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A note on constant geodesic curvature curves on surfaces

Taoniu Sun [1,2]

University of Science and Technology of China, China and University Paris-Est, France

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Abstract

In this paper we are concerned with the structure of curves on surfaces whose geodesic curvature is a large constant. We first discuss the relation between closed curves with large constant geodesic curvature and the critical points of Gauss curvature. Then, we consider the case where a curve with large constant geodesic curvature is immersed in a domain which does not contain any critical point of the Gauss curvature.
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Résumé

Dans cet article nous nous intéressons au comportement des courbes sur une surface, dont la courbure géodésique est une constante très grande. Dans un premier temps nous nous intéressons aux relations entre les courbes fermées dont la courbure géodésique est grande et les points critiques de la courbure de Gauss. Ensuite, nous nous intéressons au comportement asymptotique des courbes immergées dans un domaine de la surface qui ne contient aucun point critique de la courbure de Gauss.
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1. Introduction

Suppose (M^{n+1}, g) is a Riemannian manifold. We are interested in the structure of embedded spheres $S^n \hookrightarrow M^{n+1}$ that have constant mean curvature. In the case where s, the scalar curvature function of (M^{n+1}, g), has a non-degenerate critical point p, R. Ye has constructed constant mean curvature embedded spheres with high mean curvature which in fact form a local foliation of a neighborhood of p in [9]. When the manifold is compact, F. Pacard and X. Xu have recently generalized this result relaxing the non-degeneracy assumption but loosing some control on the fact that the embedded spheres form a foliation in [3]. All these results point out the crucial role played by the critical points of the scalar curvature in the existence of embedded spheres with large enough mean curvature. More precisely, it is natural to ask the converse question:

E-mail address: tnsun@mail.ustc.edu.cn.

[1] Present address: Department of Mathematics, University of Science and Technology of China, Hefei, Anhui 230026, PR China.

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Question 1. Assume that $p \in M$ is fixed such that there exists a sequence of constant mean curvature hyper-surfaces H_i, $i \in \mathbb{N}$, with mean curvature $m_i \to +\infty$ which converges (for the Hausdorff distance) to the point p. Is it true that p is a critical point of the scalar curvature function?

Recall that the Hausdorff distance between two sets A and B is defined to be

$$D_H(A, B) := \inf_{r>0}\{r > 0\colon\ A \subset T_r(B) \text{ and } B \subset T_r(A)\},$$

where

$$T_r(X) := \{p \in M\colon\ \mathrm{dist}(p, X) < r\}.$$

The result of O. Druet [2] tells us that the answer to this question is positive under the additional assumption that the constant mean curvature hyper-surfaces are solutions of the isoperimetric problem. In this note we give a positive answer to this question in the simplest case, that is when (M, g) is a 2-dimensional Riemannian manifold. In this case, constant mean curvature hypersurfaces are nothing but constant geodesic curvature curves and the scalar curvature function is nothing but the Gauss curvature.

Let (M, g) be an oriented 2-dimensional Riemannian manifold. Even though most of our results have straightforward generalization to the noncompact complete setting, we will always assume that M is compact to simplify the statements. We first recall the results of R. Ye in our setting.

Theorem 1. *(See [9].) Assume that p is a non-degenerate critical point of the Gauss curvature function K. Then, for all k large enough, say $k \geqslant k_*$, the geodesic circle of radius $1/k$ centered at p can be perturbed into Γ_k, a constant $(= k)$ geodesic curvature embedded curve. More precisely, Γ_k is a normal graph for some function w_k over the geodesic circle of radius $1/k$ centered at a point p_k where*

$$\|w_k\|_{\mathcal{C}^2} \leqslant ck^{-3} \quad \textit{and} \quad \mathrm{dist}(p_k, p) \leqslant ck^{-2}$$

for some constant $c > 0$ which does not depend on k. Moreover, the curves Γ_k form a local foliation of a neighborhood of p.

We also have the following general property for curves on 2-dimensional Riemannian manifolds.

Theorem 2. *(See [6].) Assume that Γ is a closed embedded curve in M with constant geodesic curvature k satisfying*

$$k^2 > -\min_M K,$$

then, $M \setminus \Gamma$ has two disjoint connected components.

This result was obtained H. Rosenberg [6] for constant mean curvature surfaces in 3-manifolds but his proof extends to any dimension. A similar argument was used in [5] for constant mean curvature surfaces in flat 3-manifolds. For the sake of completeness, we give here a short proof of the result in the case of curves on surfaces.

If Γ is a closed embedded curve in M which is the boundary of a compact domain Ω, the Gauss Bonnet theorem implies that

$$\int_\Gamma k = 2\pi - \int_\Omega K\,\mathrm{dvol}_g$$

and hence if the geodesic curvature k is bounded from below by some positive constant, we conclude that the length of Γ is bounded from above. more precisely, we have

$$|\Gamma| \min |k| \leqslant 2\pi - \int_M \min(0, K)\,\mathrm{dvol}_g\,.$$

Hence if $\min |k|$ is large enough, $|\Gamma|$ will be smaller than the injectivity radius of M, then Γ is topologically trivial.

Our main result gives, in dimension 2, a positive answer to the question raised above.

Theorem 3. *Assume that $p \in M$ is fixed and assume that there exists a sequence of embedded closed curves Γ_i, $i \in \mathbb{N}$, with constant geodesic curvature $k_i \to +\infty$, which converges (for the Hausdorff distance) to the point p. Then p is a critical point of the Gauss curvature function.*

If in addition the Gauss curvature function K is a Morse function, we have:

Theorem 4. *Assume that the Gauss curvature function K is a Morse function. There exists $c > 0$ (only depending on M and g) such that if Γ is a closed embedded curve in M with constant geodesic curvature $k > c$, then there exists p, a critical point of the Gauss curvature function, such that the Hausdorff distance between Γ and the geodesic circle of radius $1/k$ centered at p is bounded by a c/k^2.*

The result of Theorem 2 together with the last result just says that, provided $k > 0$ is large enough, Γ separates M into two different connected components, one of which is close to a geodesic disc centered at a critical point of the Gauss curvature function. Unfortunately, we do not have an expression of the constant c which appears in this result.

Next we turn our attention to nonembedded curves. Since the equation which ensures that the geodesic curvature of a curve is a second order ordinary differential equation and since we are on a compact manifold, immersions of $\mathbb{R}$ in M as a constant geodesic curvature curve (parameterized by arc length) exist in abundance. In fact once an initial point p and an initial (unit) speed $v \in T_pM$ have been chosen there exists a unique curve $\Gamma(p, v)$ passing through p with speed v. We study the behavior of these curves as their geodesic curvature tends to ∞. To be more precise, this curve is parameterized by $\gamma = \gamma(p, v, k)$ such that

$$\gamma(0) = p \quad \text{and} \quad \partial_s\gamma(0) = v.$$

We choose $r > 0$ smaller than the injectivity radius of the underlying manifold and define $I = I(p, v, k) \subset \mathbb{R}$ to be the largest interval containing 0 whose image $\tilde{\Gamma}(p, v, k)$ by γ is included in $\bar{B}_r(p)$. With these definitions, we have the following:

Theorem 5. *Assume that $\bar{B}_r(p) \cap \{q\colon K(q) = K(p)\}$ does not contain any critical point of K. Then as k tends to ∞, the sequence of constant geodesic curvature curves $\tilde{\Gamma}(p, v, k)$ converges in Hausdorff distance to the connected component of $\bar{B}_r(p) \cap \{q\colon K(q) = K(p)\}$ which passes through p.*

Roughly speaking, as k tends to ∞, the curve $\tilde{\Gamma}(p, v, k)$ looks like the trajectory of a particle circling (at unit speed) at distance $1/k$ around a center which travels along the level curve of the function K passing through p at speed $\|dK\|_g k^{-3}/8$.

