\ z \in \big(-(2 + d), 0\big)\big\}. \tag{25}$$

We remark that, given $d_{max} > 0$, there exists $l_{max} > 0$ such that $\tilde{\Omega}_{d,l_{max}} \subset \tilde{\mathcal{P}}^l$ for any $d \in [0, d_{max}]$. We assume $l_{max} > 1/2$. This is equivalent to considering a weak solution on a time-range where d_{max} is sufficiently small.

We notice that the upper boundary and the lower boundary of $\tilde{\Omega}_{d,l}$ are parametrized respectively by:

$$(r, \theta, z) \in \partial\tilde{\Omega}_{d,l} \cap \partial\tilde{\mathcal{B}}_* \quad \Leftrightarrow \quad \big\{r \in [0, l) \text{ with } z = \delta_*(r)\big\},$$

where

$$\delta_*(s) := -\sqrt{1 - s^2}, \quad \forall s \in [0, 1), \tag{26}$$

and

$$(r, \theta, z) \in \partial\tilde{\Omega}_{d,l} \cap \partial\tilde{\mathcal{B}}_d \quad \Leftrightarrow \quad \big\{r \in [0, l) \text{ with } z = \delta_d(r)\big\},$$

where, for all $d > 0$,

$$\delta_d(s) := -(2 + d) + \sqrt{1 - s^2}, \quad \forall s \in [0, 1). \tag{27}$$

Finally, the remainder of the geometry (i.e. outside $\tilde{\Omega}_{d,1/2}$) is "smooth" in the sense that, there exists a width d_0 such that, for any distance $d \in [0, d_{max}]$ the annuli with width d_0 which surround the boundaries of $\tilde{\mathcal{B}}_*$ and the hole $\tilde{\mathcal{B}}_d$ are subsets of $\tilde{\mathcal{F}}_d$. To turn this into a quantitative information, we define:

$$d_0 = \frac{1}{2} \inf_{0 \leqslant d \leqslant d_{max}} \operatorname{dist}\big(\partial\tilde{\mathcal{B}}_* \cap (\tilde{\Omega}_{d,\frac{1}{4}})^c, \partial\tilde{\mathcal{B}}_d \cap (\tilde{\Omega}_{d,\frac{1}{4}})^c\big) = \frac{\sqrt{17/16} - 1}{2}.$$

With this choice, for a radius M of the cavity large enough, for any $d \in [0, d_{max}]$ and $\tilde{\mathbf{x}} \notin \tilde{\Omega}_{d,1/4}$, if $0 < \operatorname{dist}(\tilde{\mathbf{x}}, \tilde{\mathcal{B}}_d) < d_0$ or $0 < \operatorname{dist}(\tilde{\mathbf{x}}, \tilde{\mathcal{B}}_*) < d_0$ then $\tilde{\mathbf{x}}$ is in the fluid domain $\tilde{\mathcal{F}}_d$. Moreover, for d_{max} sufficiently small if $\operatorname{dist}(\tilde{\mathbf{x}}, \tilde{\mathcal{B}}_d) \leqslant d_0$ then $\tilde{\mathbf{x}} \in \tilde{\mathcal{P}}^l$.

3.1. Parallel component

We first construct a velocity field which is equal to $\tilde{\mathbf{e}}_1$ on $\tilde{\mathcal{B}}_*$. At first, this velocity field is computed in the local frame, i.e., with coordinates $\tilde{\mathbf{x}}$. Between the two spheres, our potential vector field reads, in cylindrical coordinates:

$$\tilde{\mathbf{a}}^{ns}_{/\!/}(r, \theta, z) = \left(0, \phi_{/\!/}(r, z), \frac{1}{2} r \sin(\theta)\right), \quad \forall (r, \theta, z) \in \tilde{\Omega}_{d,1/2}.$$

Hence, the components of $\tilde{\mathbf{w}}^{ns}_{/\!/}[d] = \operatorname{curl} \tilde{\mathbf{a}}^{ns}_{/\!/}[d]$ read:

$$\tilde{\mathbf{w}}^{ns}_{/\!/}(r, \theta, z) = \left(\frac{1}{2} - \partial_z \phi_{/\!/}(r, z), 0, \cos(\theta)\partial_r \phi_{/\!/}(r, z)\right), \quad \forall (r, \theta, z) \in \tilde{\Omega}_{d,1/2}. \tag{28}$$

In this expression, *ns* stands for "non-smooth", this is the part of $\tilde{\mathbf{w}}$ which becomes singular when d goes to 0, and $\phi_{/\!/}$ is a truncation function enabling $\tilde{\mathbf{w}}^{ns}_{/\!/}$ to go from $(1, 0, 0)$ on $\partial\tilde{\mathcal{B}}_*$ to $(0, 0, 0)$ on $\partial\tilde{\mathcal{B}}_d$. We set, in order to fit with these boundary conditions (this is critical in Lemma 3):

$$\phi_{/\!/}(r, z) = -\frac{\chi_{/\!/}(\lambda(r, z))}{4}\big(\delta_*(r) - \delta_d(r)\big) + \frac{2 + d}{4}, \tag{29}$$

with

$$\chi_{/\!/}(s) = 2s^2 - 2s + 1, \quad \forall s \in (0, 1), \tag{30}$$

and where λ is the normalized vertical distance do $\partial\tilde{\mathcal{B}}_d$:

$$\lambda(r, z) = \frac{z - \delta_d(r)}{\delta_*(r) - \delta_d(r)}.$$

In the complement of $\tilde{\Omega}_{d,1/2}$ we set:

$$\tilde{\mathbf{a}}^{s}_{/\!/} = \frac{\eta_{d_0}(|\tilde{\mathbf{x}} + (0, 0, 2 + d)| - 1)}{2}\big(0, (z + 2 + d)/2, r\sin(\theta)/2\big) + \frac{\eta_{d_0}(|\tilde{\mathbf{x}}| - 1)}{2}(\tilde{\mathbf{e}}_1 \times \tilde{\mathbf{x}}), \quad \forall \tilde{\mathbf{x}} \in \mathbb{R}^3.$$

Here and in what follows, we denote by $\eta : [0, \infty) \to [0, 1]$ a smooth function such that

$$\eta(s) = \begin{cases} 1, & \text{if } s < \frac{1}{2}, \\ 0, & \text{if } s > 1, \end{cases}$$

and, we set $\eta_\alpha = \eta(\cdot/\alpha)$ for all parameter $\alpha > 0$. By definition of d_0, if $\tilde{\mathbf{x}} \notin \tilde{\Omega}_{d,1/4}$, then at most one of the functions $\eta_{d_0}(|\tilde{\mathbf{x}} + (0, 0, 2 + d)| - 1)$ and $\eta_{d_0}(|\tilde{\mathbf{x}}| - 1)$ is different from 0.

Finally, we set:

$$\tilde{\mathbf{a}}_{/\!/} = \begin{cases} \eta_{1/2}(r)\tilde{\mathbf{a}}^{ns}_{/\!/} + (1 - \eta_{1/2}(r))\tilde{\mathbf{a}}^{s}_{/\!/}, & \text{in } \tilde{\Omega}_{d,1/2}, \\ \tilde{\mathbf{a}}^{s}_{/\!/}, & \text{in } \mathbb{R}^3 \setminus (\tilde{\Omega}_{d,1/2} \cup \tilde{\mathcal{B}}_* \cup \tilde{\mathcal{B}}_d), \end{cases}$$

and

$$\tilde{\mathbf{w}}_{/\!/}[d] = \begin{cases} \operatorname{curl} \tilde{\mathbf{a}}_{/\!/}, & \text{in } \mathbb{R}^3 \setminus (\tilde{\mathcal{B}}_* \cup \tilde{\mathcal{B}}_d), \\ \tilde{\mathbf{e}}_1, & \text{in } \tilde{\mathcal{B}}_*, \\ 0, & \text{in } \tilde{\mathcal{B}}_d. \end{cases}$$

This family of functions $(\tilde{\mathbf{w}}_{/\!/}[d])_{d>0}$ satisfies the following result.

