Abstract
Starting from a mass transportation proof of the Brunn–Minkowski inequality on convex sets, we improve the inequality showing a sharp estimate about the stability property of optimal sets. This is based on a Poincaré-type trace inequality on convex sets that is also proved in sharp form.
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1. Introduction

We deal with the Brunn–Minkowski inequality: given $E$ and $F$ non-empty subsets of $\mathbb{R}^n$, we have
\[
|E + F|^{1/n} \geq |E|^{1/n} + |F|^{1/n},
\]
where $E + F = \{x + y : x \in E, y \in F\}$ is the Minkowski sum of $E$ and $F$, and where $|\cdot|$ stands for the (outer) Lebesgue measure on $\mathbb{R}^n$. The central role of this inequality in many branches of Analysis and Geometry, and especially in the theory of convex bodies, is well explained in the excellent survey [11] by R. Gardner. Concerning the case $E$ and $F$ are open bounded convex sets (shortly: convex bodies), it may be proved (see [4,14]) that equality holds in (1) if and only if $E$ and $F$ are homothetic, i.e.
\[
\exists \lambda > 0, \ x_0 \in \mathbb{R}^n : \ E = x_0 + \lambda F.
\]
Theorem 1 provides a refined Brunn–Minkowski inequality on convex bodies, in the spirit of [7,12,18,17]. We define the relative asymmetry of $E$ and $F$ as
\[
A(E,F) := \inf_{x_0 \in \mathbb{R}^n} \left\{ \frac{|E \Delta (x_0 + \lambda F)|}{|E|} : \lambda = \left( \frac{|E|}{|F|} \right)^{1/n} \right\},
\]
and the relative size of $E$ and $F$ as
\[ \sigma(E, F) := \max \left\{ \frac{|F|}{|E|}, \frac{|E|}{|F|} \right\}. \tag{4} \]

We note that $A(E, F) = A(F, E)$ and $\sigma(E, F) = \sigma(F, E)$.

**Theorem 1.** If $E$ and $F$ are convex bodies, then
\[ |E + F|^{1/n} \geq \left( \left| \frac{E}{n} \right| + \frac{|F|}{|E|} \right) \left( 1 + \frac{A(E, F)^2}{C_0(n)\sigma(E, F)^{1/n}} \right). \tag{5} \]

In [10], inequality (5) was derived as a corollary of the sharp quantitative Wulff inequality, with a constant $C_0(n) \approx n^2$ and with explicit examples proving the sharpness of decay rate of $A(E, F)$ and $\sigma(E, F)$ in the regime $\beta(E, F) \to 0$. Here, we introduce the Brunn–Minkowski deficit of the pair $(E, F)$ by setting
\[ \beta(E, F) := \frac{n}{|E + F|^{1/n}} \left( \left| \frac{E}{n} \right| + \frac{|F|}{|E|} \right) - 1, \]
so that (5) becomes equivalent to
\[ C_0(n) \sqrt{\beta(E, F)\sigma(E, F)^{1/n}} \geq A(E, F). \tag{6} \]

As in [10], our approach to (5) is based on the theory of mass transportation. A one-dimensional mass transportation argument is at the basis of the beautiful proof of (1) by Hadwiger and Ohmann [13], see [9, 3.2.41] and [11, Proof of Theorem 4.1]. The impact of mass transportation theory in the field of sharp functional-geometric inequalities is now widely recognized, with many old and new inequalities treated from a unified and elegant viewpoint (see [19, Chapter 6] for an introduction). A proof of the Brunn–Minkowski inequality in this framework is already contained in the seminal paper by McCann [16], see also Step two in the proof of Theorem 1.

In Section 3 of this note we present a direct proof of (5), independent from the structure theory for sets of finite perimeter that was heavily used in [10]. As a technical drawback, this approach does not provide a polynomial bound on $C_0(n)$, but only an exponential behavior in $n$. However, we believe this proof is more broadly accessible and substantially simpler. A technical element of this proof that we believe of independent interest is the Poincaré-type trace inequality on convex sets proved in Section 2, with a constant having sharp dependence on the dimension $n$ and on the ratio between the in-radius and the out-radius of the set (see Remark 3).