The above analysis leaves the possibility of having an immersed constant geodesic curvature curve circling around a critical point of the Gauss curvature function. To shed light over what is going on in this case, we restrict our attention to curves which are immersed in a simply connected domain of M for which it makes sense to define the degree of the curve. First of all, it is not surprising that the result of Theorem 4 holds, namely:

Theorem 6. *Let Ω be a simply connected domain in M over which K has only non-degenerate critical points and let $d \in \mathbb{N}$ be fixed. There exists $c > 0$ such that, if Γ is a closed curve of degree d immersed in Ω with constant geodesic curvature $k > 0$, then there exists p, a critical point of the Gauss curvature function, such that the Hausdorff distance between Γ and the geodesic circle of radius $1/k$ centered at p is bounded by a c/k^2.*

More surprising is the following result we obtain.

Theorem 7. *Let Ω be a simply connected domain in M over which K has only non-degenerate critical points and let $d \in \mathbb{N}$ be fixed. There exists $k_* > 0$ such that, if Γ is a closed curve of degree d immersed in Ω with constant geodesic curvature $k > k_*$, then Γ is a d-cover of an embedded constant geodesic curvature curve.*

Now we outline the organization of this note briefly. In the beginning, we do some fundamental calculations about the metric and the geodesic curvature of the curves, these build Section 2. Then, we will prove Theorems 3 and 4 in Section 3 while Appendix A is devoted to Theorem 2. We study the immersed curves with large constant geodesic curvature in Section 4, the main result there is Theorem 5. The closed constant geodesic curvature curves immersed in a simple connected domain with degree d are considered in Section 5, where the main aim is to prove Theorem 7.

2. The geodesic curvature

We first give the expansion of the metric in polar geodesic coordinates. Next, we recall the expression of the geodesic curvature.

Given $p \in M$, we choose $\{e_1, e_2\}$ an orthonormal basis of T_pM. To parameterize a neighborhood of p, we use either geodesic normal coordinates $(x_1, x_2) \in \mathbb{R}^2$ via the exponential map

$$\Phi(x_1, x_2) := \mathrm{Exp}_p(x_1 e_1 + x_2 e_2),$$

or polar coordinates $(r, \theta) \in [0, \infty) \times S^1$ via

$$\Psi(r, \theta) := \mathrm{Exp}_p\big(r(\cos\theta e_1 + \sin\theta e_2)\big).$$

It will be convenient to define

$$\Theta(\theta) := \cos\theta e_1 + \sin\theta e_2 \in T_pM.$$

Gauss's Lemma implies that, in polar geodesic coordinates, the metric g can be written as

$$\Psi^* g = dr^2 + f^2(r, \theta)\, d\theta^2.$$

Recall [1] the expression of K, the Gauss curvature, at the point of coordinates (r, θ) is given in terms of f by

$$K \circ \Psi = -\frac{\partial_r^2 f}{f}. \tag{1}$$

We now recall the Taylor expansion of the function f in powers of r.

Proposition 8. *(See [7,8].) The following expansion holds*

$$f(r, \theta) = r - \frac{1}{6}K(p)r^3 - \frac{1}{12}\nabla_\Theta K(p)r^4 + \frac{1}{120}\big(K(p)^2 - 3\nabla_\Theta^2 K(p)\big)r^5 + \mathcal{O}_p\big(r^6\big), \tag{2}$$

where the subscript p in $\mathcal{O}_p(r^6)$ is meant to remind the reader that this is a function of p.

Proof. By definition of geodesic coordinates, we have $f(0, \theta) = 0$, and $\partial_r f(0, \theta) = 1$. Also the formula of the Gauss curvature tells us that

$$\partial_r^2 f = -Kf,$$

where we write for short K instead of $K \circ \Psi$. Hence $\partial_r^2 f(0, \theta) = 0$. We take the derivative of (1) with respect to r and evaluate the result at $r = 0$ to find

$$\partial_r^3 f = -K\partial_r f - \nabla_\Theta K f.$$

So $\partial_r^3 f(0, \theta) = -K(p)$. Taking twice the derivative of (1) with respect to r and evaluating the result at $r = 0$, we get

$$\partial_r^4 f = -K\partial_r^2 f - 2\nabla_\Theta K\partial_r f - \nabla_\Theta^2 K(p) f,$$

therefore $\partial_r^4 f(0, \theta) = -2\nabla_\Theta K(p)$. Similarly

$$\partial_r^5 f(0) = -K\partial_r^3 f - 3\nabla_\Theta K\partial_r^2 f - 3\nabla_\Theta^2 K\partial_r f - \nabla_\Theta^3 K f,$$

so that $\partial_r^5 f(0, \theta) = K(p)^2 - 3\nabla_\Theta^2 K(p)$. Collecting these, we have completed the proof of the expansion. □

We recall that formula for the geodesic curvature of a smooth curve Γ, which is parameterized in geodesic polar coordinates centered at p by $\theta \mapsto (r(\theta), \theta)$.

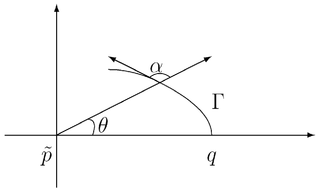

Fig. 1. Local graph of Γ.

Lemma 9. *The geodesic curvature k_g of Γ at the point $\Psi(r(\theta),\theta)$, is given by*

$$k_g = \frac{1}{(r'^2+f^2)^{3/2}}\big(r'\partial_\theta f + 2r'^2\partial_r f - r'' f + \partial_r f f^2\big), \tag{3}$$

where $'$ stands for ∂_θ and where f is computed at the point $(r(\theta),\theta)$.

Proof. First we recall Liouville's formula in [1,4] for the computation of the geodesic curvature: Suppose that (u,v) are isothermal coordinates on the surface M so that the metric can be written as $g = E\,du^2 + G\,dv^2$, where E and G depend on u and v. Further assume that $C(s) := (u(s), v(s))$ is an immersed curve on M parameterized by arc-length. Let α denote the angle between the velocity vector $\partial_s C$ and ∂_u. Then the geodesic curvature of C is given by the formula

$$k_g = \frac{d\alpha}{ds} - \frac{1}{2\sqrt{G}}\frac{\partial \ln E}{\partial v}\cos\alpha + \frac{1}{2\sqrt{E}}\frac{\partial \ln G}{\partial u}\sin\alpha.$$

In our case, we obtain

$$k_g = \frac{d\alpha}{ds} + \frac{\partial_r f}{f}\sin\alpha,$$

where α denotes the angle between r-line and curve Γ. One can see Fig. 1.

