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Singly periodic solutions of a semilinear equation

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Abstract

We consider the solutions of the equation $-\varepsilon^2\Delta u + u - |u|^{p-1}u = 0$ in $S^1 \times \mathbb{R}$, where ε and p are positive real numbers, $p > 1$. We prove that the set of the positive bounded solutions even in x_1 and x_2, decreasing for $x_1 \in]-\pi, 0[$ and tending to 0 as x_2 tends to $+\infty$ is the first branch of solutions constructed by bifurcation from the ground-state solution $(\varepsilon, w_0(\frac{x_2}{\varepsilon}))$. We prove that there exists a positive real number $\varepsilon_\star$ such that for every $\varepsilon \in]0, \varepsilon_\star]$ there exists a finite number of solutions verifying the above properties and none such solution for $\varepsilon > \varepsilon_\star$. The proves make use of compactness results and of the Leray–Schauder degree theory.

Résumé

Nous étudions l'équation $-\varepsilon^2\Delta u + u - |u|^{p-1}u = 0$ dans $S^1 \times \mathbb{R}$, où ε et p sont des nombres réels strictement positifs, $p > 1$. Nous identifions l'ensemble des solutions (ε, u) où u est une fonction positive, paire en x_1 et x_2, décroissante en x_1 dans $[-\pi, 0]$ et tendant vers 0 quand x_2 tend vers $+\infty$, comme la première branche de solutions issue d'une bifurcation à partir de l'état fondamental $(\varepsilon, w_0(\frac{x_2}{\varepsilon}))$. Nous prouvons qu'il existe un réel $\varepsilon_\star$ tel que pour tout $\varepsilon \in]0, \varepsilon_\star]$ il y a un nombre fini de solutions vérifiant les propriétés énoncées ci-dessus, et aucune telle solution pour $\varepsilon > \varepsilon_\star$. Les preuves utilisent des résultats de compacité et la théorie du degré de Leray–Schauder.

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1. Introduction

Let ε and p be positive real numbers, $p > 1$. We consider the positive bounded solutions of the equation

$$-\varepsilon^2\Delta u + u - |u|^{p-1}u = 0 \quad \text{in } S^1 \times \mathbb{R} \tag{E}$$

that are 2π periodic in the first variable x_1 and that tend to 0, as $|x_2|$ tends to $+\infty$, uniformly in $x_1 \in S^1$. We know that these solutions are symmetric in x_2, around a real number t_0, and decreasing for $x_2 > t_0$. This can be proved

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by an application of the moving plane method [13,3,5]. Let us recall, for $n = 1, 2$, the existence of positive bounded solutions of the equation

$$-\Delta u + u - u^p = 0 \quad \text{in } \mathbb{R}^n.$$

The existence and the uniqueness of such solutions that are radially symmetric with respect to 0 is proved in [15]. We will denote them by w_0 for $n = 1$ and w_1 for $n = 2$. So the function $x_2 \to w_0(x_2/\varepsilon)$ is the unique positive bounded solution of (E), up to translations, that depends only on the second variable x_2. But there exist solutions that depend on the variable x_1. Such solutions are constructed by Dancer in [10], by bifurcation from the bounded positive solution $x_2 \to w_0(x_2/\varepsilon)$, by the use of a Crandall–Rabinowitz theorem in a convenient Banach space. More precisely, there exists a value, that we denote by $\varepsilon_\star$, of the parameter ε for which for all $k \in \mathbb{N}^\star$ curves of new solutions bifurcate from the solutions $(\varepsilon_\star/k, w_0(kx_2/\varepsilon_\star))$, while the solutions $(\varepsilon, w_0(x_2/\varepsilon))$, for $\varepsilon \neq \varepsilon_\star/k$ are locally unique. We refer to Malchiodi and Montenegro [16], for an analysis of the eigenvalues of the linearized operator $-\varepsilon^2\Delta + I - pw_0^{p-1}(x_2/\varepsilon)I$. We may consider only the positive bounded solutions of (E) that are even in x_2, the other solutions being deduced by translations. For the bifurcation we will consider the bounded positive solutions of Eq. (E) in the domain $S^1 \times \mathbb{R}^+$ that verify the Neumann boundary condition $\frac{\partial u}{\partial \nu} = 0$ on $S^1 \times \{0\}$, that are even in x_1 and such that u tends to 0 as x_2 tends to $+\infty$, uniformly in x_1. The other branches can be deduced by translations of the variable x_1. Let us call a trivial solution any solution of the form $(\varepsilon, w_0(x_2/\varepsilon))$. Let $\mathcal{S}$ be the closure of the set of the non-trivial solutions in the convenient Banach space. For all $k \in \mathbb{N}^\star$ we consider the component of $\mathcal{S}$ to which $(\varepsilon_\star/k, w_0(kx_2/\varepsilon_\star))$ belongs, that is the maximal connected set containing this solution. We call it the kth continuum of solutions and we denote it by Σ_k. It is proved in [10], by the maximum principle, that the solutions in Σ_k are positive. Moreover, by a continuity argument that uses the fact that, by its definition, Σ_1 is connected, it is proved that for all $(\varepsilon, u) \in \Sigma_1$ we have $\frac{\partial u}{\partial x_1} > 0$ in $]-\pi, 0[\, \times \mathbb{R}^+$ and $\frac{\partial u}{\partial x_1} < 0$ in $]0, \pi[\, \times \mathbb{R}^+$. In particular, all solution in Σ_1 is of minimal period 2π. If $(\varepsilon, u) \in \Sigma_1$, then we extend it to $[-k\pi, k\pi] \times \mathbb{R}+$ by 2π-periodicity and we define $v(x_1, x_2) = u(kx_1, kx_2)$. We can deduce from the construction of Σ_k that $(\varepsilon/k, v)$ belongs to Σ_k and that this rescaling gives every element of Σ_k. Consequently for all (ε, u) in Σ_k the minimal period of u is $2\pi/k$ and this implies that $\Sigma_1 \cap \Sigma_k = \emptyset$ for all $k \neq 1$. This is an important tool, following a global bifurcation theorem of Rabinowitz [19] in the proof of the existence of solutions (ε, u) in Σ_1 for all $0 < \varepsilon < \varepsilon_\star$. So we will focus our interest on the first continuum Σ_1. The results in [10] are in fact more general that what we summarized here. They concern bounded positive solutions of (E) in $S^1 \times \mathbb{R}^{n-1}$, $n \geqslant 2$ and the variable x_2 is replaced by the radius r of the polar coordinates in $\mathbb{R}^{n-1}$. But the case $n = 2$ is particular. In this case, for all $p > 1$, the solutions in Σ_1 are bounded in $L^\infty(S^1 \times \mathbb{R})$. Consequently, for $n = 2$, we have that if $(\varepsilon, u) \in \Sigma_1$, with $\varepsilon \to 0$ and if $\tilde{u}_\varepsilon$ is defined in $\frac{S^1}{\varepsilon} \times \mathbb{R}$ by $\tilde{u}_\varepsilon(x_1, x_2) = u_\varepsilon(\varepsilon x_1, \varepsilon x_2)$, then $\tilde{u}_\varepsilon$ tends to w_1, as ε tends to 0, i.e. the norm of $\tilde{u}_\varepsilon - w_1$ in $L^\infty(S^1/\varepsilon \times \mathbb{R})$ tends to 0.

In [16] the function w_0 and the linearized operator are used in view of the construction of positive solutions of $-\varepsilon^2\Delta u + u - u^p = 0$ in a bounded domain with a Neumann condition at the boundary. Many other authors studied the same equation in a bounded domain or in $\mathbb{R}^n$ [1,11,12, ...] or related equations [4].

In [2], we have proved the following theorem.

Theorem 1.1. *There exists $\bar{\varepsilon} > 0$ such that for $\varepsilon > \bar{\varepsilon}$ any positive solution of* (E) *that tends to* 0 *as x_2 tends to infinity, uniformly in $x_1 \in S^1$, can only be a function of the variable x_2.*

In this paper we will prove that the first continuum Σ_1 is in fact the set of all the positive solutions of (E) even in x_1 and x_2 that tend to 0 as x_2 tends to $+\infty$ and that verify $\frac{\partial u}{\partial x_1} > 0$ in $]-\pi, 0[\, \times \mathbb{R}^+$ and $\frac{\partial u}{\partial x_1} < 0$ in $]0, \pi[\, \times \mathbb{R}^+$. We will also prove that for each $\varepsilon \in]0, \varepsilon_*[$ there exists a finite number of such solutions and that there exists $\varepsilon_0 > 0$ for which for $\varepsilon < \varepsilon_0$ such a solution is unique. We do not know whether the $\bar{\varepsilon}$ in Theorem 1.1 is equal to $\varepsilon_\star$ or not, but we will prove that the first continuum Σ_1 is contained in $\{(\varepsilon, u), \varepsilon \leqslant \varepsilon_\star\}$. Thus so are the continua Σ_k for all $k \in \mathbb{N}^\star$ and all the sets of solutions that can be deduced from them by translations. For $p \in \mathbb{N}$, $p \geqslant 2$, when the function $u \mapsto u^p$ is analytic, we can describe more precisely the continuum Σ_1 as a finite number of curves that admit local analytic parameterizations.

First we will prove the following propositions

Proposition 1.1. *If $(\varepsilon, u) \in \mathbb{R} \times L^\infty(S^1 \times \mathbb{R})$ is a solution of* (E)*, $u > 0$, u even in x_2 and $\lim_{x_2 \to \infty} u = 0$ uniformly in x_1, then there exist positive real numbers C_1 and C_2, depending on u and on ε, such that for all $(x_1, x_2) \in S^1 \times \mathbb{R}+$,*

$C_1 e^{-x_2/\varepsilon} \leqslant u(x_1, x_2) \leqslant C_2 e^{-x_2/\varepsilon}$. More, given $\varepsilon_1 < \varepsilon_2$ in $]0, +\infty[$, for every set $\mathcal{A}$ of positive solutions of (E) as above that is included in $[\varepsilon_1, \varepsilon_2] \times L^\infty(S^1 \times \mathbb{R})$, there exists $C > 0$ such that for all $(\varepsilon, u) \in \mathcal{A}$ and all $(x_1, x_2) \in S^1 \times \mathbb{R}+$, $u(x_1, x_2) \leqslant C e^{-x_2/\varepsilon}$. Moreover, the set $\mathcal{A}$ is relatively compact for the topology associated to the norm defined as follows $\|u\| = \|u\|_{H^2(S^1\times\mathbb{R}+)} + \|u e^{x_2/\varepsilon_2}\|_{L^\infty(S^1\times\mathbb{R}+)}$.

Proposition 1.2. *Let $(\varepsilon, u) \in \mathbb{R} \times L^\infty(S^1 \times \mathbb{R})$ be a solution of (E), $u > 0$, u even in x_1 and x_2, $\lim_{x_2\to\infty} u = 0$ and such that $\frac{\partial u}{\partial x_1}$ has a constant sign in $]0, \pi[\, \times \mathbb{R}+$. Then the kernel of the linearized operator $-\varepsilon^2\Delta + I - pu^{p-1}I$ in $\{\phi \in H^1(S^1 \times \mathbb{R}), \phi$ even in x_1 and $x_2\}$ has the dimension 0 or 1.*

Let us summarize the results of the present paper in the following theorem

Theorem 1.2.

(i) *For $p > 1$, the first continuum Σ_1 of positive bounded solutions even in x_1 and x_2 of $-\varepsilon^2\Delta u + u - u^p = 0$ bifurcating from $(\varepsilon_\star, w_0(x_2/\varepsilon_\star))$ is composed of $(\varepsilon_\star, w_0(x_2/\varepsilon_\star))$ and of all the solutions (ε, u) of (E) such that $u > 0$, u even in x_1 and x_2, $\lim_{x_2\to\infty} u = 0$ and $\frac{\partial u}{\partial x_1} > 0$ in $]0, \pi[\, \times \mathbb{R}+$.*

(ii) *There exists a bounded subset $\mathcal{A}$ of $L^\infty(S^1 \times \mathbb{R}+)$ such that the set Σ_1 is entirely contained in $]0, \varepsilon_\star] \times \mathcal{A}$.*

(iii) *For each $(\varepsilon, u) \in \Sigma_1$, u is an isolated point of $\{v \in L^\infty(S^1 \times \mathbb{R}+);\ v$ even in x_1 and x_2; (ε, v) solution of $(E)\}$. For every $\varepsilon > 0$, $\varepsilon < \varepsilon_\star$, there exists a finite number of solutions (ε, u) in Σ_1.*

(iv) *There exists ε_0 such that for all $0 < \varepsilon < \varepsilon_0$ this continuum is a curve that has a one-to-one $\mathcal{C}^1$ parameterization $\varepsilon \to (\varepsilon, u_\varepsilon)$. Moreover, for each $(\varepsilon_1, u) \in \Sigma_1$ there exists a continuous map $\varepsilon \mapsto (\varepsilon, u_\varepsilon)$ from $]0, \varepsilon_\star]$ to Σ_1 such that $u_{\varepsilon_1} = u$. For $p \in \mathbb{N}$, the continuum is constituted by a finite number of curves that admit local analytic parameterizations.*

We have to define Banach spaces of functions that are suitable for our purposes. We will use the following notations:

$$B = \Big\{u \in L^\infty\big(S^1 \times \mathbb{R}\big);\ u \text{ even in } x_1 \text{ and } x_2;\ \lim_{|x_2|\to\infty} u(x_1, x_2) = 0 \text{ uniformly in } x_1\Big\},$$

$$X = H^1\big(S^1 \times \mathbb{R}\big) \cap B$$

and

$$U = \left\{u \in X;\ u > 0;\ \frac{\partial u}{\partial x_1} > 0 \text{ in }]-\pi, 0[\, \times \mathbb{R}^+ \text{ and } \frac{\partial u}{\partial x_1} < 0 \text{ in }]0, \pi[\, \times \mathbb{R}^+\right\}.$$

The vector space X is a Banach space for the norm $\|u\|_X = \|u\|_{H^1(S^1\times\mathbb{R}^+)} + \|u\|_{L^\infty(S^1\times\mathbb{R}^+)}$. We consider that the first continuum Σ_1 is obtained by bifurcation from the solution $(\varepsilon_\star, w_0(x_2/\varepsilon_\star))$ in the space X. We will recall the beginning of the construction of Σ_1 in Section 5. Let us define, for $\varepsilon > 0$

$$Y_\varepsilon = \big\{u \in X;\ u e^{x_2/\varepsilon} \in L^\infty\big(S^1 \times \mathbb{R}^+\big);\ \Delta u \in L^2\big(S^1 \times \mathbb{R}^+\big)\big\}.$$

The vector space Y_ε is a Banach space for the norm $\|u\|_{Y_\varepsilon} = \|u\|_{H^1(S^1\times\mathbb{R}^+)} + \|\Delta u\|_{L^2(S^1\times\mathbb{R}^+)} + \|u e^{x_2/\varepsilon}\|_{L^\infty(S^1\times\mathbb{R}^+)}$.

