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Random data Cauchy problem for supercritical Schrödinger equations

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Abstract

In this paper we consider the Schrödinger equation with power-like nonlinearity and confining potential or without potential. This equation is known to be well-posed with data in a Sobolev space $\mathcal{H}^s$ if s is large enough and strongly ill-posed is s is below some critical threshold s_c. Here we use the randomisation method of the inital conditions, introduced by N. Burq and N. Tzvetkov, and we are able to show that the equation admits strong solutions for data in $\mathcal{H}^s$ for some $s < s_c$.

Résumé

Dans cet article on s'intéresse à l'équation de Schrödinger avec non-linéarité polynômiale et potentiel confinant ou sans potentiel. Cette équation est bien posée pour des données dans un espace de Sobolev $\mathcal{H}^s$ si s est assez grand, et fortement instable si s est sous un certain seuil critique s_c. Grâce à une randomisation des conditions initiales, comme l'ont fait N. Burq et N. Tzvetkov, on est capable de construire des solutions fortes pour des données dans $\mathcal{H}^s$ avec des $s < s_c$.

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Mots-clés : Équation de Schrödinger non linéaire ; Potentiel ; Équation surcritique

1. Introduction

In this paper we are concerned with the following nonlinear Schrödinger equations

$$\begin{cases} i\partial_t u + \Delta u = \pm |u|^{r-1}u, & (t,x) \in \mathbb{R} \times \mathbb{R}^d, \\ u(0,x) = f(x), \end{cases} \tag{1.1}$$

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and

$$\begin{cases} i\partial_t u + \Delta u - V(x)u = \pm |u|^{r-1}u, & (t,x) \in \mathbb{R} \times \mathbb{R}^d, \\ u(0,x) = f(x), \end{cases} \tag{1.2}$$

where r is an odd integer, and where V is a confining potential which satisfies the following assumption.

Assumption 1. We suppose that $V \in \mathcal{C}^\infty(\mathbb{R}^d, \mathbb{R}_+)$, and that there exists $k \geqslant 2$ so that:

(i) There exists $C > 1$ so that for $|x| \geqslant 1$, $\frac{1}{C}\langle x\rangle^k \leqslant V(x) \leqslant C\langle x\rangle^k$.
(ii) For any $j \in \mathbb{N}^d$, there exists $C_j > 0$ so that $|\partial_x^j V(x)| \leqslant C_j \langle x\rangle^{k-|j|}$.

In the following, H will stand for the operator,

$$H = -\Delta + V(x).$$

It is well known that under Assumption 1, the operator H has a self-adjoint extension on $L^2(\mathbb{R}^d)$ (still denoted by H) and has eigenfunctions $(e_n)_{n\geqslant 1}$ which form a hilbertian basis of $L^2(\mathbb{R}^d)$ and satisfy

$$He_n = \lambda_n^2 e_n, \quad n \geqslant 1, \tag{1.3}$$

with $\lambda_n \to +\infty$, when $n \to +\infty$.

For $s \in \mathbb{R}$ and $p \geqslant 1$, we define the Sobolev spaces based on the operator H

$$\mathcal{W}^{s,p} = \mathcal{W}^{s,p}(\mathbb{R}^d) = \{u \in \mathcal{S}'(\mathbb{R}^d):\ \langle H\rangle^{\frac{s}{2}} u \in L^p(\mathbb{R}^d)\},$$

and the Hilbert spaces

$$\mathcal{H}^s = \mathcal{H}^s(\mathbb{R}^d) = \mathcal{W}^{s,2}(\mathbb{R}^d) = \{u \in \mathcal{S}'(\mathbb{R}^d):\ \langle H\rangle^{\frac{s}{2}} u \in L^2(\mathbb{R}^d)\}, \tag{1.4}$$

where $\langle H\rangle = (1 + H^2)^{\frac{1}{2}}$.

In our paper we either consider the case $k = 2$ in all dimension or the case $d = 1$ and any $k \geqslant 2$. As we will see, we crucially use the L^p bounds for the eigenfunctions e_n which are only known in these cases.

Our results for the Cauchy problem (1.1) will be deduced from the study of (1.2) with the harmonic oscillator, thanks to a suitable transformation.

Let's recall some results about the Cauchy problems (1.1) and (1.2).

1.1. Previous deterministic results

Here we mainly discuss the results concerning the problem (1.2). The numerology for (1.1) is the same as (1.2) with a quadratic potential ($k = 2$). See [14] for more references for the problem (1.1).

Assume here that $d \geqslant 1$ and $k \geqslant 2$.

The linear Schrödinger flow enjoys Strichartz estimates, with loss of derivatives in general and without loss in the special case $k = 2$.

We say that the pair (p, q) is admissible, if

$$\frac{2}{p} + \frac{d}{q} = \frac{d}{2}, \quad 2 \leqslant p, q \leqslant \infty,\ (d, p, q) \neq (2, 2, \infty). \tag{1.5}$$

Let $0 < T \leqslant 1$ and assume that the pair (p, q) is admissible, then the solution u of the equation

$$i\partial_t u - Hu = 0, \qquad u(0,x) = f(x), \quad (t,x) \in \mathbb{R} \times \mathbb{R}^d,$$

satisfies

$$\|u\|_{L^p(0,T;L^q(\mathbb{R}^d))} \lesssim \|f\|_{\mathcal{H}^\rho(\mathbb{R}^d)}, \tag{1.6}$$

with loss

$$\rho = \rho(p,k) = \begin{cases} 0, & \text{if } k = 2, \\ \frac{2}{p}(\frac{1}{2} - \frac{1}{k}) + \eta, \text{ for any } \eta > 0, & \text{if } k > 2. \end{cases} \tag{1.7}$$

In the case $k = 2$, these estimates follow from the dispersion properties of the Schrödinger–Hermite group, obtained thanks to an explicit integral formula. Then (1.6) follows from the standard TT^* argument of J. Ginibre and G. Velo [10], and the endpoint is obtained with the result of M. Keel and T. Tao [11].

In the case $k > 2$, the result is due to K. Yajima and G. Zhang [18].

Thanks to the estimates (1.6), K. Yajima and G. Zhang [18] are able to use a fixed point argument in a Strichartz space and show that the problem (1.2) is well-posed (with uniform continuity of the flow map) in $\mathcal{H}^s$ for $s \geqslant 0$ so that

$$s > \frac{d}{2} - \frac{2}{r-1}\left(\frac{1}{2} + \frac{1}{k}\right).$$

The next statement shows that the problem (1.2) is ill-posed below the threshold $s = \frac{d}{2} - \frac{2}{r-1}$. In particular when $k = 2$, the well-posedness result is sharp.

Theorem 1.1 *(Ill-posedness). (See [7,14].) Assume that* $\frac{d}{2} - \frac{2}{r-1} > 0$ *and let* $0 < \sigma < \frac{d}{2} - \frac{2}{r-1}$. *Then there exist a sequence* $f_n \in \mathcal{C}^\infty(\mathbb{R}^d)$ *of Cauchy data and a sequence of times* $t_n \to 0$ *such that*

$$\|f_n\|_{\mathcal{H}^\sigma} \to 0, \quad \text{when } n \to +\infty,$$

and such that the solution u_n *of* (1.1) *or* (1.2) *satisfies*

$$\left\|u_n(t_n)\right\|_{\mathcal{H}^\rho} \to +\infty, \quad \text{when } n \to +\infty, \text{ for all } \rho \in \left]\frac{\sigma}{\frac{r-1}{2}(\frac{d}{2} - \sigma)}, \sigma\right].$$

Remark 1.2. Indeed we proved this result in [14] for the laplacian without potential. But the counterexamples constructed in the proof are functions which concentrate exponentially at the point 0, so that a regular potential plays no role.

This result shows that the flow map (if it exists) is not continuous at $u = 0$, and that there is even a loss of regularity in the Sobolev scale. For this range of σ, we cannot solve the problems (1.1) or (1.2) with a classical fixed point argument, as the uniform continuity of the flow map is a corollary of such a method.

The index $s_c := \frac{d}{2} - \frac{2}{r-1}$ can be understood in the following way. Assume that u is solution of the equation

$$i\partial_t u + \Delta u = |u|^{r-1}u, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^d, \tag{1.8}$$

then for all $\lambda > 0$, $u_\lambda : (t,x) \mapsto u_\lambda(t,x) = \lambda^{\frac{2}{r-1}} u(\lambda^2 t, \lambda x)$ is also solution of (1.8). The homogenous Sobolev space which is invariant with respect to this scaling is $\dot{H}^{s_c}(\mathbb{R}^d)$.

Hence, for $s < s_c$, we say that the problems (1.1) and (1.2) are *supercritical*.

Now we show that we can break this threshold in some probabilistic sense.

1.2. Randomisation of the initial condition

Let $(\Omega, \mathcal{F}, \mathbf{p})$ be a probability space. In the sequel we consider a sequence of random variables $(g_n(\omega))_{n\geqslant 1}$ which satisfy

Assumption 2. The random variables are independent and identically distributed and are either

(i) Bernoulli random variables: $\mathbf{p}(g_n = 1) = \mathbf{p}(g_n = -1) = \frac{1}{2}$, or
(ii) complex Gaussian random variables $g_n \in \mathcal{N}_{\mathbb{C}}(0,1)$.

