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On the two-phase membrane problem with coefficients below the Lipschitz threshold ☆

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Abstract

We study the regularity of the two-phase membrane problem, with coefficients below the Lipschitz threshold. For the Lipschitz coefficient case one can apply a monotonicity formula to prove the $C^{1,1}$-regularity of the solution and that the free boundary is, near the so-called branching points, the union of two C^1-graphs. In our case, the same monotonicity formula does not apply in the same way. In the absence of a monotonicity formula, we use a specific scaling argument combined with the classification of certain global solutions to obtain $C^{1,1}$-estimates. Then we exploit some stability properties with respect to the coefficients to prove that the free boundary is the union of two Reifenberg vanishing sets near so-called branching points.

MSC: 35J70; 35J60; 35J85

Keywords: Obstacle problem; Elliptic equation; Regularity

1. Introduction and main result

1.1. Problem

Given two strictly positive functions λ_1 and λ_2 and boundary data $g \in H^1(B_1) \cap L^\infty(B_1)$ we study the minimizer of the functional

$$J(u) = \int_{B_1} \frac{|\nabla u|^2}{2} + \lambda_1(x)u^+ + \lambda_2(x)u^- \,\mathrm{d}x \tag{1.1}$$

over the set $\{u\colon\ u - g \in H_0^1(B_1)\}$, with its corresponding Euler–Lagrange equation

$$\Delta u = \lambda_1(x)\chi_{\{u>0\}} - \lambda_2(x)\chi_{\{u<0\}} \quad \text{in } B_1. \tag{1.2}$$

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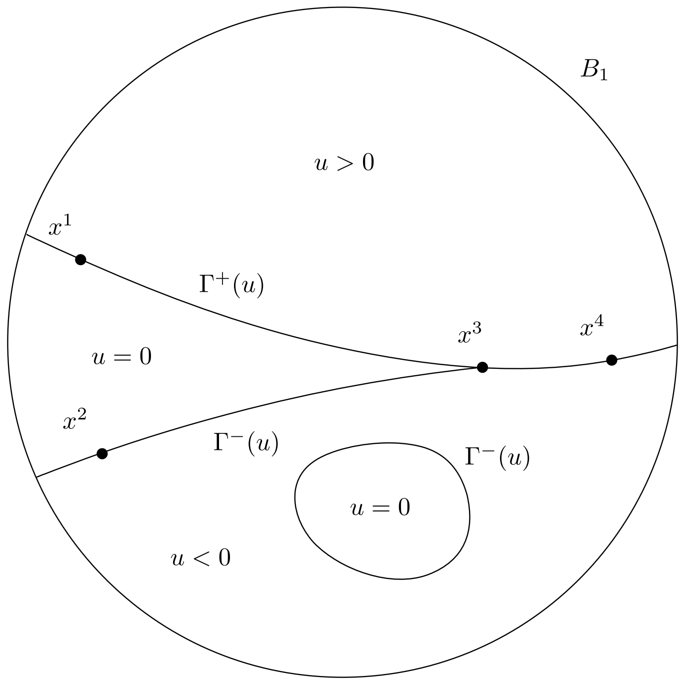

Fig. 1. x^1 a positive one-phase point, x^2 a negative one-phase point, x^3 a branching point and x^4 a non-branching two-phase point.

The existence and uniqueness of minimizers of (1.1) and solutions of (1.2) can be obtained by standard methods, see [16]. In general, if g attains both positive and negative values, Δu will have a jump where u changes sign, i.e. across the set $\partial\{u \neq 0\}$. We call this set the free boundary, and also we denote its two parts $\partial\{u > 0\}$ and $\partial\{u < 0\}$ by $\Gamma^+(u)$ and $\Gamma^-(u)$ respectively (see Fig. 1). The main purpose of the present paper is to study the regularity of u and $\Gamma(u)$. Note that in the sets $\{u > 0\}$ and $\{u < 0\}$ the regularity of u is completely determined by the regularity of the coefficients λ_i. The difficulty that arises, when we have coefficients that are not Lipschitz continuous, is that we cannot directly apply the monotonicity formula from [6]. In the absence of a monotonicity formula, we use a scaling argument combined with the classification of certain global solutions to obtain $C^{1,1}$-estimates. These arguments are elaborated versions of those introduced in [8]. Once the optimal regularity is settled we exploit some stability properties with respect to the coefficients to prove that the free boundary is the union of two Reifenberg vanishing sets near so-called branching points. The stability arguments here are based on the ideas in [3].

1.2. Known results

The one-phase case, i.e. when the minimizer is assumed not to change sign, is the classical obstacle problem and it has been well-studied before. In [5] it is proved that minimizers are locally $C^{1,1}$ and that the free boundary is locally a $C^{1,\alpha}$-graph if the coefficients are Lipschitz and if the zero set satisfies a certain thickness assumption. Later, in [3] Blank proved that minimizers are locally $C^{1,1}$ and that the free boundary is locally a C^1-graph under similar thickness assumptions when the coefficients are only assumed to be Dini continuous.

The two-phase case (when the minimizer is allowed to change sign) has also been studied before under stronger assumptions on the coefficients. In [15] the $C^{1,1}$-regularity of minimizers is proved when the coefficients are assumed to be constant. This result was extended in [12] to the case when the coefficients are assumed to be Lipschitz. Moreover, in [13] global solutions of (1.2) is considered and classified in the case of constant coefficients. This result is then later used in [14] to prove that in the case of Lipschitz coefficients, the free boundary is the union of two C^1-graphs close to branching points, i.e. points close to the set $\partial\{u > 0\} \cap \partial\{u < 0\} \cap \{|\nabla u| = 0\}$.

1.3. Main result

The main result of this paper is that solutions to (1.2) are locally $C^{1,1}$ if the coefficients λ_i are Hölder continuous and that the free boundary, $\Gamma(u)$, is the union of two Reifenberg vanishing sets close to branching points under the weaker assumption that the λ_i's are merely continuous. In order to state our main theorems in their precise forms we must define the class of solutions that we consider in this paper.

Definition 1.1. We say $u \in P_1(M, M_0, \omega)$ if

(1) $\Delta u = \lambda_1(x)\chi_{\{u>0\}} - \lambda_2(x)\chi_{\{u<0\}}$ in B_1, in the sense of distributions.
(2) $\sup_{B_1} |u| \leqslant M_0$.
(3) $\inf \lambda_i \geqslant 1/M > 0$.
(4) $\sup \lambda_i \leqslant M$.
(5) λ_1 and λ_2 are uniformly continuous with ω as modulus of continuity.
(6) $0 \in \partial\{u > 0\} \cap \partial\{u < 0\} \cap \{|\nabla u| = 0\}$.