It is easy to see that

$$\cos\alpha = \frac{r'}{\sqrt{r'^2+f^2}} \quad \text{and} \quad \sin\alpha = \frac{f}{\sqrt{r'^2+f^2}}, \tag{4}$$

where f is computed at the point $(r(\theta),\theta)$. Differentiating the first formula with respect to θ and using the second formula, we get

$$\frac{d\alpha}{d\theta} = \frac{r'(\partial_\theta f + \partial_r f r') - f r''}{r'^2+f^2}. \tag{5}$$

Hence, we conclude that

$$\frac{d\alpha}{ds} = \frac{1}{(r'^2+f^2)^{3/2}}\big(r'(\partial_\theta f + \partial_r f r') - f r''\big).$$

We can now use Liouville's formula

$$k_g = \frac{d\alpha}{ds} + \frac{\partial_r f}{f}\sin\alpha = \frac{1}{(r'^2+f^2)^{3/2}}\big(r'\partial_\theta f + 2r'^2\partial_r f - r'' f + f^2\partial_r f\big).$$

This completes the proof of the lemma. □

We now specialize the previous general formula to curves $\Gamma_{p,\epsilon,w}$ which, in polar coordinates centered at the point p, are parameterized by

$$r(\theta) = \epsilon\big(1 - w(\theta)\big), \tag{6}$$

where $\epsilon > 0$ is a small parameter and w is small (smooth enough) function. We expand the geodesic curvature of this curve in powers of ϵ and w. According to (3), the geodesic curvature of $\Gamma_{p,\epsilon,w}$ reads:

$$k_g(p,\epsilon,w) = \left(1 + \frac{\epsilon^2 w'^2}{f^2}\right)^{-3/2}\left(\frac{\partial_r f}{f} + \frac{\epsilon w''}{f^2} - \frac{\epsilon w' \partial_\theta f}{f^3} + \frac{2\epsilon^2 w'^2 \partial_r f}{f^3}\right). \tag{7}$$

In order to make notations shorter, it will be convenient to use the following notations. An expression of the form $L_{p,\epsilon}(w)$ will denote a linear second order differential operator such that, there exists a constant $c > 0$ independent of $p \in M$ and $\epsilon \in (0,1)$ such that

$$\left\| L_{p,\epsilon}(w) \right\|_{\mathcal{C}^0(S^1)} \leqslant c \| w \|_{\mathcal{C}^2(S^1)}$$

for all $w \in \mathcal{C}^2(S^1)$. Similarly, given $a \in \mathbb{N}$, any expression of the form $Q^{(a)}_{p,\epsilon}(w)$ denotes a nonlinear second order differential operator such that, $Q^{(a)}_{p,\epsilon}(0) = 0$ and there exists a constant $c > 0$ independent of $p \in M$ and $\epsilon \in (0,1)$

$$\left\| Q^{(a)}_{p,\epsilon}(w_2) - Q^{(a)}_{p,\epsilon}(w_1) \right\|_{\mathcal{C}^0(S^1)} \leqslant c \left(\| w_2 \|_{\mathcal{C}^2(S^1)} + \| w_1 \|_{\mathcal{C}^2(S^1)} \right)^{a-1} \| w_2 - w_1 \|_{\mathcal{C}^2(S^1)}$$

provided $\| w_j \|_{C^1(S^1)} \leqslant 1$, $j = 1, 2$.

The following result gives the Taylor expansion of $k_g(p, \epsilon, w)$ in powers of w and ϵ:

Proposition 10. *The geodesic curvature $k_g(p, \epsilon, w)$ of the curve $\Gamma_{p,\epsilon,w}$ can be expanded as*:

$$\begin{aligned} \epsilon k_g(p, \epsilon, w) = {} & 1 - \frac{1}{3} K(p) \epsilon^2 - \frac{1}{4} \nabla_\Theta K(p) \epsilon^3 - \left(\frac{1}{45} K^2(p) + \frac{1}{10} \nabla^2_\Theta K(p) \right) \epsilon^4 \\ & + \mathcal{O}_p\left(\epsilon^5\right) + \left(1 + \frac{1}{3} K(p) \epsilon^2 \right) \left(\partial^2_\theta + 1 \right) w + \epsilon^3 L_{p,\epsilon}(w) \\ & + w^2 + \frac{1}{2} w'^2 + 2 w w'' + Q^{(3)}_{p,\epsilon}(w) + \epsilon^2 Q^{(2)}_{p,\epsilon}(w). \end{aligned} \tag{8}$$

The subscript p in $\mathcal{O}_p(\epsilon^5)$ is meant to remind the reader that this is a function of p bounded by a constant times ϵ^5.

Proof. Using (2) with $r = \epsilon(1 - w)$, we can write

$$\begin{aligned} & \frac{\epsilon^2 w'^2}{f^2} = w'^2 + Q^{(3)}_{p,\epsilon}(w) + \epsilon^2 Q^{(2)}_{p,\epsilon}(w), \\ & \frac{\epsilon w' \partial_\theta f}{f^3} = \epsilon^2 L_p(w) + \epsilon^2 Q^{(2)}_{p,\epsilon}(w), \\ & \frac{\epsilon^3 w'^2 \partial_r f}{f^3} = w'^2 + Q^{(3)}_{p,\epsilon}(w) + \epsilon^2 Q^{(2)}_{p,\epsilon}(w), \\ & \frac{\epsilon^2 w''}{f^2} = w'' + 2 w w'' + \frac{1}{3} K(p) \epsilon^2 w'' + Q^{(3)}_{p,\epsilon}(w) + \epsilon^2 Q^{(2)}_{p,\epsilon}(w). \end{aligned}$$

Using once more (2), we see that

$$\frac{\partial_r f}{f}(r, \theta) = \frac{1}{r} - \frac{1}{3} K(p) r - \frac{1}{4} \nabla_\Theta K(p) r^2 - \left(\frac{1}{45} K^2(p) + \frac{1}{10} \nabla^2_\Theta K(p) \right) r^3 + \mathcal{O}\left(r^4\right)$$

at the point (r, θ). Taking $r = \epsilon(1 - w)$, we get

$$\begin{aligned} \epsilon \frac{\partial_r f_r}{f} = {} & 1 - \frac{1}{3} K(p) \epsilon^2 - \frac{1}{4} \nabla_\Theta K(p) \epsilon^3 - \left(\frac{1}{45} K^2(p) + \frac{1}{10} \nabla^2_\Theta K(p) \right) \epsilon^4 + \mathcal{O}\left(\epsilon^5\right) \\ & + \left(1 + \frac{1}{3} K(p) \epsilon^2 \right) w + \epsilon^3 L_{\epsilon,p}(w) + w^2 + Q^{(3)}_{\epsilon,p}(w) + \epsilon^2 Q^{(2)}_{\epsilon,p}(w). \end{aligned}$$

Inserting these into (7), this completes the proof of the result. □

3. Constant geodesic curvature curves

In this section, we assume that Γ is an embedded closed curve in M with constant geodesic curvature $k = 1/\epsilon$. We assume that k is large enough so that the result of Theorem 2 holds true.

We now show that Γ is in fact a normal graph over a geodesic circle of radius $1/k$. For the sake of simplicity, let us assume that (M, g) is compact.