The paper is organized as follows. In Section 2 we prove or recall various preliminary lemmas and we establish compactness results, especially Proposition 1.1. Section 3 is devoted to properties of the linearized operator and to some uniqueness results for the solutions (ε, u), $u \in U$. It contains the proof of Proposition 1.2. We complete the proof of Theorem 1.2 in Section 4. In Section 5 we present some calculation that proves directly that, for $p \geqslant 2$, the bifurcation from the solution $(\varepsilon_\star, w_0(\frac{x_2}{\varepsilon_\star}))$ is not vertical and goes in the sense of the decreasing ε. We give the value of $\varepsilon_\star$ and of the first eigenfunction.

2. Preliminary lemmas and compactness results

We begin by the useful following two lemmas.

Lemma 2.1. *Let $C_\star = (\frac{p+1}{2})^{\frac{1}{p-1}}$. Let (ε, u) be a positive solution of* (E). *Then $\|u\|_{L^\infty(S^1\times\mathbb{R}^+)} \geqslant C_\star$, the equality being true only for $u(x_1,x_2) = w_0(\frac{x_2}{\varepsilon})$.*

Proof. Multiplying (E) by $\frac{\partial u}{\partial x_2}$ and integrating on $S^1 \times \mathbb{R}^+$ we get the identity

$$\frac{\varepsilon^2}{2}\int_0^{2\pi}\left(\frac{\partial u}{\partial x_1}\right)^2(x_1,0)\,dx_1 = \int_0^{2\pi}\left(-\frac{u^2}{2}+\frac{u^{p+1}}{p+1}\right)(x_1,0)\,dx_1.$$

Consequently if $\|u\|_{L^\infty(S^1\times\mathbb{R}^+)} \leqslant C_\star$ we have that $\frac{\partial u}{\partial x_1}(x_1,0) = 0$ for all $x_1 \in S^1$. But as it is proved in [2] this implies that $\frac{\partial u}{\partial x_1}(x_1,x_2) = 0$ for all $(x_1,x_2) \in S^1 \times \mathbb{R}^+$. Then $u = w_0(\frac{x_2}{\varepsilon})$ and we know that $\|w_0\|_{L^\infty(\mathbb{R})} = w_0(0) = C_\star$. □

Lemma 2.2. *Let (ε_1, u_1) be a solution of* (E). *If $u_1 > 0$ and if (ε, u) is any solution of* (E) *that is sufficiently closed to (ε_1, u_1) for the norm of $\mathbb{R} \times L^\infty(S^1 \times \mathbb{R}+)$, then $u > 0$. If $u_1 \in U$ and if (ε, u) is any solution of* (E) *sufficiently closed to (ε_1, u_1) for the norm of $\mathbb{R} \times X$, then $u \in U$.*

Proof. First, we have that $u > 0$ for any solution (ε, u) of (E) closed to (ε_1, u_1) in $\mathbb{R} \times L^\infty(S^1 \times \mathbb{R}+)$. The proof follows from the maximum principle and is given in [10], Section 1. Now, as in [10], Section 2, we have that $\frac{\partial u}{\partial x_1}$ has a constant sign in $]0,\pi[$ and in $]-\pi,0[$, since, if it is closed to (ε_1,u_1), it is the first eigenfunction of the eigenvalue problem $-\varepsilon^2\Delta v + v - pu^{p-1}v = \alpha v$. But for (ε, u) sufficiently closed to (ε_1, u_1) in $\mathbb{R} \times X$, these constant signs have to be those of (ε_1, u_1). Indeed, if the signs of $\frac{\partial u}{\partial x_1}$ and of $\frac{\partial u_1}{\partial x_1}$ in $]0,\pi[$ are not the same, we have $\int_{[0,\pi]\times\mathbb{R}+}(\frac{\partial u}{\partial x_1} - \frac{\partial u_1}{\partial x_1})^2 \geqslant \int_{[0,\pi]\times\mathbb{R}+}(\frac{\partial u_1}{\partial x_1})^2$, that cannot be true if we suppose that $\|u-u_1\|_X^2 < \int_{[0,\pi]\times\mathbb{R}+}(\frac{\partial u_1}{\partial x_1})^2$. □

Lemma 2.3.

(i) *There exists $M > 0$ such that for every solution (ε, u) of* (E), *with $u > 0$, $\lim_{|x_2|\to\infty} u(x_1,x_2) = 0$ uniformly in x_1 and $u \in L^\infty(S^1 \times \mathbb{R})$, we have $\|u\|_{L^\infty(S^1\times\mathbb{R})} \leqslant M$.*
(ii) *Let (ε_k, u_k) be a solution of* (E), *defined for $\varepsilon_k \to 0$ such that $u_k \in U$. Then $\tilde u_k : x \mapsto u_k(\varepsilon_k x)$ tends to w_1 as $\varepsilon_k \to 0$, i.e. $\|\tilde u_k - w_1\|_{L^\infty(S^1/\varepsilon_k\times\mathbb{R}+)} \to 0$.*
(iii) *If (ε, u) and (ε, v) are solutions of* (E), *u and v in X, $u \neq v$, then $u - v$ has not a constant sign in $S^1 \times \mathbb{R}+$.*

Proof. (i) Let (ε, u) be solutions of (E). We may suppose that u is even in x_2 and decreasing in x_2. Up to a translation in x_1, we may suppose that the maximum of u is attained at $(0,0)$. Let us recall why u is bounded in $L^\infty(S^1 \times \mathbb{R}^+)$ (here ε tends to 0 or not). Let α be a positive real number to be chosen later. We set $v(x) = u(\alpha x)/\|u\|_{L^\infty(S^1\times\mathbb{R}^+)}$. It verifies in $S^1/\alpha \times \mathbb{R}^+$

$$-\Delta v + (\alpha^2/\varepsilon^2)v - (\alpha^2/\varepsilon^2)\|u_\varepsilon\|^{p-1}_{L^\infty(S^1\times\mathbb{R}^+)}v^p = 0.$$

If $\|u\|_{L^\infty(S^1\times\mathbb{R}^+)}$ tends to $+\infty$, we choose α/ε that tends to 0 such that $(\alpha^2/\varepsilon^2)\|u\|^{p-1}_{L^\infty(S^1\times\mathbb{R}^+)}$ tends to 1. We obtain by standard estimates that v is bounded in $H^1(K)$ for all compact subset K of $\mathbb{R}^2$, and consequently a subsequence of v tends to a limit $\bar v$ that is a non-negative bounded solution in $\mathbb{R}^2$ of

$$-\Delta v - v^p = 0.$$

But such a solution is identically null. (Let $v_0(r) = \frac{1}{2\pi}\int_0^{2\pi}\bar v(r,\theta)\,d\theta$ then $-v_0'' - \frac{v_0'}{r} \geqslant 0$, $v_0 \geqslant 0$, that gives $rv_0'(r) \leqslant 0$ for all $r > 0$. But if there exists $r_0 > 0$ such that $v_0'(r_0) < 0$, then v_0 tends to $-\infty$ as r tends to $+\infty$. Thus $v_0' = 0$. Consequently $\int_0^{2\pi}\bar v^p = 0$ and $\bar v = 0$.) On the other hand, v attains its maximum in $x = 0$, then v tends uniformly on the compact sets of $\mathbb{R}^2$ to a limit that is not identically null. This contradiction proves that $\|u\|_{L^\infty(S^1\times\mathbb{R}^+)}$ is bounded.

(ii) If ε_k tends to 0, $\tilde u_k$ is bounded in $L^\infty(\frac{S^1}{\varepsilon_k} \times \mathbb{R}^+)$, attains its maximum at $(0,0)$, $\tilde u_k(0,0) \geqslant C_\star$ and verifies the equation $-\Delta u + u - u^p = 0$. Then, by standard elliptic arguments [14] $\tilde u_k$ converges uniformly in the compact subsets of $\mathbb{R}^2$ to a limit w such that $-\Delta w + w - w^p = 0$, $\|w\|_\infty \geqslant C_\star$, w is decreasing in x_1 and x_2 and $w \geqslant 0$. It is proved in [10] that $w = w_1$.

Now let $A_k = (a_k, b_k) \in S^1/\varepsilon_k \times R+$ be such that $\|\tilde{u}_k - w_1\|_{L^\infty(S^1/\varepsilon_k \times \mathbb{R}+)} = (\tilde{u}_k - w_1)(A_k)$ and, say, $a_k \to +\infty$ and b_k bounded. For all $x_1 \in \mathbb{R}+$ there exists K such that for all $k > K$ we have $x_1 \in S^1/\varepsilon_k$, $x_1 < a_k$. Then $\tilde{u}_k(a_k, b_k) \leqslant \tilde{u}_k(x_1, b_k)$. Up to a subsequence, we have that b_k tends to a limit b, thus $\lim \tilde{u}_k(a_k, b_k) \leqslant \lim \tilde{u}_k(x_1, b_k) = w_1(x_1, b)$. But $w_1(x_1, b)$ tends to 0 as x_1 tends to $+\infty$, thus $\tilde{u}_k(a_k, b_k)$ tends to 0 as k tends to $+\infty$, so $\|\tilde{u}_k - w_1\|_{L^\infty(S^1/\varepsilon_k \times \mathbb{R}+)}$ tends to 0. The same proof works if a_k and b_k tend to $+\infty$ or if a_k is bounded while b_k tends to $+\infty$.

(iii) Let (ε, u) and (ε, v) be two solutions, u and v in X, $u > 0$, $v > 0$. Let $w = u - v$. Then

$$-\varepsilon^2 \Delta w + w - \frac{u^p - v^p}{u - v} w = 0.$$

Let us suppose that $w < 0$. A convexity inequality gives

$$\frac{u^p - v^p}{u - v} > p u^{p-1}.$$

Multiplying the equation above by u we get

$$\int\limits_{S^1 \times \mathbb{R}^+} \varepsilon^2 \nabla u \cdot \nabla w + u w < p \int\limits_{S^1 \times \mathbb{R}^+} u^p w.$$

Multiplying (E), that is verified by u, by w we obtain

$$\int\limits_{S^1 \times \mathbb{R}^+} \varepsilon^2 \nabla u \cdot \nabla w + u w = \int\limits_{S^1 \times \mathbb{R}^+} u^p w.$$

Consequently,

$$\int\limits_{S^1 \times \mathbb{R}^+} u^p w < \int\limits_{S^1 \times \mathbb{R}^+} p u^p w$$

that is impossible, since $u > 0$, $w < 0$ and $p > 1$. □

We have now the following propositions, that will permit to rely the topologies of X and Y_ε.