Proposition 3. *For any $d > 0$, the following holds*:

(1) $\tilde{\mathbf{w}}_{/\!/}[d] \in \mathcal{C}(\mathbb{R}^3)$, *with*:

$$\tilde{\mathbf{w}}_{/\!/}[d] = \tilde{\mathbf{e}}_1 \quad \text{on } \tilde{\mathcal{B}}_*, \qquad \tilde{\mathbf{w}}_{/\!/}[d] = 0 \quad \text{on } \tilde{\mathcal{B}}_d.$$

(2) *In a neighborhood of $\partial\tilde{\mathcal{P}}^l$*

$$\tilde{\mathbf{w}}_{/\!/}[d](\tilde{\mathbf{x}}) = \operatorname{curl}_{\tilde{\mathbf{x}}}\left(\frac{\eta_{d_0}(|\tilde{\mathbf{x}}| - 1)}{2} \tilde{\mathbf{e}}_1 \times \tilde{\mathbf{x}}\right). \tag{31}$$

Proof. As $d > 0$, the only difficulty to obtain (1) is to prove continuity through $\partial\tilde{\mathcal{B}}_d$ and $\partial\tilde{\mathcal{B}}_*$. In the following we drop arguments in λ and we denote by subscripts its differentiations. For example,

$$\lambda_z = \frac{1}{\delta_*(r) - \delta_d(r)}, \qquad \lambda_r = -\frac{\delta_d'(r)}{\delta_*(r) - \delta_d(r)} - \lambda \frac{\delta_*'(r) - \delta_d'(r)}{\delta_*(r) - \delta_d(r)}.$$

Differentiating $\phi_{/\!/}$, this yields

$$\partial_z \phi_{/\!/}(r, z) = -\frac{\chi_{/\!/}'(\lambda)}{4} \lambda_z \big(\delta_*(r) - \delta_d(r)\big) = -\frac{\chi_{/\!/}'(\lambda)}{4}, \tag{32}$$

$$\partial_r \phi_{/\!/}(r, z) = -\frac{\chi_{/\!/}'(\lambda)}{4} \lambda_r \big(\delta_*(r) - \delta_d(r)\big) - \frac{\chi_{/\!/}(\lambda)}{4} \big(\delta_*'(r) - \delta_d'(r)\big), \tag{33}$$

where

$$\chi_{/\!/}(0) = \chi_{/\!/}(1) = 1, \qquad \chi_{/\!/}'(0) = -2, \qquad \chi_{/\!/}'(1) = 2.$$

As a consequence for $\lambda = 0$ ($z = \delta_d(r)$):

$$\tilde{\mathbf{a}}_{/\!/}^{ns}(\tilde{\mathbf{x}}) = \big(0, \big(z + (2 + d)\big)/2, r\sin(\theta)/2\big) \quad \text{and} \quad \tilde{\mathbf{w}}_{/\!/}^{ns}(\tilde{\mathbf{x}}) = \mathbf{0}, \quad \forall \tilde{\mathbf{x}} \in \partial\tilde{\mathcal{B}}_d \cap \partial\tilde{\Omega}_{d,1/4}, \tag{34}$$

and for $\lambda = 1$ ($z = \delta_*(r)$):

$$\tilde{\mathbf{a}}_{/\!/}^{ns}(\tilde{\mathbf{x}}) = (\tilde{\mathbf{e}}_1 \times \tilde{\mathbf{x}})/2 \quad \text{and} \quad \tilde{\mathbf{w}}_{/\!/}^{ns}(\tilde{\mathbf{x}}) = \tilde{\mathbf{e}}_1, \quad \forall \tilde{\mathbf{x}} \in \partial\tilde{\mathcal{B}}_* \cap \partial\tilde{\Omega}_{d,1/4}. \tag{35}$$

Concerning the smooth part, we recall that we chose d_0 so that outside $\tilde{\Omega}_{d,1/4}$, we have:

$$\left.\begin{aligned} &\eta_{d_0}\big(\big|\tilde{\mathbf{x}} - \big(0, 0, -(2 + d)\big)\big| - 1\big) = 1, \qquad \eta_{d_0}\big(|\tilde{\mathbf{x}}| - 1\big) = 0, \\ &\eta_{d_0}'\big(\big|\tilde{\mathbf{x}} - \big(0, 0, -(2 + d)\big)\big| - 1\big) = 0, \qquad \eta_{d_0}'\big(|\tilde{\mathbf{x}}| - 1\big) = 0, \end{aligned}\right\} \quad \forall \tilde{\mathbf{x}} \in \partial\tilde{\mathcal{B}}_d,$$

and

$$\left.\begin{aligned} &\eta_{d_0}\big(\big|\tilde{\mathbf{x}} - \big(0, 0, -(2 + d)\big)\big| - 1\big) = 0, \qquad \eta_{d_0}\big(|\tilde{\mathbf{x}}| - 1\big) = 1, \\ &\eta_{d_0}'\big(\big|\tilde{\mathbf{x}} - \big(0, 0, -(2 + d)\big)\big| - 1\big) = 0, \qquad \eta_{d_0}'\big(|\tilde{\mathbf{x}}| - 1\big) = 0, \end{aligned}\right\} \quad \forall \tilde{\mathbf{x}} \in \partial\tilde{\mathcal{B}}_*.$$

Consequently, outside $\tilde{\Omega}_{d,1/4}$:

$$\begin{cases} \tilde{\mathbf{a}}_{/\!/}^{s}(\tilde{\mathbf{x}}) = \big(0, (z + 2 + d)/2, r\sin(\theta)/2\big), & \tilde{\mathbf{w}}_{/\!/}^{s}(\tilde{\mathbf{x}}) = \mathbf{0}, \quad \text{on } \partial\tilde{\mathcal{B}}_d, \\ \tilde{\mathbf{a}}_{/\!/}^{s}(\tilde{\mathbf{x}}) = (\tilde{\mathbf{e}}_1 \times \tilde{\mathbf{x}})/2, & \tilde{\mathbf{w}}_{/\!/}^{s}(\tilde{\mathbf{x}}) = \tilde{\mathbf{e}}_1, \quad \text{on } \partial\tilde{\mathcal{B}}_*. \end{cases} \tag{36}$$

The continuity of $\tilde{\mathbf{w}}_{/\!/}[d]$ through $\partial\tilde{\mathcal{B}}_*$ and $\partial\tilde{\mathcal{B}}_d$ yields by interpolation of (34)–(35) and (36).