Proposition 11. *There exists $k_* > 0$ and $c > 0$ such that, if Γ is an embedded closed curve with constant geodesic curvature $k = 1/\epsilon \geqslant k_*$, then there exist a point $p \in M$ such that Γ can be parameterized in polar coordinates centered at p by $r(\theta) = \epsilon(1 - w(\theta))$ where the function $w \in \mathcal{C}^2(S^1)$ satisfies*

$$\|w\|_{\mathcal{C}^2} \leqslant c\epsilon^2,$$

and

$$\int_0^{2\pi} w(\theta)\cos\theta\, d\theta = \int_0^{2\pi} w(\theta)\sin\theta\, d\theta = 0. \tag{9}$$

Proof. The proof goes as follows. We first show that, there exists $\tilde{p} \in M$ such that Γ can be written as a normal graph over the geodesic circle of radius ϵ centered at $\tilde{p}$, for some function which is bounded by a constant times ϵ^3. Obviously, there is no uniqueness in the choice of $\tilde{p}$ and next, we show that, moving the point $\tilde{p}$ if this is necessary, one can arrange in such a way that the function satisfies the orthogonality condition (9).

We pick a point $q \in \Gamma$ and consider the point $\tilde{p}$ defined as follows: The point $\tilde{p}$ is at distance $\epsilon = 1/k$ from q along the geodesic starting at q with velocity the normal vector about Γ (see Fig. 1). We assume that k is large enough so that ϵ is less than the cut locus of $\tilde{p}$ and we denote by $\tilde{\Gamma}$ the geodesic circle of radius $\epsilon = 1/k$ centered at $\tilde{p}$. Clearly, near q, the curve Γ can be written as a normal graph over $\tilde{\Gamma}$ and hence we can parameterize Γ near q using geodesic polar coordinates centered at $\tilde{p}$, namely

$$\theta \mapsto \big(r(\theta), \theta\big)$$

with $r(0) = \epsilon$ and $r'(0) = 0$. Let $\tilde{\theta} \in (0, \pi]$ be the largest value such that

$$\big|r(\theta) - \epsilon\big| \leqslant \epsilon^2 \quad \text{and} \quad \big|r'(\theta)\big| \leqslant \epsilon^2,$$

for all $\theta \in [-\tilde{\theta}, \tilde{\theta}]$. Obviously $\tilde{\theta} > 0$.

Since $k_g(\Gamma) = k$, by Lemma 9, we know that r is a solution of the following second order ordinary differential equation

$$r'' = f\partial_r f - \frac{1}{\epsilon} f^2 + \frac{r'}{f}\partial_\theta f + 2\left(\frac{r'}{f}\right)^2 f\partial_r f - \frac{f^2}{\epsilon}\left(\left(1 + \left(\frac{r'}{f}\right)^2\right)^{3/2} - 1\right).$$

Thanks to the expansion of the function f given in (2), we obtain the following estimates for $\theta \in [-\tilde{\theta}, \tilde{\theta}]$.

$$\frac{r'}{f}\partial_\theta f = \mathcal{O}\big(\epsilon^5\big), \qquad f\partial_r f - \frac{1}{\epsilon} f^2 = r - \frac{r^2}{\epsilon} + \mathcal{O}\big(\epsilon^3\big),$$
$$2\left(\frac{r'}{f}\right)^2 f\partial_r f = \mathcal{O}\big(\epsilon^3\big), \qquad -\frac{f^2}{\epsilon}\left(\left(1 + \left(\frac{r'}{f}\right)^2\right)^{3/2} - 1\right) = \mathcal{O}\big(\epsilon^3\big).$$

Therefore, we conclude that $\theta \mapsto r(\theta)$ is a solution of the equation

$$r'' = r - \frac{r^2}{\epsilon} + \mathcal{O}\big(\epsilon^3\big),$$

with $r(0) = \epsilon$ and $r'(0) = 0$. We set $r = \epsilon(1 - w)$, so that

$$w'' = -w + w^2 + \mathcal{O}\big(\epsilon^2\big),$$

with $w(0) = w'(0) = 0$. It is easy to see $w = \mathcal{O}(\epsilon^2)$, $w' = \mathcal{O}(\epsilon^2)$ and hence $w'' = \mathcal{O}(\epsilon^2)$. Going back to the original function, we see that

$$r = \epsilon + \mathcal{O}\big(\epsilon^3\big), \qquad r' = \mathcal{O}\big(\epsilon^3\big).$$

This implies that $\tilde{\theta} = \pi$, for ϵ small enough. In addition, the curve Γ being an embedded curve, we conclude easily that

$$r(-\pi) = r(\pi)$$

provided ϵ is small enough and this completes the proof of the first part of the result. It remains to prove that, modulo some small change in the position of $\tilde{p}$, one can ensure that (9) holds.

As already mentioned, the point $\tilde{p}$ is not unique and in fact once that we know that Γ is a normal graph over the geodesic circle of radius ϵ centered at $\tilde{p}$, for some function $w_{\tilde{p}}$, we conclude that the same is true if instead of $\tilde{p}$, we choose any point $\hat{p}$ close enough to $\tilde{p}$. We claim that it is possible to choose $\hat{p}$ in such a way that (9) is fulfilled. This follows at once from the following argument.

It is easy to check that there exists $c > 0$ small enough such that, for all $\tilde{v} \in T_{\tilde{p}}M$ and all $\tilde{w} \in \mathcal{C}^2(S^1)$ satisfying

$$\|\tilde{v}\|_g \leqslant c \quad \text{and} \quad \|\tilde{w}\|_{\mathcal{C}^2} \leqslant c$$

the curve $\Gamma(\tilde{p}, \epsilon, \tilde{w})$ can also be written as the normal graph over the geodesic circle of radius ϵ centered at $p = \mathrm{Exp}_{\tilde{p}}(\epsilon\tilde{v})$ for some function $w = w_{\tilde{p},\tilde{v},\tilde{w}}$. In other words, we can write

$$\Gamma(\tilde{p}, \epsilon, \tilde{w}) = \Gamma(p, \epsilon, w_{\tilde{p},\tilde{v},\tilde{w}}).$$

We define

$$P(\epsilon, \tilde{v}, \tilde{w}) = \frac{1}{\pi}\int_0^{2\pi} w_{\tilde{p},\tilde{v},\tilde{w}}\Theta\, d\theta \in T_{\tilde{p}}M.$$

It is easy to check that P is depends smoothly on v and ϵ (at least when $\epsilon > 0$ is small enough) and extends smoothly to $\epsilon = 0$. Moreover, $P(0,0,0) = 0$ and

$$D_{\tilde{v}}P_{(0,0,0)}(\tilde{v}) = \tilde{v}.$$

The implicit function theorem implies that, for all $\epsilon > 0$ and $\|\tilde{w}\|_{\mathcal{C}^2}$ small enough, there exists a vector $\tilde{v}_\epsilon \in T_{\tilde{p}}M$ such that $P(\epsilon, \tilde{v}_\epsilon, \tilde{w}) = 0$. In addition $\mathrm{dist}(p, \tilde{p}) \leqslant c\epsilon\|\tilde{w}\|_{\mathcal{C}^2}$ if $p = \mathrm{Exp}_{\tilde{p}}(\epsilon\tilde{v})$. This completes the proof of the result. □

We keep the notations, assumptions and conclusions of Proposition 11. Making use of Proposition 10, we get:

Proposition 12. *There exists a constant $c > 0$ such that*

$$\big\|dK(p)\big\|_g \leqslant c\epsilon^2$$

provided $k = 1/\epsilon \geqslant k_*$.