A complex Gaussian $X \in \mathcal{N}_{\mathbb{C}}(0,1)$ can be understood as

$$X(\omega) = \frac{\sqrt{2}}{2}\big(X_1(\omega) + iX_2(\omega)\big),$$

where $X_1, X_2 \in \mathcal{N}_{\mathbb{R}}(0,1)$ are independent.

Each $f \in \mathcal{H}^s$ can be written in the hilbertian basis $(e_n)_{n\geqslant 1}$ defined in (1.3)

$$f(x) = \sum_{n\geqslant 1} \alpha_n e_n(x),$$

and we can consider the map

$$\omega \mapsto f^\omega(x) = \sum_{n\geqslant 1} \alpha_n g_n(\omega) e_n(x), \tag{1.9}$$

from $(\Omega, \mathcal{F})$ to $\mathcal{H}^s$ equipped with the Borel sigma algebra. The map (1.9) is measurable and $f^\omega \in L^2(\Omega; \mathcal{H}^s)$. The random variable f^ω is called the randomisation of f.

The map (1.9) was introduced by N. Burq and N. Tzvetkov [5,6] in the context of the wave equation. More precisely the authors study the problem

$$\begin{cases} \big(\partial_t^2 u - \Delta\big)u + u^3 = 0, \quad (t,x) \in \mathbb{R} \times M, \\ \big(u(0,x), \partial_t u(0,x)\big) = \big(f_1(x), f_2(x)\big) \in H^s(M) \times H^{s-1}(M), \end{cases} \tag{1.10}$$

where M is a three-dimensional compact manifold.

This equation is $H^{\frac{1}{2}} \times H^{-\frac{1}{2}}$ critical, and known to be well-posed for $s \geqslant \frac{1}{2}$ and ill-posed for $s < \frac{1}{2}$. Using that the randomised initial condition (f_1^ω, f_2^ω) is almost surely more regular than (f_1, f_2) in L^p spaces, N. Burq and N. Tzvetkov are able to show that the problem (1.10) admits a.s. strong solutions for $s \geqslant \frac{1}{4}$ (resp. $s \geqslant \frac{8}{21}$) if $\partial M = \emptyset$ (resp. $\partial M \neq \emptyset$).

Some authors have used random series to construct invariant Gibbs measures for dispersive PDEs, in order to get long-time dynamic properties of the flow map, see J. Bourgain [1,2], P. Zhidkov [20], N. Tzvetkov [17,16,15], N. Burq and N. Tzvetkov [4]. See also our forthcoming paper [3]. However, to the best of the author's knowledge, [5,6] is the first work in which stochastic methods are used in the proof of existence itself of solutions for a dispersive PDE. But above all, it is the only well-posedness result for a supercritical dispersive equation.

In this paper, we adapt these ideas to the study of the problem (1.1).

1.3. The main results

1.3.1. The cubic Schrödinger equation with quadratic potential

Our first result deals with the case $V(x) \sim \langle x\rangle^2$ in all dimension, for the cubic equation

$$\begin{cases} i\partial_t u + \Delta u - V(x)u = \pm|u|^2 u, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^d, \\ u(0,x) = f(x). \end{cases} \tag{1.11}$$

Theorem 1.3. *Let V satisfy Assumption 1 with $k = 2$.*

Assume that $d \geqslant 2$. Let $\sigma > \frac{d}{2} - 1 - \frac{1}{d+3}$ and $f \in \mathcal{H}^\sigma(\mathbb{R})$. Consider the function $f^\omega \in L^2(\Omega; \mathcal{H}^\sigma(\mathbb{R}))$ given by the randomisation (1.9). Then there exists $s > \frac{d}{2} - 1$ such that: for almost all $\omega \in \Omega$ there exist $T_\omega > 0$ and a unique solution to (1.11) with initial condition f^ω of the form

$$u(t,\cdot) = e^{-itH} f^\omega + \mathcal{C}\big([0,T_\omega]; \mathcal{H}^s\big(\mathbb{R}^d\big)\big) \bigcap_{(p,q)\ \text{admissible}} L^p\big([0,T_\omega]; \mathcal{W}^{s,q}\big(\mathbb{R}^d\big)\big). \tag{1.12}$$

More precisely: For every $0 < T \leqslant 1$ there exists an event Ω_T so that

$$\mathbf{p}(\Omega_T) \geqslant 1 - Ce^{-c/T^\delta}, \quad C, c, \delta > 0,$$

and so that for all $\omega \in \Omega_T$, there exists a unique solution to (2.12) in the class (1.12).

In the case $d = 1$, the same conclusion holds for $\sigma > -\frac{1}{4}$ with an $s \geqslant 0$.

Remark 1.4. Our method allows to treat every power-like nonlinearity. The gauge invariance structure of the nonlinearity plays no role, as we only work in Strichartz spaces.

Remark 1.5. As is [5], we can replace Assumption 2 made on $(g_n)_{n\geqslant 1}$ by any sequence of independent, centred random variables which satisfy some integrability conditions. However the event Ω_T in Theorem 1.3 will generally be of the form

$$\mathbf{p}(\Omega_T) \geqslant 1 - CT^{\delta}.$$

Remark 1.6. Let $\varepsilon > 0$ and $s \in \mathbb{R}$. If $f \in \mathcal{H}^s$ is such that $f \notin \mathcal{H}^{s+\varepsilon}$, then for almost all $\omega \in \Omega$, $f^{\omega} \in \mathcal{H}^s$ and $f^{\omega} \notin \mathcal{H}^{s+\varepsilon}$, hence the randomisation has no regularising effect in the L^2 scale. See [5, Lemma B.1] for a proof of this fact.

1.3.2. The cubic Schrödinger equation

We are also able to consider the case of the cubic Schrödinger equation without potential

$$\begin{cases} i\partial_t u + \Delta u = \pm|u|^2 u, & (t,x) \in \mathbb{R} \times \mathbb{R}^d, \\ u(0,x) = f(x). \end{cases} \tag{1.13}$$

Denote by $\mathcal{H}^s(\mathbb{R}^d)$ the space defined in (1.4) when $V(x) = |x|^2$ (the harmonic oscillator). Then we have

Theorem 1.7. *Let $d \geqslant 2$. Let $\sigma > \frac{d}{2} - 1 - \frac{1}{d+3}$ and $f \in \mathcal{H}^{\sigma}(\mathbb{R})$. Consider the function $f^{\omega} \in L^2(\Omega; \mathcal{H}^{\sigma}(\mathbb{R}))$ given by the randomisation* (1.9). *Then there exists $s > \frac{d}{2} - 1$ such that*: *for almost all $\omega \in \Omega$ there exist $T_{\omega} > 0$, $u_0 \in \mathcal{C}([0,T_{\omega}]; \mathcal{H}^{\sigma}(\mathbb{R}^d))$ and a unique solution to* (1.13) *with initial condition f^{ω} in a space continuously embedded in*

$$Y_{\omega} = u_0 + \mathcal{C}\big([0,T_{\omega}]; \mathcal{H}^s\big(\mathbb{R}^d\big)\big).$$

In the case $d = 1$, the same conclusion holds for $\sigma > -\frac{1}{4}$ with an $s \geqslant 0$.

Remark 1.8. In fact u_0 can be written $u_0(t,\cdot) = \mathcal{L}\mathrm{e}^{-itH_2} f^{\omega}$, where $\mathcal{L}$ is a linear operator defined in (6.1) and (6.4), and $H_2 = -\Delta + |x|^2$ is the harmonic oscillator.

Remark 1.9. Denote by $H^s(\mathbb{R}^d)$ the usual Sobolev space on $\mathbb{R}^d$. For $s > 0$, we have $\mathcal{H}^s(\mathbb{R}^d) \subset H^s(\mathbb{R}^d)$, hence our result cannot be compared to the classical deterministic well-posedness results. However, in the case of the dimensions $d = 1, 2$, we obtain a result for initial conditions with negative regularity. Moreover we can observe that $H^s(\mathbb{R}^d) \subset \mathcal{H}^s(\mathbb{R}^d)$ for $s \leqslant 0$, therefore we can read the previous result in the usual setting.

1.3.3. The Schrödinger equation in dimension 1

Our second result concerns the case $V(x) \sim \langle x \rangle^k$, in dimension 1.

$$\begin{cases} i\partial_t u + \Delta u - V(x)u = \pm|u|^{r-1} u, & (t,x) \in \mathbb{R} \times \mathbb{R}, \\ u(0,x) = f(x). \end{cases} \tag{1.14}$$

Theorem 1.10. *Let V satisfy Assumption* 1 *with $k \geqslant 2$. Let $r \geqslant 9$ be an odd integer. Let $\sigma > \frac{1}{2} - \frac{2}{r-1}(\frac{1}{2} + \frac{1}{k}) - \frac{1}{2k}$ and $f \in \mathcal{H}^{\sigma}(\mathbb{R})$. Consider the function $f^{\omega} \in L^2(\Omega; \mathcal{H}^{\sigma}(\mathbb{R}))$ given by the randomisation* (1.9). *Then there exists $s > \frac{1}{2} - \frac{2}{r-1}(\frac{1}{2} + \frac{1}{k})$ such that*: *for almost all $\omega \in \Omega$ there exist $T_{\omega} > 0$ and a unique solution to* (1.14) *with initial condition f^{ω} in a space continuously embedded in*

$$Y_{\omega} = \mathrm{e}^{-itH} f^{\omega} + \mathcal{C}\big([0,T_{\omega}]; \mathcal{H}^s(\mathbb{R})\big). \tag{1.15}$$

More precisely: *For every $0 < \varepsilon < 1$ and $0 < T \leqslant 1$ there exists an event Ω_T so that*

$$\mathbf{p}(\Omega_T) \geqslant 1 - C\mathrm{e}^{-c_0/T^{\delta}}, \quad C, c_0, \delta > 0,$$

and so that for all $\omega \in \Omega_T$, there exists a unique solution to (1.14) *in the class* (1.15).