Lemma 2.4. *Let $\varepsilon_1, \varepsilon_2$, $0 < \varepsilon_1 < \varepsilon_2$, and $\mathcal{A}$ be a bounded set of $L^\infty(S^1 \times \mathbb{R})$. The sets of positive solutions of* (E)*, such that u is even in x_2 and $u(x_1, x_2)$ tends to* 0 *when $|x_2|$ tends to $+\infty$ uniformly in x_1, that are included in $[\varepsilon_1, \varepsilon_2] \times \mathcal{A}$ are relatively compact for the topology of $\mathbb{R} \times L^\infty(S^1 \times \mathbb{R})$.*

Proof. Let (ε_m, u_m) be a sequence of solutions of (E), in $[\varepsilon_1, +\infty[\times \mathcal{A}$, as above. It follows from standard elliptic theory that we extract a subsequence, still denoted by (ε_m, u_m), such that ε_m tends to a limit $\varepsilon > 0$, u_m tends to a limit u uniformly on the compact subsets of $S^1 \times \mathbb{R}^+$ and $\|u\|_\infty \geqslant C_\star$. Let us explain why u tends to 0 as x_2 tends to $+\infty$. If it is not true it is not difficult to see that for all ε_1 small enough there exist two sequences $x_{1,m} \in S^1$ and $x_{2,m} \to +\infty$ such that $u_m(x_{1,m}, x_{2,m}) = \varepsilon_1$. Now we apply the proof in [10]. Let us recall it for completeness. We set $v_m(x_1, s) = u_m(x_1, x_{2,m} + s)$ in $S^1 \times]-x_{2,m}, +\infty[$. We have $v_m(x_{1,m}, 0) = \varepsilon_1$. There exists v solution of $-\varepsilon^2 \Delta v + v - v^p = 0$ in $S^1 \times \mathbb{R}$, v non-increasing in s, such that v_m tends to v uniformly in all compact sets of $S^1 \times \mathbb{R}$ and if $\lim x_{1,m} = \alpha$, $v(\alpha, 0) = \varepsilon_1$. As in [10] (page 547), we see, by the maximum principle and the Harnack inequality, that if ε_1 is chosen small enough, then v cannot depend only on one variable. Let $v_-(x_1) = \lim_{s \to -\infty} v(x_1, s)$ and $v_+(x_1) = \lim_{s \to +\infty} v(x_1, s)$. The functions v_+ and v_- are bounded in S^1, verify the ordinary equation $-\varepsilon^2 g'' + g - g^p = 0$, $\|v_+\|_\infty \leqslant \varepsilon_1$ and $\|v_-\|_\infty \geqslant \varepsilon_1$. If ε_1 is chosen small enough, then $v_+ = 0$, by the above principle. If v_- is not a constant function, then $-\frac{\varepsilon^2}{2} v_-'^2 + \frac{1}{2} v_-^2 - \frac{1}{p+1} v_-^{p+1}$ is a constant function. It follows that both the maximum value and the minimum value of v_- are less than $C_\star$. But the energy of v_+ and v_- is the same, that gives $\int_{S^1} (\frac{v_-'^2}{2} + \frac{v_-^2}{2} - \frac{v_-^{p+1}}{p+1}) = 0$. Thus $v_-' = 0$, $\frac{v_-^2}{2} - \frac{v_-^{p+1}}{p+1} = 0$ and $v_- = 0$ or 1. That gives $v_- = 0$, that is a contradiction with $\|v_-\|_\infty \geqslant \varepsilon_1$. This gives a contradiction, so u tends to 0 as x_2 tends to $+\infty$.

Let us prove that the convergence is uniform in $S^1 \times \mathbb{R}^+$. Let us suppose that $A_m = (a_m, b_m) \in S^1 \times \mathbb{R}^+$ is such that $\|u_m - u\|_{L^\infty(S^1\times\mathbb{R}^+)} = (u_m - u)(A_m)$ and that b_m tends to $+\infty$. For all x_2, we have, for m large enough, $u_m(a_m, x_2) > u_m(a_m, b_m)$. Let $x_2 > 0$ be given, let $m \to +\infty$ and suppose (up to a subsequence) that a_m tends to a, we obtain that, for all $x_2 > 0$, $u(a, x_2) \geqslant \overline{\lim}_{m\to+\infty} u_m(A_m)$. But u tends to 0 as x_2 tends to $+\infty$. Consequently $u_m(A_m)$ tends to 0, and then $\|u_m - u\|_{L^\infty(S^1\times\mathbb{R}^+)}$ tends to 0. □

Proposition 2.3. *Let $\varepsilon > 0$ be given and let u be a positive solution of* (E)*, such that u is even in x_2 and $u(x_1, x_2)$ tends to* 0 *when $|x_2|$ tends to $+\infty$ uniformly in x_1. Then there exist two positive real numbers C_1 and C_2, depending on u and on ε, such that for all $x_2 > 0$ and for all $x_1 \in S^1$,*

$$C_1 e^{-x_2/\varepsilon} \leqslant u(x_1, x_2) \leqslant C_2 e^{-x_2/\varepsilon}.$$

Moreover, if $(\varepsilon_1, u_1) \in \mathbb{R} \times L^\infty(S^1 \times \mathbb{R}+)$ is a solution of (E)*, $u_1 > 0$, there exists $\delta > 0$ and $C > 0$ such that for all solution (ε, u) of* (E) *in $\mathbb{R} \times L^\infty(S^1 \times \mathbb{R}+)$ that verifies $|\varepsilon - \varepsilon_1| + \|u_1 - u\|_\infty < \delta$, we have for all $(x_1, x_2) \in S^1 \times \mathbb{R}+$, $u(x_1, x_2) \leqslant C e^{-x_2/\varepsilon}$.*

Proof. Let us define $\Psi(x_2) = \frac{1}{2\pi}\int_0^{2\pi} u(x_1, x_2)\, dx_1$. We will prove first that

$$u \leqslant C_2 e^{-x_2/\varepsilon}.$$

The first step will be to prove that for all $0 < \beta < 1/\varepsilon$ there exists a positive real number C_0 such that for all $x_2 > 0$

$$\Psi(x_2) \leqslant C_0 e^{-\beta x_2} \tag{2.1}$$

and then to deduce the same inequality for u, by use of the Harnack inequalities.

Integrating (E) on $[0, 2\pi]$ we obtain

$$-\varepsilon^2 \Psi''(x_2) + \Psi(x_2) - \frac{1}{2\pi}\int_0^{2\pi} u^p\, dx_1 = 0. \tag{2.2}$$

Let us choose a real number α such that $1 - \alpha^{p-1} > 0$. There exists $A > 0$ such that for all $x_1 \in S^1$ and all $x_2 > A$ we have $u(x_1, x_2) \leqslant \alpha$. We remark that if u is sufficiently closed to u_1 for the uniform norm, we may choose $0 < \alpha < 1$ and A independent of u. Let us define β by $\beta^2 = \frac{1}{\varepsilon^2}(1 - \alpha^{p-1})$. We have by (2.2)

$$-\Psi''(x_2) + \beta^2 \Psi(x_2) \leqslant 0 \quad \text{for all } x_2 > A$$

that gives, by the maximum principle, for all $x_2 > A$,

$$\Psi(x_2) \leqslant \Psi(A) e^{-\beta(x_2 - A)}.$$

Then $w = e^{\beta x_2}\Psi(x_2)$ is bounded for large x_2 and we obtain (2.1). It is easy to verify that the constant C_0 in (2.1) may be chosen independently from (ε, u), for (ε, u) sufficiently closed to (ε_1, u_1) for the norm of $\mathbb{R} \times L^\infty$.

Now we will verify the following Harnack inequality. For all $R > 0$ there exists C such that, for all $y \in \mathbb{R}^2$, we have

$$\sup_{B_R(y)} u \leqslant C\Big(\inf_{B_R(y)} u + \Big(\inf_{B_R(y)} u\Big)^p\Big) \tag{2.3}$$

where the constant C depends on R and ε, does not depend on y. Indeed, we use first Theorem 8.17 in [14] for $L = \Delta$ and for the equation $\Delta u = \frac{1}{\varepsilon^2}(u - u^p)$ and then we use Theorem 8.18 in [14] for the equation $L = \varepsilon^2 \Delta u - u$ and for $Lu \leqslant 0$. The two inequalities that we obtain give (2.3). Moreover C decreases in ε ([2]). By (2.1) we have for all $x_2 > 0$

$$\inf_{x_1\in S^1} u(x_1, x_2) \leqslant C_0 e^{-\beta x_2}. \tag{2.4}$$

Let $x_2 > 0$ and $y = (0, x_2)$ and $R = \pi$. We have

$$\inf_{B_R(y)} u \leqslant \inf_{x_1\in S^1} u(x_1, x_2) \leqslant C_0 e^{-\beta x_2}.$$

This inequality, together with (2.3) gives a constant C_0' such that for all $x_2 > 0$ and all $x_1 \in S^1$ we have

$$u(x_1, x_2) \leqslant C_0' e^{-\beta x_2}. \tag{2.5}$$

Once more, the constant C_0' does not depend on (ε, u), chosen in a neighborhood of (ε_1, u_1).

The second step will be to prove that there exists a constant $C > 0$ such that for all x_2

$$\Psi(x_2) \leqslant C e^{-x_2/\varepsilon} \tag{2.6}$$

and to deduce the same inequality for u.

From (2.2) we deduce that for $x_2 > 0$ we have

$$-\varepsilon^2 \Psi'' + \Psi \leqslant C e^{-p\beta x_2}$$

and $\Psi'(0) = 0$. This implies that $\Psi \leqslant \phi$, where ϕ is the bounded solution of

$$-\varepsilon^2 \phi'' + \phi = C e^{-p\beta x_2}, \quad \phi'(0) = 0.$$

But we may suppose that $p\sqrt{1 - \alpha^{p-1}} > 1$ and consequently we have

$$\phi = A e^{-p\beta x_2} + B e^{-x_2/\varepsilon},$$

with

$$A = C/\big(1 - \big(p^2\beta^2\varepsilon^2\big)\big) \quad \text{and} \quad B = -Ap\beta\varepsilon.$$

Thus we have proved (2.6). We can deduce by the same proof as above the existence of a constant C_2 such that for $x_2 > 0$ and for all $x_1 \in S^1$

$$u(x_1, x_2) \leqslant C_2 e^{-x_2/\varepsilon}.$$

Once more we may choose C_2 independent from (ε, u) closed to (ε_1, u_1).

Let us prove now that

$$u \geqslant C_1 e^{-x_2/\varepsilon}.$$

The first step will be to prove that this is true for Ψ. From (2.2) we have

$$-\varepsilon^2 \Psi'' + \Psi \geqslant 0 \quad \text{for } x_2 > 0$$

and then we have by the maximum principle

$$\Psi(x_2) \geqslant \Psi(0) e^{-x_2/\varepsilon} \quad \text{for } x_2 > 0.$$

Now we can write:

$$u = \sum_{j \geqslant 0} a_j(x_2) \cos(jx_1) + \sum_{j \geqslant 1} b_j(x_2) \sin(jx_1)$$

with $\Psi = a_0$. We have for $j \geqslant 1$,

$$-\varepsilon^2 a_j'' + \big(1 + j^2\big) a_j = \frac{1}{\pi} \int_0^{2\pi} u^p \cos(jx_1)\, dx_1$$

then

$$\big|-\varepsilon^2 a_j'' + \big(1 + j^2\big) a_j\big| \leqslant C e^{-px_2/\varepsilon}.$$

Let us choose $1 < \tilde{p} < \min\{p, \sqrt{2}\}$. Therefore, $|a_j| \leqslant \phi_j$ where ϕ_j is the solution of

$$-\varepsilon^2 \phi_j'' + \big(1 + j^2\big)\phi_j = C e^{-\tilde{p}x_2/\varepsilon}, \quad \phi_j'(0) = 0, \quad \phi_j \longrightarrow 0 \text{ at infinity}.$$

The function ϕ_j is a combination of $e^{-\tilde{p}x_2/\varepsilon}$ and $e^{-\sqrt{1+j^2}x_2/\varepsilon}$, then $a_j = o(e^{-x_2/\varepsilon})$, for $j \geqslant 1$. We have a similar result for b_j, $j \geqslant 1$. We infer that $u \sim \Psi$ at infinity, in the sense that u/ψ tends to 1 as x_2 tends to $+\infty$, uniformly in x_1. $\square$

Corollary 2.1. *Let $\varepsilon_1, \varepsilon_2$, $0 < \varepsilon_1 < \varepsilon_2$, and $\mathcal{A}$ be a bounded set of $L^\infty(S^1 \times \mathbb{R})$. The sets of positive solutions of* (E), *such that u is even in x_2 and $u(x_1, x_2)$ tends to 0 when $|x_2|$ tends to $+\infty$ uniformly in x_1, that are included in $[\varepsilon_1, \varepsilon_2] \times \mathcal{A}$ are relatively compact for the topology of $\mathbb{R} \times H^1(S^1 \times \mathbb{R})$.*

Proof. Returning to the proof of Lemma 2.4, it remains to verify that u_m tends to u for the $H^1(S^1 \times \mathbb{R}+)$-norm. We know that the sequence (u_m) tends to u for the $L^\infty(S^1 \times \mathbb{R}^+)$ norm and that (ε_m, u_m) is a solution of (E) for all m, $u_m > 0$. By Proposition 2.3 there exists a positive constant C such that for all $x \in S^1 \times \mathbb{R}^+$ and for all m we have $u_m(x) \leqslant Ce^{-\frac{x_2}{\varepsilon_m}}$. This implies that u_m tends to u in $H^1(S^1 \times \mathbb{R}+)$, since we have

$$\int\limits_{S^1\times\mathbb{R}^+} \varepsilon_m^2 \big|\nabla(u_m - u)\big|^2 + \int\limits_{S^1\times\mathbb{R}+} \big(\big(\varepsilon^2 - \varepsilon_m^2\big)\Delta u(u_m - u) + (u_m - u)^2\big) = \int\limits_{S^1\times\mathbb{R}^+} \frac{u_m^p - u^p}{u_m - u}(u_m - u)^2,$$

and the right member is less than

$$pC^{p-1} \int\limits_{S^1\times\mathbb{R}^+} e^{(-p+1)x_2/\varepsilon}(u_m - u)^2,$$

that tends to 0 by the Lebesgue theorem. Consequently u_m tends to u for the $H^1(S^1 \times \mathbb{R}^+)$-norm. □

Remark 2.1. Let (ε_m, u_m) be a sequence of solutions of (E), such that $u_m \in U$ for all m, that tends to a limit (ε, u) in $\mathbb{R} \times X$. We have $\|u_m\|_{L^\infty(S^1\times\mathbb{R}^+)} \geqslant C_*$, thus the limit u is not 0. We deduce easily that either $u \in U$ or $u(x_1, x_2) = w_0(\frac{x_2}{\varepsilon})$.