### 2. A Poincaré-type trace inequality on convex sets

In this section we aim to prove the following Poincaré-type trace inequality for a convex body:

**Lemma 2.** Let $E$ be a convex body such that $B_r \subset E \subset B_R$, for $0 < r < R$. Then
\[ \int \frac{n \sqrt{2}}{\log(2)} \frac{R}{r} \left| \nabla f \right| \geq \inf_{c \in \mathbb{R}} \int_E \left| f - c \right| d\mathcal{H}^{n-1}, \tag{7} \]

for every $f \in C^\infty(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$.

It is quite easy to prove (7) by a contradiction argument, if we allow to replace $n(R/r)$ by a constant generically depending on $E$. However, in order to prove Theorem 1, we need to express this dependence just in terms of $n$ and $R/r$, and thus require a more careful approach. Let us also note that, by a standard density argument, (7) holds true for every $f \in BV(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ (see [1,8]), in the form
\[ \frac{n \sqrt{2}}{\log(2)} \frac{R}{r} |Df|(E) \geq \inf_{c \in \mathbb{R}} \int_{\partial E} \left| f - c \right| d\mathcal{H}^{n-1}, \]
where $|Df|$ denotes the total variation measure of $Df$ and where $\text{tr}_E(f)$ is the trace of $f$ on $\partial E$, defined as an element of $L^1(\mathcal{H}^{n-1}(\partial E))$ (see [1, Theorem 3.87]). However, we shall not need this stronger form of the inequality.
Given a convex body \( E \) containing the origin in its interior, we introduce a weight function on directions defined for \( \nu \in S^{n-1} \) as
\[
\| \nu \|_E := \sup\{ x \cdot \nu : x \in E \}.
\]
When \( F \) is a set with Lipschitz boundary and outer unit normal \( \nu_F \), we define the anisotropic perimeter of \( F \) with respect to \( E \) as
\[
P_E(F) := \int_{\partial F} \| \nu_F(x) \|_E d\mathcal{H}^{n-1}(x),
\]
and recall that \( P_E(E) = n|E| \). Then, the anisotropic isoperimetric inequality, or Wulff inequality,
\[
P_E(F) \geq n|E|^{1/n} |F|^{(n-1)/n},
\]
holds true, as it can be shown starting from (1) (see [11, Section 3]).

**Proof of Lemma 2.** Let us set
\[
\tau(E) := \inf_{F} \frac{\mathcal{H}^{n-1}(E \cap \partial F)}{\mathcal{H}^{n-1}(F \cap \partial E)}
\]
where \( F \) ranges over the class of open sets of \( \mathbb{R}^n \) with smooth boundary such that \( |E \cap F| \leq |E|/2 \). Then, fixed \( f \in C^\infty(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \), we set \( F_t = \{ x \in \mathbb{R}^n : f(x) > t \} \) for every \( t \in \mathbb{R} \). The proof of the lemma is then achieved on combining the following two statements.

**Step one:** We have that
\[
\int_{E} |\nabla f| \geq \tau(E) \int_{\partial E} |f - m| d\mathcal{H}^{n-1},
\]
where \( m \) is a median of \( f \) in \( E \), i.e.
\[
|F_t \cap E| \leq \frac{|E|}{2}, \quad \forall t \geq m,
\]
\[
|F_t \cap E| > \frac{|E|}{2}, \quad \forall t < m.
\]
Indeed, let \( g = \max\{f - m, 0\} \) and let \( G_t = \{ x \in \mathbb{R}^n : g(x) > t \} \). Then by the Coarea Formula, the choice of \( m \) and the definition of \( \tau(E) \) (note that \( F_t \) is admissible in \( \tau(E) \) for a.e. \( t \geq m \) by Morse–Sard Lemma)
\[
\int_{E \cap F_m} |\nabla f| = \int_{E} |\nabla g| = \int_{0}^{\infty} \mathcal{H}^{n-1}(E \cap \partial G_t) dt \geq \tau(E) \int_{0}^{\infty} \mathcal{H}^{n-1}(G_t \cap \partial E) dt = \tau(E) \int_{\partial E} g d\mathcal{H}^{n-1} = \tau(E) \int_{\partial E} \max\{f - m, 0\} d\mathcal{H}^{n-1}.
\]
The choice of \( m \) allows to argue similarly with \( \max\{m - f, 0\} \) in place of \( g \) and to eventually achieve the proof of Step one.