Finally, equality (31) holds outside $\tilde{\Omega}_{d,1/2}$ and at distance larger than d_0 of $\tilde{\mathcal{B}}_d$. Due to our choice for d_0 and because we assume $1/2 < l_{max}$, this equality holds in particular in a neighborhood of $\partial\tilde{\mathcal{P}}^l$. $\square$

3.2. Normal component

Now, we construct a velocity field which is equal to $\tilde{\mathbf{e}}_3$ on $\tilde{\mathcal{B}}_*$. This is the direction along which the ball $\tilde{\mathcal{B}}_*$ gets closer to the hole $\tilde{\mathcal{B}}_d$. This construction is completely similar to the one in [11]. We only change the value of λ by using the formula of the previous section. Hence, our potential vector field reads, in cylindrical coordinates:

$$\tilde{\mathbf{a}}_{\perp}^{ns}(r,\theta,z) = (-\phi_{\perp}\sin\theta, \phi_{\perp}\cos\theta, 0), \quad \forall (r,\theta,z) \in \tilde{\Omega}_{d,1/2},$$

where

$$\phi_{\perp}(r,z) = r\chi_{\perp}(\lambda), \tag{37}$$

with

$$\chi_{\perp}(s) = \frac{s^2(3-2s)}{2} \quad \bigl(s \in (0,1)\bigr).$$

Consequently, for all $(r,\theta,z) \in \tilde{\Omega}_{d,1/2}$:

$$\tilde{\mathbf{w}}_{\perp}^{ns}(r,\theta,z) = \operatorname{curl} \tilde{\mathbf{a}}_{\perp}^{ns} = \left(-\partial_z\phi_{\perp}\cos\theta, -\partial_z\phi_{\perp}\sin\theta, \partial_r\phi_{\perp} + \frac{\phi_{\perp}}{r}\right). \tag{38}$$

In the complement of $\tilde{\Omega}_{d,1/2}$, we set:

$$\tilde{\mathbf{a}}_{\perp}^{s} = \frac{\eta_{d_0}(|\tilde{\mathbf{x}}|-1)}{2}(\tilde{\mathbf{e}}_3 \times \tilde{\mathbf{x}}), \quad \forall \tilde{\mathbf{x}} \in \mathbb{R}^3$$

and we obtain the final potential by interpolation:

$$\tilde{\mathbf{a}}_{\perp} = \begin{cases} \eta_{1/2}(r)\tilde{\mathbf{a}}_{\perp}^{ns} + (1-\eta_{1/2}(r))\tilde{\mathbf{a}}_{\perp}^{s}, & \text{in } \tilde{\Omega}_{d,1/2}, \\ \tilde{\mathbf{a}}_{\perp}^{s}, & \text{in } \mathbb{R}^2 \setminus (\tilde{\Omega}_{d,1/2} \cup \tilde{\mathcal{B}}_d \cup \tilde{\mathcal{B}}_*). \end{cases}$$

Finally, we set:

$$\tilde{\mathbf{w}}_{\perp}[d] = \begin{cases} \operatorname{curl} \tilde{\mathbf{a}}_{\perp}, & \text{in } \mathbb{R}^3 \setminus (\tilde{\mathcal{B}}_* \cup \tilde{\mathcal{B}}_d), \\ \tilde{\mathbf{e}}_3, & \text{in } \tilde{\mathcal{B}}_*, \\ 0, & \text{in } \tilde{\mathcal{B}}_d. \end{cases}$$

Proposition 4. *For any $d > 0$, the following holds*:

(1) $\tilde{\mathbf{w}}_{\perp}[d] \in \mathcal{C}(\mathbb{R}^3)$, *with*:

$$\tilde{\mathbf{w}}_{\perp}[d] = \tilde{\mathbf{e}}_3 \quad \text{on } \tilde{\mathcal{B}}_*, \qquad \tilde{\mathbf{w}}_{\perp}[d] = 0 \quad \text{on } \tilde{\mathcal{B}}_d.$$

(2) *In a neighborhood of $\partial\tilde{\mathcal{P}}^l$*

$$\tilde{\mathbf{w}}_{\perp}[d](\tilde{\mathbf{x}}) = \operatorname{curl}_{\tilde{\mathbf{x}}}\left(\frac{\eta_{d_0}(|\tilde{\mathbf{x}}|-1)}{2}\tilde{\mathbf{e}}_3 \times \tilde{\mathbf{x}}\right).$$

Proof. The proof is exactly the same as for the parallel component. We let the reader refer to [11] for technical details. □

3.3. Complete test function

We recall that, by definition:

$$\mathbf{e}_3 = \cos(\alpha)\tilde{\mathbf{e}}_3 - \sin(\alpha)\tilde{\mathbf{e}}_1, \qquad \tilde{\mathbf{e}}_3 = Q_{-\alpha}\mathbf{e}_3, \qquad \tilde{\mathbf{e}}_1 = Q_{-\alpha}\mathbf{e}_1,$$

with $\alpha \in (0, \pi/2)$ given by

$$\sin(\alpha) = \frac{2}{2+d}, \qquad \cos(\alpha) = \frac{\sqrt{d^2+4d}}{2+d}. \tag{39}$$

Hence, in order to obtain a velocity field with rigid velocity $\mathbf{e}_3$, we set

$$\tilde{\mathbf{w}}[d](\tilde{\mathbf{x}}) = \cos\alpha\, \tilde{\mathbf{w}}_\perp[d](\tilde{\mathbf{x}}) - \sin\alpha\, \tilde{\mathbf{w}}_{/\!/}[d](\tilde{\mathbf{x}}). \tag{40}$$

In the global frame (the one without tildes), this velocity field reads

$$\mathbf{w}[d](\mathbf{x}) = Q_{-\alpha}\tilde{\mathbf{w}}[d]\big(Q_\alpha(\mathbf{x}-\mathbf{G}_d)\big) \tag{41}$$

for all $\mathbf{x} \in \mathcal{P}^l$, or more precisely:

$$\mathbf{w}[d](\mathbf{x}) = \cos\alpha\, Q_{-\alpha}\tilde{\mathbf{w}}_\perp[d]\big(Q_\alpha(\mathbf{x}-\mathbf{G}_d)\big) - \sin\alpha\, Q_{-\alpha}\tilde{\mathbf{w}}_{/\!/}[d]\big(Q_\alpha(\mathbf{x}-\mathbf{G}_d)\big).$$

As mentioned above, we obtain our test-velocity field in the remainder of the geometry by symmetry

$$\mathbf{w}[d](\mathbf{x}) = S_D\big[\mathbf{w}[d]\big](\mathbf{x}), \quad \forall \mathbf{x} \in \mathcal{P}^r.$$

The family $(\mathbf{w}[d])_{d>0}$ constructed this way satisfies Proposition 1. The only difficulty to prove this, is to obtain that $\mathbf{w}$ is smooth in a neighborhood of $\partial\mathcal{P}^l = \partial\mathcal{P}^r$. But, in a neighborhood of $\partial\mathcal{P}^l$ inside $\mathcal{P}^l$, we have by substitution:

$$\begin{aligned}\mathbf{w}[d](\mathbf{x}) &= \operatorname{curl}_{\mathbf{x}}\left(\eta_{d_0}\big(|\mathbf{x}-\mathbf{G}_d|-1\big)\frac{(\mathbf{e}_3 \times (\mathbf{x}-\mathbf{G}_d))}{2}\right) \\ &= \eta'_{d_0}\big(|\mathbf{x}-\mathbf{G}_d|-1\big)\frac{\mathbf{x}-\mathbf{G}_d}{|\mathbf{x}-\mathbf{G}_d|} \times \frac{(\mathbf{e}_3 \times (\mathbf{x}-\mathbf{G}_d))}{2} + \eta_{d_0}\big(|\mathbf{x}-\mathbf{G}_d|-1\big)\mathbf{e}_3.\end{aligned}$$

As $\mathbf{e}_3$ is symmetric with respect to D, the same formula holds in the other half space. Therefore, $\mathbf{w}$ is smooth in the whole fluid domain as long as $d \neq 0$. We also emphasize that $\mathbf{w}$ is symmetric with respect to D so that we only estimate the restriction of $\mathbf{w}$ to $\mathcal{P}^l$ in what follows.