Our two main theorems are the following:

Theorem 1.2. *Assume that $u \in P_1(M, M_0, \omega)$ where $\omega(r) \leqslant Mr^\alpha$ for some $0 < \alpha < 1$. Then there are r_0 and C depending on M, M_0 and the dimension such that*

$$\|u\|_{C^{1,1}(B_{r_0})} \leqslant C.$$

Remark 1.3. The observant reader might ask if condition (3) is really necessary for the $C^{1,1}$-regularity. Indeed, if $\lambda_1 = \lambda_2 = 0$ then the function would be harmonic and therefore analytic. However, the technique used in this paper relies heavily on a classification of global solution which would fail if $\lambda_i = 0$.

Theorem 1.4. *Assume $u \in P_1(M, M_0, \omega)$. Then there are r_0 and γ depending on M, M_0 and ω such that $|\nabla u(y)| \leqslant \gamma$ and $\operatorname{dist}(y, \Gamma^\pm(u)) \leqslant \gamma$ implies that $\Gamma(u)^+ \cap B_{r_0}(y)$ and $\Gamma(u)^- \cap B_{r_0}(y)$ are both Reifenberg vanishing sets. In particular they both admit Hölder parameterizations.*

Remark 1.5. Interesting here is to note that the assumptions on ω are weaker in Theorem 1.4 than in Theorem 1.2. The reason for that is that there are some technical difficulties in Proposition A.1 in Appendix A which we need in order to obtain $C^{1,1}$-estimates. See also Remark 2.9.

2. $C^{1,1}$-estimates

In this section we prove Theorem 1.2. This is done by first proving a weaker type of quadratic growth away from branching points. The method used here is a contradictory scaling argument very similar to the one used in [8]. After that we use some properties of coercive elliptic systems to obtain $C^{1,1}$-regularity close to points on the free boundary where the gradient does not vanish. Gluing these pieces together we finally obtain $C^{1,1}$-estimates close to branching points.

2.1. Non-degeneracy

Here we present the well known-result that the solutions do not grow too slow around free boundary points; the proof is standard.

Proposition 2.1. *Let $u \in P_1(M, M_0, \omega)$. Then there is a constant c depending on the dimension and M such that for all $y \in \Gamma(u) \cap B_1$*

$$\sup_{B_r(y)\cap\{u>0\}} u \geqslant cr^2$$

and

$$\inf_{B_r(y)\cap\{u<0\}} u \leqslant -cr^2,$$

for $r < \operatorname{dist}(y, \partial B_1)$.

Proof. Let $y \in \{u > 0\} \cap B_1$ and $w(x) = u(x) - t|x - y|^2$ and r so small that $B_r(y) \subset B_1$. Then $\Delta w \geqslant 0$ in $\{u > 0\} \cap B_1$ if $t = 2n \inf \lambda_1$. Since $w(y) > 0$ and w is subharmonic, there is $x_y \in \partial(B_r(y) \cap \{u > 0\})$ such that $w(x_y) > 0$. Now on $\Gamma^+(u)$ we have $w \leqslant 0$, hence $x_y \in \partial B_r(y) \cap \{u > 0\}$. In other words

$$\sup_{\partial B_r(y)\cap\{u>0\}} w > 0.$$

Taking $z \in \Gamma^+(u)$ we can find a sequence of points $y_n \in \{u > 0\}$ such that $y_n \to z$, then by continuity we obtain

$$\sup_{B_r(z)\cap\{u>0\}} w \geqslant 0,$$

which implies the desired result. On $\Gamma^-(u)$ we can argue similarly. □

2.2. *Classification of global solutions of a related problem*

In order to go through with our scaling argument we will need a classification of global solutions to the equation

$$\Delta u = \lambda_1 \chi_{\{u>-ax_1\}} - \lambda_2 \chi_{\{u<-ax_1\}},$$

where λ_1 and λ_2 are constants. One major difference here is that one cannot apply the monotonicity formula from [1] to the pair of functions $(\partial_e u)^\pm$ for all directions e, only for those directions orthogonal to e_1. Of course, translating the function u linearly with ax_1 yields a global solution to the two-phase obstacle problem. However the gradient will not vanish at the origin, so we cannot apply the classification from [13] directly.

Lemma 2.2. *Let u be a solution of the following problem*

$$\begin{cases} \Delta u = \lambda_1 \chi_{\{u>-ax_1\}} - \lambda_2 \chi_{\{u<-ax_1\}} & \text{in } \mathbb{R}^n, \\ u(0) = \left|\nabla u(0)\right| = 0, \\ 0 \in \partial\{u \neq 0\}, \end{cases} \tag{2.1}$$

where $a > 0$ *and* λ_1 *and* λ_2 *are constants. Assume also that for some* $C > 0$

$$\sup_{B_r} |u| \leqslant Cr^2 \tag{2.2}$$

whenever $r > 1$. *Then one of the following holds*

(1) $u(x) = \frac{\lambda_1}{2}(x_1^+)^2 - \frac{\lambda_2}{2}(x_1^-)^2$,
(2) $u(x) = \frac{\lambda_1}{2}(x_1^+)^2$,
(3) $u(x) = -\frac{\lambda_2}{2}(x_1^-)^2$.

In particular

$$\sup_{B_1} |u| \leqslant \frac{1}{2}\max(\lambda_1, \lambda_2).$$

Remark 2.3. A more general form of this lemma with the right-hand side equal to $\lambda_1 \chi_{\{u>-L(x)\}} - \lambda_2 \chi_{\{u<-L(x)\}}$ with L being a linear function can easily be obtained by a change of coordinates.

Proof of Lemma 2.2. We observe that

$$\Delta(\partial_e u) = \partial_e \Delta u = \frac{\lambda_1 + \lambda_2}{|\nabla(u + ae_1)|}(\partial_e u + ae \cdot e_1)\mathcal{H}^{n-1}\lfloor\{u = -ax_1\}.$$

Therefore with $v^\pm = (\partial_e u)^\pm$ for $e \cdot e_1 = 0$ we have

(1) $v^+ \cdot v^- = 0$,
(2) $\Delta(v^\pm) \geqslant 0$,
(3) $v^\pm(0) = 0$.

