

Two dimensional incompressible ideal flow around a thin obstacle tending to a curve

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Abstract

In this work we study the asymptotic behavior of solutions of the incompressible two dimensional Euler equations in the exterior of a single smooth obstacle when the obstacle becomes very thin tending to a curve. We extend results by Iftimie, Lopes Filho and Nussenzweig Lopes, obtained in the context of an obstacle tending to a point, see [D. Iftimie, M.C. Lopes Filho, H.J. Nussenzweig Lopes, Two dimensional incompressible ideal flow around a small obstacle, *Comm. Partial Differential Equations* 28 (1–2) (2003) 349–379].

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1. Introduction

The purpose of this work is to study the influence of a thin material obstacle on the behavior of two dimensional incompressible ideal flows. More precisely, we consider a family of obstacles Ω_ε which are smooth, bounded, open, connected, simply connected subsets of the plane, contracting to a smooth curve Γ as $\varepsilon \rightarrow 0$. Given the geometry of the exterior domain $\mathbb{R}^2 \setminus \Omega_\varepsilon$, a velocity field (divergence free and tangent to the boundary) on this domain is uniquely determined by the two following (independent) quantities: vorticity and circulation of velocity on the boundary of the obstacle. Throughout this paper we assume that initial vorticity ω_0 is independent of ε , smooth, compactly supported outside the obstacles Ω_ε and that γ , the circulation of the initial velocity on the boundary, is independent of ε . From the work of K. Kikuchi [4], we know that there exists $u^\varepsilon = u^\varepsilon(x, t)$ a unique global solution to the Euler equation in the exterior domain $\mathbb{R}^2 \setminus \Omega_\varepsilon$ associated to the initial data described above. Our aim is to determine the limit of u^ε as $\varepsilon \rightarrow 0$. As a consequence, we also obtain the existence of a solution of the Euler equations in the exterior of the curve Γ .

The study of incompressible fluid flows in presence of small obstacles was initiated by Iftimie, Lopes Filho and Nussenzweig Lopes [2,3]. The paper [2] treats the same problem as above but with obstacles that shrink homothetically to a point P , instead of a curve. The case of Navier–Stokes is considered in [3]. In the inviscid case, these authors prove that if the circulation γ vanishes, then the limit velocity verifies the Euler equation in \mathbb{R}^2 (with the same initial

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vorticity). If the circulation is non-zero, then the limit equation involves a new term that looks that a fixed Dirac mass in the point P of strength γ ; the initial vorticity also acquires a Dirac mass in P . In the case of Navier–Stokes, the limit equation is always Navier–Stokes but the initial vorticity of the limit equation still has an additional Dirac mass in P .

Here we will show that, in the inviscid case, the limit equation is the Euler equation in $\mathbb{R}^2 \setminus \Gamma$. The initial velocity for the limit equation is a velocity field which is divergence free in \mathbb{R}^2 , tangent to Γ such that the curl computed in $\mathbb{R}^2 \setminus \Gamma$ is ω_0 and the curl computed in \mathbb{R}^2 is $\omega_0 + g_{\omega_0} \delta_\Gamma$ where g_{ω_0} is a density given explicitly in terms of ω_0 and γ . Alternatively, g_{ω_0} is the jump of the tangential velocity across Γ .

More precisely, let Φ_ε be a cut-off function in a small ε -neighborhood of the boundary (the precise definition of Φ_ε is given in Section 4.2) and set $\omega_\varepsilon = \text{curl } u^\varepsilon$. Our main theorem may be stated as follows:

Theorem 1.1. *There exists a subsequence $\varepsilon = \varepsilon_k \rightarrow 0$ such that*

- (a) $\Phi^\varepsilon u^\varepsilon \rightarrow u$ strongly in $L^2_{\text{loc}}(\mathbb{R}_+ \times \mathbb{R}^2)$;
- (b) $\Phi^\varepsilon \omega^\varepsilon \rightarrow \omega$ weak-* in $L^\infty(\mathbb{R}_+; L^4_{\text{loc}}(\mathbb{R}^2))$;
- (c) u is related to ω by means of relation (5.2);
- (d) u and ω are weak solutions of $\omega_t + u \cdot \nabla \omega = 0$ in $\mathbb{R}^2 \times (0, \infty)$.