4. Estimating the test functions

This section is devoted to Proposition 2. The method is similar for all inequalities. First, we reduce these computations in the global framework to inequalities in the local one. Then, in the local framework, we prove such inequalities hold by using the analytic formulas defining $\tilde{\mathbf{w}}[d]$. The computations are done for $d < 1$ so that the result holds in particular for $d < d_{max}$.

For example computing the left-hand side of (20), we have:

$$\int_\Omega \rho_d \mathbf{u}\cdot\mathbf{w}[d]\,\mathrm{d}\mathbf{y} = \int_{\mathcal{P}^l} \rho_d \mathbf{u}\cdot\mathbf{w}[d]\,\mathrm{d}\mathbf{y} + \int_{\mathcal{P}^r} \rho_d \mathbf{u}\cdot\mathbf{w}[d]\,\mathrm{d}\mathbf{y}.$$

We focus on the term in $\mathcal{P}^l$. The other domination is computed by symmetry. We split:

$$\int_{\mathcal{P}^l} \rho_d \mathbf{u}\cdot\mathbf{w}[d]\,\mathrm{d}\mathbf{y} = \int_{\Omega_{d,1/4}} \rho_d \mathbf{u}\cdot\mathbf{w}[d]\,\mathrm{d}\mathbf{y} + \int_{\mathcal{P}^l\setminus\Omega_{d,1/4}} \rho_d \mathbf{u}\cdot\mathbf{w}[d]\,\mathrm{d}\mathbf{y}.$$

By construction,

$$\tilde{\mathbf{a}}[d](\tilde{\mathbf{x}}) := \cos\alpha(t)\tilde{\mathbf{a}}_\perp[d](\tilde{\mathbf{x}}) - \sin\alpha(t)\tilde{\mathbf{a}}_{/\!/}[d](\tilde{\mathbf{x}})$$

is continuous in d, smooth in the spatial variable and with compact support in

$$\big\{(d,\mathbf{x}) \in [0,1]\times\mathbb{R}^3;\ \mathbf{x} \notin \tilde{\Omega}_{d,1/4}\big\}.$$

Thus, there exists a constant $C = C(\beta)$ independent of d such that

$$\|\tilde{\mathbf{a}}\|_{H^\beta(\tilde{\mathcal{F}}_d\setminus\tilde{\Omega}_{d,1/4})} \leqslant C, \quad \forall d < 1.$$

As a consequence, we only focus on

$$\left| \int_{\Omega_{d,1/4}} \rho_d \mathbf{u} \cdot \mathbf{w}[d] \,\mathrm{d}\mathbf{y} \right| \leqslant C \|\mathbf{u}\|_{L^2(\Omega)} \left\|\mathbf{w}[d]\right\|_{L^2(\Omega_{d,1/4})}.$$

Using that Q_α is a unit transformation, the proof of (20), as other dominations in Proposition 2, is reduced to estimate $\tilde{\mathbf{w}}[d]$ in $\tilde{\Omega}_{d,1/4}$.

First, to end up the proof of (20) and in preparation for (23), we obtain the following result.

Proposition 5. *The function* $\tilde{\mathbf{w}}[d]$ *defined in* (40) *satisfies*

$$\left\|\tilde{\mathbf{w}}[d]\right\|_{L^2(\tilde{\Omega}_{d,1/4})} \leqslant C, \tag{42}$$

$$\left\|\nabla\tilde{\mathbf{w}}[d]\right\|_{L^2(\tilde{\Omega}_{d,1/4})} \leqslant C\sqrt{\ln(1/d)}, \tag{43}$$

for any $d > 0$, *with a constant* C *independent of* d.

Proof. To prove (42), we apply Lemmata 3 and 4. This yields that, for any $(r, \theta, z) \in \tilde{\Omega}_{d,1/4}$, and $d \in (0, 1)$,

$$\left|\tilde{\mathbf{w}}_\perp^{ns}\right| \leqslant |\partial_z \phi_\perp| + |\partial_r \phi_\perp| + \frac{|\phi_\perp|}{r} \leqslant C\left(1 + \frac{r}{\delta_*(r) - \delta_d(r)}\right), \tag{44}$$

$$\left|\tilde{\mathbf{w}}_{/\!/}^{ns}\right| \leqslant \frac{1}{2} + |\partial_z \phi_{/\!/}| + |\partial_r \phi_{/\!/}| \leqslant C. \tag{45}$$

Hence, we obtain:

$$\left\|\tilde{\mathbf{w}}_\perp^{ns}\right\|^2_{L^2(\tilde{\Omega}_{d,1/4})} + \left\|\tilde{\mathbf{w}}_{/\!/}^{ns}\right\|^2_{L^2(\tilde{\Omega}_{d,1/4})} \leqslant C \int_0^r \int_{\delta_d(r)}^{\delta_*(r)} \left(1 + \frac{r}{\delta_*(r) - \delta_d(r)}\right)^2 r \,\mathrm{d}z \,\mathrm{d}r,$$

so that Lemma 1 implies (42) holds true.

To prove (43), we first notice that for any $\mathbf{v}$,

$$|\nabla \mathbf{v}| \leqslant C\left(|\partial_r \mathbf{v}| + \frac{|\partial_\theta \mathbf{v}|}{r} + |\partial_z \mathbf{v}|\right). \tag{46}$$

From (38) and Lemma 3, we deduce

$$\left|\partial_r \tilde{\mathbf{w}}_\perp^{ns}\right| \leqslant C\left(|\partial_{rz}\phi_\perp| + |\partial_{rr}\phi_\perp| + \left|\frac{\partial_r \phi_\perp}{r} - \frac{\phi_\perp}{r^2}\right|\right) \leqslant \frac{C}{\delta_* - \delta_d}, \tag{47}$$

$$\frac{|\partial_\theta \tilde{\mathbf{w}}_\perp^{ns}|}{r} \leqslant \frac{|\partial_z \phi_\perp|}{r} \leqslant \frac{C}{\delta_* - \delta_d}, \tag{48}$$

$$\left|\partial_z \tilde{\mathbf{w}}_\perp^{ns}\right| \leqslant C\left(|\partial_{zz}\phi_\perp| + |\partial_{rz}\phi_\perp| + \left|\frac{\partial_z \phi_\perp}{r}\right|\right) \leqslant C\left(\frac{r}{(\delta_* - \delta_d)^2} + \frac{1}{\delta_* - \delta_d}\right). \tag{49}$$

Gathering (46)–(49) yields

$$\left|\nabla \tilde{\mathbf{w}}_\perp^{ns}\right| \leqslant C\left(\frac{r}{(\delta_* - \delta_d)^2} + \frac{1}{\delta_* - \delta_d}\right). \tag{50}$$

From (39) and Lemma 1, we obtain

$$\left\|\cos(\alpha)\nabla\tilde{\mathbf{w}}_\perp^{ns}\right\|_{L^2(\tilde{\Omega}_{d,1/4})} \leqslant C\sqrt{d}\left(\sqrt{\ln 1/d} + \frac{1}{\sqrt{d}}\right) \leqslant C.$$