Hence, the Alt–Caffarelli–Friedman monotonicity formula applies to $v^\pm$. It states that with

$$\phi(r, v) = \frac{1}{r^4} \int_{B_r} \frac{|\nabla v^+|^2}{|x|^{n-2}}\,\mathrm{d}x \int_{B_r} \frac{|\nabla v^-|^2}{|x|^{n-2}}\,\mathrm{d}x$$

we have $\phi'(r, v) \geqslant 0$ for all r. Moreover, if $\phi'(r, v) = 0$ for all r then one of the functions $v^\pm$ vanishes.

Define $u_r(x) = u(rx)/r^2$. Now we wish to study the behavior of u_r as r tends to ∞ and 0. Since the Laplacian is invariant under the quadratic scaling, Δu_r is uniformly bounded independently of r. We observe that the function $v = u + ax_1$ is a solution to the usual two-phase membrane problem. Therefore $v \in C^{1,1}_{\mathrm{loc}}(\mathbb{R}^n)$ by the result in [12] and [15]. Together with the hypotheses this implies that we have uniform quadratic growth as $r \to 0$ and as $r \to \infty$. This allows us to extract a subsequence r_j tending either to 0 or to ∞ such that u_{r_j} converges to a limit in $C^{1,\alpha}_{\mathrm{loc}} \cap W^{2,p}_{\mathrm{loc}}$ for any $0 < \alpha < 1$ and $1 < p < \infty$.

We first consider the case when $r \to \infty$: We claim that u_{r_j} tends to u_∞ which is a global solution to the ordinary two-phase obstacle problem.

Indeed, in $\{u_\infty > 0\}$ we have $\Delta u_\infty = \lambda_1$ and in $\{u_\infty < 0\}$, $\Delta u_\infty = -\lambda_2$. Therefore it remains to determine what happens in the set $\{u_\infty = 0\}$. Now we can use the implicit function theorem to conclude that $\{u_\infty = 0\} \cap \{|\nabla u_\infty| \neq 0\}$ is locally a C^1-surface and thus it has zero measure. When $|\nabla u_\infty| = 0$ we use that $u_\infty \in W^{2,1}_{\mathrm{loc}}$ and obtain $\Delta u_\infty = 0$ a.e. in this set.

By the $C^{1,\alpha}$-convergence it follows that $u_\infty(0) = |\nabla u_\infty(0)| = 0$ and also by Proposition 2.1 the origin must be a two-phase point also for u_∞. Hence, by the classification done in [13] u_∞ is a one-dimensional solution in some direction ν and must be of form (1). Therefore, for any direction e we have $\phi(1, \partial_e u_\infty) = 0$ which together with the ACF monotonicity formula mentioned above implies that $\phi(r, \partial_e v) = 0$ for all $r > 0$ and e orthogonal to e_1. Hence $\partial_e u \geqslant 0$ (or $\leqslant 0$) for any direction e such that $e \cdot e_1 = 0$. There are $(n-1)$ such directions, and thus by calculus we can reduce the dependence of u in those directions to one. So u can only depend on two directions, say e_1 and e_2. Moreover we know that $\partial_{e_2} u \geqslant 0$.

Now we consider instead the case when $r \to 0$: We know that u_{r_j} solves

$$\Delta u_{r_j} = \lambda_1 \chi_{\{u_{r_j} > -(a/r_j)x_1\}} - \lambda_2 \chi_{\{u_{r_j} < -(a/r_j)x_1\}}.$$

In $\{u_{r_j} > -(a/r_j)x_1\}$, $\Delta u_{r_j} = \lambda_1$ and we can use standard estimates to conclude that there $u_{r_j} \in C^\infty$ uniformly. This is also true in the set $\{u_{r_j} < -(a/r_j)x_1\}$.

Now consider the remaining set, $\{u_{r_j} = -(a/r_j)x_1\}$. We know that we have quadratic growth and therefore u_{r_j} is bounded on every compact set. Thus in every compact subset of $\{x_1 > 0\}$ we have that $u_{r_j} > -a/r_j x_1$ for r_j small enough. Using the same argument in $\{x_1 < 0\}$ with $\{u_{r_j} < -(a/r_j)x_1\}$ we can therefore conclude that $u_{r_j} \to u_0$, where

$$\Delta u_0 = \lambda_1 \chi_{\{x_1 > 0\}} - \lambda_2 \chi_{\{x_1 < 0\}}.$$

We also observe that by the reasoning above we have that $u_{r_j} \to u_0$ in $C^\infty(K)$ for any compact K contained in $\{x_1 \neq 0\}$. Let h be the one-dimensional solution in the x_1-direction given by

$$h(x) = \frac{\lambda_1}{2}\left(x_1^+\right)^2 - \frac{\lambda_2}{2}\left(x_1^-\right)^2.$$

Then $\Delta(u_0 - h) = 0$. Moreover u_0 has quadratic growth at infinity, $0 = |\nabla u_0(0)| = u_0(0)$ and $0 \in \partial\{u_0 \neq 0\}$. This implies that $u_0 - h$ is a quadratic harmonic polynomial of two variables $P(x_1, x_2)$ and therefore P is of the form $C(x_1^2 - x_2^2) + Ax_1x_2$. Since we also know that $\partial_{e_2} u_0 \geqslant 0$ we must have

$$0 \leqslant \partial_{e_2} P(x_1, x_2) = -2Cx_2 + Ax_1$$

which can only be true if $C = A = 0$. Thus we can conclude that $P = P(e_1)$ so $P = 0$ since it must be quadratic and harmonic. Moreover, the fact that u_0 only depends on x_1 and that $u_{r_j} \to u_0$ in $C^2_{\mathrm{loc}}(\mathbb{R}^n \setminus \{x_1 = 0\})$ implies

$$0 = \partial_1\partial_2 u_0(x) = \lim_{r_j \to 0} \partial_1\partial_2 u(r_j x), \tag{2.3}$$

for any $x \notin \{x_1 = 0\}$. Assume now that u does really depend on x_2. By the implicit function theorem, the free boundary is near the origin at the graph of a $C^{1,\alpha}$-function. Hence we can apply the Hopf's lemma from [17]. We observe that $\partial_2 u(0) = 0$ and $\partial_2 u \geqslant 0$, moreover $\partial_2 u$ is harmonic away from the free boundary, which has normal e_1. Hopf's lemma says that

$$\liminf_{x \to 0} \partial_1\partial_2 u(x) > 0$$

which contradicts (2.3). Therefore u is one-dimensional and the following hold:

(1) $u(x_1) = \frac{\lambda_1}{2}(x_1^+)^2 - \frac{\lambda_2}{2}(x_1^-)^2$,
(2) $u(x_1) = \frac{\lambda_1}{2}(x_1^+)^2$,
(3) $u(x_1) = -\frac{\lambda_2}{2}(x_1^-)^2$.