The limit velocity u is explicitly given in terms of ω and γ (see Theorem 5.6) and can be viewed as the divergence free vector field which is tangent to Γ , vanishing at infinity, with curl in $\mathbb{R}^2 \setminus \Gamma$ equal to ω and with circulation around the curve Γ equal to γ . This velocity is blowing up at the endpoints of the curve Γ as the inverse of the square root of the distance and has a jump across Γ . Moreover, we have $\text{curl } u = \omega + g_\omega(s) \delta_\Gamma$ in $\mathbb{R}^2 \times [0, \infty)$, where δ_Γ is the Dirac function of the curve Γ , and the g_ω which is defined in Lemma 5.8 depends on ω and the circulation γ . The function g_ω is continuous on Γ and blows up at the endpoints of the curve Γ as the inverse of the square root of the distance. One can also characterize g_ω as the jump of the tangential velocity across Γ . The presence of the additional term g_ω in the expression of $\text{curl } u$, compared of the Euler equation in the full plane, is compulsory to obtain a vector field tangent to the curve, with a circulation γ around the curve.

There is a sharp contrast between the behavior of ideal flows around a small and thin obstacle. In [2], the authors studied the vanishing obstacle problem when the obstacle tends homothetically to a point P . Their main result is that the limit vorticity satisfies a modified vorticity equation of the form $\omega_t + u \cdot \nabla \omega = 0$, with $\text{div } u = 0$ and $\text{curl } u = \omega + \gamma \delta(x - P)$. In other words, for small obstacles the correction due to the vanishing obstacle appears as time-independent additional convection centered at P , whereas in the thin obstacle case, the correction term depends on the time. Although treating a related problem, the present work requires a different approach. Indeed, in [2], the proofs are simplified by the fact that the obstacles are homothetic to a fixed domain. Indeed, an easy change of variables $y = x/\varepsilon$ allows in that case to return to a fixed obstacle and to deduce the required estimates. This argument clearly does not work here and a considerable amount of work is needed to characterize the conformal mapping that sends the exterior of a small obstacle into the exterior of the unit disk. Moreover, in [2] the authors use the div–curl Lemma to obtain strong convergence for velocity. This is made possible by the validity of some bounds on the divergence and the curl of the velocity. A consequence of our work is that these estimates are no longer valid in our case, so this approach cannot work. We will be able to prove directly strong convergence for the velocity through several applications of the Lebesgue dominated convergence theorem. We finally observe that, in contrast to the case of [2], the vanishing of the circulation γ plays no role in our result. The limit velocity will always verify the same type of equation.

We also mention that Lopes Filho treated in [5] the case of several obstacles with one of the obstacles tending to a point, but the author had to work on a bounded domain. In this case, we do not have explicit formulas anymore, and the conformal mapping technique is replaced by qualitative analysis using elliptic techniques, including variational methods and the maximum principle.

The remainder of this work is organized in five sections. We introduce in Section 2 a family of conformal mappings between the exterior of Ω_ε and the exterior of the unit disk, allowing the use of explicit formulas for basic harmonic fields and the Biot–Savart law, which will be really helpful to obtain sharp estimations. In the third part, we precisely formulate the flow problem in the exterior of a vanishing obstacle. In Section 4, we collect a priori estimates in order to find the equation limit in Section 5 of this article. The last subsection concerns an existence result of the Euler equations on the exterior of a curve.

For the sake of clarity, the main notations are listed in an appendix at the end of the paper.

2. The Laplacian in an exterior domain

2.1. Conformal maps

Let $D = B(0, 1)$ and $S = \partial D$. In what follows we identify \mathbb{R}^2 with the complex plane \mathbb{C} . We begin this section by recalling some basic definitions on the curve.

Definition 2.1. We call a *Jordan arc* a curve C given by a parametric representation $C: \varphi(t), 0 \leq t \leq 1$, with φ an injective (= one-to-one) function, continuous on $[0, 1]$. An *open Jordan arc* has a parametrization $C: \varphi(t), 0 < t < 1$, with φ continuous and injective on $(0, 1)$.