From (28) and Lemma 4, we get

$$\left|\partial_r \tilde{\mathbf{w}}_{/\!/}^{ns}\right| \leqslant C\big(|\partial_{rz}\phi_{/\!/}| + |\partial_{rr}\phi_{/\!/}|\big) \leqslant C\left(\frac{r}{\delta_* - \delta_d} + 1\right), \tag{51}$$

$$\frac{|\partial_\theta \tilde{\mathbf{w}}_{/\!/}^{ns}|}{r} \leqslant \frac{|\partial_r \phi_{/\!/}|}{r} \leqslant C, \tag{52}$$

$$\left|\partial_z \tilde{\mathbf{w}}_{/\!/}^{ns}\right| \leqslant C\big(|\partial_{zz}\phi_{/\!/}| + |\partial_{rz}\phi_{/\!/}|\big) \leqslant C\left(\frac{1}{\delta_* - \delta_d} + \frac{r}{\delta_* - \delta_d}\right). \tag{53}$$

Gathering (46) and (51)–(53), we deduce

$$\left|\nabla \tilde{\mathbf{w}}_{/\!/}^{ns}\right| \leqslant \frac{C}{\delta_* - \delta_d}. \tag{54}$$

From (39) and Lemma 1, we conclude

$$\left\|\sin(\alpha)\nabla \tilde{\mathbf{w}}_{/\!/}^{ns}\right\|_{L^2(\tilde{\Omega}_{d,1/4})} \leqslant C\sqrt{\ln 1/d}. \qquad \square$$

Concerning (21) and (22), we split as previously and this yields

$$\int_{\mathcal{P}^l} \rho_d \mathbf{u} \cdot \partial_d \mathbf{w}[d]\, \mathrm{d}\mathbf{y} = I_1 + I_2, \qquad \int_{\mathcal{P}^l} \mathbf{u} \otimes \mathbf{u} \cdot D(\mathbf{w})[d]\, \mathrm{d}\mathbf{y} = J_1 + J_2,$$

where

$$|I_1| = \left|\int_{\mathcal{P}^l \setminus \Omega_{d,1/4}} \rho_h \mathbf{u} \cdot \partial_d \mathbf{w}[d]\, \mathrm{d}\mathbf{y}\right| \leqslant \|\mathbf{u}\|_{L^2(\Omega)} \left\|\partial_d \mathbf{w}[d]\right\|_{L^2(\mathcal{P}^l \setminus \Omega_{d,1/4})},$$

$$|J_1| \leqslant \|\mathbf{u}\|_{L^2(\Omega)}^2 \left\|\mathbf{w}[d]\right\|_{H^3(\mathcal{P}^l \setminus \Omega_{d,1/4})},$$

and, with the same technique as in [11, Lemma 3.1]:

$$|I_2| \leqslant M_2 \|\nabla \mathbf{u}\|_{L^2(\Omega)} \qquad |J_2| \leqslant M_2^\infty \|\nabla \mathbf{u}\|_{L^2(\Omega)},$$

where

$$M_2 = \left[\int_0^{\frac{1}{4}} \left(\big(\delta_d(r) - \delta_*(r)\big)^2 \left[\int_{\delta_*(r)}^{\delta_d(r)} \sup_{\theta \in (0,2\pi)} \left|\partial_d \mathbf{w}(r,\theta,z)\right|^2 \mathrm{d}z\right]\right) r\, \mathrm{d}r\right]^{\frac{1}{2}},$$

$$M_2^\infty = \sup_{r \in (0,\frac{1}{4})} \big(\delta_d(r) - \delta_*(r)\big)^{3/2} \left[\int_{\delta_*(r)}^{\delta_d(r)} \sup_{\theta \in (0,2\pi)} \left|\nabla \mathbf{w}(r,\theta,z)\right|^2 \mathrm{d}z\right]^{\frac{1}{2}}.$$

Consequently, (21) and (22) are consequences of the following result.

Proposition 6. *There exists a positive constant C such that*

$$\int_0^{\frac{1}{4}} \left(\big(\delta_d(r) - \delta_*(r)\big)^2 \left[\int_{\delta_*(r)}^{\delta_d(r)} \sup_{\theta \in (0,2\pi)} \left|\partial_d \mathbf{w}(r,\theta,z)\right|^2 \mathrm{d}z\right]\right) r\, \mathrm{d}r \leqslant \frac{C}{d}, \tag{55}$$

$$\left\|\partial_d \mathbf{w}[d]\right\|_{L^2(\mathcal{P}^l \setminus \Omega_{d,1/4})} \leqslant \frac{C}{\sqrt{d}}, \tag{56}$$

$$\sup_{r \in (0,\frac{1}{4})} \left(\big(\delta_d(r) - \delta_*(r)\big)^{3/2} \left[\int_{\delta_*(r)}^{\delta_d(r)} \sup_{\theta \in (0,2\pi)} \left|\nabla \mathbf{w}(r,\theta,z)\right|^2 \mathrm{d}z\right]^{\frac{1}{2}}\right) \leqslant C, \tag{57}$$

for any $d \in (0,1)$.

Proof. To prove (55), we remark that in $\mathcal{P}^l$:

$$\begin{aligned}\partial_d \mathbf{w} &= \partial_d\big[Q_{-\alpha}\tilde{\mathbf{w}}[d]\big(Q_\alpha(\mathbf{x}-\mathbf{G}_d)\big)\big]\\ &= M_d^\top \tilde{\mathbf{w}}[d] + Q_{-\alpha}\big(M_d(\mathbf{x}-\mathbf{G}_d) - Q_\alpha \partial_d \mathbf{G}_d\big)\cdot \nabla\tilde{\mathbf{w}}[d] + Q_{-\alpha}\partial_d\tilde{\mathbf{w}}[d],\end{aligned} \tag{58}$$

with

$$\partial_d\tilde{\mathbf{w}}[d] = \partial_d(\cos\alpha)\tilde{\mathbf{w}}_\perp[d] + \cos\alpha\,\partial_d\tilde{\mathbf{w}}_\perp[d] - \partial_d(\sin\alpha)\tilde{\mathbf{w}}_{/\!/}[d] - \sin\alpha\,\partial_d\tilde{\mathbf{w}}_{/\!/}[d].$$

Due to (39) and (18), there exists a universal constant C for which:

$$|\partial_d\cos\alpha| \leqslant \frac{C}{\sqrt{d}}, \qquad |\partial_d \sin\alpha| \leqslant C, \qquad |\partial_d \mathbf{G}_d| \leqslant \frac{C}{\sqrt{d}}.$$

Moreover, outside $\tilde{\Omega}_{d,1/4}$, $\tilde{\mathbf{w}}_{/\!/}$ and $\tilde{\mathbf{w}}_\perp$ are smooth functions of all its arguments. Consequently, the only singular terms in $\partial_d\mathbf{w}$, outside $\Omega_{d,1/4}$, are $\partial_d\cos\alpha$ and $\partial_d\mathbf{G}_d$ so that the above control leads to (56).