Remark 1.11. In the case $r = 3$, $r = 5$ or $r = 7$, the gain of derivative is less that $\frac{1}{2k}$. We do not write the details.

1.4. Notations and plan of the paper

Notations. In this paper c, C denote constants the value of which may change from line to line. These constants will always be universal, or uniformly bounded with respect to the parameters $p, q, \kappa, \varepsilon, \omega, \ldots$. We use the notations $a \sim b$, $a \lesssim b$ if $\frac{1}{C}b \leqslant a \leqslant Cb$, $a \leqslant Cb$ respectively.

The notation L_T^p stands for $L^p(0,T)$, whereas $L^q = L^q(\mathbb{R}^d)$, and $L_T^p L^q = L^p(0,T; L^q(\mathbb{R}^d))$. For $1 \leqslant p \leqslant \infty$, the number p' is so that $\frac{1}{p} + \frac{1}{p'} = 1$.

The abbreviation r.v. is meant for random variable.

In this paper we follow the strategy initiated by N. Burq and N. Tzvetkov [5,6].

In Section 2 we recall the L^p estimates for the Hermite functions and we show a smoothing effect in L^p spaces for the linear solution of the Schrödinger equation, yield by the randomisation. We also show how some a priori deterministic estimates imply the main results.

In Section 3 we recall some deterministic estimates in Sobolev spaces.

In Section 4 we prove the estimates of Section 2 in the case $k = 2$, and conclude the proof of Theorem 1.3.

In Section 5 we consider the case $d = 1$ with any potential under Assumption 1 and conclude the proof of Theorem 1.10.

In Section 6 we are concerned with NLS without potential.

Remark 1.12. In our forthcoming paper [3], thanks to the construction of an invariant Gibbs measure, we will show that the following Schrödinger equations

$$\begin{cases} i\partial_t u + \Delta u - |x|^2 u = \kappa_0 |u|^{r-1} u, & (t,x) \in \mathbb{R} \times \mathbb{R}, \\ u(0,x) = f(x), \end{cases} \tag{1.16}$$

with ($\kappa_0 = -1$ and $r = 3$) or ($\kappa_0 = 1$ and $r \geqslant 3$) admit a large set of rough (supercritical) initial conditions leading to global solutions.

2. Stochastic estimates

In the following we will take profit on the L^p bounds for the eigenfunctions of H. This result is due to Yajima and Zhang [19] in the case $(d,k) = (1,k)$ and to Koch and Tataru [12] when $(d,k) = (d,2)$.

Theorem 2.1. *(See [19,12].) Let $k \geqslant 2$. Then the eigenfunctions e_n defined by* (1.3) *satisfy the bound*

$$\|e_n\|_{L^q(\mathbb{R}^d)} \lesssim \lambda_n^{-\theta(q,k,d)} \|e_n\|_{L^2(\mathbb{R}^d)}, \tag{2.1}$$

where θ is defined by

$$\theta(q,k,1) = \begin{cases} \frac{2}{k}\left(\frac{1}{2} - \frac{1}{q}\right) & \text{if } 2 \leqslant q < 4, \\ \frac{1}{2k} - \eta \text{ for any } \eta > 0 & \text{if } q = 4, \\ \frac{1}{2} - \frac{2}{3}\left(1 - \frac{1}{q}\right)\left(1 - \frac{1}{k}\right) & \text{if } 4 < q < \infty, \\ \frac{4-k}{6k} & \text{if } q = \infty, \end{cases} \tag{2.2}$$

and

$$\theta(q,2,d) = \begin{cases} \frac{1}{2} - \frac{1}{q} & \text{if } 2 \leqslant q < \frac{2(d+3)}{d+1}, \\ \frac{1}{d+3} - \eta \text{ for any } \eta > 0 & \text{if } q = \frac{2(d+3)}{d+1}, \\ \frac{1}{3} - \frac{d}{3}\left(\frac{1}{2} - \frac{1}{q}\right) & \text{if } \frac{2(d+3)}{d+1} < q \leqslant \frac{2d}{d-2}, \\ 1 - d\left(\frac{1}{2} - \frac{1}{q}\right) & \text{if } \frac{2d}{d-2} \leqslant q \leqslant \infty. \end{cases} \tag{2.3}$$

Notice that θ can be negative, but its maximum is always positive, attained for

$$q_*(d) = q_* = \frac{2(d+3)}{d+1}. \tag{2.4}$$

Let $f \in \mathcal{H}^\sigma$ and consider f^ω given by the randomisation (1.9).

Observe that the linear solution to the linear Schrödinger equation with initial condition f^ω is

$$u_f^\omega(t,x) = \mathrm{e}^{-itH} f^\omega(x) = \sum_{n\geqslant 1} \alpha_n g_n(\omega) \mathrm{e}^{-i\lambda_n^2 t} e_n(x).$$

Now we state the main stochastic tool of the paper. See [5] for two different proofs of this result, one based on explicit computations, and one based on large deviation estimates.

Lemma 2.2. *(See [5].) Let $(g_n(\omega))_{n\geqslant 1}$ be a sequence of random variables which satisfies Assumption 2. Then for all $r \geqslant 2$ and $(c_n) \in l^2(\mathbb{N}^*)$ we have*

$$\Big\| \sum_{n\geqslant 1} c_n g_n(\omega) \Big\|_{L^r(\Omega)} \lesssim \sqrt{r} \Big(\sum_{n\geqslant 1} |c_n|^2 \Big)^{\frac{1}{2}}.$$

Thanks to this result we will obtain

Proposition 2.3. *Let $d \geqslant 1$, $2 \leqslant q \leqslant p \leqslant r < \infty$, $\sigma \in \mathbb{R}$ and $0 < T \leqslant 1$. Let $f \in \mathcal{H}^\sigma$ and let f^ω be its randomisation given by* (1.9)*. Then*

$$\|\mathrm{e}^{-itH} f^\omega\|_{L^r(\Omega) L^p(0,T) \mathcal{W}^{\theta(q)+\sigma,q}(\mathbb{R}^d)} \lesssim \sqrt{r} T^{\frac{1}{p}} \|f\|_{\mathcal{H}^\sigma(\mathbb{R}^d)}, \tag{2.5}$$

where $\theta(q) = \theta(q,k,d)$ is the function defined in (2.2)*.*

As a consequence, if we set

$$E_{\lambda,f}(p,q,\sigma) = \big\{\omega \in \Omega \colon \ \|\mathrm{e}^{-itH} f^\omega\|_{L^p(0,T)\mathcal{W}^{\theta(q)+\sigma,q}} \geqslant \lambda\big\},$$

then there exist $c_1, c_2 > 0$ such that for all $p \geqslant q \geqslant 2$, all $\lambda > 0$ and $f \in \mathcal{H}^\sigma$

$$\mathbf{p}\big(E_{\lambda,f}(p,q,\sigma)\big) \leqslant \exp\left(c_1 p T^{\frac{2}{p}} - \frac{c_2 \lambda^2}{\|f\|^2_{\mathcal{H}^\sigma}} \right). \tag{2.6}$$

Remark 2.4. The previous estimate can be compared to the known deterministic estimate

$$\big\| \langle H \rangle^{\frac{\theta(q,k,1)}{2}} \mathrm{e}^{-itH} f \big\|_{L^p(\mathbb{R};L^2(0,T))} \lesssim \|f\|_{L^2(\mathbb{R})}, \tag{2.7}$$

which is proved by K. Yajima and G. Zhang in [19].