This is a consequence of solving (2.1) in one dimension:

$$u'' = \lambda_1 \chi_{\{u > -ax_1\}} - \lambda_2 \chi_{\{u < -ax_1\}} \quad \text{in } \mathbb{R}.$$

In $\{u > -ax_1\}$ every solution is of the form $u(x_1) = \lambda_1 x_1^2/2 + Ax + B$. Using the continuity we know that u and u' tend to zero as $x_1 \to 0^+$, thus $A = B = 0$ and $u(x_1) = \lambda_1 x_1^2/2$ is the only solution. The same arguments applied in $\{u < -ax_1\}$ imply that $u(x_1) = -\lambda_2 x_1^2/2$ is the only possible solution. In the remaining set $\{u = -ax_1\}$ we only have the zero solution. The zero solution itself does not contain a free boundary point at the origin and thus, as stated above we need to combine the zero solution with at least one of the first two solutions to obtain a global solution. □

2.3. *Quadratic growth*

Now we are ready to prove a certain type of quadratic growth which validity depends on the modulus of the gradient. Before proving that we need a result that says that the gradient have to vanish on the free boundary for global solutions having one-phase points.

Lemma 2.4. *Let λ_1 and λ_2 be constants. Suppose u satisfies*

(1) $\sup_{B_\rho} |u| \leqslant C\rho^2$ *for* $\rho > 1$,
(2) $\Delta u = \lambda_1 \chi_{\{u>0\}} - \lambda_2 \chi_{\{u<0\}}$ *in* $\mathbb{R}^n$,
(3) $0 \in \Gamma^+(u) \setminus \Gamma^-(u)$,

for some constant C. Then $|\nabla u| = 0$ on $\Gamma(u)$.

Proof. If the statement of the lemma does not hold then there must be a point $y \in \Gamma(u)$ where the gradient does not vanish. Then y must be a two-phase point. Since we have a solution to the two-phase problem with constant coefficients the Alt–Caffarelli–Friedman monotonicity formula applies to $w_e^\pm = (\partial_e v_0^C)^\pm$ as in [13]. Now as in Lemma 2.2 we study the limits of the scalings

$$v_j(x) = u(r_j x)/r_j^2$$

for sequences $r_j \to \infty$. The functions v_j satisfy

(1) $\sup_{B_1} |v_j| \geqslant c$ from Proposition 2.1,
(2) $\sup_{B_\rho} |v_j| \leqslant C\rho^2$ for $\rho > 1$,
(3) $\Delta v_j = \lambda_1 \chi_{\{v_j>0\}} - \lambda_2 \chi_{\{v_j<0\}}$ in $\mathbb{R}^n$,
(4) $0 \in \Gamma^+(v_j) \setminus \Gamma^-(v_j)$.

So as before we have, passing to a subsequence if necessary, that $v_j \to v_0$ in $C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^n)$ where v_0 satisfies

(1) $\sup_{B_1} |v_0| \geqslant c$,
(2) $\sup_{B_\rho} |v_0| \leqslant C\rho^2$ for $\rho > 1$,
(3) $v_0(0) = |\nabla v_0(0)| = 0$,
(4) $\Delta v_0 = \lambda_1 \chi_{\{v_0>0\}} - \lambda_2 \chi_{\{v_0<0\}}$ in $\mathbb{R}^n$.

Also, since there is a two-phase point y, any such limit v_0 must by Proposition 2.1 have a two-phase point at the origin. Hence we can apply the classification of global solutions from [13] and conclude that it must be one-dimensional. This in turn implies that the functions $\phi(r, w_e)$ in the monotonicity formula vanishe for all r and all directions e, which as in Lemma 2.2 gives that u must be one-dimensional. If u is one-dimensional it is easy to see that ∇u vanishes on $\Gamma(u)$. □

Now we can proceed with the growth result.

Proposition 2.5. *Let $u \in P_1(M, M_0, \omega)$. For $A > 0$ there are an $\varepsilon = \varepsilon_A$ and an $r_0 = r_0(A)$ such that if* $\sup \omega < \varepsilon$ *then for any $y \in B_{1/4} \cap \partial\{u \neq 0\} \cap \{0 < |\nabla u| < Ar\}$ we have*

$$S_r(y, u) \leqslant C M_0 r^2,$$

for all $r < r_0$. Here

$$S_r(y, u) = \sup_{x \in B_r} \big| u(y + x) - \nabla u(y) \cdot x \big|$$

and C depends on M and the dimension.

Proof. We argue by contradiction. Assume that $u_j \in P_1(M, M_0, \omega)$ with $\sup \omega < \varepsilon_j \to 0$ so that the statement of the proposition fails for u_j. Then for a subsequence, again labeled u_j, we know that u_j converges to a solution u_0 of the two-phase obstacle problem with constant coefficients. For this problem there are local $C^{1,1}$-estimates (see [12] and [15]). Therefore $S_r(y, u_0) \leqslant C_1(\lambda_i(0)) M_0 r^2$ for any $y \in B_{\frac{1}{4}}$ and any $r \leqslant 1/4$. This implies

$$S_r(y, u_j) \leqslant C_1 M_0 r^2 + \tau(j). \tag{2.4}$$

In particular, since $\tau \to 0$ as $j \to \infty$ we have

$$S_r(y, u_j) \leqslant 2C_1 M_0 r^2$$

for j large enough at least when $r = 1/4$. For any sequence $y_j \in B_{\frac{1}{4}} \cap \partial\{u_j \neq 0\}$ and for a constant $C > C_1$ to be chosen later define

$$r_j = \sup\big\{r:\ S_r(y_j, u_j) > 2C M_0 r^2\big\}.$$

If the assertion fails for C then there is a convergent sequence $y_j \to y_0$ such that $r_j = r_j(y_j) \to 0$, since we otherwise would have a contradiction to (2.4). Moreover, we know that $r_j \leqslant 1/4$. Let

$$v_j(x) = \frac{u_j(r_j x + y_j) - \nabla u_j(y_j) \cdot r_j x}{r_j^2}.$$

Then

(1) $\sup_{B_1} |v_j| = 2C M_0$,
(2) $\sup_{B_\rho} |v_j| \leqslant 2C M_0 \rho^2$ for $\rho > 1$,

(3) $v_j(0) = |\nabla v_j(0)| = 0$,
(4) $\Delta v_j = \lambda_1(r_j x + y_j)\chi_{\{v_j > -\nabla u(y_j)\cdot x/r_j\}} - \lambda_2(r_j x + y_j)\chi_{\{v_j < -\nabla u(y_j)\cdot x/r_j\}}$ in B_{1/r_j}.