Proposition 2.4. *Let $\varepsilon > 0$ be given and let $u \in Y_\varepsilon$, $u > 0$. Then the operator*

$$L = -\varepsilon^2\Delta + I - pu^{p-1}I$$

is a Fredholm operator of index 0 from the Banach space Y_ε to its topological dual space Y'_ε.

Proof. There exist results for the Fredholm property for general linear elliptic problems in unbounded domains [20]. Let us give the proof that we did for this particular problem. We will prove that $v \to (-\varepsilon^2\Delta + I)^{-1}(pu^{p-1}v)$ is a compact operator from Y_ε to Y_ε, where $t = (-\varepsilon^2\Delta + I)^{-1}(pu^{p-1}v)$ is defined as the solution in $H^1(S^1 \times \mathbb{R}^+)$ of the equation

$$-\varepsilon^2\Delta t + t = pu^{p-1}v \quad \text{in } S^1 \times \mathbb{R}^+, \qquad \frac{\partial t}{\partial \nu} = 0 \quad \text{in } S^1 \times \{0\}.$$

We use the Fourier expansion

$$t = \sum_{j=0}^{+\infty} t_j(x_2)\cos(jx_1).$$

We get for all j

$$-\varepsilon^2 t_j'' + \big(1 + \varepsilon^2 j^2\big)t_j = \frac{1}{\pi}\int\limits_0^{2\pi} pu^{p-1}v\cos(jx_1)\,dx_1, \qquad t_j'(0) = 0.$$

Now we are going to prove that $t \in Y_\varepsilon$ together with an estimate of $\|e^{x_2/\varepsilon}t\|_{L^\infty(S^1\times\mathbb{R}+)}$. There exists a constant $C > 0$ such that

$$\left|\int\limits_0^{2\pi} pu^{p-1}v\cos(jx_1)\,dx_1\right| \leqslant Ce^{(-p+1)x_2/\varepsilon}\int\limits_0^{2\pi} |v|\,dx_1.$$

For any positive real number $m > 1$ there exists another constant C such that

$$\left|\int_0^{2\pi} pu^{p-1} v\cos(jx_1)\,dx_1\right| \leqslant Ce^{(-p+1/m)x_2/\varepsilon}\left\|e^{x_2/\varepsilon}v\right\|^{1-1/m}_{L^\infty(S^1\times\mathbb{R}+)}\|v\|^{1/m}_{L^\infty(S^1\times\mathbb{R}+)}.$$

Let $\tilde{p} = \min\{p, \sqrt{1+\varepsilon^2}\}$. Let us choose $m > 1$ such that $1 < \tilde{p} - \frac{1}{m} < \sqrt{1+\varepsilon^2}$. Thus we have $\tilde{p} - \frac{1}{m} \neq \sqrt{1+\varepsilon^2 j^2}$ for all $j \in \mathbb{N}$. We define

$$f(x_2) = e^{(-\tilde{p}+1/m)x_2/\varepsilon}.$$

The bounded solution of the equation

$$-\varepsilon^2\phi'' + \left(1+\varepsilon^2 j^2\right)\phi = f, \quad \phi'(0) = 0$$

is

$$\phi_j(x_2) = \frac{e^{(-\tilde{p}+1/m)x_2/\varepsilon}}{1+\varepsilon^2 j^2 - (-\tilde{p}+\frac{1}{m})^2} + \frac{(-\tilde{p}+\frac{1}{m})e^{-\sqrt{1+\varepsilon^2 j^2}x_2/\varepsilon}}{\sqrt{1+\varepsilon^2 j^2}(1+\varepsilon^2 j^2 - (-\tilde{p}+\frac{1}{m})^2)}.$$

We obtain $|t_j| \leqslant \phi_j$ and then, for all $x_2 \in \mathbb{R}$,

$$\left|e^{x_2/\varepsilon}t_j(x_2)\right| \leqslant C\left\|e^{x_2/\varepsilon}v\right\|^{1-1/m}_{L^\infty(S^1\times\mathbb{R}+)}\|v\|^{1/m}_{L^\infty(S^1\times\mathbb{R}+)}\left(e^{x_2/\varepsilon}\phi_j(x_2)\right).$$

The summation over j shows that for our choice of $m > 1$ such that $-\tilde{p} + 1 + \frac{1}{m} < 0$, there exists a constant C such that for all x_2

$$\left|e^{x_2/\varepsilon}t(x_2)\right| \leqslant C\left\|e^{x_2/\varepsilon}v\right\|^{1-1/m}_{L^\infty(S^1\times\mathbb{R}+)}\|v\|^{1/m}_{L^\infty(S^1\times\mathbb{R}+)}. \tag{2.7}$$

Thus it is clear that $t \in Y_\varepsilon$. More, if (v_m) is a bounded sequence in Y_ε, then the corresponding sequence (t_m) is bounded in Y_ε. Let us prove that the operator $v \to t$ is compact from Y_ε to Y_ε. We remark first that the injection of Y_ε into $L^q(S^1 \times \mathbb{R}+)$ is compact for all $q \geqslant 1$, including $q = +\infty$. Indeed, any bounded set in Y_ε is compact in $L^2(K)$, where K is any compact subset of $S^1 \times \mathbb{R}+$. If v_m is in a bounded subset of Y_ε, we construct a subsequence v_m that tends to a limit v in $L^2_{\mathrm{loc}}(S^1 \times \mathbb{R}+)$, by a standard diagonal process. But there exists a constant C such that, for all m, $\|e^{x_2/\varepsilon}v_m\|_{L^\infty(S^1\times\mathbb{R}+)} \leqslant C$. Consequently, v_m tends to v in $L^q(S^1 \times \mathbb{R}+)$, $q \geqslant 1$, by the Lebesgue theorem. Moreover (v_m) is bounded in $H^2(S^1 \times \mathbb{R}+)$, since it is bounded in $H^1(S^1 \times \mathbb{R}+)$ while Δv_m is bounded in $L^2(S^1 \times \mathbb{R}+)$. (This can be easily verified. For example we write $v = \sum v_j \cos(jx_1)$ and $-\Delta v = \sum w_j \cos(jx_1)$ and we estimate the L^2-norms of v_j'', v_j' and v_j with respect to the L^2-norm of w_j). We deduce that (v_m) tends to its limit v uniformly in the compact sets of $S^1 \times \mathbb{R}+$. But there exists C such that for all m and all (x_1, x_2), $v_m(x_1, x_2)$ and $v(x_1, x_2) \leqslant Ce^{-x_2/\varepsilon}$. Consequently (v_m) tends to v uniformly in $S^1 \times \mathbb{R}+$.

So, if v_m is in a bounded subset of Y_ε, there exists a subsequence of t_m, still denoted by t_m that tends to a limit t in $L^2(S^1 \times \mathbb{R}^+)$ and it is not difficult to see that t_m tends to t in $H^1(S^1 \times \mathbb{R}+)$. At the same time there exists a subsequence of v_m, still denoted by v_m that tends to a limit v in $L^\infty(S^1 \times \mathbb{R}+)$ and we have

$$-\Delta(t_m - t) + t_m - t = pu^{p-1}(v_m - v).$$

By (2.7), in which we substitute $t_m - t$ to t and $v_m - v$ to v and using the fact that $v_m - v$ is bounded in Y_ε, we get that $\lim\|t_m - t\|_{Y_\varepsilon} = 0$. □

Proposition 2.5. *Let $0 < \varepsilon_1 < \varepsilon_2$ be given. If $u \in Y_{\varepsilon_2}$ and $\varepsilon_1 \leqslant \varepsilon \leqslant \varepsilon_2$, then $(-\varepsilon^2\Delta + I)^{-1}(|u|^{p-1}u) \in Y_{\varepsilon_2}$ and the operator*

$$\begin{aligned}&[\varepsilon_1, \varepsilon_2] \times Y_{\varepsilon_2} \to Y_{\varepsilon_2},\\ &\qquad(\varepsilon, u) \to \left(-\varepsilon^2\Delta + I\right)^{-1}\left(|u|^{p-1}u\right)\end{aligned}$$

is compact.

Proof. The function $t = (-\varepsilon^2\Delta + I)^{-1}(|u|^{p-1}u)$ is the solution in $H^1(S^1 \times \mathbb{R}^+)$ of

$$-\varepsilon^2\Delta t + t = |u|^{p-1}u \quad \text{in } S^1 \times \mathbb{R}^+, \qquad \frac{\partial t}{\partial \nu} = 0 \quad \text{in } S^1 \times \{0\}.$$

Let us prove that $t \in Y_{\varepsilon_2}$. We write $t = \sum_{j=0}^{+\infty} t_j(x_2)\cos(jx_1)$ where t_j is the bounded solution of

$$-\varepsilon^2 t_j'' + \left(1+\varepsilon^2 j^2\right)t_j = \frac{1}{\pi}\int_0^{2\pi} |u|^{p-1}u\cos(jx_1)\,dx_1, \qquad t_j'(0) = 0.$$

There exists $C > 0$ such that $|u(x_1,x_2)| \leqslant Ce^{-x_2/\varepsilon_2}$, so there exists another constant C such that

$$\left|-\varepsilon^2 t_j'' + \left(1+\varepsilon^2 j^2\right)t_j\right| \leqslant Ce^{-px_2/\varepsilon_2}.$$

Now, if $\varepsilon < \varepsilon_2$, we will replace p by $\tilde{p} = 1$ and we have, for all $j \in \mathbb{N}$, $\sqrt{1+\varepsilon^2 j^2} \neq \frac{\tilde{p}\varepsilon}{\varepsilon_2}$. If $\varepsilon = \varepsilon_2$, the existence of $j_0 \in \mathbb{N}$ such that $\sqrt{1+\varepsilon^2 j_0^2} = p\frac{\varepsilon}{\varepsilon_2}$ is possible. In this case we take $\tilde{p} < p$ such that $\sqrt{1+\varepsilon^2(j_0-1)^2} < \tilde{p}\frac{\varepsilon}{\varepsilon_2} < \sqrt{1+\varepsilon^2 j_0^2}$, else, $\tilde{p} = p$, in order to have $\sqrt{1+\varepsilon^2 j^2} \neq \tilde{p}\frac{\varepsilon}{\varepsilon_2}$ for all $j \in \mathbb{N}$. By comparison with the bounded solution of

$$-\varepsilon^2\phi'' + \left(1+\varepsilon^2 j^2\right)\phi = Ce^{-x_2\tilde{p}/\varepsilon_2} \qquad \phi'(0) = 0,$$

we deduce that

$$|t_j| \leqslant \frac{-C\varepsilon\varepsilon_2\tilde{p}}{(\varepsilon_2^2(1+j^2\varepsilon^2) - \varepsilon^2\tilde{p}^2)\sqrt{1+\varepsilon^2 j^2}}e^{-x_2\sqrt{1+\varepsilon^2 j^2}/\varepsilon} + \frac{C\varepsilon_2^2}{\varepsilon_2^2(1+j^2\varepsilon^2) - \varepsilon^2\tilde{p}^2}e^{-x_2\tilde{p}/\varepsilon_2}.$$

We sum over j and we obtain that $t \in Y_{\varepsilon_2}$.

Now if (ε_m, u_m) is bounded in $[\varepsilon_1, \varepsilon_2] \times Y_{\varepsilon_2}$, then up to a subsequence, we suppose that ε_m tends to a limit $\varepsilon > 0$. We note first, that by the proof above, $t_m e^{x_2/\varepsilon_2}$ is bounded in $L^\infty(S^1 \times \mathbb{R}^+)$. But u_m being bounded in Y_{ε_2} we have as in the proof of Proposition 2.4 that a subsequence tends to a limit u in $L^q(S^1 \times \mathbb{R}+)$ for all $1 \leqslant q \leqslant +\infty$. Since t_m is defined by

$$-\varepsilon_m^2\Delta t_m + t_m = |u_m|^{p-1}u_m$$

we deduce that t_m is bounded in $H^1(S^1 \times \mathbb{R}^+)$ and consequently in Y_{ε_2}. Thus a subsequence of t_m tends to a limit t in $L^q(S^1 \times \mathbb{R}^+)$ for all $1 \leqslant q \leqslant \infty$. By the Lebesgue theorem we have that $t_m - t$ tends to 0 in $H^1(S^1 \times \mathbb{R}^+)$ and Δt_m tends to Δt in $L^2(S^1 \times \mathbb{R}+)$. To make the end of the proof easier we will prove only that for all $0 < \varepsilon \leqslant \varepsilon_2$ if (u_m) tends to u in Y_{ε_2} and if $t_m = (-\varepsilon^2\Delta + I)^{-1}(|u_m|^{p-1}u_m)$ then (t_m) tends to t in Y_{ε_2}. We have

$$-\varepsilon^2\Delta(t_m - t) + (t_m - t) = |u_m|^{p-1}(u_m - u) + \left(|u_m|^{p-1} - |u|^{p-1}\right)(u - u_m).$$

Let $C > 0$ be such that for all m $|u_m| \leqslant Ce^{-x_2/\varepsilon_2}$. Letting

$$t_m - t = \sum_{j=0}^{+\infty} v_j(x_2)\cos(jx_1)$$

we get an other constant C such that for all m and all j

$$\left|-\varepsilon^2 v_j'' + \left(1+\varepsilon^2 j^2\right)v_j\right| \leqslant Ce^{(-p+1)x_2/\varepsilon_2}\int_0^{2\pi} |u_m - u|.$$