We call a *Jordan curve* a curve C given by a parametric representation $C: \psi(t), t \in \mathbb{R}$, 1-periodic, with ψ an injective function on $[0, 1)$, continuous on \mathbb{R} .

Thus a Jordan curve is closed ($\varphi(0) = \varphi(1)$) whereas a Jordan arc has distinct endpoints. If J is a Jordan curve in \mathbb{C} , then the Jordan Curve Theorem states that $\mathbb{C} \setminus J$ has exactly two components G_0 and G_1 , and these satisfy $\partial G_0 = \partial G_1 = J$.

The Jordan arc (or curve) is of class $C^{n,\alpha}$ ($n \in \mathbb{N}^*, 0 < \alpha \leq 1$) if its parametrization φ is n times continuously differentiable, satisfying $\varphi'(t) \neq 0$ for all t , and if $|\varphi^{(n)}(t_1) - \varphi^{(n)}(t_2)| \leq C|t_1 - t_2|^\alpha$ for all t_1 and t_2 .

Let $\Gamma: \Gamma(t), 0 \leq t \leq 1$, be a Jordan arc. Then the subset $\mathbb{R}^2 \setminus \Gamma$ is connected and we will denote it by Π . The purpose of this part is to obtain some properties of a biholomorphism $T: \Pi \rightarrow \text{int } D^c$. After applying a homothetic transformation, a rotation and a translation, we can suppose that the endpoints of the curve are $-1 = \Gamma(0)$ and $1 = \Gamma(1)$.

Proposition 2.2. *If Γ is a C^2 Jordan arc, such that the intersection with the segment $[-1, 1]$ is a finite union of segments and points, then there exists a biholomorphism $T: \Pi \rightarrow \text{int } D^c$ which verifies the following properties:*

- T^{-1} and DT^{-1} extend continuously up to the boundary, and T^{-1} maps S to Γ ,
- DT^{-1} is bounded,
- T and DT extend continuously up to Γ with different values on each side of Γ , except at the endpoints of the curve where T behaves like the square root of the distance and DT behaves like the inverse of the square root of the distance,
- DT is bounded in the exterior of the disk $B(0, R)$, with $\Gamma \subset B(0, R)$,
- DT is bounded in $L^p(\Pi \cap B(0, R))$ for all $p < 4$ and $R > 0$.

Proof. We first study the case where the arc is the segment $[-1, 1]$. We can have here an explicit formula for T . Indeed, the Joukowski function

$$G(z) = \frac{1}{2} \left(z + \frac{1}{z} \right)$$

is a biholomorphism between the exterior of the disk and the exterior of the segment. It maps the circle $C(0, R)$ on the ellipse parametrized by $\frac{1}{2}(R + 1/R) \cos \theta + \frac{1}{2}(R - 1/R)i \sin \theta$ with $\theta \in [0, 2\pi)$, and it maps the unit circle on the segment.

Remarking that $G(z) = G(1/z)$ we can conclude that G is also a biholomorphism between the interior of the disk minus 0 and the exterior of the segment.

Therefore, any $z \notin [-1, 1]$ has one antecedent of G in D and another one in $\text{int } D^c$. For $z \in [-1, 1]$ the antecedents are $\exp(\pm i \arccos z) = z \pm i\sqrt{1 - z^2}$. Therefore, there are exactly two antecedents except for -1 and 1 . In fact, we have considered the double covering G from $\mathbb{C}^* \rightarrow \mathbb{C}$, which is precisely ramified over -1 and 1 .

Let \tilde{T} be the biholomorphism between the exterior of the segment and $\text{int } D^c$, such that $\tilde{T}^{-1} = G$, and let $\tilde{T}_{\text{int}} = 1/\tilde{T}$ be the biholomorphism between the exterior of the segment and $D \setminus \{0\}$, such that $\tilde{T}_{\text{int}}^{-1} = G$.