**Step two:** We have that
\[
\tau(E) \geq \frac{r}{R} \left( 1 - \frac{1}{2^{1/n}} \right).
\]
To prove this, let us consider an admissible set $F$ for $\tau(E)$ and set for simplicity
\[ \lambda := \frac{\mathcal{H}^{n-1}(E \cap \partial F)}{\mathcal{H}^{n-1}(F \cap \partial E)}. \] (9)

On denoting $F_1 = F \cap E$ and $F_2 = E \setminus \overline{F}$, we have that
\[ E \cap \partial F_1 = E \cap \partial F_2 = E \cap \partial F, \quad \text{with } \nu_F = -\nu_{F_1} \text{ on } E \cap \partial F. \]

Therefore
\[
P_E(E) \geq P_E(F_1) + P_E(F_2) - \int_{E \cap \partial F_1} \|\nu_{F_1}\|_E d\mathcal{H}^{n-1} - \int_{E \cap \partial F_2} \|\nu_{F_2}\|_E d\mathcal{H}^{n-1}
\]
\[
\geq P_E(F_1) + P_E(F_2) - 2\mathcal{R}\mathcal{H}^{n-1}(E \cap \partial F)
\]
\[
= P_E(F_1) + P_E(F_2) - 2\mathcal{R}\mathcal{H}^{n-1}(F \cap \partial E)
\]
\[
\geq P_E(F_1) + P_E(F_2) - 2\mathcal{R}\mathcal{H}^{n-1}(\partial F_1)
\]
\[
\geq \left(1 - 2\lambda \frac{R}{r}\right)P_E(F_1) + P_E(F_2),
\] (10)

where we have used (9) and the elementary inequality
\[ r \leq \|\nu\|_E \leq R, \]
for every $\nu \in S^{n-1}$. On combining (10), the anisotropic isoperimetric inequality (8) and the fact that $P_E(E) = n|E|$, we come to
\[ n|E| \geq n|E|^{1/n} \left(1 - 2\lambda \frac{R}{r}\right)|F_1|^{1/n'} + |F_2|^{1/n'}, \]
i.e. we have proved that
\[ \lambda t^{1/n'} \geq \frac{R}{2} (t^{1/n'} + (1 - t)^{1/n'}) - 1, \]
where $t = |F_1|/|E|$. As $t \in (0, 1/2]$ by construction and
\[ s^{1/n'} + (1 - s)^{1/n'} - 1 \geq 2(2^{1/n'})s^{1/n'}, \quad \forall s \in (0, 1/2], \]
the proof of Step two is easily concluded. \( \square \)

**Remark 3.** Let us point out that the dependence on $n$ and $R/r$ given in the above result, that is $n(R/r)$, is sharp. In $\mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}$, it suffices to consider the box $E$ defined as
\[ E = Q \times [-R_0, R_0], \quad Q = \left[-\frac{r}{2}, \frac{r}{2}\right]^{n-1}. \]

We clearly have that $B_r \subset E \subset B_R$, with $R = \sqrt{R_0^2 + (n-1)r^2}$. Now, let us consider as a test set for the trace constant the half-space $F = \mathbb{R}^{n-1} \times (0, \infty)$, so that
\[ \partial F \cap E = Q \times \{0\}, \quad \partial E \cap F = (\partial Q \times (0, R_0)) \cup (Q \times \{R_0\}). \]
The boundary $\partial Q$ is the union of $2(n-1)$ cubes of dimension $(n-2)$ and size $r$. Thus,
\[ \mathcal{H}^{n-1}(\partial F \cap E) = r^{n-1}, \quad \mathcal{H}^{n-1}(\partial E \cap F) = 2(n-1)R_0 r^{n-2} + r^{n-1}. \]
For $R_0 \gg \sqrt{n-1}r$ we have $R \approx R_0$, and therefore
\[ \frac{n \sqrt{2}}{\log(2)} \frac{R}{r} \leq \tau(E) \leq \frac{2(n-1)R_0 r^{n-2} + r^{n-1}}{r^{n-1}} \approx n \frac{R_0}{r} \approx n \frac{R}{r}. \]
This shows the sharpness of our trace constant, up to a numeric factor.
3. Proof of Theorem 1

This section is devoted to the proof of Theorem 1. We consider two convex bodies $E$ and $F$, and we aim to prove (6). Without loss of generality, we may assume that $|E| \geq |F|$. By approximation, we can also assume that $E$ and $F$ are smooth and uniformly convex. Eventually, we can directly consider the case

$$\beta(E, F) \sigma(E, F)^{1/n} \leq 1. \quad (11)$$

Indeed, as we always have $A(E, F) \leq 2$, if $\beta(E, F) \sigma(E, F)^{1/n} > 1$ then (6) holds trivially with $C_0(n) = 2$. Observe further that, since $\sigma(E, F) \geq 1$, (11) implies

$$\beta(E, F) \leq 1. \quad (12)$$

We divide the proof in several steps.