Exactly as in the proof of Proposition 2.4 we deduce that there exists a constant C such that for all $(x_1,x_2) \in S^1 \times \mathbb{R}+$

$$\left|(t_m - t)(x_1,x_2)\right|e^{x_2/\varepsilon_2} \leqslant C\|u_m - u\|_{L^\infty(S^1\times\mathbb{R}^+)}$$

that tends to 0 as m tends to $+\infty$. Thus t_m tends to t in Y_{ε_2}. The proof of the proposition follows. □

Proof of Proposition 1.1. Let $\mathcal{A} \subset [\varepsilon_1, \varepsilon_2] \times L^\infty(S^1 \times \mathbb{R})$ be a set of positive solutions (ε, u) of (E), such that u is even in x_2 and $u(x_1, x_2)$ tends to 0 when $|x_2|$ tends to $+\infty$ uniformly in x_1. By Lemma 2.3(i), $\mathcal{A}$ is bounded in $\mathbb{R} \times L^\infty(S^1 \times \mathbb{R}+)$. By Lemma 2.4, its closure $\bar{\mathcal{A}}$ is compact in $\mathbb{R} \times L^\infty(S^1 \times \mathbb{R}+)$. Thus it can be recovered by a finite number of balls $B_i = B((\varepsilon_i, u_i), \delta_i)$, the δ_i being chosen, thanks to Proposition 2.3, such that for all i there exists $C_i > 0$ for which for all $(\varepsilon, u) \in B_i$, (ε, u) being a positive solution of (E), $u \leqslant C_i e^{-x_2/\varepsilon}$. Consequently, $\mathcal{A}$ is bounded in $[\varepsilon_1, \varepsilon_2] \times \tilde{Y}_{\varepsilon_2}$ where $\tilde{Y}_{\varepsilon_2} = \{u \in H^1(S^1 \times \mathbb{R}),\, u \text{ even in } x_2,\, \lim_{|x_2| \to +\infty} u = 0 \text{ uniformly in } x_1,\, \Delta u \in L^2(S^1 \times \mathbb{R})\}$ with the norm $\|u\|_{H^1(S^1 \times \mathbb{R}^+)} + \|\Delta u\|_{L^2(S^1 \times \mathbb{R}^+)} + \|u e^{x_2/\varepsilon}\|_{L^\infty(S^1 \times \mathbb{R}^+)}$. We remark that Proposition 2.5 is still valid if we replace Y_{ε_2} by $\tilde{Y}_{\varepsilon_2}$. It follows that $\bar{\mathcal{A}}$ is compact in $\mathbb{R} \times \tilde{Y}_{\varepsilon_2}$. For the solutions of (E), the norm of $\tilde{Y}_{\varepsilon_2}$ is equivalent to $\|u\|_{H^2(S^1 \times \mathbb{R}^+)} + \|u e^{x_2/\varepsilon}\|_{L^\infty(S^1 \times \mathbb{R}^+)}$. □

3. The linearized operator and some uniqueness results

Proposition 3.6. *There exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in]0, \varepsilon_0[$, there exists at most one solution (ε, u) of (E), with $u \in U$. Moreover, for all $0 < \varepsilon < \varepsilon_0$ and all solution (ε, u), $u \in U$, the operator $L = -\varepsilon^2 \Delta + I - p u^{p-1} I$ is an isomorphism from X to its topological dual space.*

Proof. Let us suppose that there exists a sequence ε_k that tends to 0 and two sequences of solutions (ε_k, u_k) and (ε_k, v_k), $u_k \in U$ and $v_k \in U$, $u_k \not\equiv v_k$ for all k. Let $z_k(x_1, x_2) = \tilde{u}_k(x_1, x_2) - \tilde{v}_k(x_1, x_2)$ be defined in $S^1/\varepsilon_k \times \mathbb{R}+$ with $\tilde{u}_k(x) = u_k(\varepsilon_k x)$ and the same for $\tilde{v}_k$. We know that $\tilde{u}_k$ and $\tilde{v}_k$ tend to w_1 as k tends to $+\infty$, uniformly in the compact subsets of $\mathbb{R}^2$. We have

$$\Delta z_k = \bigl(1 - \bigl(\tilde{u}_k^p - \tilde{v}_k^p\bigr)/\bigl(\tilde{u}_k - \tilde{v}_k\bigr)\bigr) z_k \quad \text{in } S^1/\varepsilon_k \times \mathbb{R}. \tag{3.8}$$

As, for all k, $z_k \not\equiv 0$, we know that z_k takes some positive values and some negative values. Then z_k attains its positive maximum in $S^1/\varepsilon_k \times \mathbb{R}+$ at a point denoted by M_k and its negative minimum at a point denoted by m_k. There exists $A > 0$ such that $1 - p w_1^{p-1}(r) > 0$ for $r \geqslant A$. Let us prove that $\|M_k\|_{\mathbb{R}^2} \leqslant A$, for k large enough. As $\tilde{u}_k$ tends to w_1, uniformly for $r = \sqrt{x_1^2 + x_2^2} = A$, there exists $K > 0$ such that for $k > K$ and for $r = A$, $1 - p\tilde{u}_k^{p-1}(x_1, x_2) > 0$. But, for $r \leqslant \frac{\pi}{\varepsilon_k}$, $\tilde{u}_k$ decreases in the both variables $|x_1|$ and $|x_2|$. Consequently, $1 - p\tilde{u}_k^{p-1}(x_1, x_2) > 0$ for all $(x_1, x_2) \in S^1/\varepsilon_k \times \mathbb{R}+$ such that $A \leqslant r \leqslant \frac{\pi}{\varepsilon_k}$. Now a convexity inequality gives

$$\bigl(\tilde{u}_k^p - \tilde{v}_k^p\bigr)/\bigl(\tilde{u}_k - \tilde{v}_k\bigr) \leqslant p\tilde{u}_k^{p-1},$$

when $z_k > 0$. Thus, in the domains where $z_k > 0$ we have

$$\Delta z_k \geqslant z_k\bigl(1 - p\tilde{u}_k^{p-1}\bigr).$$

Consequently, for $k > K$, we have

$$\Delta z_k > 0,$$

in any domain contained in $S^1/\varepsilon_k \times \mathbb{R}+$ where $z_k > 0$ and $r \geqslant A$. But we have $z_k > 0$ in a neighborhood of M_k, and the maximum principle gives that for $k > K$, the norm of M_k is less than A. By a similar proof, there exists $A > 0$ such that for all k the norm of m_k is less than A.

We normalize z_k by

$$y_k = z_k / \|z_k\|_{L^\infty(S^1 \times \mathbb{R}^+)}.$$

Thus $\|y_k\|_{L^\infty(\mathbb{R}^2)} = 1$. By a standard limit argument, we deduce from (3.8) that, up to a subsequence, y_k tends to a limit y, uniformly in the compact sets of $\mathbb{R}^2$ and that $\|y\|_{L^\infty(S^1 \times \mathbb{R}^+)} = 1$, thus $y \neq 0$. Consequently, y is a non-trivial bounded solution of the equation

$$-\Delta u + u - p w_1^{p-1} u = 0. \tag{3.9}$$

But we claim that (3.9) has no bounded solution in $\mathbb{R}^2$, except a vector space of solutions spanned by the two solutions $w_1'(r)\cos(\theta)$ and $w_1'(r)\sin(\theta)$, where (r, θ) are the polar coordinates. That claim about Eq. (3.9) seems to be well known, but we have not found a direct reference for it. So let us now give a justification of this claim. We search solutions of (3.9), $u \in H^1(\mathbb{R}^2)$, in the form

$$u = u_0(r) + \sum_{i \geqslant 1} \big(u_i(r)\cos(i\theta) + v_i(r)\sin(i\theta)\big) \tag{3.10}$$

where u_i and v_i satisfy the equation for the appropriate i

$$-\phi'' - \frac{\phi'}{r} + i^2 \frac{\phi}{r^2} + \phi - p w_1^{p-1}\phi = 0 \quad \text{in }]0, +\infty[\tag{3.11}$$

and

$$\int_0^{+\infty} \left(r\phi^2 + r\phi'^2 + \frac{\phi^2}{r} \right) < +\infty. \tag{3.12}$$

Let us remark that if ϕ is any bounded solution of (3.11) in $\mathbb{R}+$, then, as w_1 decreases exponentially at $+\infty$, $\phi\cos(i\theta)$ and $\phi\sin(i\theta)$ are solutions of (3.9) in $H^1(\mathbb{R}^2)$ and consequently, ϕ verifies the condition (3.12). But w_1' is a bounded solution of (3.11) for $i = 1$ and has a constant sign. Thus 0 is the first eigenvalue for the problem

$$-\phi'' - \frac{\phi'}{r} + \frac{\phi}{r^2} + \phi - p w_1^{p-1}\phi = \mu\phi \tag{3.13}$$

with the condition $\int_0^{+\infty} (r\phi^2 + r\phi'^2 + \frac{\phi^2}{r}) < +\infty$.

Consequently, for $i = 1$ the only bounded solution of (3.11) is w_1' and for $i > 1$ Eq. (3.11) has no solutions that are bounded both in 0 and $+\infty$. Now the proof that (3.11) has no bounded solution for $i = 0$ appears in Kwong [15], in the course of the proof of the uniqueness of the ground state w_1. More precisely, it is proved there that

$$\lim_{r \to +\infty} \frac{\partial \phi}{\partial \alpha}(\alpha_0, r) = -\infty, \tag{3.14}$$

where α_0 is the unique $\alpha > 0$ such that the solution of

$$-\phi'' - \frac{\phi'}{r} + \phi - \phi^p = 0 \quad \text{in }]0, +\infty[, \qquad \phi(0) = \alpha, \qquad \phi'(0) = 0 \tag{3.15}$$

is positive and has the limit 0 as r tends to $+\infty$. (See (4.7) in [15] and the lemmas which follow.) Now, since $\frac{\partial\phi}{\partial\alpha}|_{\alpha_0}$ is a solution of (3.11) with $i = 0$ and $\frac{\partial\phi}{\partial\alpha}|_{\alpha_0}(0) = 1$, we conclude that (3.11) has no bounded solution for $i = 0$. The above claim is proved. Now $w_1'\cos\theta$ and $w_1'\sin\theta$ are not available for being y, since y is even in x_1 and in x_2. The first part of the proposition is proved.

The second part of the proposition can be proved by the same arguments than the first part. Let us suppose that for some sequence ε_k tending to 0 there exist $u_k \in U$ and $\xi_k \neq 0$ such that

$$\varepsilon_k^2 \Delta\xi_k = \xi_k - p u_k^{p-1}\xi_k \quad \text{in } S^1 \times \mathbb{R}+. \tag{3.16}$$

Let $\tilde{\xi}_k(x_1, x_2) = \xi_k(\varepsilon_k x_1, \varepsilon_k x_2)$. We have

$$\Delta\tilde{\xi}_k = \tilde{\xi}_k - p\tilde{u}_k^{p-1}\tilde{\xi}_k \quad \text{in } S^1/\varepsilon_k \times \mathbb{R}+. \tag{3.17}$$

But ξ_k has not a constant sign, since 0 is not the first eigenvalue. We proceed exactly as in the above proof to show that we can extract a subsequence of $\xi_k/\|\xi_k\|_{L^\infty}$ that tends uniformly in all compact set of $\mathbb{R}^2$ to a non-trivial bounded solution of (3.9), that gives a contradiction. □

Let us now turn to the kernel of the operator $-\varepsilon^2\Delta + I - pu^{p-1}I$, for u in U.

Lemma 3.5. *Let (ε, u) be a solution of* (E), *$u \in U$. Let suppose that there exists a non-trivial solution ξ in X of*

$$-\varepsilon^2\Delta\xi + \xi - pu^{p-1}\xi = 0. \tag{3.18}$$

Then $\xi(0, 0) \neq 0$.

Proof. Let $f(u) = u - u^p$. We have $\varepsilon^2 \Delta\xi = f'(u)\xi$. The maximum of u being attained at $(0,0)$, we have $f(u(0,0)) \leqslant 0$ and consequently we have $f'(u) < 0$ near $(0,0)$. Let us suppose by contradiction that $\xi(0,0) = 0$. If ξ has a constant sign in a neighborhood of $(0,0)$, then $\Delta\xi$ has the sign of $-\xi$. This is in contradiction with the maximum principle, thus ξ has not a constant sign near $(0,0)$. The function $x \to f'(u(x))$ being $\mathcal{C}^\infty$, the structure of the nodal lines and of the nodal domains is described in [7], Theorem 2.5 and [8]. There exists a finite number of nodal lines through $(0,0)$ and, as $\nabla\xi(0,0) = 0$, there exists at least two nodal lines trough $(0,0)$ and we know that in this case they form an equiangular system at $(0,0)$. More, let A be on the x_2 axis and near 0. If $[0, A]$ would be contained in a nodal line, it would be a part of the boundary of a domain in which, say, $\xi > 0$ and $\Delta\xi < 0$, thus we would have $\frac{\partial\xi}{\partial x_1} \neq 0$ on $[0, A]$, by the Hopf maximum theorem (see [17], Chapter 2), that is in contradiction with $\xi \in X$. We deduce that there exists A with $x_1(A) = 0$ and for instance $x_2(A) > 0$, such that ξ has a constant sign in $[O, A]$.