Note that $|\nabla u_j(y_j)|/r_j < A$, which means that for a subsequence $\nabla u_j(y_j)/r_j \to \nu$ where $|\nu| \leqslant A$. We consider two cases: $\nu = 0$ or $\nu \neq 0$.

Case 1. $\nu \neq 0$. Since $|\Delta v_j|$ is uniformly bounded on B_{1/r_j} this implies that for a subsequence $v_j \to v_0$ in $C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^n) \cap W^{2,p}_{\mathrm{loc}}(\mathbb{R}^n)$ for some function v_0 which satisfies

(1) $\sup_{B_1} |v_0| = 2CM_0$,
(2) $\sup_{B_\rho} |v_0| \leqslant 2CM_0\rho^2$ for $\rho > 1$,
(3) $v_0(0) = |\nabla v_0(0)| = 0$,
(4) $\Delta v_0 = \lambda_1(y_0)\chi_{\{v_0 > -\nu\cdot x\}} - \lambda_2(y_0)\chi_{\{v_0 < -\nu\cdot x\}}$ in $\mathbb{R}^n$.

By Lemma 2.2

$$\sup_{B_1} |v_1| \leqslant \frac{1}{2}\max\bigl(\lambda_1(y_0), \lambda_2(y_0)\bigr),$$

which will contradict (1) if $4CM_0 > \max(\lambda_1(y_0), \lambda_2(y_0))$.

The second case needs to be split into two subcases: 1) In the limit v_0 has a two-phase point at the origin or 2) in the limit v_0 has a one-phase point at the origin.

Case 2a. $\nu = 0$ and in the limit we have a two-phase point at the origin. Since $|\Delta v_j|$ is uniformly bounded on B_{1/r_j} this implies that for a subsequence $v_j \to v_0$ in $C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^n)$ for some function v_0 which satisfies

(1) $\sup_{B_1} |v_0| = 2CM_0$,
(2) $\sup_{B_\rho} |v_0| \leqslant 2CM_0\rho^2$ for $\rho > 1$,
(3) $v_0(0) = |\nabla v_0(0)| = 0$,
(4) $\Delta v_0 = \lambda_1(y_0)\chi_{\{v_0 > 0\}} - \lambda_2(y_0)\chi_{\{v_0 < 0\}}$ in $\mathbb{R}^n$,
(5) $0 \in \Gamma^+(v_0) \cap \Gamma^-(v_0)$.

Now we can apply the classification of global solutions done in [13] and obtain

$$\sup_{B_1} |v_0| \leqslant \frac{1}{2}\max\bigl(\lambda_1(y_0), \lambda_2(y_0)\bigr),$$

which will contradict (1) if $4CM_0 > \max(\lambda_1(y_0), \lambda_2(y_0))$.

Case 2b. $\nu = 0$ and in the limit we have a one-phase point at the origin. We treat the case when this is a positive one-phase point. As before we have $v_j \to v_0$ in $C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^n)$ for some function v_0 which now will satisfy

(1) $\sup_{B_1} |v_0| = 2CM_0$,
(2) $\sup_{B_\rho} |v_0| \leqslant 2CM_0\rho^2$ for $\rho > 1$,
(3) $v_0(0) = |\nabla v_0(0)| = 0$,
(4) $\Delta v_0 = \lambda_1(y_0)\chi_{\{v_0 > 0\}} - \lambda_2(y_0)\chi_{\{v_0 < 0\}}$ in $\mathbb{R}^n$,
(5) $0 \in \Gamma^+(v_0) \setminus \Gamma^-(v_0)$.

By Lemma 2.4 this implies that ∇v_0 vanishes on $\Gamma(v_0)$. Hence $v_0^{\pm}$ are solutions to the one-phase problem, i.e. to the obstacle problem, which in turn implies that we have

$$\Delta v_0^+ = \lambda_1(y_0)\chi_{\{v_0^+ > 0\}} \quad \text{in } \mathbb{R}^n,$$

and

$$\Delta v_0^- = \lambda_2(y_0)\chi_{\{v_0^- > 0\}} \quad \text{in } \mathbb{R}^n.$$

Moreover, we know from (5) that $0 \in \Gamma(v_0)$. Then Lemma 2.2 from [3] applies when $|v_0|$ is considered to be a solution in B_2. This gives

$$\sup_{B_1} |v_0| \leqslant C'\big(\lambda_1(y_0) + \lambda_2(y_0)\big),$$

where C' depends only on the dimension. From (1) we know

$$\sup_{B_1} |v_0| = 2CM_0.$$

This is contradiction if C is chosen big enough. □

Proposition 2.5 only gives estimates for points outside a special neighborhood of the free boundary. In order to control the growth around free boundary where the gradient does not vanish, we need the following proposition.

Proposition 2.6. *Let $u \in P_1(M, M_0, \omega)$ where $\omega(r) \leqslant Mr^\alpha$. Then there are constants t_0 and C depending on M, M_0 and the dimension such that*

$$\|u\|_{C^{1,1}(B_{t_0|\nabla u(y)|}(y))} \leqslant C$$

for all $y \in \Gamma(u) \cap \{0 < |\nabla u(y)| \leqslant r_0\}$, where $r_0 = r_0(1)$ is from Proposition 2.5.

Proof. Take $y \in \Gamma(u) \cap \{|\nabla u| > 0\}$ and define $r_y = |\nabla u(y)|$. Let

$$v(x) = \frac{u(r_y x + y)}{r_y^2}.$$

Then Proposition 2.5 along with simple calculations give

(1) $\Delta v(x) = \lambda_1(r_y x + y)\chi_{\{v>0\}} - \lambda_2(r_y x + y)\chi_{\{v<0\}}$,
(2) $\sup_{B_1} |v| \leqslant CM_0$ for $r_y < r_0$,
(3) $|\nabla v(0)| = 1$,
(4) $0 \in \Gamma^+(v) \cap \Gamma^-(v)$.

By Proposition A.1 in Appendix A this implies that $\|v\|_{C^{1,1}(B_{t_0})} \leqslant C(M, M_0)$, which after rescaling implies the desired result. □

2.4. $C^{1,1}$-estimates

The result in the previous section actually implies local $C^{1,1}$-estimates under the assumption that the functions λ_i are $C^{1,1}$-potentials, that is, there are functions $f_i \in C^{1,1}$ such that $\lambda_i = \Delta f_i$ which is of course the case if the λ_i's are supposed to be C^α. First we need the following lemma.