**Step one: John’s normalization.** A classical result in the theory of convex bodies by F. John [15] ensures the existence of a linear map $L : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that

$$B_1 \subset L(E) \subset B_n.$$ We note that

$$\beta(E, F) = \beta(L(E), L(F)), \quad A(E, F) = A(L(E), L(F)), \quad |L(E)| \geq |L(F)|.$$ Therefore in the proof of Theorem 1 we may also assume that

$$B_1 \subset E \subset B_n. \quad (13)$$

In particular, under this assumption one has $1 \leq r \leq R \leq n$, so that by Lemma 2 we can write

$$\frac{n^2 \sqrt{2}}{\log(2)} \int_E |\nabla f| \geq \inf_{c \in \mathbb{R}} \int_{\partial E} |f - c| d\mathcal{H}^{n-1}$$

for every $f \in C^\infty(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. 

**Step two: Mass transportation proof of Brunn–Minkowski.** We prove the Brunn–Minkowski inequality by mass transportation. By the Brenier Theorem [2,3], there exists a convex function $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$ such that its gradient $T = \nabla \varphi$ defines a map $T \in BV(\mathbb{R}^n, F)$ pushing forward $|E|^{-1} 1_E(x) dx$ to $|F|^{-1} 1_F(x) dx$, i.e.

$$\frac{1}{|F|} \int_{\mathcal{F}} h(y) dy = \frac{1}{|E|} \int_{E} h(T(x)) dx,$$ for every Borel function $h : \mathbb{R}^n \rightarrow [0, \infty)$. As shown by Caffarelli [5,6], under our assumptions the Brenier map is smooth up to the boundary, i.e. $T \in C^\infty(\mathcal{E}, \mathcal{F})$. Moreover, the push-forward condition (15) takes the form

$$\det \nabla T(x) = \frac{|F|}{|E|}, \quad \forall x \in E. \quad (16)$$

We are going to consider the eigenvalues $\{\lambda_k(x)\}_{k=1,\ldots,n}$ of $\nabla T(x) = \nabla^2 \varphi(x)$, ordered so that $\lambda_k \leq \lambda_{k+1}$ for $1 \leq k \leq n - 1$. We also define, for every $x \in E$,

$$\lambda_A(x) = \frac{\sum_{k=1}^{n} \lambda_k(x)}{n}, \quad \lambda_G(x) = \left( \prod_{k=1}^{n} \lambda_k(x) \right)^{1/n}.$$ Thanks to (16) we have

$$\lambda_G(x) = \left( \frac{|F|}{|E|} \right)^{1/n}$$
for every \( x \in E \). We are in the position to prove the Brunn–Minkowski inequality. Let \( S(x) := x + T(x) \), then \( S(E) \subset E + F \). As \( \det \nabla S = \prod_{k=1}^{n} (1 + \lambda_k) > 1 \), we have \( |\det \nabla S| = \det \nabla S \). Thus

\[
|E + F|^{1/n} \geq |S(E)|^{1/n} = \left( \int_{E} \det \nabla S \right)^{1/n} = \left( \int_{E} \prod_{k=1}^{n} (1 + \lambda_k) \right)^{1/n}.
\]

We observe that

\[
\prod_{k=1}^{n} (1 + \lambda_k) = 1 + \sum_{m=1}^{n} \sum_{\{1 \leq i_1 \ldots i_m \leq n\}} \prod_{j=1}^{m} \lambda_{i_j},
\]

Note that the set of indexes \((i_1, \ldots, i_m)\) with \(1 \leq i_j < i_{j+1} \leq n\) counts \( \binom{n}{m} \) elements. For each fixed \( m \geq 1 \), the arithmetic–geometric mean inequality implies that

\[
\sum_{\{1 \leq i_1 \ldots i_m \leq n\}} \prod_{j=1}^{m} \lambda_{i_j} \geq \left( \prod_{k=1}^{n} \lambda_{m} \right)^{m}.
\]