Proof. By Proposition 11, Γ can be parameterized by $r(\theta) = \epsilon(1 - w(\theta))$ in polar coordinates centered at p. In addition, we know that $\|w\|_{\mathcal{C}^2} = \mathcal{O}(\epsilon^2)$. Using this information in (8), we conclude that the function w is a solution of

$$1 := \epsilon k_g(\epsilon, w) = 1 - \frac{1}{3}K(p)\epsilon^2 + \big(\partial_\theta^2 + 1\big)w + \mathcal{O}\big(\epsilon^3\big).$$

In particular, we get

$$\big(\partial_\theta^2 + 1\big)w = \frac{1}{3}K(p)\epsilon^2 + \mathcal{O}\big(\epsilon^3\big),$$

moreover we know that, by construction, w is $L^2(S^1)$-orthogonal to the functions $\cos\theta$ and $\sin\theta$. Hence we conclude that

$$w = \frac{1}{3}K(p)\epsilon^2 + \mathcal{O}\big(\epsilon^3\big).$$

Therefore, we get

$$\int_0^{2\pi}\left(w^2 + \frac{1}{2}w'^2 + 2ww''\right)\cos\theta\, d\theta = \mathcal{O}\big(\epsilon^5\big).$$

Obviously,

$$\int_0^{2\pi} \cos\theta\left(\frac{1}{3}K(p)\epsilon^2 + \left(\frac{1}{45}K^2(p) + \frac{1}{10}\nabla^2_\Theta K(p)\right)\epsilon^4\right) d\theta = 0,$$

and, thanks to (9)

$$\left(1 + \frac{1}{3}K(p)\epsilon^2\right)\int_0^{2\pi} \cos\theta\left(\partial_\theta^2 + 1\right)w\, d\theta = 0.$$

Multiplying (8) by $\cos\theta$, using the fact that $\epsilon k_g(\epsilon, w) = 1$, and integrating the result over $(0, 2\pi)$, we conclude that

$$\frac{1}{3}\epsilon^3\int_0^{2\pi} \nabla_\Theta K(p)\cos\theta\, d\theta = \mathcal{O}\left(\epsilon^5\right).$$

Similarly, we get

$$\frac{1}{3}\epsilon^3\int_0^{2\pi} \nabla_\Theta K(p)\sin\theta\, d\theta = \mathcal{O}\left(\epsilon^5\right)$$

which implies that

$$\left\|dK(p)\right\|_g \leqslant c\epsilon^2$$

as claimed. This completes the proof of the result. □

We are now in a position to prove both Theorems 3 and 4.

Proof of Theorem 3. By assumption, we have a sequence of closed embedded curves Γ_i with constant geodesic curvature $k_i \to +\infty$. According to the result of Proposition 12, when i is large enough, we can write Γ_i as a normal graph, for some function bounded by a constant times $1/k_i^3$, over a geodesic circle of radius $1/k_i$ centered at a point p_i with

$$\left\|dK(p_i)\right\|_g \leqslant c/k_i^2.$$

The fact that Γ_i converges to p forces p_i to converge to p. Passing to the limit, as $i \to +\infty$, we conclude that $dK(p) = 0$. This completes the proof of Theorem 3. □

Proof of Theorem 4. We keep the notations of the previous paragraph. The novelty being that K is now assumed to be a Morse function on M. In particular it has finite number of critical points which are all isolated. The curve Γ is known to be a normal graph (for some function bounded by a constant times $1/k^3$) over a geodesic circle of radius $1/k$ centered at a point $\tilde{p}$ such that $\|dK(\tilde{p})\|_g = \mathcal{O}(1/k^2)$. Since K is a Morse function, we conclude that $\tilde{p}$ is at most at distance a constant times $1/k^2$ from one of the critical points of K. This completes the proof of the result. □

4. Limit of constant geodesic curvature curves as their curvature tends to infinity

In this section we consider curves with large constant geodesic curvature which are immersed in some open domain Ω which does not contain any critical point of the Gauss curvature. Without loss of generality, we can assume that this curve is parameterized by $s \in I \mapsto \gamma(s) \in M$ where s is the arc length and I the maximal interval for whose image by γ lies in Ω.

As usual we set

$$\epsilon = 1/k.$$

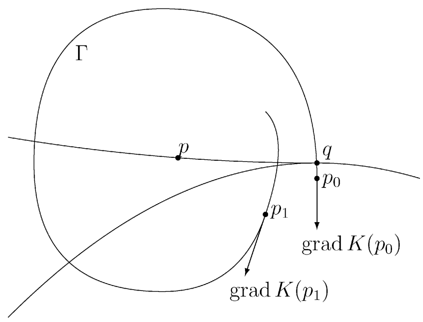

Fig. 2. Local description of Γ.

We pick a point $q \in \Gamma$ and consider the point p which is at distance ϵ from q along the geodesic starting at q with velocity the normal vector about Γ. By Proposition 11, Γ can be parameterized by $r(\theta) = \epsilon(1 - w(\theta))$ in polar coordinates centered at p. (The point q corresponds to $\theta = 0$.) In addition, we know that $\|w\|_{C^2} = \mathcal{O}(\epsilon^2)$ on any interval of fixed length and also that $w(0) = w'(0) = 0$ (observe that here we do not assume that w satisfies (9)).

Using this information in (8), we conclude that the function w is a solution of

$$\left(\partial_\theta^2 + 1\right)w = \frac{1}{3}K(p)\epsilon^2 + \frac{1}{4}\left(\partial_{x_1}K(p)\cos\theta + \partial_{x_2}K(p)\sin\theta\right)\epsilon^3 + \mathcal{O}\left(\epsilon^4\right).$$

It is easy to check that

$$\begin{aligned} w(\theta) = {} & \frac{1}{3}K(p)\epsilon^2(1-\cos\theta) \\ & + \frac{1}{8}\epsilon^3\left(\theta\left(\partial_{x_1}K(p)\sin\theta - \partial_{x_2}K(p)\cos\theta\right) + \partial_{x_2}K(p)\sin\theta\right) + \mathcal{O}\left(\epsilon^4\right) \end{aligned}$$

and in particular, we conclude that

$$w(\theta + 2\pi) = w(\theta) + \frac{\pi}{4}\epsilon^3\left(\partial_{x_1}K(p)\sin\theta - \partial_{x_2}K(p)\cos\theta\right) + \mathcal{O}\left(\epsilon^4\right). \tag{10}$$

Since the metric can be written as

$$\Psi^* g = dr^2 + f^2\, d\theta^2,$$

we get

$$\operatorname{grad} K = \partial_r K\, \partial_r + \frac{1}{f^2}\partial_\theta K\, \partial_\theta.$$

If v is a tangent vector to M, we denote $v^\perp$ the vector obtained from v by rotation of angle $\pi/2$. We have the formula

$$\operatorname{grad} K^\perp = \frac{1}{f}(\partial_r K\, \partial_\theta - \partial_\theta K\, \partial_r).$$

If Γ is parameterized in geodesic polar coordinates by $\gamma : \theta \to (\theta, r(\theta))$ with $r(\theta) = \epsilon(1 - w(\theta))$, we have

$$\partial_\theta \gamma = \partial_\theta - \epsilon w' \partial_r.$$

Therefore,

$$g\left(\operatorname{grad} K^\perp, \partial_\theta\gamma\right) = f\,\partial_r K + \frac{1}{f}\epsilon w' \partial_\theta K.$$

The first lemma ensures that, in I, the set of s such that $\partial_s\gamma$ is colinear to grad $K(\gamma)$, with opposite orientation, is a sequence of isolated points whose mutual distance are roughly multiple of $2\pi\epsilon$.