To find an explicit formula of \tilde{T} , we have to solve an equation of degree two. We find two solutions:

$$\tilde{T}_+ = z + \sqrt{z^2 - 1} \quad \text{and} \quad \tilde{T}_- = z - \sqrt{z^2 - 1}. \tag{2.1}$$

We consider that the function square-root is defined by $\sqrt{z} = \sqrt{|z|}e^{i\theta/2}$ with θ , the argument of z , verifying $-\pi < \theta \leq \pi$. It is easy to observe that $\tilde{T} = \tilde{T}_+$ on $\{z \mid \Re(z) > 0\} \cup i\mathbb{R}_+$ and $\tilde{T} = \tilde{T}_-$ on $\{z \mid \Re(z) < 0\} \cup i\mathbb{R}_-$. Despite this, \tilde{T} is $C^\infty(\mathbb{C} \setminus [-1, 1], \text{int } D^c)$ because $\tilde{T} = G^{-1}$.

Therefore in the segment case, $T = \tilde{T}$ and the first two points are straightforward. An obvious calculation allows us to find an explicit formula for \tilde{T}' :

$$\tilde{T}'(z) = 1 \pm \frac{z}{\sqrt{z^2 - 1}}, \tag{2.2}$$

with the choice of sign as above. This form shows us that $D\tilde{T}$ blows up at the endpoints of the segment as the inverse of the square root of the distance, which is bounded in L^p_{loc} for $p < 4$. Moreover, a mere verification can be done to find that for every $x \in (-1, 1)$, we have

$$\lim_{z \rightarrow x, \Im(z) > 0} T(z) = x + i\sqrt{1 - x^2} = \tilde{T}_+(x)$$

even if $\Re(z) < 0$, and

$$\lim_{z \rightarrow x, \Im(z) < 0} T(z) = x - i\sqrt{1 - x^2} = \tilde{T}_-(x).$$

In the same way, DT extends continuously up to each side of Γ , which concludes the proposition in the segment case.

Now, we come back to our problem, with any curve Γ . We consider the curve $\tilde{\Gamma} \equiv \overline{\tilde{T}(\Gamma)} \cup \overline{\tilde{T}_{\text{int}}(\Gamma)} = \tilde{T}_+(\Gamma) \cup \tilde{T}_-(\Gamma)$. We now show that $\tilde{\Gamma}$ is a $C^{1,1}$ Jordan curve.

We consider first the case where Γ does not intersect the segment $(-1, 1)$. Then $\gamma_1 \equiv \tilde{T}(\Gamma) \subset D^c$ and $\gamma_2 \equiv \tilde{T}_{\text{int}}(\Gamma) \subset D$ are C^2 open Jordan arcs, with the endpoints on -1 and 1 (see Fig. 1). So $\tilde{T}(-1) = -1 = \tilde{T}_{\text{int}}(-1)$ and we can observe that $\tilde{\Gamma}$ is a Jordan curve.

We wrote *open* Jordan curve because the problem with -1 and 1 is that $\tilde{T}'(\pm 1)$ is not defined. However, if we use the arclength coordinates

$$s(t) = \int_0^t |\gamma_1'(\tau)| d\tau = \int_0^t |\tilde{T}'(\Gamma(\tau))| |\Gamma'(\tau)| d\tau, \tag{2.3}$$

which are well-defined and bounded because \tilde{T}' is bounded in L^1_{loc} , then we have $\frac{d\gamma_1}{ds} = \frac{\gamma_1'}{|\gamma_1'|}$. So to prove the derivative continuity, we should show that $\lim_{t \rightarrow 0} \frac{\gamma_1'}{|\gamma_1'|}$ and $\lim_{t \rightarrow 0} \frac{\gamma_2'}{|\gamma_2'|}$ exist and are opposite. For that, we will use the following lemma:

Lemma 2.3. *If there exists a neighborhood of 0 where $\Gamma(t)$ does not intersect the segment $(-1, 1)$, then $\frac{\tilde{T}'(\Gamma)}{|\tilde{T}'(\Gamma)|}(t)$ has a limit as $t \rightarrow 0$.*