Proof of Proposition 2.3. Let $f = \sum_{n\geqslant 1} \alpha_n e_n \in \mathcal{H}^\sigma$. Then we have the explicit computation

$$\langle H \rangle^{\frac{\theta(q)+\sigma}{2}} \mathrm{e}^{-itH} f^\omega = \sum_{n\geqslant 1} \alpha_n g_n(\omega) \mathrm{e}^{-it\lambda_n^2} \big\langle \lambda_n^2 \big\rangle^{\frac{\theta(q)+\sigma}{2}} e_n.$$

Then by Lemma 2.2 we deduce

$$\big\| \langle H \rangle^{\frac{\theta(q)+\sigma}{2}} \mathrm{e}^{-itH} f^\omega \big\|_{L^r(\Omega)} \lesssim \sqrt{r} \Big(\sum_{n\geqslant 1} |\alpha_n|^2 \lambda_n^{2(\theta(q)+\sigma)} |e_n|^2 \Big)^{\frac{1}{2}}.$$

Now, for $2 \leqslant q \leqslant r$ take the $L^q(\mathbb{R}^d)$ norm of the previous estimate. By Minkowski and by the bounds (2.1), we obtain

$$\begin{aligned}
\big\|\mathrm{e}^{-itH}f^\omega\big\|_{L^r(\Omega)\mathcal{W}^{\theta(q)+\sigma,q}(\mathbb{R}^d)} &= \big\|\langle H\rangle^{\frac{\theta(q)+\sigma}{2}}\mathrm{e}^{-itH}f^\omega\big\|_{L^r(\Omega)L^q(\mathbb{R}^d)}\\
&\lesssim \big\|\langle H\rangle^{\frac{\theta(q)+\sigma}{2}}\mathrm{e}^{-itH}f^\omega\big\|_{L^q(\mathbb{R}^d)L^r(\Omega)}\\
&\lesssim \sqrt{r}\Big(\sum_{n\geqslant 1}|\alpha_n|^2\lambda_n^{2(\theta(q)+\sigma)}\|e_n\|_{L^q}^2\Big)^{\frac12}\\
&\lesssim \sqrt{r}\Big(\sum_{n\geqslant 1}|\alpha_n|^2\lambda_n^{2\sigma}\Big)^{\frac12} = \sqrt{r}\|f\|_{\mathcal{H}^\sigma}.
\end{aligned} \tag{2.8}$$

For $2\leqslant q\leqslant p\leqslant r$ we now take the $L^p(0,T)$ norm of (2.8), and by Minkowski again

$$\begin{aligned}
\big\|\mathrm{e}^{-itH}f^\omega\big\|_{L^r(\Omega)L^p(0,T)\mathcal{W}^{\theta(q)+\sigma,q}} &\lesssim \big\|\mathrm{e}^{-itH}f^\omega\big\|_{L^p(0,T)L^r(\Omega)\mathcal{W}^{\theta(q)+\sigma,q}}\\
&\lesssim \sqrt{r}T^{\frac1p}\|f\|_{\mathcal{H}^\sigma},
\end{aligned}$$

which is the estimate (2.5).

By the Bienaymé–Tchebychev inequality, there exists $C_0>0$ such that

$$\mathbf{p}\big(E_{\lambda,f}(p,q,\sigma)\big)=\mathbf{p}\big(\big\|\langle H\rangle^{\frac{\theta(q)+\sigma}{2}}\mathrm{e}^{-itH}f^\omega\big\|^r_{L^p(0,T)L^q(\mathbb{R})}\geqslant\lambda^r\big)\leqslant\bigg(\frac{C_0\sqrt{r}T^{\frac1p}\|f\|_{\mathcal{H}^\sigma}}{\lambda}\bigg)^r.$$

Either $\lambda>0$ is such that

$$\frac{\lambda}{\|f\|_{\mathcal{H}^\sigma}}\leqslant C_0\sqrt{p}T^{\frac1p}\mathrm{e}, \tag{2.9}$$

then inequality (2.6) holds for $c_1>0$ large enough.

Or we define

$$r:=\bigg(\frac{\lambda}{C_0T^{\frac1p}\|f\|_{\mathcal{H}^\sigma}\mathrm{e}}\bigg)^2\geqslant p, \tag{2.10}$$

then

$$\mathbf{p}\big(E_{\lambda,f}(p,q,\sigma)\big)\leqslant \mathrm{e}^{-r}=\exp\bigg(-\frac{c\lambda^2}{\|f\|^2_{\mathcal{H}^\sigma}}\bigg),$$

hence the result. □

Recall the notation (2.4) and define the event

$$E_{\lambda,f}=E_{\lambda,f}(M,q_*,\varepsilon), \tag{2.11}$$

where M is a large positive number which will be fixed in Sections 4 and 5.

We now show how the proof of the local existence of the Cauchy problem (1.2) with randomised data can be reduced to a priori deterministic estimates.

In fact we want to solve the equation

$$\begin{cases} i\partial_t u - Hu = |u|^{r-1}u, & (t,x)\in\mathbb{R}\times\mathbb{R}^d,\\ u(0)=f^\omega\in L^2\big(\Omega;\mathcal{H}^\sigma\big), \end{cases} \tag{2.12}$$

where $\sigma\in\mathbb{R}$ and the operator H satisfies Assumption 1.

This problem has the integral formulation

$$u(t,\cdot)=u_f^\omega(t)-i\int_0^t \mathrm{e}^{-i(t-\tau)H}|u|^{r-1}u(\tau,\cdot)\,\mathrm{d}\tau,$$

where u_f^ω stands for $\mathrm{e}^{-itH}f^\omega$.

Write $u = u_f^\omega + v$. Therefore, v satisfies the integral equation

$$v(t,\cdot) = -i\int_0^t \mathrm{e}^{-i(t-\tau)H}\left|u_f^\omega + v\right|^{r-1}\left(u_f^\omega + v\right)(\tau,\cdot)\,\mathrm{d}\tau,$$

thus we are reduced to find a fixed point of the map

$$K_f^\omega\colon\ v \mapsto -i\int_0^t \mathrm{e}^{-i(t-\tau)H}\left|u_f^\omega + v\right|^{r-1}\left(u_f^\omega + v\right)(\tau,\cdot)\,\mathrm{d}\tau.$$

Indeed the next proposition shows how a priori estimates on K imply the local well-posedness results.

Proposition 2.5. *(See [5].) Let $0 < T \leqslant 1$ and $\sigma \in \mathbb{R}$. Let $f \in \mathcal{H}^\sigma$ and $f^\omega \in L^2(\Omega; \mathcal{H}^\sigma)$ be its randomisation. Assume there exist $s \geqslant \sigma$ and a space $X_T^s \subset \mathcal{C}([0,T]; \mathcal{H}^s)$ and constants $\kappa > 0$, $C > 0$ so that for every $v, v_1, v_2 \in X_T^s$, $\lambda > 0$ and $\omega \in E_{\lambda,f}^c$ we have*

$$\left\|K_f^\omega(v)\right\|_{X_T^s} \leqslant CT^\kappa\left(\lambda^r + \|v\|_{X_T^s}^r\right), \tag{2.13}$$

and

$$\left\|K_f^\omega(v_1) - K_f^\omega(v_2)\right\|_{X_T^s} \leqslant CT^\kappa\|v_1 - v_2\|_{X_T^s}\left(\lambda^{r-1} + \|v_1\|_{X_T^s}^{r-1} + \|v_2\|_{X_T^s}^{r-1}\right). \tag{2.14}$$

Then for every $0 < T \leqslant 1$ there exists an event Ω_T so that

$$\mathbf{p}(\Omega_T) \geqslant 1 - C\mathrm{e}^{-c_0/T^\delta}, \quad C, c_0, \delta > 0,$$

and so that for all $\omega \in \Omega_T$, there exists a unique solution to (2.12) of the form

$$u(t,\cdot) = \mathrm{e}^{-itH} f^\omega + X_T^s.$$

Proof. Here we can follow the proof given in [5].

Let $0 < \mu < 1$ be small. Define $\delta = \frac{\kappa}{r^2}$, where κ is given by Proposition 2.5, and let $0 < T \leqslant 1$ be such that $T^\delta \leqslant \mu$. Take also $\omega \in E_{\lambda,f}^c$.

In a first time, we will show that the application K is a contraction on the ball $B(0, 2C\lambda^r)$ in X_T^s for $\lambda = \mu T^{-\delta}$ ($\geqslant 1$), if μ is chosen small enough, depending only on the absolute constant C.

By (2.13) and (2.14), to have a contraction, it suffices to find $\mu > 0$ such that the following inequalities hold

$$CT^\kappa\left(\lambda^r + \left(2C\lambda^r\right)^r\right) \leqslant 2C\lambda^r \quad \text{and} \quad CT^\kappa\left(\lambda^{r-1} + 2\left(2C\lambda^r\right)^{r-1}\right) \leqslant \frac{1}{2},$$

which is the case for $\mu \leqslant \mu(C)$, with our choice of the parameter $\lambda \geqslant 1$.

Now define

$$\Omega_T = E_{\lambda=\mu T^{-\delta},T,f}^c \quad \text{and} \quad \Sigma = \bigcup_{n\geqslant n_0} \Omega_{\frac{1}{n}},$$

where n_0 is such that $n_0^{-\delta} \leqslant \mu$. Then we deduce that

$$\mathbf{p}(\Omega_T) \geqslant 1 - C\mathrm{e}^{-c/T^{2\delta}} \quad \text{and} \quad \mathbf{p}(\Sigma) = 1,$$

which ends the proof of Theorem 1.3. □

3. Deterministic estimates in the space $\mathcal{W}^{s,p}$

We will need the following technical lemmas.

Lemma 3.1 *(Sobolev embeddings). Let $1 \leqslant q_1 \leqslant q_2 \leqslant \infty$ and $s \in \mathbb{R}$. The following inequalities hold*

$$\|v\|_{L^{q_2}} \lesssim \|v\|_{\mathcal{W}^{s,q_1}} \quad \text{for } s = d\left(\frac{1}{q_1} - \frac{1}{q_2}\right), \text{ when } q_2 < \infty, \tag{3.1}$$

$$\|v\|_{L^\infty} \lesssim \|v\|_{\mathcal{W}^{s,q_1}} \quad \text{for } s > \frac{d}{q_1}. \tag{3.2}$$

The results (3.1) and (3.2) are classical and left here.