Proposition 13. *There exists $k_* > 0$ such that, if the geodesic curvature of Γ is constant and larger than k_* then the set of parameters θ for which $\partial_\theta \gamma$ and $\operatorname{grad} K(\gamma)$ are co-linear and have opposite orientations is a finite sequence of points $\theta_0 < \theta_1 < \cdots < \theta_m$ (which depend on Γ). In addition*

$$|\theta_{j+1} - \theta_j - 2\pi| \leqslant c\epsilon^4,$$

for some constant $c > 0$ only depending on Ω.

Proof. The proof of this result follows from the implicit function theorem which is applied to the function

$$\phi := g\big(\operatorname{grad} K^{\perp}(\gamma), \partial_\theta \gamma\big).$$

We find

$$f^{-1}\phi = \left(\partial_{x_1} K(\gamma) + \frac{1}{f^2}\epsilon^2 w'(1-w)\partial_{x_2} K(\gamma)\right)\cos\theta + \left(\partial_{x_2} K(\gamma) - \frac{1}{f^2}\epsilon^2 w'(1-w)\partial_{x_1} K(\gamma)\right)\sin\theta.$$

One has to be careful that the set of zeros of ϕ is not exactly the set we are looking but the set of points where $\partial_s \gamma$ and $\operatorname{grad} K(\gamma)$ are colinear, independently of the choice of orientation. It is easy to check that

$$\frac{1}{f^2}\epsilon^2 w'(1-w) = \mathcal{O}\big(\epsilon^2\big) \quad \text{and} \quad \partial_\theta\left(\frac{1}{f^2}\epsilon^2 w'(1-w)\right) = \mathcal{O}\big(\epsilon^2\big).$$

Moreover

$$\partial_\theta\big(\partial_{x_1} K(\gamma)\big) = \mathcal{O}(\epsilon) \quad \text{and} \quad \partial_\theta\big(\partial_{x_2} K(\gamma)\big) = \mathcal{O}(\epsilon).$$

Therefore,

$$f^{-1}\phi = \partial_{x_1} K(\gamma)\cos\theta + \partial_{x_2} K(\gamma)\sin\theta + \mathcal{O}(\epsilon)$$

and

$$\partial_\theta\big(f^{-1}\phi\big) = -\partial_{x_1} K(\gamma)\sin\theta + \partial_{x_2} K(\gamma)\cos\theta + \mathcal{O}(\epsilon). \tag{11}$$

Let $\theta_0 \in (-\pi, \pi]$ such that

$$\partial_{x_1} K(p)\cos\theta_0 + \partial_{x_2} K(p)\sin\theta_0 = 0.$$

Since the distance between p and γ can be estimated by a constant times ϵ, we see that the zeros of ϕ are given by

$$\theta_n = \theta_0 + n\pi + \mathcal{O}(\epsilon),$$

where $n \in \mathbb{Z}$. But using (11) we can estimate $\partial_\theta(f^{-2}\phi)$ at any zero of ϕ and show that the zeros of ϕ are isolated and the distance between two consecutive zeros is about π. Taking into account the fact that we are only interested in points where $\partial_s \gamma$ and $\operatorname{grad} K(\gamma)$ have opposite orientation, we conclude that the distance between the points we are interested in is about 2π. We can assume that θ_0 is chosen so that these points correspond to θ_{2n} for $n \in \mathbb{Z}$.

To get a better estimate for the distance between two such zeros of ϕ we use (10) which implies that

$$w(\theta + 2\pi) = w(\theta) + \frac{\pi}{4}\epsilon^3\big(\partial_{x_1} K\big(\gamma(\theta)\big)\sin\theta - \partial_{x_2} K\big(\gamma(\theta)\big)\cos\theta\big) + \mathcal{O}\big(\epsilon^4\big)$$

since the distance between $\gamma(\theta)$ and p is estimated by a constant times ϵ. We also get

$$w'(\theta + 2\pi) = w'(\theta) + \frac{\pi}{4}\epsilon^3\big(\partial_{x_1} K\big(\gamma(\theta)\big)\cos\theta + \partial_{x_2} K\big(\gamma(\theta)\big)\sin\theta\big) + \mathcal{O}\big(\epsilon^4\big)$$

since $\partial_\theta \partial_{x_j} K(\gamma(\theta)) = \mathcal{O}(\epsilon)$. Using these informations we estimate

$$\begin{aligned}&\big(f^{-1}\phi\big)(\theta + 2\pi) - \big(f^{-1}\phi\big)(\theta)\\ &\quad = \frac{\pi}{4}\epsilon^3\big(\partial_{x_1} K\big(\gamma(\theta)\big)\cos\theta + \partial_{x_2} K\big(\gamma(\theta)\big)\sin\theta\big)\big(\partial_{x_2} K\big(\gamma(\theta)\big)\cos\theta - \partial_{x_1} K\big(\gamma(\theta)\big)\sin\theta\big) + \mathcal{O}\big(\epsilon^4\big)\end{aligned}$$

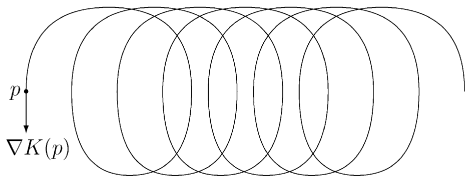

Fig. 3. "Spring".

which can also be written as

$$\left(f^{-1}\phi\right)(\theta+2\pi)=\left(1+\frac{\pi}{4}\epsilon^3\partial_\theta\left(f^{-1}\phi\right)(\theta)\right)\left(f^{-1}\phi\right)(\theta)+\mathcal{O}\left(\epsilon^4\right).$$

This together with the estimate of the derivative of $f^{-1}\phi$ implies that

$$\theta_{2n+2}=\theta_{2n}+2\pi+\mathcal{O}\left(\epsilon^4\right).$$

This completes the proof of the result by dividing the subscript of θ by two. □

We set

$$p_j=\gamma(\theta_j).$$

Since

$$w(\theta+2\pi)=w(\theta)+\frac{\pi}{4}\epsilon^3\left(\partial_{x_1}K(p)\sin\theta-\partial_{x_2}K(p)\cos\theta\right)+\mathcal{O}\left(\epsilon^4\right)$$

we conclude that

$$p_{j+1}=\operatorname{Exp}_{p_j}(V_j),$$

where $V_j=\frac{\pi}{4}\epsilon^4\operatorname{grad}K(p_j)^\perp+\mathcal{O}(\epsilon^5)$. This completes the proof of Theorem 5.