Proof. First, since $\tilde{T}'(z) = 1 - z/\sqrt{z^2 - 1}$ in a neighborhood of -1 , we compute

$$\frac{\tilde{T}'(\Gamma)}{|\tilde{T}'(\Gamma)|}(t) = \frac{|\sqrt{\Gamma^2 - 1}|}{\sqrt{\Gamma^2 - 1}}(t) \frac{\sqrt{\Gamma^2 - 1} - \Gamma}{|\sqrt{\Gamma^2 - 1} - \Gamma|}(t).$$

The second fraction tends to 1 as $t \rightarrow 0$. We have $\Gamma'(0) \neq 0$, so we can write $\Gamma^2(t) = 1 + at + o(t)$ for $a \in \mathbb{C}^*$. If $a \notin \mathbb{R}^-$ then for t small enough $\{\Gamma^2(t) - 1\} \subset \mathbb{C} \setminus \mathbb{R}^-$ and

$$\lim_{t \rightarrow 0} \frac{\tilde{T}'(\Gamma)}{|\tilde{T}'(\Gamma)|}(t) = \frac{\sqrt{|a|}}{\sqrt{a}} = e^{-i\theta/2} \quad \text{with } \theta \equiv \arg a \in (-\pi, \pi).$$

For $a \in \mathbb{R}^-$, as $a = (\Gamma^2)'(0) = 2\Gamma(0)\Gamma'(0)$, then we have $\Gamma'(0) \in \mathbb{R}^+$ and the curve is tangent to the segment $[-1, 1]$. We have here two cases:

- if Γ is over the segment on the neighborhood, then $\Im(\Gamma^2(t) - 1) < 0$ and

$$\lim_{t \rightarrow 0} \frac{\tilde{T}'(\Gamma)}{|\tilde{T}'(\Gamma)|}(t) = i,$$

- if Γ is below the segment on the neighborhood, then $\Im(\Gamma^2(t) - 1) > 0$ and

$$\lim_{t \rightarrow 0} \frac{\tilde{T}'(\Gamma)}{|\tilde{T}'(\Gamma)|}(t) = -i. \quad \square$$

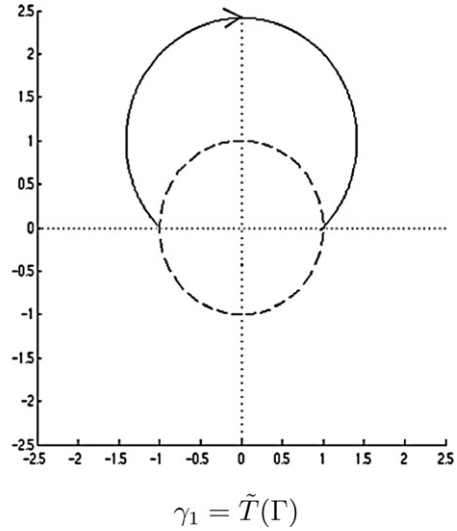
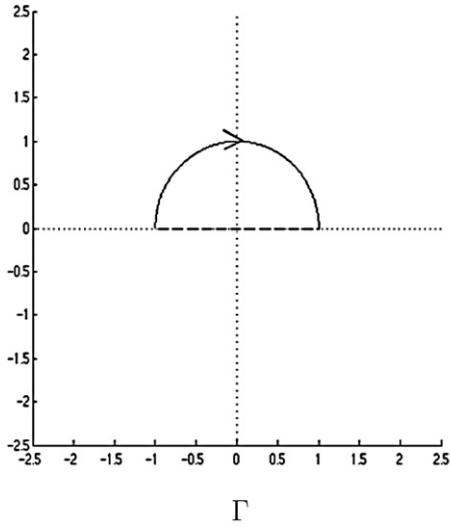
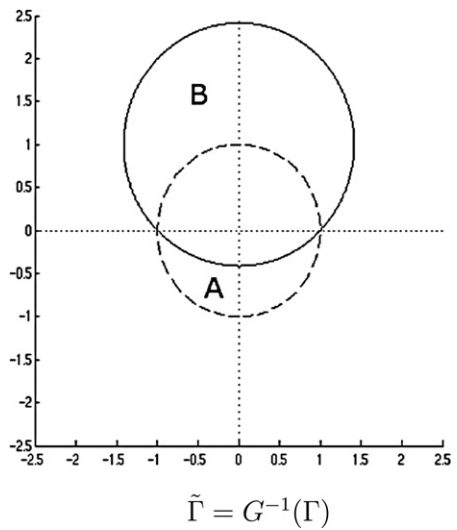
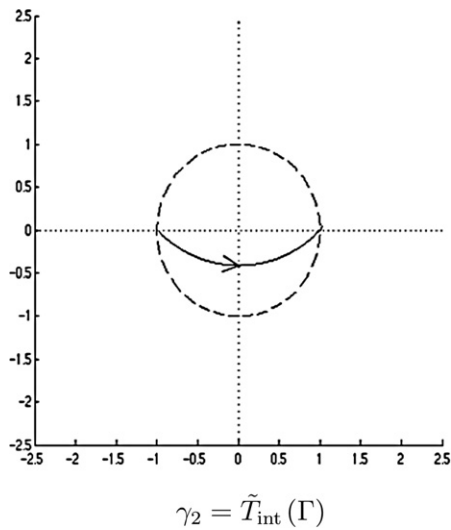