We will use the fact that any nodal domain cannot be entirely contained in the half-planes $x_2 > 0$, $x_2 < 0$ or in the domains $0 < x_1 < \pi$ or $-\pi < x_1 < 0$. This property can be proved by multiplying successively (3.18) by $\frac{\partial u}{\partial x_1}$ and $\frac{\partial u}{\partial x_2}$ and by use of the Green formula. Indeed, if $\mathcal{D}$ is a nodal domain for ξ we obtain

$$-\varepsilon^2 \int\limits_{\partial\mathcal{D}} \frac{\partial\xi}{\partial\nu}\frac{\partial u}{\partial x_i} = 0, \quad i = 1, 2.$$

But $\frac{\partial\xi}{\partial\nu}$ and $\frac{\partial u}{\partial x_1}$ have constant signs on $\partial\mathcal{D}$ and the conclusion follows. □

Let $\mathcal{D}$ be a nodal domain which contains a segment $[O, A]$ where A belongs to the x_2-axis. For example, $x_2(A) > 0$ and $\xi > 0$ in $\mathcal{D}$. As $\mathcal{D}$ is not contained in $x_2 > 0$, there exists B in $\mathcal{D} \cap \{x_2 = 0\} \cap \{0 < |x_1| \leqslant \pi\}$. Let $\Gamma : t \mapsto (x_1(t), x_2(t))$ a path in $\mathcal{D}$ from A to B. Due to the x_1 and x_2-symmetries of ξ, $\tilde{\Gamma} : t \mapsto (|x_1(t)|, |x_2(t)|)$ is a path in $\mathcal{D} \cap \{x_1 \geqslant 0\} \cap \{x_2 \geqslant 0\}$ and we can build a closed path $\mathcal{C}$ in $\mathcal{D} \cap \{x_1 \geqslant 0\}$ with $\tilde{\Gamma} \cup [O, A]$ and its symmetric set with respect to the axis $x_2 = 0$. Now, due to the existence of at least two nodal lines through $(0,0)$, we know that there exists a nodal domain $\mathcal{D}'$ such that $O \in \partial\mathcal{D}'$ and $\xi < 0$ in $\mathcal{D}'$. So $\mathcal{D}'$ is surrounded by $\mathcal{C}$ or its symmetric curve with respect to the axis $x_1 = 0$. Therefore $\mathcal{D}'$ is in the domain $0 < x_1 < \pi$ or in the domain $-\pi < x_1 < 0$, which is not possible.

The proof of Proposition 1.2. follows from Lemma 3.5.

Lemma 3.6. *Let $\varepsilon > 0$ be given and let u and v in U be such that (ε, u) and (ε, v) are solutions of* (E). *If $\|u\|_{L^\infty(S^1\times\mathbb{R}+)} = \|v\|_{L^\infty(S^1\times\mathbb{R}+)}$, then u=v.*

Proof. Let $w = u - v$. The maximum of u and the maximum of v are both attained at the point $(0,0)$. Thus we have $w(0,0) = 0$. Let us prove that $w \equiv 0$. As $p > 1$ the function defined by $x \to \frac{u^p - v^p}{u - v}(x)$ for $u(x) \neq v(x)$ and $x \to pu^{p-1}(x)$ for $u(x) = v(x)$ is $\mathcal{C}^\infty$. The function w verifies the equation

$$\varepsilon^2 \Delta w = \left(1 - \frac{u^p - v^p}{u - v}\right) w. \tag{3.19}$$

If $w \not\equiv 0$, the considerations over the nodal lines and the nodal domains of w are the same as for those of ξ in Lemma 3.5. The only point that we have to verify is that any nodal domain of w cannot lay entirely in a half-plane $x_i > 0$ or $x_i < 0$, $i = 1, 2$. For $i = 1, 2$, we multiply Eq. (3.19) by $\frac{\partial u}{\partial x_i}$ and we use the Green formula to get

$$\varepsilon^2 \int\limits_{\partial\mathcal{D}} \frac{\partial w}{\partial\nu}\frac{\partial u}{\partial x_i} = \int\limits_{\mathcal{D}} \frac{\partial u}{\partial x_i} w\left(pu^{p-1} - \frac{u^p - v^p}{u - v}\right).$$

In any nodal domain $\mathcal{D}$ where $w > 0$, a convexity inequality gives

$$\left(u^p - v^p\right)/(u - v) \leqslant pu^{p-1}.$$

So if $w > 0$ and $\frac{\partial u}{\partial x_i} < 0$ we get

$$\varepsilon^2 \int\limits_{\partial\mathcal{D}} \frac{\partial w}{\partial\nu}\frac{\partial u}{\partial x_i} < 0,$$

that is false if $\partial\mathcal{D}$ is a nodal line for w, since $\frac{\partial w}{\partial \nu}$ has a constant sign. The other cases follow by the same proof. $\square$

4. The proof of the main theorem

Let us define

$$M(\varepsilon,u)=-\varepsilon^2\Delta u+u-|u|^{p-1}u.$$

Proposition 4.7. *For a given $\varepsilon>0$, the solutions (ε,u) of* (E) *such that $u\in U$ are isolated for the norm of Y_ε. More precisely, (ε,u) being a solution of* (E) *such that $u\in U$, there exists $\eta>0$ such that if (ε,v) is any solution of* (E), *$v\in Y_\varepsilon$, $v\neq u$, then $\|u-v\|_{Y_\varepsilon}\geqslant\eta$.*

Proof. If $-\varepsilon^2\Delta+I-pu^{p-1}I$ is an isomorphism from Y_ε to Y'_ε then u is the only v such that (ε,v) is a solution of (E), v in a Y_ε-neighborhood of u.

Let us suppose that $-\varepsilon^2\Delta+I-pu^{p-1}I$ is not an isomorphism from Y_ε to Y'_ε. Then the dimension of its kernel is one. Let ξ be a basis of the kernel. The operator $-\varepsilon^2\Delta+I-pu^{p-1}I$ is a Fredholm operator of index 0 from Y_ε to Y'_ε so there exist two Banach spaces Z and K such that

$$Y_\varepsilon=\langle\xi\rangle\oplus Z\quad\text{and}\quad Y'_\varepsilon=\mathcal{R}\big(M_u(\varepsilon,u)\big)\oplus K.$$

Let us search solutions near (ε,u) of the form

$$(\varepsilon,u+\alpha\xi+z),$$

where α is a real number and $z\in Z$. There exists $C_0>0$ and $C_1>0$ such that for all $(x_1,x_2)\in S^1\times\mathbb{R}+$,

$$u(x_1,x_2)\geqslant C_0e^{-x_2/\varepsilon}$$

and

$$\big|\xi(x_1,x_2)\big|\leqslant C_1e^{-x_2/\varepsilon}.$$

Let z be such that $\|z\|_{Y_\varepsilon}<\eta$. For $|\alpha|$ and η sufficiently small, we have for all $(x_1,x_2)\in S^1\times\mathbb{R}+$

$$\big|(\alpha\xi+z)(x_1,x_2)\big|<u(x_1,x_2).$$

But for all $(x_1,x_2)\in S^1\times\mathbb{R}+$ the function $h\to(u(x_1,x_2)+h)^p$ is analytic for any complex number such that $|h|<u(x_1,x_2)$. Consequently the function

$$V\to(u+V)^p$$

of the complex valued function $V=V_1+iV_2$, V_1, $V_2\in Y_\varepsilon$, is analytic in a neighborhood of $V=0$. Let

$$F(\alpha,z)=M(\varepsilon,u+\alpha\xi+z).$$

By the above considerations the function F is analytic for $|\alpha|$ and $\|z\|_{Y_\varepsilon}$ sufficiently small. Let E be the projection onto $\mathcal{R}(M_u(\varepsilon,u))$. Let us solve first

$$EF(\alpha,z)=0.$$

The partial derivative with respect to z at the point $(0,0)$ is $EM_u(\varepsilon,u)$ that is an isomorphism from Z to $\mathcal{R}(M_u(\varepsilon,u))$. The implicit function theorem gives a function $\alpha\mapsto z(\alpha)$, that is analytic from a neighborhood of 0 to a neighborhood of $z=0$ in Z. We may suppose that $u+\alpha\xi+z(\alpha)>0$, so the function h defined by

$$h(\alpha)=\big\langle M\big(\varepsilon,u+\alpha\xi+z(\alpha)\big),\xi\big\rangle$$

is analytic. We set

$$v(\alpha)=u+\alpha\xi+z(\alpha).$$

In an $\mathbb{R}\times Y_\varepsilon$-neighborhood of the solution (ε,u) all the solutions of (E) of the form (ε,v) are the $v(\alpha)$ where α are the zeroes of the analytic function h, so either they are isolated or h is identically null. Let us prove that this last possibility cannot occur. Suppose that h is identically null, then there exists a $\mathcal{C}^1$ curve of solutions $(\varepsilon, v(\alpha))$. Thanks to the maximum principle and to a continuity argument [10] we have that $v(\alpha) \in U$. Let us denote by $\mathcal{S}_\varepsilon$ the set of the solutions (ε, v) of (E), $v \in U$ or $v = w_0(\frac{x_2}{\varepsilon})$. Let us define

$$\mathcal{A} = \{\|v\|_\infty, (\varepsilon, v) \in S_\varepsilon\}$$

and let $\mathcal{C}$ be the component in $\mathbb{R}$ of $\mathcal{A}$ to which $\|u\|_\infty$ belongs. The function $\alpha \mapsto \|v(\alpha)\|_{L^\infty(S^1\times\mathbb{R}+)}$ is continuous and injective, by Lemma 3.6. Consequently $\mathcal{C}$ is an interval of $\mathbb{R}$ that contains $\|u\|_\infty$ and that is not equal to $\{\|u\|_\infty\}$. By Lemma 2.3 we know that $\mathcal{C}$ is bounded, thus it is compact. Let u_0 be such that $(\varepsilon, u_0) \in S_\varepsilon$ and

$$\|u_0\|_\infty = \sup \mathcal{C}.$$

The operator $-\varepsilon^2\Delta + I - pu_0^{p-1}I$ is not an isomorphism, since we can deduce from $\mathcal{C} \neq \{\|u\|_\infty\}$ and from Lemmas 2.1 and 3.6 that there exists a sequence of distinct $u_m \in U$ that tends to u_0 in X and such that $(\varepsilon, u_m) \in S_\varepsilon$. Let ξ_0 be a basis of its kernel in Y_ε. Let us use the proof above to define an analytic function $\tilde{h}$ and a function z_0 such that all the solutions (ε, v), v in a Y_ε-neighborhood of u_0, are the $v(\alpha) = u_0 + \alpha\xi_0 + z_0(\alpha)$, where α are the zeroes, near 0, of the analytic function $\tilde{h}$. We may suppose as above that $\alpha \mapsto \|v(\alpha)\|_\infty$ increases and, as $\tilde{h}$ is null for $\alpha < 0$, α near 0 and $\tilde{h}$ is analytic, then $\tilde{h}$ is identically null. Thus there exists $\alpha > 0$ such that $(\varepsilon, v(\alpha)) \in \mathcal{C}$ and $\|v(\alpha)\|_\infty > \|u_0\|_\infty$, that is in contradiction with the definition of u_0. We conclude that the analytic function h above has isolated zeroes and the conclusion of the proposition follows since, using the projection on $\langle\xi\rangle$ we obtain a constant $C > 0$ such that $\|v(\alpha) - u\|_{Y_\varepsilon} \geqslant C|\alpha|$. □

Remark 4.2. The above proof is still valid if we replace ε by $\varepsilon_\star$ and u by $w_0(\frac{x_2}{\varepsilon_\star})$.

Corollary 4.2. *Let $\varepsilon > 0$ be given. Let (ε, u) be a solution of* (E)*, $u \in U$. There exists $\eta > 0$ such that for all solution (ε, v), $v \in B$, $v \neq u$, we have $\|u - v\|_{L^\infty(S^1\times\mathbb{R}^+)} > \eta$. This is still true if we replace (ε, u) by $(\varepsilon_\star, w_0(\frac{x_2}{\varepsilon_\star}))$. Moreover there exists at most a finite number of solutions (ε, u) of* (E)*, $u \in U$.*

Proof. Let us suppose that a sequence (u_m) of distinct functions in B tends to u for the $L^\infty(S^1 \times \mathbb{R}^+)$ norm and that (ε, u_m) is a solution of (E) for all m. For m large enough we have that $u_m > 0$ and we deduce from Proposition 1.1 that $\{u_m;\ m \in \mathbb{N}\}$ is relatively compact in Y_ε. Thus (u_m) tends to u for the Y_ε-norm, that is not possible, by Proposition 4.7.