Finally, inside $\tilde{\Omega}_{d,1/4}$, we already estimated $\tilde{\mathbf{w}}[d]$ and $\nabla\tilde{\mathbf{w}}[d]$. Combining these dominations with:

$$\big|M_d(\mathbf{x}-\mathbf{G}_d) - Q_\alpha\partial_d\mathbf{G}_d\big| \leqslant \frac{C}{\sqrt{d}}, \quad \forall \mathbf{x}\in\Omega_{d,1/4},$$

and the above control on $\partial_d\cos\alpha$, this yields

$$\big|M_d^\top\tilde{\mathbf{w}}[d] + Q_{-\alpha}\big(M_d(\mathbf{x}-\mathbf{G}_d) - Q_\alpha\partial_d\mathbf{G}_d\big)\cdot\nabla\tilde{\mathbf{w}}[d]\big| \leqslant C\left(1 + \frac{1}{\sqrt{d}(\delta_* - \delta_d)} + \frac{r}{(\delta_*-\delta_d)^2}\right).$$

In $\partial_d\tilde{\mathbf{w}}[d]$ the same right-hand side dominates:

$$\big|\partial_d(\cos\alpha)\tilde{\mathbf{w}}_\perp[d] - \partial_d(\sin\alpha)\tilde{\mathbf{w}}_{/\!/}[d]\big|.$$

Finally, from Lemmata 3 and 4, we compute that

$$\big|\cos\alpha\,\partial_d\tilde{\mathbf{w}}_\perp[d] - \sin\alpha\,\partial_d\tilde{\mathbf{w}}_{/\!/}[d]\big| \leqslant C\left(\sqrt{d}\left[\frac{1}{\delta_*-\delta_d} + \frac{r}{(\delta_*-\delta_d)^2}\right] + \frac{r}{\delta_*-\delta_d} + \frac{1}{\delta_*-\delta_d}\right).$$

The above dominations reduce to:

$$|\partial_d\mathbf{w}| \leqslant C\left(1 + \frac{1}{\sqrt{d}(\delta_*-\delta_d)} + \frac{r}{(\delta_*-\delta_d)^2}\right) \quad \text{in } \tilde{\Omega}_{d,1/4}.$$

From Lemma 1, we finally obtain (55).

To prove, (57), we use (50) and (54):

$$\big(\delta_d(r) - \delta_*(r)\big)^3 \int_{\delta_*(r)}^{\delta_d(r)} \sup_{\theta\in(0,2\pi)} \big|\nabla\mathbf{w}(r,\theta,z)\big|^2\,\mathrm{d}z \leqslant C\big((\delta_*-\delta_d)^2 + r^2\big). \qquad \square$$

In order to prove (23), we first construct a suitable pressure field:

Proposition 7. *Given $d > 0$, there exists a smooth pressure-field $\tilde{q}[d]$ such that*

$$-\Delta\tilde{\mathbf{w}}[d] + \nabla\tilde{q}[d] = \tilde{\mathbf{f}}^1 + \tilde{\mathbf{f}}^2, \quad \text{in } \tilde{\mathcal{P}}^l \tag{59}$$

with

$$\int_0^{\frac{1}{4}} \left(\big(\delta_d(r) - \delta_*(r)\big)^2\left[\int_{\delta_*(r)}^{\delta_d(r)} \sup_{\theta\in(0,2\pi)} \big|\tilde{\mathbf{f}}^1(r,\theta,z)\big|^2\,\mathrm{d}z\right]\right) r\,\mathrm{d}r \quad \textit{and} \quad \|\tilde{\mathbf{f}}^2\|_{L^{6/5}(\tilde{\Omega}_{d,1/4})} \tag{60}$$

uniformly bounded for $d \in (0,1)$.

Proof. With arguments similar to those in [11, Lemma 3.8], we first construct a pressure field $q_\perp[d]$ such that

$$-\Delta\tilde{\mathbf{w}}_\perp + \nabla\tilde{q}_\perp = \tilde{\mathbf{f}}_\perp \quad \text{in } \tilde{\mathcal{P}}^l$$

with

$$\int_0^{\frac14}\left(\big(\delta_d(r)-\delta_*(r)\big)^2\left[\int_{\delta_*(r)}^{\delta_d(r)} \sup_{\theta\in(0,2\pi)}\left|\tilde{\mathbf{f}}_\perp(r,\theta,z)\right|^2 \mathrm{d}z\right]\right) r\,\mathrm{d}r$$

uniformly bounded.

Then by definition of $\tilde{\mathbf{w}}_{/\!/}$ we have

$$-\Delta\tilde{\mathbf{w}}_{/\!/} = \begin{pmatrix} \Delta(\partial_z\phi_{/\!/}) \\ 0 \\ -\Delta(\cos(\theta)\partial_r\phi_{/\!/}) \end{pmatrix}.$$

First

$$\Delta(\partial_z\phi_{/\!/}) = \partial_{rrz}\phi_{/\!/} + \frac{1}{r}\partial_{rz}\phi_{/\!/} + \partial_{zzz}\phi_{/\!/}.$$

Using that $\partial_{zzz}\phi_{/\!/} = 0$ and Lemma 4, we deduce

$$\left|\Delta(\partial_z\phi_{/\!/})\right| \leqslant \frac{C}{\delta_* - \delta_d}.$$

Second

$$\Delta\big(\cos(\theta)\partial_r\phi_{/\!/}\big) = \cos(\theta)\left(\partial_{rrr}\phi_{/\!/} + \frac{1}{r}\partial_{rr}\phi_{/\!/} - \frac{1}{r^2}\partial_r\phi_{/\!/} + \partial_{zzr}\phi_{/\!/}\right).$$

Using again Lemma 4, we obtain that

$$\left|\cos(\theta)(\partial_{rrr}\phi_{/\!/} + \partial_{zzr}\phi_{/\!/})\right| \leqslant C\frac{r}{(\delta_* - \delta_d)^2}$$

and

$$\left|\cos(\theta)\left(\frac{1}{r}\partial_{rr}\phi_{/\!/} - \frac{1}{r^2}\partial_r\phi_{/\!/}\right)\right| \leqslant \frac{C}{r}.$$

From Lemma 1,

$$\tilde{\mathbf{f}}^1_{/\!/} = \begin{pmatrix} \Delta(\partial_z\phi_{/\!/}) \\ 0 \\ -\cos(\theta)(\partial_{rrr}\phi_{/\!/} + \partial_{zzr}\phi_{/\!/}) \end{pmatrix} \quad \text{and} \quad \tilde{\mathbf{f}}^2_{/\!/} = \begin{pmatrix} 0 \\ 0 \\ -\cos(\theta)(\frac{1}{r}\partial_{rr}\phi_{/\!/} - \frac{1}{r^2}\partial_r\phi_{/\!/}) \end{pmatrix}$$

satisfy

$$\int_0^{\frac14}\left(\big(\delta_d(r)-\delta_*(r)\big)^2\left[\int_{\delta_*(r)}^{\delta_d(r)} \sup_{\theta\in(0,2\pi)}\left|\tilde{\mathbf{f}}^1_{/\!/}(r,\theta,z)\right|^2 \mathrm{d}z\right]\right) r\,\mathrm{d}r \leqslant C,$$

and

$$\left\|\tilde{\mathbf{f}}^2_{/\!/}\right\|_{L^{6/5}(\tilde{\Omega}_{d,1/4})} \leqslant C.$$

Finally, we set $\tilde{q}[d] = \cos\alpha\,\tilde{q}_\perp[d]$ so that $\tilde{\mathbf{f}}^1 = \sin\alpha\,\tilde{\mathbf{f}}^1_{/\!/} + \cos\alpha\,\tilde{\mathbf{f}}_\perp$ and $\tilde{\mathbf{f}}^2 = \sin\alpha\,\tilde{\mathbf{f}}^2_{/\!/}$. This ends up the proof. □