In Fig. 3 is the global picture of this process, the curve goes along the level curve of K like a spring.

5. Constant geodesic curvature d-circles: the proof of Theorem 7

In this section we study the case of immersed closed curves with constant geodesic curvature. We start with:

Definition 1. Given $d\in\mathbb{N}^+$, a closed curve immersed in M is called a *d-circle* if it is a degree d curve immersed in a simple connected domain of M.

The proof of Theorem 7 is based on the following idea: It is easy to check that the results of Theorems 6 and 4 hold when "embedded curves" are replaced by "d-circles", with d fixed. Therefore, if Γ is a constant geodesic curvature d-circle, it is a graph over the geodesic circle of radius $\epsilon=1/k$ centered at a point q which is at distance c/k^2 from p a critical point of the Gauss curvature function, for some $2\pi d$-periodic function which can be estimated by $c\epsilon^3$ in $\mathcal{C}^2$ topology.

We now prove that, for all ϵ small enough, there exists a unique normal graph over the geodesic circle of radius $\epsilon=1/k$ centered at a point q at distance c/k^2 from p for some $2\pi d$-periodic function which can be estimated by $c\epsilon^3$ in $\mathcal{C}^2$ topology. Since the d-cover of the embedded curve obtained by R. Ye in Theorem 1 has constant geodesic curvature equal to k, Γ has to be the d-cover of this embedded curve.

From now on, we focus our attention on proving Theorem 7. We will use the fixed point argument to derive an uniqueness property which is enough to obtain the theorem. Assume that we have Γ_1 and Γ_2 two d-circles with constant geodesic curvature. It follows from the result of Theorem 4 that Γ_j a normal multi-graph for some $2\pi d$-periodic function ϵw_j over the geodesic circle of radius $\epsilon=1/k$ centered at the point p_j. Furthermore, we can assume that

$$\int_0^{2d\pi}w_j\cos\theta\,d\theta=\int_0^{2d\pi}w_j\sin\theta\,d\theta=0$$

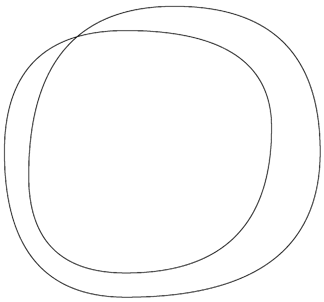

Fig. 4. A 2-circle.

and that

$$\operatorname{dist}(p_j, p) + \|w_j\|_{\mathcal{C}^2(S^1)} \leqslant c\epsilon^2, \tag{12}$$

where p is a non-degenerate critical point of K.

Let $k_g(q, \epsilon, w)$ denote the geodesic curvature of the curve parameterized by $(\epsilon(1 - w(\theta)), \theta)$ in geodesic polar coordinates centered at q. We use the result of Proposition 10 to get the expansion

$$\begin{aligned}\epsilon k_g(q, \epsilon, w) = {} & 1 - \frac{1}{3}K(q)\epsilon^2 - \frac{1}{4}\nabla_\Theta K(q)\epsilon^3 + \mathcal{O}_q\big(\epsilon^4\big) \\ & + \big(\partial_\theta^2 + 1\big)w + \epsilon^2 L_{q,\epsilon}(w) + Q^{(2)}_{q,\epsilon}(w),\end{aligned} \tag{13}$$

where the subscript q in $\mathcal{O}_q(\epsilon^4)$ means that this is a function of q. We denote

$$F(\epsilon, q) := -\frac{1}{3}K(q) - \frac{1}{4}\nabla_\Theta K(q)\epsilon + \mathcal{O}_q\big(\epsilon^2\big).$$

Since $\epsilon k_g(p_j, \epsilon, w_j) = 1$ we get by substraction

$$\begin{aligned}\big(\partial_\theta^2 + 1\big)(w_2 - w_1) = {} & \big(F(\epsilon, p_1) - F(\epsilon, p_2)\big)\epsilon^2 + \epsilon^2(L_{p_1,\epsilon} w_1 - L_{p_2,\epsilon} w_2) \\ & + \big(Q^{(2)}_{p_1,\epsilon}(w_1) - Q^{(2)}_{p_2,\epsilon}(w_2)\big).\end{aligned}$$

Since $w_2 - w_1$ is L^2-orthogonal to $\cos\theta$ and $\sin\theta$, we conclude easily that

$$\|w_2 - w_1\|_{\mathcal{C}^2(S^1)} \leqslant c\epsilon^2\big(\operatorname{dist}(p_2, p_1) + \|w_2 - w_1\|_{\mathcal{C}^2(S^1)}\big),$$

where we have implicitly used (12). Hence, for ϵ small enough, we conclude that

$$\|w_2 - w_1\|_{\mathcal{C}^2(S^1)} \leqslant c\epsilon^2 \operatorname{dist}(p_2, p_1). \tag{14}$$

Now, we project onto $\cos\theta$, the identity $k_g(p_2, \epsilon, w_2) - k_g(p_1, \epsilon, w_1) = 0$. Using the arguments already used in the proof of Proposition 12 we get

$$\int_0^{2d\pi} \big(\partial_\theta^2 + 1\big)(w_2 - w_1)\cos\theta \, d\theta = \int_0^{2d\pi} \big(K(p_2) - K(p_1)\big)\cos\theta \, d\theta = 0.$$

Moreover, using (12), we easily conclude that

$$\epsilon^2 \left| \int_0^{2d\pi} (L_{p_2,\epsilon} w_2 - L_{p_1,\epsilon} w_1)\cos\theta \, d\theta \right| \leqslant c\big(\epsilon^4 \operatorname{dist}(p_2, p_1) + \epsilon^2\|w_2 - w_1\|_{\mathcal{C}^2(S^1)}\big)$$

and

$$\left| \int_0^{2d\pi} \big(Q^{(2)}_{p_2,\epsilon}(w_2) - Q^{(2)}_{p_1,\epsilon}(w_1)\big)\cos\theta \, d\theta \right| \leqslant c\big(\epsilon^4 \operatorname{dist}(p_2, p_1) + \epsilon^2\|w_2 - w_1\|_{\mathcal{C}^2(S^1)}\big).$$

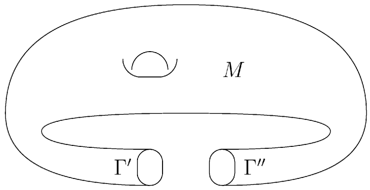

Fig. 5. Graph of $M\backslash\Gamma$.

Therefore, we conclude that

$$\epsilon^3\left|\int_0^{2d\pi}\big(\nabla_\Theta K(p_2)-\nabla_\Theta K(p_1)\big)\cos\theta\,d\theta\right|\leqslant c\big(\epsilon^4\operatorname{dist}(p_2,p_1)+\epsilon^2\|w_2-w_1\|_{\mathcal{C}^2(S^1)}\big).$$

Similarly, we have

$$\epsilon^3\left|\int_0^{2d\pi}\big(\nabla_\Theta K(p_2)-\nabla_\Theta K(p_1)\big)\sin\theta\,d\theta\right|\leqslant c\big(\epsilon^4\operatorname{dist}(p_2,p_1)+\epsilon^2\|w_2-w_1\|_{\mathcal{C}^2(S^1)}\big).$$

This implies that

$$\epsilon\operatorname{dist}(p_2,p_1)\leqslant c\|w_2-w_1\|_{\mathcal{C}^2(S^1)},\tag{15}$$

provided ϵ is small enough.