We will also need the following lemma. See [13, Proposition 1.1, p. 105].

Lemma 3.2 *(Product rule). Let $s \geqslant 0$, then the following estimates hold*

$$\|uv\|_{\mathcal{W}^{s,q}} \lesssim \|u\|_{L^{q_1}} \|v\|_{\mathcal{W}^{s,\overline{q_1}}} + \|v\|_{L^{q_2}} \|u\|_{\mathcal{W}^{s,\overline{q_2}}}, \tag{3.3}$$

with $1 < q < \infty$, $1 < q_1, q_2 \leqslant \infty$ and $1 \leqslant \overline{q_1}, \overline{q_2} < \infty$ so that

$$\frac{1}{q} = \frac{1}{q_1} + \frac{1}{\overline{q_1}} = \frac{1}{q_2} + \frac{1}{\overline{q_2}}.$$

In particular

$$\|uv\|_{\mathcal{W}^{s,q}} \lesssim \|u\|_{L^\infty} \|v\|_{\mathcal{W}^{s,q}} + \|v\|_{L^\infty} \|u\|_{\mathcal{W}^{s,q}}, \tag{3.4}$$

for any $1 < q < \infty$.

4. Proof of Theorem 1.3

In this section, we consider the cubic Schrödinger equation with quadratic potential.

In the case $k = 2$, there is no loss of derivative in the Strichartz estimates (1.6). Then, thanks to the Christ–Kiselev lemma, we deduce that the solution to the problem

$$\begin{cases} i\partial_t u - Hu = F, & (t,x) \in \mathbb{R} \times \mathbb{R}^d, \\ u(0) = f \in \mathcal{H}^s, \end{cases}$$

satisfies

$$\|u\|_{L^{p_1}(0,T;\mathcal{W}^{s,q_1}(\mathbb{R}^d))} \lesssim \|f\|_{\mathcal{H}^s} + \|F\|_{L^{p_2'}(0,T;\mathcal{W}^{s,q_2'}(\mathbb{R}^d))}, \tag{4.1}$$

where $0 < T \leqslant 1$ and (p_1, q_1), (p_2, q_2) are any admissible pairs, in the sense of (1.5).

Denote by

$$X_T^s = \mathcal{C}\big([0,T];\mathcal{H}^s\big) \cap L^p\big([0,T];\mathcal{W}^{s,q}\big), \tag{4.2}$$

where the intersection is meant over all admissible pairs (p, q).

Recall that $E_{\lambda,f} = E_{\lambda,f}(M, q_*, \sigma)$ which is defined in (2.11). Then for M large enough, independent of λ and T we have the following proposition.

Proposition 4.1. *Let $0 < T \leqslant 1$, $d \geqslant 1$ and $\sigma > \frac{d}{2} - 1 - \frac{1}{d+3}$. Let $f \in \mathcal{H}^\sigma$. Then there exist $s > \frac{d}{2} - 1$, $\kappa > 0$ and $C > 0$ so that for every $v, v_1, v_2 \in X_T^s$, $\lambda > 0$ and $\omega \in E_{\lambda,f}^c$ we have*

$$\big\|K_f^\omega(v)\big\|_{X_T^s} \leqslant CT^\kappa\big(\lambda^3 + \|v\|_{X_T^s}^3\big), \tag{4.3}$$

and

$$\big\|K_f^\omega(v_1) - K_f^\omega(v_2)\big\|_{X_T^s} \leqslant CT^\kappa \|v_1 - v_2\|_{X_T^s}\big(\lambda^2 + \|v_1\|_{X_T^s}^2 + \|v_2\|_{X_T^s}^2\big). \tag{4.4}$$

For the proof of Proposition 4.1, we distinguish the cases $d = 1$, $d = 2$ and $d \geqslant 3$.

4.1. Case $d \geqslant 3$

Denote by $q_d = \frac{2d}{d-2}$, so that $(2, q_d)$ is the end point in the Strichartz estimates (4.1). Then the resolution space X_T^s defined in (4.2) reads

$$X_T^s = \mathcal{C}\big([0,T]; \mathcal{H}^s\big) \cap L^2\big([0,T]; \mathcal{W}^{s,q_d}\big).$$

Let q_* be defined by (2.4), then as $2 \leqslant q_* \leqslant q_d$, the following inclusion holds

$$X_T^s \subset L^{p_*}\big([0,T]; \mathcal{W}^{s,q_*}\big),$$

with $p_* \geqslant 2$ so that (p_*, q_*) is an admissible pair, i.e. $p_* = \frac{2(d+3)}{d}$.

Proof of Proposition 4.1, case $d \geqslant 3$. In this proof, we will write $u = u_f^\omega$.

The term $|u+v|^2(u+v)$ is a homogenous polynomial of degree 3. We expand it, and for sake of simplicity in the notations, we forget the complex conjugates. Hence

$$|u+v|^2(u+v) = \mathcal{O}\bigg(\sum_{0 \leqslant j \leqslant 3} u^j v^{3-j}\bigg). \tag{4.5}$$

By (4.1), we only have to estimate each term of the right-hand side in $L_T^1\mathcal{H}^s + L_T^2\mathcal{W}^{s,q_d'}$, with $q_d' = \frac{2d}{d+2}$.

Let $\varepsilon > 0$ so that

$$\sigma = \frac{d}{2} - 1 - \frac{1}{d+3} + \varepsilon.$$

Recall that $\theta(q_*) = \frac{1}{d+3} - \eta$, for any $\eta > 0$. In the following we choose $\eta = \varepsilon/2$ and we set

$$s = \theta(q_*) + \sigma = \frac{d}{2} - 1 + \frac{\varepsilon}{2}. \tag{4.6}$$

With this choice of s, by (3.2), the following embedding holds

$$\|u\|_{L^{q_0}} \lesssim \|u\|_{\mathcal{W}^{s,q_*}}, \quad \text{with } q_0 = \frac{1}{3}d(d+3). \tag{4.7}$$

Moreover, as $s > \frac{d}{q_d}$, by (3.2), it is straightforward to check that there exists $\kappa > 0$ so that

$$\|v\|_{L_T^2 L^\infty} \lesssim T^\kappa \|v\|_{L^2\mathcal{W}^{s,q_d}} \lesssim T^\kappa \|v\|_{X_T^s}. \tag{4.8}$$

Now assume that $\omega \in E_{\lambda,f}^c$ and turn to the estimation of each term in the r.h.s. of (4.5).

- We estimate the term v^3 in $L_T^1\mathcal{H}^s = L_T^1\mathcal{W}^{s,2}$. Use the inequality (3.4) with $q = 2$

$$\big\|v^3\big\|_{\mathcal{H}^s} \lesssim \|v\|_{L^\infty}^2 \|v\|_{\mathcal{H}^s},$$

and thus by (4.8)

$$\big\|v^3\big\|_{L_T^1\mathcal{H}^s} \lesssim \|v\|_{L_T^2 L^\infty}^2 \|v\|_{L_T^\infty \mathcal{H}^s} \lesssim T^\kappa \|v\|_{X_T^s}^3. \tag{4.9}$$

- We estimate the term uv^2 in $L_T^1\mathcal{H}^s$. By (3.3)

$$\begin{aligned}\big\|uv^2\big\|_{W^{s,2}} &\lesssim \|u\|_{L^{q_1}} \big\|v^2\big\|_{\mathcal{W}^{s,\overline{q_1}}} + \big\|v^2\big\|_{L^{q_2}} \|u\|_{\mathcal{W}^{s,\overline{q_2}}} \\ &\lesssim \|u\|_{L^{q_1}} \|v\|_{L^\infty} \|v\|_{\mathcal{W}^{s,\overline{q_1}}} + \|v\|_{L^{2q_2}}^2 \|u\|_{\mathcal{W}^{s,\overline{q_2}}}.\end{aligned}$$

Define $A_1 = \|u\|_{L^{q_1}} \|v\|_{L^\infty} \|v\|_{\mathcal{W}^{s,\overline{q_1}}}$.

We choose $q_1 = 2d$, then $\overline{q_1} = \frac{2d}{d-1}$. Observe that $2 < \overline{q_1} < q_d$ and for $d \geqslant 3$, $q_1 \leqslant q_0$, where q_0 in given by (4.7). Therefore by (4.8) we infer

$$\|A_1\|_{L_T^1} \lesssim T^\kappa \lambda \|v\|_{L_T^2 L^\infty} \|v\|_{L_T^{\overline{p_1}}\mathcal{W}^{s,\overline{q_1}}} \lesssim T^\kappa \lambda \|v\|_{X_T^s}^2, \tag{4.10}$$

where $\overline{p_1}$ is such that $(\overline{p_1}, \overline{q_1})$ is admissible.

Define $B_1 = \|v\|^2_{L^{2q_2}} \|u\|_{\mathcal{W}^{s,\overline{q_2}}}$.