To complete the proof of (23), let $\mathbf{v} \in \mathbb{H}(\mathbf{G}_d)$ with $\mathbf{v} = \ell\mathbf{e}_3$ on $\mathcal{B}_d$, and consider

$$I = \int_\Omega D(\mathbf{v}) : D\big(\mathbf{w}[d]\big)\,\mathrm{d}\mathbf{y}.$$

We split this integral as previously $I = I^l + I^r$ with obvious notations. Then we introduce $\tilde{\mathbf{v}} := Q_\alpha \mathbf{v}(Q_{-\alpha} \cdot + \mathbf{G}_d)$. Because Q_α is a unit transformation, we have:

$$I^l = \int_{\tilde{\mathcal{F}}_d \cap \tilde{\mathcal{P}}^l} D(\tilde{\mathbf{v}}) : D(\tilde{\mathbf{w}}[d])\,\mathrm{d}\tilde{\mathbf{y}}$$

with $\tilde{\mathbf{w}}[d]$ as defined in Section 3.3. Integrating by parts, this yields

$$I^l = \int_{\partial(\tilde{\mathcal{F}}_d \cap \tilde{\mathcal{P}}^l)} \big(2D(\tilde{\mathbf{w}}[d])\mathbf{n} - \tilde{q}[d]\mathbf{n}\big) \cdot \tilde{\mathbf{v}}\,\mathrm{d}\tilde{\sigma} - \int_{\tilde{\mathcal{F}}_d \cap \tilde{\mathcal{P}}^l} \big(\Delta \tilde{\mathbf{w}}[d] - \nabla \tilde{q}[d]\big) \cdot \tilde{\mathbf{v}}\,\mathrm{d}\tilde{\mathbf{y}}.$$

For symmetry reasons, after compensation with I^r the only relevant boundary integral is:

$$\int_{\partial \tilde{\mathcal{B}}_*} \big(2D(\tilde{\mathbf{w}}[d])\mathbf{n} - \tilde{q}[d]\mathbf{n}\big) \cdot \tilde{\mathbf{v}}\,\mathrm{d}\tilde{\sigma}.$$

This last integral is linear with respect to $\mathbf{v}$ and fixed by the value of $\mathbf{v}$ on $\partial\mathcal{B}_d$. So, we can rewrite it $\ell b(d)$ with some function b to be made precise. Moreover, applying the previous proposition, and similar technique to [11, Lemma 3.9], we obtain that

$$\int_{\tilde{\mathcal{F}}_d \cap \tilde{\mathcal{P}}^l} \big(\Delta \tilde{\mathbf{w}}[d] - \nabla \tilde{q}[d]\big) \cdot \tilde{\mathbf{v}}\,\mathrm{d}\tilde{\mathbf{y}} \leqslant C\|\nabla \tilde{\mathbf{v}}\|_{L^2(\tilde{\mathcal{P}}^l \cap \tilde{\Omega})} \leqslant C\|\nabla \mathbf{v}\|_{L^2(\Omega)}.$$

Computing similarly $\mathcal{P}^r$, we finally obtain $I = \ell b(d) + R$ with $|R| \leqslant C\|\nabla \mathbf{v}\|_{L^2(\Omega)}$ where C is an absolute constant.

Taking in particular $\mathbf{v} = \mathbf{w}[d]$ we might compute our integral in the same way. This yields

$$b(d) = \int_\Omega \big|D(\mathbf{w})\big|^2 + R \quad \text{with } |R| \leqslant C\|\nabla \mathbf{w}\|_{L^2(\Omega)}.$$

From the control on this $L^2(\Omega)$ norm obtained in Proposition 5, we finally obtain that

$$b(d) \leqslant C\|\nabla \mathbf{w}\|^2_{L^2(\Omega)} + C\|\nabla \mathbf{w}\|_{L^2(\Omega)} \leqslant C|\ln d| + C\sqrt{|\ln d|} \leqslant C|\ln d|.$$

Appendix A. Detailed description of potentials $\phi_{/\!/}$ and $\phi_\perp$

This appendix is very similar to the one given in [11], however there are some differences since we estimate not only the size of $\phi_\perp$ and its derivatives, but also the size of $\phi_{/\!/}$ and its derivatives. However, since the proofs are completely similar, we only state the results used in this paper.

We emphasize that $\phi_{/\!/}$ and $\phi_\perp$ depend on d, even if the dependency is not explicitly mentioned. In order to compare such functions in what follows, we introduce the following conventions. Given families $(f_d : \tilde{\Omega}_{d,1/4} \to \mathbb{R})_{d\in(0,1)}$ and $(g_d : \tilde{\Omega}_{d,1/4} \to \mathbb{R})_{d\in(0,1)}$, we denote by $f_d \prec g_d$ if there exists an absolute constant C such that

$$\big|f_d(\mathbf{x})\big| \leqslant C g_d(\mathbf{x}), \quad \forall \mathbf{x} \in \tilde{\Omega}_{d,1/4} \text{ and } d < 1.$$

Given non-negative functions $f : (0,1) \to \mathbb{R}^+$ and $g : (0,1) \to \mathbb{R}^+$, we also denote by

$$f(s) \sim g(s), \quad \forall s \in (0,1),$$

if there exist two positive constants c and C such that

$$cf(s) \leqslant g(s) \leqslant Cf(s), \quad \forall s \in (0,1).$$

First, we compute typical $L^1(0,1/4)$-sizes of functions $r \mapsto r^\alpha/(\delta_*(r) - \delta_d(r))^\beta$.

Lemma 1. *Given $(\alpha,\beta)\in(\mathbb{R}_+)^2$, we have the following estimations for all $d\in(0,1)$:*

$$\int_0^{\frac{1}{4}}\frac{r^\alpha}{(\delta_*(r)-\delta_d(r))^\beta}\,\mathrm{d}r\sim\begin{cases}1, & \text{if } \alpha>2\beta-1,\\ \ln(1/d), & \text{if } \alpha=2\beta-1,\\ d^{\frac{(\alpha+1)-2\beta}{2}}, & \text{if } \alpha<2\beta-1.\end{cases}$$

We now compare $\lambda(r,z)=\frac{z-\delta_d(r)}{\delta_*(r)-\delta_d(r)}$ and its derivatives to functions $(r,\theta,z)\mapsto r^\alpha/(\delta_*(r)-\delta_d(r))^\beta$ in $\tilde{\Omega}_{d,1/4}$ where we recall that:

$$\delta_*(r)=-\sqrt{1-r^2},\qquad \delta_d(r)=-(2+d)+\sqrt{1-r^2},\quad \forall r\in(0,1),\ \forall d>0.$$

Lemma 2. *We have the following sizes*:

$$\begin{array}{llll}\lambda\prec 1, & \lambda_r\prec r/(\delta_*-\delta_d), & \lambda_z\prec 1/(\delta_*-\delta_d), & \lambda_d\prec 1/(\delta_*-\delta_d),\\ \lambda_{rd}\prec r/(\delta_*-\delta_d)^2, & \lambda_{zd}\prec 1/(\delta_*-\delta_d)^2, & \lambda_{rr}\prec 1/(\delta_*-\delta_d), & \lambda_{rz}\prec r/(\delta_*-\delta_d)^2,\\ \lambda_{rrz}\prec 1/(\delta_*-\delta_d)^2, & \lambda_{rrr}\prec r/(\delta_*-\delta_d)^2. & & \end{array}$$

Then we obtain the following lemmata.