This last term is equal to

\[
\binom{n}{m} \prod_{k=1}^{n} \lambda_{m}^{(n-1)/(\binom{n}{m})} = \binom{n}{m} \lambda_{m}^{n}. \tag{20}
\]

On putting (18), (19) and (20) together, and applying the binomial formula to \((1 + \lambda_G)^n\) we come to

\[
\prod_{k=1}^{n} (1 + \lambda_k) - (1 + \lambda_G)^n = \sum_{m=1}^{n} \Gamma_m,
\]

where \( \Gamma_m \) denotes the difference between the left- and the right-hand side of (19). We observe that \( \Gamma_m \geq 0 \) whenever \( 1 \leq m \leq n \), and in particular \( \Gamma_1 = n(\lambda_A - \lambda_G) \). On combining this with (17), (16), and \( \lambda_G = (\det \nabla T)^{1/n} \), we find that

\[
|E + F|^{1/n} \geq \left( \int_{E} (1 + \lambda_G)^n \right)^{1/n} = |E|^{1/n} \left( 1 + \left( \frac{|F|}{|E|} \right)^{1/n} \right) = |E|^{1/n} + |F|^{1/n},
\]

i.e. we prove the Brunn–Minkowski inequality for \( E \) and \( F \).

**Step three: Lower bounds on the deficit.** In this step we aim to prove

\[
\frac{1}{|E|} \int_{E} |\nabla T(x) - \lambda_G \Id| dx \leq C(n) \sqrt{\beta(E, F)} \sqrt{\beta(E, F) + \sigma(E, F)^{-1/n}}. \tag{22}
\]

Let us set, for the sake of brevity,

\[
s = \frac{1}{|E|} \int_{E} \det \nabla S, \quad t = (1 + \lambda_G)^n.
\]

From Step two we deduce that

\[
\frac{|E + F|^{1/n} - (|E|^{1/n} + |F|^{1/n})}{|E|^{1/n}} \geq s^{1/n} - t^{1/n} = \sum_{h=1}^{n} s^{(n-h)/nt(h-1)/n}.
\]

As \( t \leq s \) and \( |E|s = |S(E)| \leq |E + F| \),

\[
\sum_{h=1}^{n} s^{(n-h)/nt(h-1)/n} \leq ns^{(n-1)/n} \leq n\left( \frac{|E + F|}{|E|} \right)^{(n-1)/n} = n\left( (1 + \beta(E, F)) \frac{|E|^{1/n} + |F|^{1/n}}{|E|^{1/n}} \right)^{n-1} \leq C(n), \tag{24}
\]

\[
\frac{1}{|E|} \int_{E} |\nabla T(x) - \lambda_G \Id| dx \leq C(n) \sqrt{\beta(E, F)} \sqrt{\beta(E, F) + \sigma(E, F)^{-1/n}}.
\]
where we have also made use of (12) and of the fact that $|F| \leq |E|$. A similar argument shows that the left-hand side of (23) is controlled by $2\beta(E, F)$, and therefore we conclude that

$$C(n)\beta(E, F) \geq s - t = \frac{1}{|E|} \int_E \left( \prod_{k=1}^n (1 + \lambda_k) - (1 + \lambda_G)^n \right) dx. \quad (25)$$

Then, by (25) and (21), as $\Gamma_m \geq 0$ whenever $1 \leq m \leq n$ and $\Gamma_1 = n(\lambda_A - \lambda_G)$, we get

$$C(n)\beta(E, F) \geq \frac{1}{|E|} \int_E \left( \prod_{m=1}^n \Gamma_m(x) \right) dx \geq \frac{1}{|E|} \int_E \Gamma_1(x) dx = \frac{n}{|E|} \int_E (\lambda_A - \lambda_G). \quad (26)$$

An elementary quantitative version of the arithmetic–geometric mean inequality proved in [10, Lemma 2.5], ensures that

$$7n^2(\lambda_A - \lambda_G) \geq \frac{1}{\lambda_n^n} \sum_{k=1}^n (\lambda_k - \lambda_G)^2. \quad (27)$$

By Hölder inequality

$$\frac{1}{|E|} \int_E (\lambda_n - \lambda_1) dx \leq C(n) \sqrt{\beta(E, F) \frac{1}{|E|} \int_E \lambda_n}. \quad (28)$$

As $\lambda_1 \leq (|F|/|E|)^{1/n} = \sigma(E, F)^{-1/n}$, from (28) we come to

$$\frac{1}{|E|} \int_E \lambda_n \leq C(n) \sqrt{\beta(E, F) \frac{1}{|E|} \int_E \lambda_n + \sigma(E, F)^{-1/n}}. \quad (29)$$

which easily implies

$$\frac{1}{|E|} \int_E (\lambda_n - \lambda_1) dx \leq C(n) \sqrt{\beta(E, F) \beta(E, F) + \sigma(E, F)^{-1/n}}. \quad (30)$$

Then (22) follows immediately.