We choose $\overline{q_2} = q_*$. Then $q_2 = d+3$, and by (4.8),

$$\|B_1\|_{L^1_T} \lesssim T^\kappa \lambda \|v\|^2_{X^s_T}. \tag{4.11}$$

From (4.10) and (4.11), we deduce

$$\|uv^2\|_{L^1_T W^{s,2}} \lesssim T^\kappa \lambda \|v\|^2_{X^s_T}. \tag{4.12}$$

- We estimate the term $u^2 v$ in $L^1_T \mathcal{H}^s$. By (3.3)

$$\|u^2 v\|_{W^{s,2}} \lesssim \|u^2\|_{L^{q_1}} \|v\|_{\mathcal{W}^{s,\overline{q_1}}} + \|v\|_{L^{q_2}} \|u^2\|_{\mathcal{W}^{s,\overline{q_2}}}.$$

Define $A_2 = \|u^2\|_{L^{q_1}} \|v\|_{\mathcal{W}^{s,\overline{q_1}}}$ and choose $\overline{q_1} = q_d$. Thus $q_1 = d$. As $d \geqslant 3$, $2d \leqslant q_0$, and by (4.7)

$$\|A_2\|_{L^1_T} \lesssim \|u\|^2_{L^4_T L^{2d}} \|v\|_{L^2_T \mathcal{W}^{s,q_d}} \lesssim T^\kappa \lambda^2 \|v\|_{X^s_T}. \tag{4.13}$$

Define $B_2 = \|v\|_{L^{q_2}} \|u^2\|_{\mathcal{W}^{s,\overline{q_2}}}$. We choose $q_2 = \infty$, and for all $\frac{1}{q_3} + \frac{1}{\overline{q_3}} = \frac{1}{2}$, by (3.3) we obtain

$$B_2 \lesssim \|v\|_{L^\infty} \|u\|_{L^{q_3}} \|u\|_{\mathcal{W}^{s,\overline{q_3}}}.$$

We take $\overline{q_3} = q_*$. Thus $q_3 = d+3$. To conclude, we only have to check that $q_3 \leqslant q_0$, which is satisfied when $d \geqslant 3$. Therefore

$$\|B_2\|_{L^1_T} \lesssim T^\kappa \lambda^2 \|v\|_{L^2_T L^\infty} \lesssim T^\kappa \lambda^2 \|v\|_{X^s_T}, \tag{4.14}$$

and by (4.13) and (4.14) we have

$$\|u^2 v\|_{L^1_T W^{s,2}} \lesssim T^\kappa \lambda^2 \|v\|_{X^s_T}. \tag{4.15}$$

- We estimate the term u^3 in $L^2_T \mathcal{W}^{s,q'_d}$. By Lemma 3.2

$$\|u^3\|_{\mathcal{W}^{s,q'_d}} \lesssim \|u^2\|_{L^{\overline{q_*}}} \|u\|_{\mathcal{W}^{s,q_*}} = \|u\|^2_{L^{2\overline{q_*}}} \|u\|_{\mathcal{W}^{s,q_*}},$$

with

$$\frac{1}{\overline{q_*}} = \frac{1}{q'_d} - \frac{1}{q_*} = \frac{d+2}{2d} - \frac{d+1}{2(d+3)} = \frac{2d+3}{d(d+3)}.$$

Observe that $2\overline{q_*} < q_0$ (where q_0 is defined in (4.7)) for $d \geqslant 2$. Hence by Hölder we deduce

$$\|u^3\|_{L^2_T \mathcal{W}^{s,q'_d}} \lesssim T^\kappa \lambda^3. \tag{4.16}$$

Collect the estimates (4.9), (4.12), (4.15) and (4.16), and by the Strichartz estimate (4.1) we obtain (4.3).

The proof of the contraction estimate (4.4) is similar and left here. □

4.2. Case $d = 2$

In this case, the resolution space (4.2) reads

$$X^s_T = \mathcal{C}([0,T]; \mathcal{H}^s) \cap L^p([0,T]; \mathcal{W}^{s,q}),$$

with intersection over all admissible pairs (p,q), i.e. $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$ with $2 < p \leqslant \infty$. In particular, for $\mu > 0$ small enough, denote by (p_μ, q_μ) the admissible pair so that

$$\frac{1}{p_\mu} = \frac{1}{2} - \mu, \qquad \frac{1}{q_\mu} = \mu, \qquad \frac{1}{p'_\mu} = \frac{1}{2} + \mu, \qquad \frac{1}{q_\mu} = 1 - \mu. \tag{4.17}$$

Notice that in the case $d = 2$, we have $q_* = \frac{10}{3}$. With the notations of (4.6) and (4.7), $s = \frac{\varepsilon}{2}$ and $q_0 = \frac{10}{3}$.

Proof of Proposition 4.1, case $d = 2$.

• The estimate

$$\|v^3\|_{L^1_T\mathcal{H}^s} \lesssim T^\kappa \|v\|^3_{X^s_T} \tag{4.18}$$

still holds.

• We estimate the term uv^2 in $L^{p'_\mu}_T \mathcal{W}^{s,q'_\mu}$. By (3.3)

$$\begin{aligned}\|uv^2\|_{W^{s,q'_\mu}} &\lesssim \|u\|_{L^{\frac{10}{3}}}\|v^2\|_{\mathcal{W}^{s,q_1}} + \|v^2\|_{L^{q_1}}\|u\|_{\mathcal{W}^{s,\frac{10}{3}}} \\ &\lesssim \|u\|_{\mathcal{W}^{s,\frac{10}{3}}}\|v\|^2_{\mathcal{W}^{s,2q_1}},\end{aligned}$$

where $\frac{1}{q_1} = \frac{7}{10} - \mu$. By time integration we deduce

$$\|uv^2\|_{L^{p'_\mu}_T\mathcal{W}^{s,q'_\mu}} \lesssim T^\kappa \lambda \|v\|^2_{X^s_T}. \tag{4.19}$$

• We estimate the term u^2v in $L^{p'_\mu}_T \mathcal{W}^{s,q'_\mu}$. By (3.3)

$$\begin{aligned}\|u^2v\|_{W^{s,q'_\mu}} &\lesssim \|v\|_{L^{q_2}}\|u^2\|_{\mathcal{W}^{s,\frac{10}{6}}} + \|u^2\|_{L^{\frac{10}{6}}}\|v\|_{\mathcal{W}^{s,q_2}} \\ &\lesssim \|u\|^2_{\mathcal{W}^{s,\frac{10}{3}}}\|v\|_{\mathcal{W}^{s,q_2}},\end{aligned}$$

where $\frac{1}{q_2} = \frac{4}{10} - \mu$. Again we conclude

$$\|u^2v\|_{L^{p'_\mu}_T\mathcal{W}^{s,q'_\mu}} \lesssim T^\kappa \lambda^2 \|v\|_{X^s_T}. \tag{4.20}$$

• The term u^3 will be estimated in $L^{\frac{10}{6}}_T \mathcal{W}^{s,\frac{10}{9}}$ (observe that the pair $(10/4, 10)$ is admissible). By (3.3)

$$\|u^3\|_{\mathcal{W}^{s,\frac{10}{9}}} \lesssim \|u^2\|_{L^{\frac{10}{6}}}\|u\|_{W^{s,\frac{10}{3}}} \lesssim \|u\|^3_{W^{s,\frac{10}{3}}},$$

and

$$\|u^3\|_{L^{\frac{10}{6}}_T\mathcal{W}^{s,\frac{10}{9}}} \lesssim T^\kappa \lambda^3. \tag{4.21}$$

The estimates (4.18)–(4.21), and the Strichartz estimate (4.1) yield the result (4.3).

The proof of (4.4) is left. □

4.3. Case $d = 1$

When $d = 1$ we work in the space

$$X_T = X^0_T = \mathcal{C}\big([0,T]; L^2\big) \cap L^4\big([0,T]; L^\infty\big).$$

Now we have $q_* = 4$.

Proof of Proposition 4.1, case $d = 1$. We can estimate the term $|u+v|^2(u+v)$ in $L^{\frac{8}{7}}_T L^{\frac{4}{3}}$. Indeed, by Hölder

$$\begin{aligned}\big\||u+v|^2(u+v)\big\|_{L^{\frac{8}{7}}_T L^{\frac{4}{3}}} &\lesssim \|u^3\|_{L^{\frac{8}{7}}_T L^{\frac{4}{3}}} + \|v^3\|_{L^{\frac{8}{7}}_T L^{\frac{4}{3}}} \\ &= \|u\|^3_{L^{\frac{24}{7}}_T L^4} + \|v\|^3_{L^{\frac{24}{7}}_T L^4} \\ &\lesssim T^\kappa\big(\lambda^3 + \|v\|^3_{X_T}\big),\end{aligned}$$

hence the result. □

5. Proof of Theorem 1.10

This proof is in the same spirit as the proof of Theorem 1.3. Here we are in dimension $d = 1$, with $k \geqslant 2$. However, the difference is that we have to deal with the losses in the Strichartz estimates (1.6).