Lemma 3. *We have the following sizes*:

$$\begin{array}{lll}\phi_\perp\prec r, & \partial_r\phi_\perp\prec 1, & \partial_z\phi_\perp\prec r/(\delta_*-\delta_d),\\ \partial_r(\phi_\perp/r)\prec r/(\delta_*-\delta_d), & \partial_d\phi_\perp\prec r/(\delta_*-\delta_d), & \partial_{rd}\phi_\perp\prec 1/(\delta_*-\delta_d),\\ \partial_{zd}\phi_\perp\prec r/(\delta_*-\delta_d)^2, & \partial_{rz}(\phi_\perp/r)\prec r/(\delta_*-\delta_d)^2, & \partial_{rr}\phi_\perp\prec r/(\delta_*-\delta_d),\\ \partial_{rz}\phi_\perp\prec 1/(\delta_*-\delta_d), & \partial_{zz}\phi_\perp\prec r/(\delta_*-\delta_d)^2, & \partial_{rrr}\phi_\perp\prec 1/(\delta_*-\delta_d),\\ \partial_{rzz}\phi_\perp\prec 1/(\delta_*-\delta_d)^2, & \partial_{rrz}\phi_\perp\prec r/(\delta_*-\delta_d)^2, & \partial_{zzz}\phi_\perp\prec r/(\delta_*-\delta_d)^3.\end{array}$$

Lemma 4. *We have the following sizes*:

$$\begin{array}{lll}\phi_{/\!/}\prec 1, & \partial_r\phi_{/\!/}\prec r, & \partial_z\phi_{/\!/}\prec 1,\\ \partial_d\phi_{/\!/}\prec 1, & \partial_{rd}\phi_{/\!/}\prec r/(\delta_*-\delta_d), & \partial_{zd}\phi_{/\!/}\prec 1/(\delta_*-\delta_d),\\ \partial_{rr}\phi_{/\!/}\prec 1, & \partial_{rz}\phi_{/\!/}\prec r/(\delta_*-\delta_d), & \partial_{zz}\phi_{/\!/}\prec 1/(\delta_*-\delta_d),\\ \partial_{rrr}\phi_{/\!/}\prec r/(\delta_*-\delta_d), & \partial_{rzz}\phi_{/\!/}\prec r/(\delta_*-\delta_d)^2, & \partial_{rrz}\phi_{/\!/}\prec 1/(\delta_*-\delta_d).\end{array}$$

Appendix B. Computation of inequality (16)

The following computations are inspired by [9]. For simplicity we consider symmetric geometries and we apply notations introduced in Section 3 (see Fig. 2 for example).

Proposition 8. *There exists a universal constant C for which, given $\mathbf{G}\in D$ and $\mathbf{u}\in H^2(\mathcal{F}_\mathbf{G})$ such that*

$$\begin{cases}\operatorname{div}\mathbf{u}(\mathbf{x})=0, & \text{in } \Omega,\\ \mathbf{u}(\mathbf{x})=0, & \text{on } \partial\Omega,\\ \mathbf{u}(\mathbf{x})=\mathbf{V}+\boldsymbol{\omega}\times(\mathbf{x}-\mathbf{G}), & \text{on } \partial\mathcal{B}_\mathbf{G},\end{cases}$$

whenever $d:=\operatorname{dist}(\mathcal{B}_\mathbf{G},\partial\Omega)<1$, there holds:

$$|\mathbf{V}_\mathbf{u}\cdot\tilde{\mathbf{e}}_3|\leqslant C|d|^{\frac{3}{2}}\Bigg[\int_{\mathcal{F}_\mathbf{G}}\big|\nabla^2\mathbf{u}(\mathbf{y})\big|^2\,\mathrm{d}\mathbf{y}\Bigg]^{\frac{1}{2}},$$

where $\tilde{\mathbf{e}}_3=\mathbf{G}-\mathbf{G}_l/|\mathbf{G}-\mathbf{G}_l|$.

Proof. For simplicity, we assume $\mathbf{u}$ is smooth in the fluid domain. The result is then obtained by a density argument. We also introduce $(\tilde{\mathbf{e}}_r, \tilde{\mathbf{e}}_\theta, \tilde{\mathbf{e}}_z)$ the orthonormal basis associated to cylindrical coordinates in $\Omega_{d,l}$.

Integrating $\operatorname{div}(\mathbf{u}) = 0$ in $\Omega_{d,l}$, this yields

$$\int_{\partial\mathcal{B}_{\mathbf{G}}\cap\partial\Omega_{d,l}} \mathbf{u}\cdot\mathbf{n}\,\mathrm{d}\sigma = -l\Phi(l),$$

where

$$\Phi(l) = \int_{\delta_d(l)}^{\delta_*(l)} \varphi(z,l)\,\mathrm{d}z, \quad \text{with } \varphi(z,l) = \int_{-\pi}^{\pi} \mathbf{u}(l,\theta,z)\cdot\tilde{\mathbf{e}}_r\,\mathrm{d}\theta.$$

We remark that no slip boundary conditions together with symmetry arguments imply

$$\varphi\big(\delta_*(l),l\big) = 0, \qquad \varphi\big(\delta_d(l),l\big) = 0, \quad \forall l \in (0,1).$$

As a straightforward consequence, we obtain

$$\big|\varphi(z,l)\big| \leqslant \big(\delta_*(l) - \delta_d(l)\big)^{\frac{3}{2}} \left[\int_{\delta_d(l)}^{\delta_*(l)} \big|\partial_{zz}\varphi(\alpha,l)\big|^2\,\mathrm{d}\alpha\right]^{\frac{1}{2}},$$

and

$$\left|\int_{\partial\mathcal{B}_{\mathbf{G}}\cap\partial\Omega_{d,l}} \mathbf{u}\cdot\mathbf{n}\,\mathrm{d}\sigma\right| \leqslant Cl\big(\delta_*(l) - \delta_d(l)\big)^{\frac{5}{2}} \left[\int_{\delta_d(l)}^{\delta_*(l)}\int_{-\pi}^{\pi} \big|\partial_{zz}\mathbf{u}(r,\theta,z)\big|^2\,\mathrm{d}r\,\mathrm{d}\theta\right]^{\frac{1}{2}}. \tag{B.1}$$

Moreover, one might compute

$$\int_{\partial\mathcal{B}_{\mathbf{G}}\cap\partial\Omega_{d,l}} \mathbf{u}\cdot\mathbf{n}\,\mathrm{d}\sigma = 2\pi l^2\mathbf{V}\cdot\tilde{\mathbf{e}}_3,$$

so that, (B.1) reads:

$$2\pi l^2|\mathbf{V}\cdot\tilde{\mathbf{e}}_3| \leqslant Cl\big(\delta_*(l) - \delta_d(l)\big)^{\frac{5}{2}} \left[\int_{\delta_d(l)}^{\delta_*(l)}\int_{-\pi}^{\pi} \big|\partial_{zz}\mathbf{u}(r,\theta,z)\big|^2\,\mathrm{d}z\,\mathrm{d}\theta\right]^{\frac{1}{2}}.$$

Finally, we integrate the above inequality with respect to l, from 0 to r. This yields

$$|\mathbf{V}\cdot\tilde{\mathbf{e}}_3| \leqslant \frac{C(\delta_*(r) - \delta_d(r))^{\frac{5}{2}}}{r^2}\big|\nabla^2\mathbf{u}\big|_{L^2(\mathcal{F}_{\mathbf{G}})}.$$

We get the optimal inequality taking $r = \sqrt{d}$. This ends up the proof. □

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