Now, if there exists a sequence of distinct solutions (ε, u_m), $u_m \in U$, then u_m is bounded in $L^\infty(S^1 \times \mathbb{R}^+)$ and consequently there exists a subsequence that tends uniformly to a limit v, (ε, v) is a solution of (E), and $v \in U$ or $v(x_1, x_2) = w_0(\frac{x_2}{\varepsilon})$ (Remark 2.1). But this is not possible, by the proof above. □

Proposition 4.8. *For every $\varepsilon < \varepsilon_\star$ there exists at least a function $u \in U$ such that $(\varepsilon, u) \in \Sigma_1$.*

Proof. Let us use a theorem of Rabinowitz [19, Theorem 1.10]. Let $\gamma > \bar{\varepsilon}$ (where $\bar{\varepsilon}$ is defined in Theorem 1.1, $\bar{\varepsilon} \geqslant \varepsilon_\star$). We have that $\Sigma_1 \subset]0, \gamma] \times Y_\gamma$. Let Φ be defined by

$$\Phi(\varepsilon, u) = \left(-\varepsilon^2\Delta + I\right)^{-1}\left(|u|^{p-1}u\right). \tag{4.1}$$

The operator $(\varepsilon, u) \to u - \Phi(\varepsilon, u)$ defined on $]0, \gamma] \times Y_\gamma$ verifies the hypothesis of Lemma 1.8 in [19], by Proposition 2.5, the parameter λ being $\frac{1}{\varepsilon}$. We deduce that either Σ_1 is "unbounded" or it contains an other trivial solution than the solution $(\varepsilon_\star, w_0(\frac{x_2}{\varepsilon_\star}))$. But $\Sigma_1 \cap \Sigma_k = \emptyset$ for all $k \neq 1$, so this last possibility is excluded. Thus Σ_1 is "unbounded". This means that either the set of all $u \in Y_\gamma$, such that there exists ε with $(\varepsilon, u) \in \Sigma_1$, is unbounded in Y_γ, or there exists a sequence (ε_m, u_m) in Σ_1 with ε_m tending to 0. Let us suppose that there exists ε_1 such that for all $(\varepsilon, u) \in \Sigma_1$, $\varepsilon \geqslant \varepsilon_1$. In this case, as Σ_1 is bounded in $[\varepsilon_1, \gamma] \times L^\infty(S^1 \times \mathbb{R}+)$, we deduce from Proposition 1.1 that Σ_1 is compact in $\mathbb{R} \times Y_\gamma$. Thus we have that $\varepsilon \geqslant \varepsilon_1$ together with the existence of M such that for all $(\varepsilon, u) \in \Sigma_1$, $\|u\|_{Y_\gamma} \leqslant M$. This cannot be true. Consequently there exists a sequence $(\varepsilon_m, u_m) \in \Sigma_1$, with $\varepsilon_m \to 0$. Now the proof of the proposition follows from the fact that Σ_1 is connected. □

Proposition 4.9. *Every solution (ε, u) of* (E) *such that $u \in U$ belongs to the first bifurcation continuum Σ_1. Moreover there does not exist solutions (ε, u), $u \in U$ for $\varepsilon \geqslant \varepsilon_\star$. In particular the bifurcation from the solution $(\varepsilon_\star, w_0(\frac{x_2}{\varepsilon_\star}))$ is not vertical.*

Proof. First let us prove that for all solution (α, v) of (E) such that $v \in U$ there exists a sequence (ε_m, u_m) of solutions of (E) in $\mathbb{R} \times X$ such that $\varepsilon_m \to \alpha$, $\varepsilon_m < \alpha$ and $u_m \to v$ in $\mathbb{R} \times X$ and another sequence of solutions still denoted by (ε_m, u_m), with $\varepsilon_m > \alpha$, $\varepsilon_m \to \alpha$ and $u_m \to v$ in $\mathbb{R} \times X$. Let us use the Leray–Schauder degree theory as in [19,18]. Let Φ be defined by (4.1). Let us choose $\gamma > \alpha$ and let us define for any $\rho > 0$

$$B_\rho = \{u \in Y_\gamma;\ \|v - u\|_{Y_\gamma} < \rho\}.$$

We choose $\beta \in]0, \alpha[$. By Proposition 2.5 the map $\Phi : [\beta, \alpha] \times \bar{B}_\rho \to Y_\gamma$ is compact. By Corollary 4.2, there exists $\rho > 0$ such that for all $u \in X$, if (α, u) is a solution of (E), $u \not\equiv v$, then

$$\|u - v\|_{Y_\gamma} > \rho.$$

We fix now such a positive real number ρ. We have that

$$\deg\bigl(I - \Phi(\alpha, \cdot), B_\rho\bigr) = \pm 1.$$

Let us prove that there exists a sequence of solutions (ε_m, u_m) in $[\beta, \alpha] \times Y_\gamma$ that tends to (α, v). If not there would exist $\tilde{\rho} \in]0, \rho[$ and $\delta \in]\beta, \alpha[$ such that for all $\varepsilon \in [\delta, \alpha[$, $\mathcal{S} \cap \bar{B}_{\tilde{\rho}} = \emptyset$. So we would have

$$\deg\bigl(I - \Phi(\delta, \cdot), B_{\tilde{\rho}}\bigr) = 0$$

and by the invariance property of the degree we would have

$$\deg\bigl(I - \Phi(\alpha, \cdot), B_{\tilde{\rho}}\bigr) = \deg\bigl(I - \Phi(\delta, \cdot), B_{\tilde{\rho}}\bigr) = 0.$$

But

$$\deg\bigl(I - \Phi(\alpha, \cdot), B_{\tilde{\rho}}\bigr) = \deg\bigl(I - \Phi(\alpha, \cdot), B_\rho\bigr) = \pm 1,$$

that is a contradiction. We deduce the existence of a sequence (ε_m, u_m) that tends to (α, v) in $\mathbb{R} \times Y_\gamma$, and consequently in $\mathcal{S} \cap [\beta, \alpha] \times X$. If we consider a real number $\beta' \in]\alpha, \gamma[$, we prove by a similar proof that there exists a sequence of solutions (ε_m, u_m) in $[\alpha, \beta'] \times Y_\gamma$ that tends to (α, v) in $\mathbb{R} \times Y_\gamma$, thus in $\mathcal{S} \cap [\alpha, \beta'] \times X$. $\square$

Now let (ε_1, u_1) be a solution of (E), $u_1 \in U$. We suppose that $(\varepsilon_1, u_1) \notin \Sigma_1$. Let us denote by $\mathcal{S}$ the closure of the non-trivial solutions of (E) in $\mathbb{R} \times X$. Let $\mathcal{C}$ be the component of $\mathcal{S} \cap \{(\varepsilon, u) \in \mathbb{R} \times X, 0 < \varepsilon \leqslant \varepsilon_1\}$ to which (ε_1, u_1) belongs. We have $\mathcal{C} \cap \Sigma_1 = \emptyset$, otherwise, $(\varepsilon_1, u_1) \in \Sigma_1$. By Proposition 3.6, we deduce that there exists $\beta > 0$ such that $\mathcal{C} \subset \mathcal{S} \cap ([\beta, \varepsilon_1] \times X)$. The component $\mathcal{C}$ being compact in $\mathbb{R} \times X$, let us choose $(\alpha, v) \in \mathcal{C}$ such that $\alpha = \inf\{\varepsilon > 0, \exists u \in U, (\varepsilon, u) \in \mathcal{C}\}$. Let us choose $\gamma > \varepsilon_1$. Let us define $\Phi(\varepsilon, u)$ and $\rho > 0$ as before. Let V be a δ-neighborhood of $\mathcal{C}$ in $\mathbb{R} \times Y_\gamma$, with $0 < \delta < \rho$. As $\partial V \cap \mathcal{C} = \emptyset$, there exist disjoint compact sets M and N such that $\bar{V} \cap \mathcal{S} = M \cup N$, $\mathcal{C} \subset M$ and $\partial V \cap \mathcal{S} \subset N$. Consequently there exists an open neighborhood $\mathcal{O}$ of $\mathcal{C}$ such that $\mathcal{S} \cap \partial\mathcal{O} = \emptyset$ and such that the only $(\alpha, u) \in \mathcal{S} \cap \mathcal{O}$ is (α, v). In view of Lemma 2.2, we may also suppose that for all $(\varepsilon, u) \in \mathcal{O} \cap \mathcal{S}$ we have $u \in U$. For all ε closed to α let us define $\mathcal{O}_\varepsilon = \{u \in Y_\gamma,\ (\varepsilon, u) \in \mathcal{O}\}$. Then we have $\deg(I - \Phi(\alpha, \cdot), \mathcal{O}_\alpha) = \pm 1$. By the invariance property of the degree we have $\deg(I - \Phi(\varepsilon, \cdot), \mathcal{O}_\varepsilon) = \pm 1$ for ε closed to α, $\varepsilon < \alpha$. Thus solutions (ε, u) exist in $\mathcal{O}$, $\varepsilon < \alpha$ and $u \in U$. Let us prove that for such parameters ε there exists a continuous curve $\varepsilon \mapsto (\varepsilon, u_\varepsilon)$ containing (α, v). We define u_ε to be the function among those that realize $\min\{|\|u\|_\infty - \|v\|_\infty|, (\varepsilon, u) \in \mathcal{S} \cap \mathcal{O}\}$ that has the least L^∞-norm. Clearly the map $\varepsilon \mapsto (\varepsilon, u_\varepsilon)$ is continuous from $\mathbb{R}$ to $\mathbb{R} \times X$. Thus there exists $\varepsilon < \alpha$ and u such that $(\varepsilon, u) \in \mathcal{C}$ and this is in contradiction with the definition of α. We have proved that $(\varepsilon_1, u_1) \in \Sigma_1$.

Let now $(\varepsilon_1, u_1) \in \mathcal{S}$, $u_1 \in U$ and let us prove that $\varepsilon_1 \leqslant \varepsilon_\star$. We know the existence of $\bar{\varepsilon} > 0$ such that for $\varepsilon > \bar{\varepsilon}$ the only solution (ε, u) of (E), $u > 0$ is $(\varepsilon, w_0(\frac{x_2}{\varepsilon}))$ (Theorem 1.1). Let $\bar{\varepsilon}_U$ be the greater $\varepsilon > 0$ for which there exists a positive function u in U such that (ε, u) is a solution of (E). We have that $\bar{\varepsilon}_U \geqslant \varepsilon_\star$ and that either there exists $u \in U$ such that $(\bar{\varepsilon}_U, u) \in \mathcal{S}$ or $u(x_1, x_2) = w_0(\frac{x_2}{\bar{\varepsilon}_U})$. If $u \in U$, the proof above gives a sequence (ε_m, u_m) of solutions such that $\varepsilon_m > \bar{\varepsilon}_U$ and by Lemma 2.2 we have that $u_m \in U$ for m large enough. That is in contradiction with the definition of $\bar{\varepsilon}_U$. Hence $\bar{\varepsilon}_U = \varepsilon_\star$ and $u(x_1, x_2) = w_0(\frac{x_2}{\varepsilon_\star})$.

Proposition 4.10. *For $p \in \mathbb{N}$, $p \geqslant 2$, the continuum Σ_1 is constituted of at most a finite number of curves that admit local analytic parameterizations.*

Proof. In this case, the function $u \mapsto u^p$ is analytic in $\mathbb{R}$. The function w_0 being analytic, the continuum Σ_1 begins by an analytic curve (see Section 5 for the construction of the beginning of Σ_1). We may have secondary bifurcations, at points (ε_1, u_1) for which the operator $M_u(\varepsilon_1, u_1) = -\varepsilon_1^2\Delta + I - pu_1^{p-1}I$ is singular, that we call singular points. For these points the kernel of $M_u(\varepsilon_1, u_1)$ is one dimensional. By use of a Lyapunov–Schmidt reduction in the analytic case (see Buffoni and Toland [6], Chapter 9), we get analytic functions $(\varepsilon, \alpha) \mapsto z(\varepsilon, \alpha)$ and $(\varepsilon, \alpha) \mapsto h(\varepsilon, \alpha)$ defined near $(\varepsilon_1, 0)$ such that the solutions of (E) in a $\mathbb{R} \times (H^1 \cap L^\infty)(S^1 \times \mathbb{R}^+)$ neighborhood of (ε_1, u_1) are the $(\varepsilon, u_{\varepsilon,\alpha})$, $u_{\varepsilon,\alpha} = u_1 + \alpha\xi + z(\varepsilon, \alpha)$, where (ε, α) are the solutions of the equation $h(\varepsilon, \alpha) = 0$. This gives a finite number of curves, intersecting locally only at (ε_1, u_1). Each curve admits a local analytic parameterizations and the critical points on these curves are isolated. Let us prove that the critical points are in finite number. If not, we may define a sequence (ε_m, u_m) of distinct solutions, $u_m \in U$ and $\varepsilon_1 < \varepsilon_m < \varepsilon_\star$. Let us suppose that ε_m tends to a limit ε. By Lemma 2.4 a subsequence of u_m tends to a limit u, uniformly in $S^1 \times \mathbb{R}^+$, $u \in U$ or $u(x_1, x_2) = w_0(\frac{x_2}{\varepsilon})$ and (ε, u) is a solution of (E). Thus (ε, u) is on the continuum (Σ_1), by Proposition 4.9 and in each $(H^1 \cap L^\infty)(S^1 \times \mathbb{R}^+)$-neighborhood of u there exists critical points in Σ_1, that is false, since the critical points are isolated on each curve. □

Remark 4.3. In the analytic case ($p \in \mathbb{N}$, $p \geqslant 2$), for degree reasons, as in the proof of Proposition 4.9, we cannot have as a part of Σ_1 a curve with a local continuous parameterization $\varepsilon \mapsto (\varepsilon, u_\varepsilon)$ that would be defined in an interval $[\alpha, \beta]$, but that would not have a local continuous prolongation for $\varepsilon < \alpha$ or for $\varepsilon > \beta$. So in this case we can represent Σ_1 as a principal analytic curve defined up to $\varepsilon \to 0$ together with a finite number of closed loops bifurcating from it and returning to it or the same thing from another loop.