Lemma 2.7. *Suppose that $\Delta v = \Delta \psi$ in B_{2R} and $\psi \in C^{1,1}(B_{2R})$, then*

$$\big\|D^2 v\big\|_{L^\infty(B_R)} \leqslant C\bigg(\frac{\|v\|_{L^\infty(B_{2R})}}{R^2} + \big\|D^2\psi\big\|_{L^\infty(B_{2R})}\bigg),$$

where C depends only on the dimension.

Proof. Put $w = v - \psi - x \cdot \nabla\psi(0) - \psi(0)$. Then w is harmonic in B_{2R} and from classical estimates for harmonic functions we have

$$\sup_{B_R}\left|D^2 w\right| \leqslant C\frac{\sup_{B_{2R}}|w|}{R^2} \leqslant C\left(\frac{\sup_{B_{2R}}|v|}{R^2} + 4\left\|D^2\psi\right\|_{L^\infty(B_{2R})}\right).$$

But $v = w + \psi$ and thus $|D^2 v| \leqslant |D^2 w| + |D^2\psi|$ which leads to

$$\left\|D^2 v\right\|_{L^\infty(B_R)} \leqslant \left\|D^2 w\right\|_{L^\infty(B_R)} + \left\|D^2\psi\right\|_{L^\infty(B_R)} \leqslant C\left(\frac{\|w\|_{L^\infty(B_{2R})}}{R^2} + \left\|D^2\psi\right\|_{L^\infty(B_{2R})}\right). \qquad \square$$

Proposition 2.8. *Let $u \in P_1(M, M_0, \omega)$ with $\omega(r) \leqslant Mr^\alpha$. Then there are constants ε, r_1 such that* $\sup\omega < \varepsilon$ *implies $u \in C^{1,1}(B_{r_1/2})$ with*

$$\|u\|_{C^{1,1}(B_{r_1/2})} \leqslant C,$$

where C depends on M, M_0 and the dimension.

Proof. Let ε be so small that Proposition 2.5 holds with $r_0 = r_0(1)$. Take $x_0 \in B_{r_1/2}$ with $r_1 < r_0$ so small that $|\nabla u| < r_0$ in B_{r_1}, and let $d = \operatorname{dist}(x_0, \Gamma(u))$. Take $y \in \partial B_d(x_0) \cap \Gamma(u)$. Let $v(x) = u(x) - \nabla u(y) \cdot x$. Take t_0 as in Proposition 2.6 and apply Proposition 2.5 to obtain

$$\sup_{B_{2d/t_0}(y)} |v| \leqslant CM_0 4d^2/t_0^2,$$

whenever $|\nabla u(y)| \leqslant 2d/t_0$ and thus also

$$\sup_{B_d(y)} |v| \leqslant CM_0 4d^2/t_0 = C' M_0 d^2 \tag{2.5}$$

for $C' = 4C/t_0^2$ whenever $|\nabla u(y)| \leqslant 2d/t_0$. Now if $|\nabla u(y)| > 2d/t_0$ then since $|\nabla(y)| < r_0$, Proposition 2.6 implies that $u \in C^{1,1}(B_{2d}(y))$ and therefore we also have (2.5) in this case.

Since Δv equals either λ_1 or λ_2 in $B_d(x)$ we can use Lemma 2.7 to obtain

$$\left\|D^2 v\right\|_{L^\infty(B_{d/2}(x_0))} \leqslant C\left(\frac{\|v\|_{L^\infty(B_d(x_0))}}{d^2} + M\right).$$

Using (2.5) and observing that $D^2 v = D^2 u$ we get

$$\left\|D^2 u\right\|_{L^\infty(B_{d/2}(x_0))} \leqslant C(C' M_0 + M). \qquad \square$$

Rescaling this result we obtain the correct regularity without the restriction on the oscillation and we can then prove Theorem 1.2.

Proof of Theorem 1.2. Take ρ so small that

$$\sup_{s<\rho} \omega(s) < \varepsilon,$$

where ε is taken so that Proposition 2.8 is valid. Define

$$v(x) = \frac{u(\rho x)}{\rho^2}.$$

Then $v \in P_1(M, M_0/\rho^2, \omega(\rho\cdot))$. Therefore we can apply Proposition 2.8 to get

$$\|v\|_{C^{1,1}(B_{r_1/2})} \leqslant C/\rho^2,$$

which is the same as

$$\|u\|_{C^{1,1}(B_{r_1\rho/2})} \leqslant C/\rho^2.$$

Obviously ρ does only depend on the modulus of continuity of the ω, the dimension, M and M_0. $\square$

Remark 2.9. We wish to mention here that the authors strongly believe that Theorem 1.2 is true under the weaker assumption that ω satisfies the Dini condition. The problem here is the proof of Proposition 2.6 which uses Proposition A.1. There we invoke Theorem 9.3 in [2] which in turn requires C^α-regularity of the coefficients. Probably this theorem is possible to extend to the case where the coefficients are only Dini continuous, which then would imply that Theorem 1.2 is true under the weaker assumption.

3. Partial regularity of the free boundary

In this section we prove that the free boundary is the union of two Reifenberg vanishing sets near so-called branching points. We use arguments similar to those in [3]. First we start out with some comparison results which allows us to estimate distances between different free boundaries.

3.1. Stability results

Proposition 3.1. *Let $u, v \in P_1(M, M_0, \omega)$ with coefficients λ_i respectively γ_i and $u = v$ on ∂B_1. Assume moreover that $\lambda_1 \leqslant \gamma_1$ and $\lambda_2 \geqslant \gamma_2$. Then $u \geqslant v$ in B_1.*

Proof. We compare Δu and Δv in the set $\{u < v\}$.

(1) When $v < 0$ then also $u < 0$. Therefore $\Delta u = -\lambda_2 \leqslant -\gamma_2 = \Delta v$.
(2) When $v > 0$ then $\Delta v = \gamma_1 \geqslant \lambda_1 = \max(0, \lambda_1, -\lambda_2) \geqslant \Delta u$.
(3) When $v = 0$ then $u < 0$ and so $\Delta v = 0 \geqslant -\lambda_2 = \Delta u$.