Fig. 1. Γ does not intersect $[-1, 1]$.



Let us continue the proof of Proposition 2.2. Lemma 2.3 allows us to observe that \tilde{T} is C^1 in -1 , because

$$\lim_{t \rightarrow 0} \frac{\gamma_1'}{|\gamma_1'|}(t) = \lim_{t \rightarrow 0} \frac{\tilde{T}'(\Gamma)}{|\tilde{T}'(\Gamma)|}(t) \frac{\Gamma'}{|\Gamma'|}(t) = - \lim_{t \rightarrow 0} - \frac{|\tilde{T}^2(\Gamma)|}{\tilde{T}^2(\Gamma)}(t) \frac{\tilde{T}'(\Gamma)}{|\tilde{T}'(\Gamma)|}(t) \frac{\Gamma'}{|\Gamma'|}(t) = - \lim_{t \rightarrow 0} \frac{\gamma_2'}{|\gamma_2'|}(t),$$

because $\tilde{T}_{\text{int}} = 1/\tilde{T}$.

To prove that \tilde{T}' is Lipschitz, we will show that γ'_1 and γ'_2 are C^1 with the arclength parametrization denoted by s and defined in (2.3) (t denotes the variable of the original parametrization). Let $f_1(s) = \frac{d\gamma_1}{ds}(s) = \frac{\gamma'_1(t)}{|\gamma'_1(t)|}$, where the primes denote derivatives with respect to t , then we need to prove that $\frac{df_1}{ds}$ has a limit when $s \rightarrow 0$. We have

$$\frac{df_1}{ds}(s) = \frac{\gamma''_1}{|\gamma'_1|^2} - \frac{\gamma'_1}{|\gamma'_1|^4} \langle \gamma'_1, \gamma''_1 \rangle = \frac{1}{|\gamma'_1|^2} \left(\gamma''_1 - \frac{\gamma'_1}{|\gamma'_1|} \left\langle \frac{\gamma'_1}{|\gamma'_1|}, \gamma''_1 \right\rangle \right) \equiv \frac{1}{|\gamma'_1|^2} A.$$

We compute

$$\begin{aligned} \gamma'_1 &= \tilde{T}'(\Gamma)\Gamma', \\ \gamma''_1 &= \tilde{T}''(\Gamma)(\Gamma')^2 + \tilde{T}'(\Gamma)\Gamma'', \end{aligned}$$

with

$$\begin{aligned} \tilde{T}(z) &= z - \sqrt{z^2 - 1}, \\ \tilde{T}'(z) &= 1 - z/\sqrt{z^2 - 1}, \\ \tilde{T}''(z) &= -1/\sqrt{z^2 - 1} + z^2/\sqrt{z^2 - 1}^3. \end{aligned}$$