5. A local analysis

In this part we recall the local construction of the bifurcation branch Σ_1 from the solution $(\varepsilon_\star, w_0(\frac{x_2}{\varepsilon_\star}))$ and we prove directly, for $p \geqslant 2$, that it is defined near $\varepsilon_\star$ in the sense of the decreasing ε. Let

$$M(\varepsilon, u) = -\varepsilon^2\Delta u + u - |u|^{p-1}u \in H^{-1}(S^1 \times \mathbb{R}+).$$

Let

$$\xi(x_1, x_2) = v\left(\frac{x_2}{\varepsilon_\star}\right)\cos x_1$$

be a basis of the kernel of $M_u(\varepsilon_\star, w_0(\frac{x_2}{\varepsilon_\star}))$, where v is a positive function that verifies $v'(0) = 0$ and

$$-v'' + (1 + \varepsilon_\star^2)v - pw_0^{p-1}v = 0.$$

(See [16].) The operator $-\varepsilon_\star^2\Delta + I - pw_0^{p-1}(\frac{x_2}{\varepsilon_\star}))I$ is a Fredholm operator of index 0 from the Banach space $H^1(S^1 \times \mathbb{R}+) \cap L^\infty(S^1 \times \mathbb{R}+)$ to its dual $(H^1(S^1 \times \mathbb{R}+) \cap L^\infty(S^1 \times \mathbb{R}+))'$, so there exist two Banach spaces Z_0 and Y_0 such that

$$H^1(S^1 \times \mathbb{R}+) \cap L^\infty(S^1 \times \mathbb{R}+) = \langle\xi\rangle \oplus Z_0$$

and

$$(H^1(S^1 \times \mathbb{R}+) \cap L^\infty(S^1 \times \mathbb{R}+))' = \mathcal{R}\left(M_u\left(\varepsilon_\star, w_0\left(\frac{x_2}{\varepsilon_\star}\right)\right)\right) \oplus Y_0.$$

As in the proof of the Crandall–Rabinowitz bifurcation theorem [9], we search for solutions in a neighborhood of $(\varepsilon_\star, w_0(\frac{x_2}{\varepsilon_\star}))$ of the form

$$w_0\left(\frac{x_2}{\varepsilon}\right) + \alpha\xi + \alpha z,$$

where $z \in Z_0$. Let us define

$$\begin{cases} f(\varepsilon, \alpha, z) = \alpha^{-1}M\left(\varepsilon, w_0\left(\frac{x_2}{\varepsilon}\right) + \alpha\xi + \alpha z\right), & \alpha \neq 0, \\ f(\varepsilon, \alpha, z) = M_u\left(\varepsilon, w_0\left(\frac{x_2}{\varepsilon}\right)\right)(\xi + z), & \alpha = 0. \end{cases}$$

We obtain that the solutions near $(\varepsilon_\star, w_0(\frac{x_2}{\varepsilon_\star}))$ are $(\varepsilon, w_0(\frac{x_2}{\varepsilon}))$ and a bifurcation branch $(\varepsilon(\alpha), u_\alpha)$, where $u_\alpha = w_0(\frac{x_2}{\varepsilon(\alpha)}) + \alpha\xi + \alpha z(\alpha)$. The function $\alpha \mapsto (\varepsilon(\alpha), u_\alpha)$, defined in a neighborhood of 0 is $\mathcal{C}^1$. Moreover, we can prove by the maximum principle that $u > 0$, when u is a solution of (E) that is near a positive solution.

Let us prove that, locally, we have $\varepsilon < \varepsilon_\star$ for the solutions on the bifurcation branch. Let us first verify that we can choose

$$v = w_0^{\frac{p+1}{2}}$$

and that

$$1 + \varepsilon_*^2 = \left(\frac{p+1}{2}\right)^2.$$

Let $k \in \mathbb{R}+^*$, we set $\tilde{v} = w_0^k$. We use the identity

$$\frac{1}{2}w_0'^2 + \frac{1}{2}w_0^2 - \frac{1}{p+1}w_0^{p+1} = 0$$

to prove that the function $\tilde{v}$ verifies the identity

$$-\tilde{v}'' + k^2\tilde{v} - k\left(1 + \frac{2k-2}{p+1}\right)\tilde{v}w_0^{p-1} = 0.$$

Let us choose $k = \frac{p+1}{2}$, in order to have $k(1 + \frac{2k-2}{p+1}) = p$. We have $\tilde{v}'(0) = 0$ and $\tilde{v}$ tends to 0 as x tends to $+\infty$. But there is a unique $\lambda \in \mathbb{R}+$ such that the positive solutions of the equation

$$-u'' + \left(\lambda - pw_0^{p-1}\right)u = 0, \quad u'(0) = 0$$

tend to 0 as x tends to $+\infty$. Consequently we have $1 + \varepsilon_*^2 = (\frac{p+1}{2})^2$ and $v = w_0^{\frac{p+1}{2}}$, up to a positive multiplicative constant. For $p \geqslant 2$ the function $\alpha \to (\varepsilon(\alpha), u_\alpha)$ is $\mathcal{C}^2$. We develop $z(\alpha)$ near $\alpha = 0$ to get $z(\alpha) = \alpha z_0 + O(\alpha^2)$ and this gives the following expansion of u_α

$$u_\alpha = w_0\left(\frac{x_2}{\varepsilon_\star}\right) + \alpha\xi + \alpha^2\left(z_0 - \frac{x_2}{2\varepsilon_\star^2}\varepsilon''(0)w_0'\left(\frac{x_2}{\varepsilon_\star}\right)\right) + O\left(\alpha^3\right)$$

and $z = z_0 - \frac{x_2}{2\varepsilon_\star^2}\varepsilon''(0)w_0'(\frac{x_2}{\varepsilon_\star})$ verifies

$$-\varepsilon_\star^2\Delta z + z - pw_0^{p-1}\left(\frac{x_2}{\varepsilon_\star}\right)z = \frac{p(p-1)}{2}w_0^{p-2}\left(\frac{x_2}{\varepsilon_\star}\right)\xi^2 - \frac{\varepsilon''(0)}{\varepsilon_\star}w_0''\left(\frac{x_2}{\varepsilon_\star}\right),$$

that gives

$$-\varepsilon_\star^2\Delta z_0 + z_0 - pw_0^{p-1}\left(\frac{x_2}{\varepsilon_\star}\right)z_0 = \frac{p(p-1)}{2}w_0^{p-2}\left(\frac{x_2}{\varepsilon_\star}\right)\xi^2.$$

We define the functions η_1 and η_2 by

$$z_0(x_1, x_2) = \eta_1\left(\frac{x_2}{\varepsilon_\star}\right) + \eta_2\left(\frac{x_2}{\varepsilon_\star}\right)\cos(2x_1).$$

The functions η_1 and η_2 verify, for $x_2 \in [0, +\infty[$,

$$-\eta_1'' + \eta_1 - pw_0^{p-1}\eta_1 = \frac{p(p-1)}{4}w_0^{p-2}v^2, \quad \eta_1'(0) = 0, \tag{5.21}$$

$$-\eta_2'' + \left(1 + 4\varepsilon_\star^2\right)\eta_2 - pw_0^{p-1}\eta_2 = \frac{p(p-1)}{4}w_0^{p-2}v^2, \quad \eta_2'(0) = 0. \tag{5.22}$$

Let us prove that $\varepsilon''(0) < 0$. An expansion of $M(\varepsilon(\alpha), w_0(\frac{x_2}{\varepsilon(\alpha)}) + \alpha\xi + \alpha z(\alpha))$ near $\alpha = 0$ at the order three, and an integral over $S^1 \times \mathbb{R}_+$ give

$$\varepsilon''(0)\int\limits_{S^1\times\mathbb{R}_+}\varepsilon_\star|\nabla\xi|^2 = p(p-1)\int\limits_{S^1\times\mathbb{R}_+}\left(w_0^{p-2}\left(\frac{x_2}{\varepsilon_\star}\right)\xi^2 z + \frac{p-2}{3!}w_0^{p-3}\left(\frac{x_2}{\varepsilon_\star}\right)\xi^4\right).$$

We deduce that

$$\frac{\varepsilon''(0)}{p(p-1)}\int\limits_{S^1\times\mathbb{R}_+}\left(\varepsilon_\star|\nabla\xi|^2+\frac{p(p-1)}{2\varepsilon_\star^2}x_2w_0^{p-2}\left(\frac{x_2}{\varepsilon_\star}\right)w_0'\left(\frac{x_2}{\varepsilon_\star}\right)\xi^2\right)$$
$$=\int\limits_{S^1\times\mathbb{R}_+}\left(w_0^{p-2}\left(\frac{x_2}{\varepsilon_\star}\right)\xi^2z_0+\frac{p-2}{3!}w_0^{p-3}\left(\frac{x_2}{\varepsilon_\star}\right)\xi^4\right). \tag{5.23}$$

We have to find the signs of the both integrals in this identity. Let us remark that the integral in the left member is

$$\left\langle M_{u,\varepsilon}\left(\varepsilon_\star,w_0\left(\frac{x_2}{\varepsilon_\star}\right)\right)\xi,\xi\right\rangle$$

and the fact that it is not null is exactly the Crandall–Rabinowitz transversality condition $M_{u,\varepsilon}(\varepsilon_\star,w_0(\frac{x_2}{\varepsilon_\star}))\xi\notin\mathcal{R}(M_u(\varepsilon_\star,w_0(\frac{x_2}{\varepsilon_\star}))$ [9]. The integral in the left member has the sign of

$$\int\limits_0^{+\infty}\left(v'^2+\varepsilon_*^2v^2+\frac{p(p-1)}{2}w_0^{p-2}w_0'x_2v^2\right).$$

Multiplying (5.21) by x_2w_0' and integrating by parts we obtain that

$$\frac{p(p-1)}{2}\int\limits_0^{+\infty}w_0^{p-2}w_0'x_2v^2=4\int\limits_0^{+\infty}\eta_1'w_0'. \tag{5.24}$$

Multiplying the equation of v by v' we obtain

$$-\frac{1}{2}\left(v'^2\right)'+\frac{1}{2}\left(1+\varepsilon_\star^2\right)\left(v^2\right)'-\frac{p}{2}\left(w_0^{p-1}v^2\right)'+\frac{p(p-1)}{2}w_0^{p-2}v^2w_0'=0.$$

We deduce that

$$v'^2-\left(1+\varepsilon_\star^2\right)v^2+pw_0^{p-1}v^2=4\int\limits_x^{+\infty}\left(\eta_1''-\eta_1+pw_0^{p-1}\eta_1\right)w_0',$$

and consequently

$$v'^2-\left(1+\varepsilon_\star^2\right)v^2+pw_0^{p-1}v^2=4\left(-\eta_1'w_0'+\eta_1w_0''\right).$$

Finally we get

$$\int\limits_0^{+\infty}v'^2+4\int\limits_0^{+\infty}w_0'\eta_1'=0. \tag{5.25}$$

We infer from (5.24) and (5.25) that

$$\int\limits_0^{+\infty}\left(v'^2+\varepsilon_*^2v^2+\frac{p(p-1)}{2}w_0^{p-2}w_0'x_2v^2\right)=\varepsilon_\star^2\int\limits_0^{+\infty}v^2,$$

that is positive.

Let us prove now that the integral in the right member is positive too. We have that

$$\int\limits_{S^1\times\mathbb{R}_+}\left(w_0^{p-2}\phi^2z_0+\frac{p-2}{3!}w_0^{p-3}\phi^4\right)=\int\limits_0^{+\infty}\left(w_0^{p-2}v^2\left(\eta_1+\frac{1}{2}\eta_2\right)+\frac{p-2}{8}w_0^{p-3}v^4\right).$$

We verify that

$$\eta_1 = -\frac{(p+1)^2}{8} w_0 + \frac{p+1}{8} w_0^p.$$

Now we define

$$\tilde{\eta} = \frac{p+1}{4} w_0 + \frac{p+1}{8} w_0^p.$$

We have

$$-\tilde{\eta}'' + \left(1 + 4\varepsilon_\star^2\right)\tilde{\eta} - p\eta_0^{p-1}\tilde{\eta} = \frac{p(p-1)}{4} w_0^{p-2} v^2 + \frac{p^2(p+1)}{8} w_0\left(p^2 + 2p - 3\right)$$

and $\tilde{\eta}'(0) = 0$ and this gives, by the maximum principle

$$\tilde{\eta} > \eta_2.$$

We recall that $v = w_0^{\frac{p+1}{2}}$, consequently we have

$$\int_0^{+\infty} \left(w_0^{p-2} v^2 \left(\eta_1 + \frac{1}{2}\eta_2 \right) + \frac{p-2}{8} w_0^{p-3} v^4 \right) < \int_0^{+\infty} \left(w_0^{2p-1} \left(\eta_1 + \frac{1}{2}\tilde{\eta} \right) + \frac{p-2}{8} w_0^{3p-1} \right).$$

We are led to search the sign of

$$\int_0^{+\infty} \left(-p(p+1) w_0^{2p} + \frac{5p-1}{2} w_0^{3p-1} \right).$$

We have

$$\int_0^{+\infty} w_0^{3p-1} = \int_0^{+\infty} w^{2p-1}\left(-w_0'' + w_0\right) = \int_0^{+\infty} w_0^{2p} + \int_0^{+\infty} w_0'^2 (2p-1) w_0^{2p-2}$$
$$= \int_0^{+\infty} w_0^{2p} + \int_0^{+\infty} \left(w_0^2 - \frac{2}{p+1} w_0^{p+1} \right)(2p-1) w_0^{2p-2}.$$

Consequently we get

$$\frac{5p-1}{2} \int_0^{+\infty} w^{3p-1} = p(p+1) \int_0^{+\infty} w^{2p}.$$

We have proved that the right member of (5.23) is negative. We have $\varepsilon''(0) < 0$, and then $\varepsilon(\alpha) < \varepsilon_\star$, for α near 0.

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