Hence $\Delta u \leqslant \Delta v$ in $\{u < v\} \cap B_1$ and therefore $\{u < v\} = \emptyset$. □

Proposition 3.2. *Let u and u_ε be the solutions of*

$$\Delta u = \lambda_1 \chi_{\{u>0\}} - \lambda_2 \chi_{\{u<0\}} \quad \text{in } B_1,$$

and

$$\Delta u_\varepsilon = (\lambda_1 + \varepsilon)\chi_{\{u_\varepsilon>0\}} - (\lambda_2 - \varepsilon)\chi_{\{u_\varepsilon<0\}} \quad \text{in } B_1,$$

with $u = u_\varepsilon = g$ on ∂B_1 where $\Gamma^\pm(u) \cap B_1$ and $\Gamma^\pm(u_\varepsilon) \cap B_1$ are all C^1-graphs. Then

$$\operatorname{dist}\big(\Gamma^\pm(u) \cap B_1, \Gamma^\pm(u_\varepsilon) \cap B_1\big) \leqslant C\sqrt{\varepsilon},$$

where C depends on the C^1-norms of $\Gamma^\pm(u) \cap B_1$ and $\Gamma^\pm(u_\varepsilon) \cap B_1$.

Proof. We treat the case $\varepsilon > 0$ and for $\Gamma^+(u)$ and $\Gamma^+(u_\varepsilon)$, the other cases can be treated similarly.

By Proposition 3.1 we have $u_\varepsilon \leqslant u$. Now we claim that $u + \varepsilon v \leqslant u_\varepsilon$ in B_1 where v is the solution to $\Delta v = 1$ in B_1 with zero boundary data on ∂B_1. Assume that $g(x_0) = u(x_0) + \varepsilon v(x_0) - u_\varepsilon(x_0) > 0$ for some $x_0 \in B_1$. Then we have the three different cases:

(1) $u(x_0) > 0$ then $\Delta g(x_0) = \lambda_1 + \varepsilon - \Delta u_\varepsilon(x_0) \geqslant 0$.
(2) $u(x_0) = 0$ then we must have $u_\varepsilon(x_0) < 0$ and so $\Delta g(x_0) = \lambda_2$.
(3) $u(x_0) < 0$ then again $u_\varepsilon(x_0) < 0$ and so $\Delta g(x_0) = -\lambda_2 + \varepsilon + \lambda_2 - \varepsilon = 0$.

So a maximum must be attained at $x_0 \in \partial B_1$ but there $g = 0$. Hence $g \leqslant 0$. Since

$$v(x) = \varepsilon \frac{|x|^2 - 1}{2n}$$

this implies that $u_\varepsilon \geqslant -C\varepsilon + u$. Also, due to the fact that $\Gamma^+(u)$ is a C^1-graph we know that $u(x) \geqslant C'(\operatorname{dist}(x, \Gamma^+(u)))^2$ for $x \in \{u > 0\}$ which gives $u_\varepsilon(x) \geqslant -C\varepsilon + C' \operatorname{dist}(x, \Gamma^+(u))^2$ for $x \in \{u > 0\}$ and thus u_ε is positive if $\operatorname{dist}(x, \Gamma^+(u)) \geqslant C''\sqrt{\varepsilon}$. Obviously $u_\varepsilon(x) \leqslant 0$ for any x such that $u(x) \leqslant 0$. This yields the desired result. □

Remark 3.3. The result corresponding to Theorem 5.4 in [3] we cannot match. There one obtains linear stability but we can only prove stability of order $\sqrt{\varepsilon}$.

3.2. Main result

Definition 3.4 *(Reifenberg-flatness).* A compact set S in $\mathbb{R}^n$ is said to be δ-Reifenberg flat if for any compact set $K \subset \mathbb{R}^n$ there exists an $R_K > 0$ such that for every $x \in K \cap S$ and every $r \in (0, R_K]$ we have a hyperplane $L(x,r)$ such that

$$\operatorname{dist}\big(L(x,r) \cap B_r(x), S \cap B_r(x)\big) \leqslant 2r\delta.$$

We define the modulus of flatness as

$$\theta_K(r) = \sup_{0<\rho\leqslant r}\left(\sup_{x\in S\cap K}\frac{\operatorname{dist}(L(x,\rho)\cap B_\rho(x), S\cap B_\rho(x))}{\rho}\right).$$

A set is called Reifenberg vanishing if

$$\lim_{r\to 0}\theta_K(r) = 0.$$

Then we have the following result:

Proposition 3.5. *There is a* $\rho > 0$ *such that*

$$\big(\Gamma^+(u) \cup \Gamma^-(u)\big) \cap \big\{\big|\nabla u(y)\big| < \sigma\rho/2\big\} \cap \big\{\max\big(\operatorname{dist}\big(y, \Gamma^+(u)\big), \big(y, \Gamma^-(u)\big)\big) < \sigma\rho/2\big\}$$

are Reifenberg vanishing. Here σ *is taken as in Theorem* 1.1 *in* [14].

Proof. Take a sequence of points $y_k \to y_0$ such that $|\nabla u(y_k)| \leqslant \sigma r_k/2$ and $\operatorname{dist}(y_k, \Gamma^{\pm}(u)) < \sigma r_k/2$ with $r_k \to 0$. Consider the functions

$$v_k(x) = \frac{u(r_k x + y_k)}{r_k^2}.$$

Then from Lemma 2.2 and Proposition 2.5 we know that v_k is bounded, and therefore for a subsequence it converges to a limit in $C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^n)$.

Now let $f_k^1 = \sup_{B_1}\lambda_1(r_k x + y_k)$, $f_k^2 = \inf_{B_1}\lambda_2(r_k x + y_k)$, $g_k^1 = \inf_{B_1}\lambda_1(r_k x + y_k)$ and $g_k^2 = \sup_{B_1}\lambda_2(r_k x + y_k)$. Take h_k to solve

$$\Delta h_k = f_k^1\chi_{\{h_k>0\}} - f_k^2\chi_{\{h_k<0\}} \quad \text{in } B_1,$$

and l_k to solve

$$\Delta l_k = g_k^1\chi_{\{l_k>0\}} - g_k^2\chi_{\{l_k<0\}} \quad \text{in } B_1,$$

where $l_k = h_k = v_k$ on ∂B_1. Then $|f_k^i - g_k^i| \leqslant 2\omega(r_k)$ and by Proposition 3.1 $h_k \geqslant v_k \geqslant l_k$. Moreover, since $h_k - v_k \to 0$ in $C^{1,\alpha}_{\mathrm{loc}}(B_1)$, $|\nabla v_k(0)| \leqslant \sigma/2$ and $\operatorname{dist}(0, \Gamma^{\pm}(v_k)) \leqslant \sigma/2$ we have for k large enough say $k > K_0$, that $|\nabla h_k(0)| \leqslant \sigma$ and $\operatorname{dist}(0, \Gamma^{\pm}(h_k)) \leqslant \sigma$. Now we can use Theorem 1.1 in [14] to say that $\Gamma^{\pm}(h_k)$ is uniformly C^1 for $k > K_0$ in B_{r_0}. The same reasoning applies to l_k as well. This means that there is a plane P_k such that $\operatorname{dist}(\Gamma^{\pm}(h_k), P_k) \leqslant \rho(r_k)r_k$ in B_{r_0} and the corresponding for l_k. Here ρ depends on the C^1-norm of $\Gamma^{\pm}(l_k)$ and $\Gamma^{\pm}(h_k)$ respectively. Moreover, by Proposition 3.2 we have

$$\operatorname{dist}\big(\Gamma^{\pm}(h_k) \cap B_{r_0}, \Gamma^{\pm}(l_k) \cap B_{r_0}\big) \leqslant C\sqrt{\omega(r_k)}.$$