Let V satisfy Assumption 1 and $0 < T < 1$. As in Yajima and Zhang [18], we define the space X_T^s by

$$X_T^s = \mathcal{C}\big([0,T];\mathcal{H}^s\big) \cap L^p\big([0,T];\mathcal{W}^{\tilde{s},q}\big),$$

with (p,q) admissible, i.e. $p, q \geqslant 2$ with $\frac{2}{p} + \frac{1}{q} = \frac{1}{2}$, and

$$p > r - 1 \quad \text{and} \quad \frac{1}{2} - \frac{2}{p} = \frac{1}{q} < \tilde{s} < s - \frac{2}{p}\left(\frac{1}{2} - \frac{1}{k}\right). \tag{5.1}$$

Under the conditions (5.1), for $T > 0$ small enough, it is possible to perform a contraction argument in the space X_T^s in order to show that the problem (1.2) with $d = 1$ is well-posed in $\mathcal{H}^s$ for

$$s > \frac{1}{2} - \frac{2}{r-1}\left(\frac{1}{2} + \frac{1}{k}\right).$$

In particular, for $\tilde{s} > \frac{1}{2} - \frac{2}{p} = \frac{1}{q}$, by (3.2), $\|v\|_{L^\infty} \lesssim \|v\|_{L^{\tilde{s},q}}$ and by Hölder in time, there exists $\kappa > 0$ so that

$$\|v\|_{L_T^{r-1}L^\infty} \lesssim T^\kappa \|v\|_{X_T^s}. \tag{5.2}$$

Now notice that here $q_*(1) = q_* = 4$ (defined in (2.4)) and that $\theta(4) = \frac{1}{2k} - \eta$, for any $\eta > 0$ (see (2.2)).

Again, we will show that the map

$$K_f^\omega : v \to -i \int_0^t \mathrm{e}^{-i(t-\tau)H} \big|u_f^\omega + v\big|^{r-1}\big(u_f^\omega + v\big)(\tau,\cdot)\,\mathrm{d}\tau,$$

is a contraction in X_T^s.

Indeed for M (independent of λ and T) large enough and $E_{\lambda,f} = E_{\lambda,f}(M,4,\sigma)$ which is defined in (2.11) we have the following proposition.

Proposition 5.1. *Let V satisfy Assumption* 1*, let $r \geqslant 9$ be an odd integer, and $0 < T \leqslant 1$. Let $\sigma > \frac{1}{2} - \frac{2}{r-1}(\frac{1}{2} + \frac{1}{k}) - \frac{1}{2k}$. Let also $f \in \mathcal{H}^\sigma$. Then there exist $s > \frac{1}{2} - \frac{2}{r-1}(\frac{1}{2} + \frac{1}{k})$, $\kappa > 0$ and $C > 0$ so that for every $v, v_1, v_2 \in X_T^s$, $\lambda > 0$ and $\omega \in E_{\lambda,f}^c$ we have*

$$\big\|K_f^\omega(v)\big\|_{X_T^s} \leqslant C T^\kappa \big(\lambda^r + \|v\|_{X_T^s}^r\big), \tag{5.3}$$

and

$$\big\|K_f^\omega(v_1) - K_f^\omega(v_2)\big\|_{X_T^s} \leqslant C T^\kappa \|v_1 - v_2\|_{X_T^s}\big(\lambda^{r-1} + \|v_1\|_{X_T^s}^{r-1} + \|v_2\|_{X_T^s}^{r-1}\big). \tag{5.4}$$

The first step of the proof of Proposition 5.1 is the following result.

Lemma 5.2. *Under the assumptions of Proposition* 5.1*, there exist $s > \frac{1}{2} - \frac{2}{r-1}(\frac{1}{2} + \frac{1}{k})$, $\kappa > 0$ and $C > 0$ so that for every $v \in X_T^s$, $\lambda > 0$ and $\omega \in E_{\lambda,f}^c$ we have*

$$\big\|\big|u_f^\omega + v\big|^{r-1}\big(u_f^\omega + v\big)\big\|_{L_T^1\mathcal{H}^s} \lesssim T^\kappa\big(\lambda^r + \|v\|_{X_T^s}^r\big).$$

Proof. In this proof, we will write $u = u_f^\omega$.

The term $|u + v|^{r-1}(u + v)$ is a homogenous polynomial of degree r. As in the proof of Proposition 4.1 we expand it, and forget the complex conjugates. Hence

$$|u+v|^{r-1}(u+v)=\mathcal{O}\Big(\sum_{0\leqslant j\leqslant r} u^j v^{r-j}\Big), \tag{5.5}$$

and we have to estimate each term of the right-hand side in $L^1_T\mathcal{H}^s$.

Now assume that $\omega\in E^c_{\lambda,f}$.

Recall that $q_*=4$, $\theta(q_*)=\frac{1}{2k}-\eta$, for any $\eta>0$. Let $\varepsilon>0$. We choose $\eta=\varepsilon/2$ and

$$\sigma=\frac{1}{2}-\frac{2}{r-1}\Big(\frac{1}{2}+\frac{1}{k}\Big)-\frac{1}{2k}+\varepsilon.$$

Then we set

$$s=\theta(q_*)+\sigma=\frac{1}{2}-\frac{2}{r-1}\Big(\frac{1}{2}+\frac{1}{k}\Big)+\frac{\varepsilon}{2}>\frac{1}{2}-\frac{2}{r-1}\Big(\frac{1}{2}+\frac{1}{k}\Big)\geqslant\frac{1}{4}.$$

Therefore as $s>\frac{1}{4}$, by the Sobolev injection (3.2) we have

$$\|u\|_{L^\infty(\mathbb{R})}\lesssim\|u\|_{\mathcal{W}^{s,4}(\mathbb{R})},$$

fact which will be used in the sequel to estimate all the terms containing u^j.

Now we turn to the estimation of (5.5) in $L^1_T\mathcal{H}^s$.

- For $j=0$, use the inequality (3.4) with $q=2$

$$\big\|v^r\big\|_{\mathcal{H}^s}\lesssim\|v\|^{r-1}_{L^\infty}\|v\|_{\mathcal{H}^s},$$

and thus by (5.2)

$$\big\|v^r\big\|_{L^1_T\mathcal{H}^s}\lesssim\|v\|^{r-1}_{L^{r-1}_T L^\infty}\|v\|_{L^\infty_T\mathcal{H}^s}\lesssim T^\kappa\|v\|^r_{X^s_T}. \tag{5.6}$$

- For $1\leqslant j\leqslant r-1$, by (3.3) in Lemma 3.2 we have

$$\begin{aligned}\big\|u^j v^{r-j}\big\|_{\mathcal{H}^s}&\lesssim\big\|u^j\big\|_{L^\infty}\big\|v^{r-j}\big\|_{\mathcal{W}^{s,2}}+\big\|v^{r-j}\big\|_{L^4}\big\|u^j\big\|_{\mathcal{W}^{s,4}}\\&\lesssim\|u\|^j_{L^\infty}\|v\|^{r-j-1}_{L^\infty}\|v\|_{\mathcal{W}^{s,2}}+\|v\|^{r-j}_{L^{4(r-j)}}\|u\|^{j-1}_{L^\infty}\|u\|_{\mathcal{W}^{s,4}}\\&\lesssim\|u\|^j_{\mathcal{W}^{s,4}}\big(\|v\|^{r-j-1}_{L^\infty}\|v\|_{\mathcal{W}^{s,2}}+\|v\|^{r-j}_{L^{4(r-j)}}\big).\end{aligned} \tag{5.7}$$

By interpolation, and by the embedding $\mathcal{W}^{s,2}\subset L^4$ (as $s>\frac{1}{4}$), for $1\leqslant j\leqslant r-1$ we have

$$\|v\|^{r-j}_{L^{4(r-j)}}\lesssim\|v\|_{L^4}\|v\|^{r-j}_{L^\infty}\lesssim\|v\|_{W^{s,2}}\|v\|^{r-j}_{L^\infty}.$$

Therefore (5.7) becomes

$$\big\|u^j v^{r-j}\big\|_{\mathcal{H}^s}\lesssim\|u\|^j_{\mathcal{W}^{s,4}}\|v\|^{r-j-1}_{L^\infty}\|v\|_{\mathcal{W}^{s,2}}.$$

By time integration and Hölder we obtain

$$\begin{aligned}\big\|u^j v^{r-j}\big\|_{L^1_T\mathcal{H}^s}&\lesssim\|u\|^j_{L^{pj}_T\mathcal{W}^{s,4}}\|v\|^{r-j-1}_{L^{p'(r-j-1)}_T L^\infty}\|v\|_{L^\infty_T\mathcal{W}^{s,2}}\\&=\|u\|^j_{L^{r-1}_T\mathcal{W}^{s,4}}\|v\|^{r-j-1}_{L^{r-1}_T L^\infty}\|v\|_{L^\infty_T\mathcal{W}^{s,2}},\end{aligned}$$

with $p=\frac{r-1}{j}$. Now, by (5.2)

$$\big\|u^j v^{r-j}\big\|_{L^1_T\mathcal{H}^s}\lesssim T^\kappa\lambda^j\|v\|^{r-j}_{X^s_T}. \tag{5.8}$$

- We now estimate the term u^r. From (3.3) we deduce

$$\big\|u^r\big\|_{\mathcal{H}^s}\lesssim\big\|u^{r-1}\big\|_{L^4}\|u\|_{\mathcal{W}^{s,4}}=\|u\|^{r-1}_{L^{4(r-1)}}\|u\|_{\mathcal{W}^{s,4}}\lesssim\|u\|^r_{\mathcal{W}^{s,4}},$$

and thus

$$\|u^r\|_{L^1_T\mathcal{H}^s} \lesssim \|u\|^r_{L^r_T\mathcal{W}^{s,4}} \lesssim T^\kappa \lambda^r. \tag{5.9}$$