Using (14) and (15), we conclude that $w_2=w_1$ and $p_1=p_2$. This completes the proof of Theorem 7.

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Appendix A. The proof of Theorem 2

We recall the proof of Theorem 2 following [6]. The proof is by contradiction. Assume that the result is not correct, so that we have an embedded curve Γ with constant geodesic curvature k such that $M\setminus\Gamma$ has only one connected component. We can consider that $M_0:=M\setminus\Gamma$ is a manifold with two boundaries Γ' and Γ''.

We consider the curve $\gamma\subset\overline{M_0}$ which minimizes the distance between a point of Γ' and a point of Γ''. Namely, γ is a solution of the problem variational problem

$$\operatorname{Inf}\big\{\operatorname{Length}(\gamma):\ \gamma:[0,\ell]\to\overline{M_0},\ \gamma(0)\in\Gamma'\text{ and }\gamma(\ell)\in\Gamma''\big\}.$$

We parameterize γ by arc-length and define $p:=\gamma(0)$ and $q:=\gamma(\ell)$ where $0<\ell:=\operatorname{Length}(\gamma)$.

We parameterize a small neighborhood of p in Γ' by arc length $s\in(-\epsilon,\epsilon)\to\phi(s)$ where $\epsilon>0$ and $\phi(0)=p$. We denote by $n(s)$ the normal vector about Γ' at the point $\phi(s)$ and assume that the orientation is chosen so that $k>0$. We further assume that the orientation of γ is chosen so that so that $\dot\gamma(0)=n$ at p (if not, just change p into q and Γ' with Γ'').

We define the map

$$\begin{aligned}A:(-\epsilon,\epsilon)\times[0,\ell]&\to M,\\(s,t)&\mapsto\operatorname{Exp}_{\phi(s)}\big(tn(s)\big).\end{aligned}$$

We claim that the following is true:

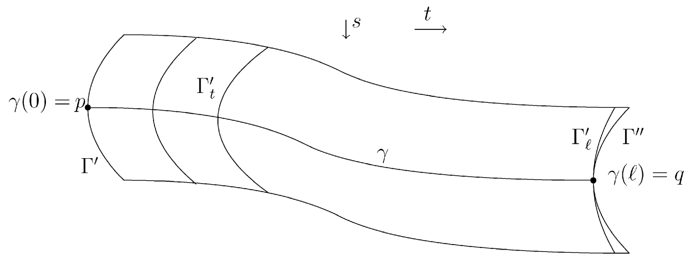

Fig. 6. Moving Γ' along γ.

Lemma 14. *We have*

$$\frac{\partial A}{\partial s}(0,t) \neq 0,$$

for all $t \in [0,\ell]$.

A straightforward application of the implicit function theorem implies that, provided $\epsilon > 0$ is chosen sufficiently small, the mapping A defined is a diffeomorphism from $(-\epsilon,\epsilon)\times[0,\ell]$ onto its image.

Let us assume that we have already proven the claim and let us complete the proof of Theorem 2.

In the image of A, we can decompose the metric g as

$$A^*g = dt^2 + f^2(s,t)\,ds^2,$$

where $f(s,0)=1$ and, because of the chosen orientation,

$$\frac{\partial_t f}{f}(s,0) = -k,$$

the (constant) geodesic curvature of Γ'.

For all $t \in [0,\ell]$, we denote by $k_g(s,t)$ the geodesic curvature of the curve $s \mapsto A(s,t)$, at the point $A(s,t)$ and we denote by $K(s,t)$ the Gauss curvature at the point $A(s,t)$. We have

$$K(s,t) = -\frac{\partial_t^2 f}{f}(s,t)$$

and, again because of the chosen orientation,

$$k_g(s,t) = -\frac{\partial_t f}{f}(s,t).$$

We compute

$$\partial_t k_g(s,t) = -\partial_t\left(\frac{\partial_t f}{f}\right)(s,t) = k_g^2(s,t) + K(s,t). \tag{A.1}$$

By assumption, we have

$$k^2 > -\min_M K,$$

so that $\partial_t k_g(s,0) > 0$, then (A.1) implies that $\partial_t k_g(s,t) > 0$ for all $t \in [0,\ell]$. In particular

$$k_g(0,\ell) > k_g(0,0) := k. \tag{A.2}$$

However, because the minimizing property of γ, the curve $s \mapsto A(s,\ell)$ is on one side of Γ'' and tangent to Γ'' at q. Therefore, we can compare the geodesic curvature of these two curves and given the chosen orientation we necessarily have

$$k \geqslant k_g(0,\ell)$$

since Γ'' has constant geodesic curvature equal to k. But this clearly contradicts (A.2). This completes the proof of Theorem 2.

Therefore, in order to complete the proof, it remains to prove Lemma 14.

Proof of Lemma 14. We argue by contradiction. Assume that the result is not correct. There would exist $\tilde{t} \in (0, \ell]$ such that

$$\partial_s A(0, \tilde{t}\,) = 0,$$

we will then find a contradiction by inspection of the second variation of the length functional about γ. Indeed, by construction γ minimizes the length between a point of Γ' and a point of Γ''. Hence, λ_0, the least eigenvalue of the Jacobi operator

$$\partial_t^2 + K$$

with Neumann boundary conditions has to be positive or equal to 0. We denote by ϕ_0 the associated eigenfunction so that

$$\left(\partial_t^2 + K\right)\phi_0 = -\lambda_0 \phi_0$$

we can assume without loss of generality that $\phi_0 > 0$. Observe that the function $\phi = g(\partial_s A(0, \cdot), \nu)$, where ν is a normal vector field to the curve parameterized by γ, is a Jacobi field in the sense that

$$\left(\partial_t^2 + K\right)\phi = 0.$$

Observe that $\phi(0) = 1$, $\partial_t \phi(0) = 0$ and, by assumption, that $\phi(\tilde{t}) = 0$. This implies that $\lambda_0 > 0$ since otherwise ϕ and ϕ_0 being solutions of a second order ordinary differential equation with 0 Neumann boundary condition at $t = 0$ would be colinear and this contradicts the fact that ϕ_0 does not vanish. Now we define $\epsilon > 0$ such that

$$\inf_{[0,\tilde{t}]} (\phi_0 - \epsilon\phi) = 0.$$

We have

$$\left(\partial_t^2 + K\right)(\phi_0 - \epsilon\phi) = -\lambda_0 \phi_0.$$

At a point where $\phi_0 - \epsilon\phi$ vanishes we conclude that $(\partial_t^2 + K)(\phi_0 - \epsilon\phi) \geqslant 0$. But the above equality implies that

$$-\lambda_0 \phi_0 < 0$$

which is a contradiction and this completes the proof of the claim. □

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