Therefore by rescaling we obtain that $\operatorname{dist}(\Gamma^{\pm}(u), P_k) \leqslant C\max(\sqrt{\omega(r_k)}, \rho(r_k))$ in $B_{r_0 r_k}(y_k)$ for $r_k < r_{K_0}$ which implies the result. □

Now we can prove Theorem 1.4.

Proof of Theorem 1.4. That $\Gamma^{\pm}$ are Reifenberg vanishing follows immediately from Proposition 3.5. The second part is then implied by Reifenbergs results in [11] and the related comments on page 386 in [7]. □

Remark 3.6. The reader might ask what is the weakest condition possible on the coefficients λ_i to get the local C^1-regularity of the free boundary. In the case of the obstacle problem (one-phase case) the optimal assumption is that the coefficient is Dini continuous. In Theorem 7.3 in [3] Blank construct examples where the free boundary is not C^1 when the coefficient has any modulus of continuity which is not Dini.

In our case, the authors guess that Dini continuity should be the optimal condition. However, the methods employed above cannot be used in order to get C^1-regularity. The reason for that is that for the two-phase membrane with constant coefficients one do not in general have $C^{1,\mathrm{dini}}$-regularity (see [13] for an example), and that is exactly what should need in order to use Theorem 6.7 in [3] which is the key to get C^1-regularity. Of course, it might be possible to obtain this result using other methods.

Also, one can construct examples where the free boundary is not C^1 when the coefficients have any modulus of continuity which is not Dini. This is done taking the example described in Remark 1.10 in [4], which is based on ideas from [3], and making an odd reflection with respect to the x_n coordinate. Then for any modulus of continuity ω which is not Dini, we obtain a solution $u \in P_1(M, M_0, \omega)$ where the free boundary is not locally a C^1-graph at the origin, even though $\Gamma^+(u)$ and $\Gamma^-(u)$ will touch tangentially there.

4. Comments and remarks

It is worthwile to mention that the results of this paper (at least Theorem 1.2) could be obtained for a more general class of elliptic operators with similar methods.

The important thing here is that the operator becomes the Laplace operator or at least something similar after the blow-ups done in Section 2.3. For more general linear operators this is of course true. The more intricate case is fully non-linear operators F. Then one could consider the perturbed operators $F_t = (1-t)\Delta + tF$. For t small enough, the methods of this paper would give the desired result. In order to prove the result for general fully non-linear operators one would have to iterate this in some way, which indeed would need some new methods since we strongly use the classification of global solutions which is not done for fully non-linear operators. This in an open problem that the authors hope to be able to treat in the future.

Appendix A

Here we present the result needed in the proof of Proposition 2.6.

Proposition A.1. *Assume that v has the following properties*:

(1) $\Delta v = f_1\chi_{\{v>0\}} - f_2\chi_{\{v<0\}}$ *in* B_1,
(2) $\sup_{B_1}|v| \leqslant C$,
(3) $|\nabla v(0)| = 1$,
(4) $0 \in \Gamma(v)$,

where $\|f_i\|_{C^\alpha} \leqslant C$ *for some* $0<\alpha<1$. *Then there is a small ball* B_{t_0} *such that*

$$\|v\|_{C^{1,1}(B_{t_0})} \leqslant C'.$$

Here C' *and* t_0 *depend on the constant* C *and the dimension.*

Proof. We note that Δv is bounded in B_1, and therefore for any $\alpha<1$ we have

$$\|v\|_{C^{1,\alpha}(B_{1/2})} \leqslant C'.$$

This implies that $|\nabla v| > 1/2$ is a small ball B_{t_0}, and therefore by the implicit function theorem, $\{v=0\}\cap B_{t_0}$ is a $C^{1,\alpha}$-surface. In what follows we will apply the methods from Section 3.1 in [9]. We can without any loss of generality assume that $\nabla v(0)$ points in the x_n-direction.

We define a new system of coordinates as $y_k = x_k$ for $k < n$ and $y_n = v(x)$. Also we introduce the functions $\psi(y) = x_n$ and $\phi(y) = \psi(y_1, \ldots, y_{n-1}, -y_n)$. This is then a one-to-one transformation of B_{t_0} onto some open set U in the new coordinates. Making the necessary computations in the coordinate transformations we obtain the two following equations for ψ and ϕ:

$$\begin{cases} -\dfrac{1}{\psi_n}\sum \psi_{kk} + \dfrac{1}{\psi_n^2}\sum \psi_k\psi_{kn} - \dfrac{1}{\psi_n^3}\Big(1+\sum \psi_k^2\Big)\psi_{nn} = f_1 & \text{in } \{y_n > 0\} \cap U, \\ -\dfrac{1}{\phi_n}\sum \phi_{kk} + \dfrac{1}{\phi_n^2}\sum \phi_k\phi_{kn} - \dfrac{1}{\phi_n^3}\Big(1+\sum \phi_k^2\Big)\phi_{nn} = f_2 & \text{in } \{y_n > 0\} \cap U. \end{cases}$$

We also have boundary conditions on $S = \{y_n = 0\} \cap U$:

$$\begin{cases} \phi = \psi, \\ \phi_n = -\psi_n. \end{cases}$$

This is a system with C^α-coefficients. In order to apply Theorem 9.3 from [2] we need thus to check that this system is coercive. This can be done in the exact same way as it is done on page 204 in [10]. Then Theorem 9.3 applied with $s_i = 0, r_1 = -2, r_2 = -1, l = l_0 = 0$ and $t_j = 2$ implies that

$$\|\psi\|_{C^{2,\alpha}(U\cap\{y_n>0\}\cup S)}, \|\phi\|_{C^{2,\alpha}(U\cap\{y_n>0\}\cup S)} \leqslant C$$

and therefore going back to our original coordinates

$$\|u\|_{C^{1,1}(B_{t_0})} \leqslant C. \qquad \square$$

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