Collect the estimates (5.6), (5.8) and (5.9) to deduce the result of Lemma 5.2. □

Similarly we have

Lemma 5.3. *Under the assumptions of Proposition* 5.1, *there exist* $s > \frac{1}{2} - \frac{2}{r-1}(\frac{1}{2}+\frac{1}{k})$ *and* $\kappa > 0$ *so that for every* $v_1, v_2 \in X^s_T$, $\lambda > 0$ *and* $\omega \in E^c_{\lambda,f}$ *we have*

$$\big\| |u^\omega_f + v_1|^{r-1}(u^\omega_f + v_1) - |u^\omega_f + v_2|^{r-1}(u^\omega_f + v_2)\big\|_{L^1_T\mathcal{H}^s} \lesssim T^\kappa \|v_1 - v_2\|_{X^s_T}\big(\lambda^{r-1} + \|v_1\|^{r-1}_{X^s_T} + \|v_2\|^{r-1}_{X^s_T}\big).$$

Proof. We have

$$\begin{aligned}|u^\omega_f + v_1|^{r-1}(u^\omega_f + v_1) - |u^\omega_f + v_2|^{r-1}(u^\omega_f + v_2) &= (v_1 - v_2)P_{r-1}\big(u^\omega_f, \bar{u}^\omega_f, v_1, \bar{v}_1, v_2, \bar{v}_2\big)\\ &\quad + (\bar{v}_1 - \bar{v}_2)Q_{r-1}\big(u^\omega_f, \bar{u}^\omega_f, v_1, \bar{v}_1, v_2, \bar{v}_2\big),\end{aligned}$$

where P_{r-1}, Q_{r-1} are homogenous polynomials of degree $r-1$. It is straightforward to check that we can perform the same computations as in the proof of Lemma 5.2. □

Proof of Proposition 5.1. Firstly, as e^{-itH} is unitary, we have

$$\begin{aligned}\|K^\omega_f(v)\|_{L^\infty_T\mathcal{H}^s} &\leqslant \int_0^T \big\| |u^\omega_f + v|^{r-1}(u^\omega_f + v)(\tau,\cdot)\big\|_{\mathcal{H}^s}(\tau,\cdot)\,\mathrm{d}\tau\\ &= \big\| |u^\omega_f + v|^{r-1}(u^\omega_f + v)\big\|_{L^1_T\mathcal{H}^s}.\end{aligned} \tag{5.10}$$

Secondly, for every admissible pair (p,q) and $\tilde{s}$ which satisfy the condition (5.1), in virtue of the Strichartz estimates (1.6)–(1.7), we have for all $F \in \mathcal{H}^s$

$$\|\mathrm{e}^{-itH}F\|_{L^p_T\mathcal{W}^{\tilde{s},p}} \lesssim \|F\|_{\mathcal{H}^s},$$

therefore we obtain

$$\begin{aligned}\|K^\omega_f(v)\|_{L^p_T\mathcal{W}^{\tilde{s},q}} &\leqslant \int_0^T \big\|\mathbf{1}_{\{\tau<t\}}\mathrm{e}^{-itH}\big(\mathrm{e}^{i\tau H}|u^\omega_f + v|^{r-1}(u^\omega_f + v)\big)(\tau,\cdot)\big\|_{L^p_T\mathcal{W}^{\tilde{s},q}}\,\mathrm{d}\tau\\ &\lesssim \big\| |u^\omega_f + v|^{r-1}(u^\omega_f + v)\big\|_{L^1_T\mathcal{H}^s}.\end{aligned} \tag{5.11}$$

Hence (5.10), (5.11) together with Lemma 5.2 yield (5.3).

The proof of the inequality (5.4) follows from Lemma 5.3. □

6. The nonlinear Schrödinger equation without potential

In this section, we show how (in our context) the study of the problem (1.1) can be reduced to the study of the problem (1.2) with harmonic potential.

In this section, the space $\mathcal{H}^s = \mathcal{H}^s(\mathbb{R}^d)$ is the Sobolev space defined in (1.4) when $V(x) = |x|^2$ is the harmonic oscillator. Let $0 < T \leqslant 1$ and consider the linear applications

$$\begin{aligned}&\mathcal{L}_0 : \mathcal{C}\big([0, \operatorname{Arctan} T]; \mathcal{H}^s(\mathbb{R}^d)\big) \to \mathcal{C}\big([0,T]; \mathcal{H}^s(\mathbb{R}^d)\big),\\ &u \mapsto \mathcal{L}_0 u,\end{aligned}$$

given by

$$\mathcal{L}_0 u(t,x) = \frac{1}{(1+t^2)^{\frac{d}{4}}}u\left(\operatorname{Arctan} t, \frac{x}{\sqrt{1+t^2}}\right)\mathrm{e}^{i\frac{|x|^2}{2}\frac{t}{1+t^2}}, \tag{6.1}$$

and for $\beta > 0$ the time-dilation

$$\mathcal{D}_\beta u(t,x) = u(\beta t, x).$$

The operator $\mathcal{L}_0$ has been used in different nonlinear problems, especially for L^2-critical Schrödinger equations. See R. Carles [8,9] and references therein.

We can check that the map $\mathcal{L}_0$ is an isomorphism and has the following property.

Assume that $v_1 \in \mathcal{C}([0, \operatorname{Arctan} T]; \mathcal{H}^s)$ solves the Cauchy problem

$$\begin{cases} i\partial_t v_1 + \frac{1}{2}\Delta v_1 - \frac{1}{2}|x|^2 v_1 = \pm\big(1+t^2\big)^\alpha |v_1|^{r-1} v_1, & (t,x) \in \mathbb{R}\times\mathbb{R}^d, \\ v_1(0,x) = f(x) \in \mathcal{H}^s, \end{cases}$$

where $\alpha = \frac{d}{4}(r-1) - 1$. Then $u_1 = \mathcal{L}_0 v_1 \in \mathcal{C}([0,T]; \mathcal{H}^s)$ solves

$$\begin{cases} i\partial_t u_1 + \frac{1}{2}\Delta u_1 = \pm|u_1|^{r-1} u_1, & (t,x) \in \mathbb{R}\times\mathbb{R}^d, \\ u_1(0,x) = \mathcal{L}_0 v_1(0,x) = f(x) \in \mathcal{H}^s. \end{cases}$$

Thus if $v \in \mathcal{C}([0, \frac{1}{2}\operatorname{Arctan}(T/2)]; \mathcal{H}^s)$ is the solution to the problem

$$\begin{cases} i\partial_t v + \Delta v - |x|^2 v = \pm 2\big(1+4t^2\big)^\alpha |v|^{r-1} v, & (t,x) \in \mathbb{R}\times\mathbb{R}^d, \\ v(0,x) = 2^{-\frac{1}{r-1}} f(x) \in \mathcal{H}^s, \end{cases} \tag{6.2}$$

then the solution $u \in \mathcal{C}([0,T]; \mathcal{H}^s)$ to the equation

$$\begin{cases} i\partial_t u + \Delta u = \pm|u|^{r-1} u, & (t,x) \in \mathbb{R}\times\mathbb{R}^d, \\ u(0,x) = f(x) \in \mathcal{H}^s, \end{cases} \tag{6.3}$$

will be given by $u = \mathcal{L} v$ with

$$\mathcal{L} = 2^{\frac{1}{r-1}} \mathcal{D}_2 \mathcal{L}_0 \mathcal{D}_{\frac{1}{2}}. \tag{6.4}$$

Denote by $H_2 = -\Delta + |x|^2$ the harmonic oscillator.

Proposition 6.1. *Let $r = 3$.*

Assume that $d \geqslant 2$. Let $\sigma > \frac{d}{2} - 1 - \frac{1}{d+3}$ and $f \in \mathcal{H}^\sigma(\mathbb{R}^d)$. Consider the function $f^\omega \in L^2(\Omega; \mathcal{H}^\sigma(\mathbb{R}^d))$ given by the randomisation (1.9). *Then there exists $s > \frac{d}{2} - 1$ such that*: *for almost all $\omega \in \Omega$ there exist $T_\omega > 0$ and a unique solution to* (6.3) *with initial condition f^ω in a space continuously embedded in*

$$Y_\omega = \mathcal{L} e^{-itH_2} f^\omega + \mathcal{C}\big([0,T_\omega]; \mathcal{H}^s\big(\mathbb{R}^d\big)\big).$$

In the case $d = 1$, the same result holds with $\sigma > -\frac{1}{4}$ and an $s \geqslant 0$.

Proof. According to the previous remarks, it is sufficient to solve the problem (6.2) with initial condition $2^{-\frac{1}{2}} f^\omega \in L^2(\Omega; \mathcal{H}^\sigma)$ (the randomisation of $2^{-\frac{1}{2}} f$). Observe that for any admissible pair (p,q) and $F \in L_T^{p'} L^{q'}$ we have

$$\big\| 2\big(1+4t^2\big)^\alpha F \big\|_{L_T^{p'} L^{q'}} \lesssim \|F\|_{L_T^{p'} L^{q'}},$$

hence we can follow step by step the proof of Proposition 4.1 with the space X_T^s defined in (4.2). This completes the proof of Theorem 1.7. □

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