**Step four: Trace inequality.** On combining (22) with (14), we conclude that, up to a translation of $F$,

$$C(n)\sqrt{\beta(E, F)} \sqrt{\beta(E, F) + \sigma(E, F)^{-1/n}} \geq \int_{\partial E} |T(x) - \lambda_G x| d\mathcal{H}^{n-1}(x).$$

If $F' = \lambda_G^{-1} F$ and $P : \mathbb{R}^n \setminus F' \to \partial F'$ denotes the projection of $\mathbb{R}^n \setminus F'$ over $F'$, then, since by construction $T$ takes value in $F'$, we get

$$C(n)\sqrt{\beta(E, F)} \sqrt{\beta(E, F) + \sigma(E, F)^{-1/n}} \geq \frac{\lambda_G}{|E|} \int_{\partial E \setminus F'} |P(x) - x| d\mathcal{H}^{n-1}(x). \quad (31)$$
We now consider the map \( \Phi : (\partial E \setminus F') \times (0, 1) \to E \setminus F' \) defined by
\[
\Phi(x, t) = tx + (1 - t)P(x).
\]
Let \( \{\epsilon_k(x)\}_{k=1}^{n-1} \) be a basis of the tangent space to \( \partial E \) at \( x \). Since \( \Phi \) is a bijection, we find
\[
|E \setminus F'| = \frac{1}{|E|} \int_{\partial E \setminus F'} (x - P(x)) \wedge \left( \bigwedge_{k=1}^{n-1} (t\epsilon_k(x) + (1 - t)dP_x(\epsilon_k(x))) \right) dH^{n-1}(x),
\]
where \( dP_x \) denotes the differential of the projection \( P \) at \( x \). As \( P \) is the projection over a convex set, it decreases distances, i.e. \( |dP_x(e)| \leq 1 \) for every \( e \in S^{n-1} \). Thus,
\[
|t\epsilon_k(x) + (1 - t)dP_x(\epsilon_k(x))| \leq 1, \quad \forall k \in \{1, \ldots, n-1\}.
\]
Recalling that \( \lambda_G = \sigma(E, F)^{-1/n} \), we combine this last inequality with (31) and (32) to get
\[
\frac{|E \setminus F'|}{|E|} \leq \frac{1}{|E|} \int_{\partial E \setminus F'} |x - P(x)| dH^{n-1}(x)
\leq C(n)\sigma(E, F)^{1/n}\sqrt{\beta(E, F)}(\beta(E, F) + \sigma(E, F)^{-1/n})
\leq C(n)\sqrt{\beta(E, F)\sigma(E, F)}^{1/n} + \beta(E, F)\sigma(E, F)^{1/n}
\leq C(n)\sqrt{\beta(E, F)\sigma(E, F)}^{1/n},
\]
where in the last inequality we have used (11). As
\[
A(E, F) \leq \frac{|E \setminus F'|}{|E|} = 2\frac{|E \setminus F'|}{|E|},
\]
this proves (6) and we achieve the proof of the theorem.

We conclude noticing that the constant \( C_0(n) \) in the above theorem can be taken to be
\[
C_0(n) \approx p(n)c_0^n,
\]
where \( p(n) \) is a polynomial in \( n \), and \( c_0 \) is any constant greater than \( \sqrt{2} \). Indeed, a quick inspection of the proof shows that all the terms to be considered for \( C(n) \) are polynomials, except for the estimate given in Step three – more precisely in (24) – which gives a term like \( nc_0^n \), with \( c > 2 \) (recall that, up to losing a numeric factor in \( C_0(n) \), we can assume from the beginning that \( \beta(E, F) \) is smaller than an arbitrarily small constant). Eventually, when applying Hölder inequality in (28) we take a square root of the constant \( C(n) \) appearing in (27), thus coming to the choice \( c_0 > \sqrt{2} \).

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References