We do some Taylor expansions in a neighborhood of zero:

$$\begin{aligned} \Gamma(t) &= -1 + at + bt^2 + O(t^3), \\ \Gamma^2(t) &= 1 - 2at + (a^2 - 2b)t^2 + O(t^3), \\ 1/\sqrt{\Gamma^2 - 1}(t) &= \frac{1}{\sqrt{-2a}} \frac{1}{\sqrt{t}} (1 - t(a^2 - 2b)/(-4a)) + O(t^{3/2}). \end{aligned}$$

The last expansion holds in any case, except when Γ is tangent to the segment ($a \in \mathbb{R}^+$) and over the segment on a neighborhood of -1 . In this last case, we should replace $1/\sqrt{-2a}$ by i instead of $-i$.

Then

$$\begin{aligned} \tilde{T}'(\Gamma) &= \frac{1}{\sqrt{-2a}} \frac{1}{\sqrt{t}} + 1 + O(t^{1/2}), \\ \tilde{T}''(\Gamma) &= \frac{1}{\sqrt{-2a}^3} \frac{1}{\sqrt{t}^3} + \frac{C_1}{\sqrt{t}} + O(t^{1/2}). \end{aligned}$$

and

$$\begin{aligned} \gamma'_1 &= \frac{a}{\sqrt{-2a}} \frac{1}{\sqrt{t}} + a + O(t^{1/2}), \\ \frac{\gamma'_1}{|\gamma'_1|} &= \frac{a}{|a|} \frac{|\sqrt{-2a}|}{\sqrt{-2a}} + C_2\sqrt{t} + O(t), \\ \gamma''_1 &= \frac{a^2}{\sqrt{-2a}^3} \frac{1}{\sqrt{t}^3} + C_3 \frac{1}{\sqrt{t}} + O(1). \end{aligned}$$

Now, we can evaluate A :

$$\gamma''_1 - \frac{\gamma'_1}{|\gamma'_1|} \left\langle \frac{\gamma'_1}{|\gamma'_1|}, \gamma''_1 \right\rangle = \frac{1}{\sqrt{t}^3} \left(\frac{a^2}{\sqrt{-2a}^3} - \frac{a}{|a|} \frac{|\sqrt{-2a}|}{\sqrt{-2a}} \left\langle \frac{a}{|a|} \frac{|\sqrt{-2a}|}{\sqrt{-2a}}, \frac{a^2}{\sqrt{-2a}^3} \right\rangle \right) + C_4 \frac{1}{t} + O(t^{-1/2}).$$

We can easily see that $\arg(a^2/\sqrt{-2a}^3) = \pm\pi + \arg(a/\sqrt{-2a})$, so

$$\frac{a^2}{\sqrt{-2a}^3} - \frac{a}{|a|} \frac{|\sqrt{-2a}|}{\sqrt{-2a}} \left\langle \frac{a}{|a|} \frac{|\sqrt{-2a}|}{\sqrt{-2a}}, \frac{a^2}{\sqrt{-2a}^3} \right\rangle = 0$$

and $df_1/ds = C_5 + O(t^{1/2})$, which means that df_1/ds has a limit as $s \rightarrow 0$. This argument holds for γ_2 , doing the calculations with $\tilde{T}_{\text{int}}(z) = z + \sqrt{z^2 - 1}$. So $d\gamma_1/ds$ and $d\gamma_2/ds$ are C^1 on $[0, 1]$ and we see that \tilde{T}' is Lipschitz because $\tilde{\Gamma} = \gamma_1 \cup \gamma_2$.

Finally, if Γ intersects $[-1, 1]$ at one point $x = \Gamma(t_0)$, then $\tilde{T}(\Gamma)$ is the union of two Jordan curves with a jump: $\tilde{T}(\Gamma(t_0^-)) = 1/\tilde{T}(\Gamma(t_0^+))$ (see Fig. 2). In this case, $\tilde{T}_{\text{int}}(\Gamma)$ is also the union of two Jordan arcs which extend $\tilde{T}(\Gamma)$, indeed

$$\tilde{T}_{\text{int}}(\Gamma(t_0^+)) = 1/\tilde{T}(\Gamma(t_0^+)) = \tilde{T}(\Gamma(t_0^-)).$$

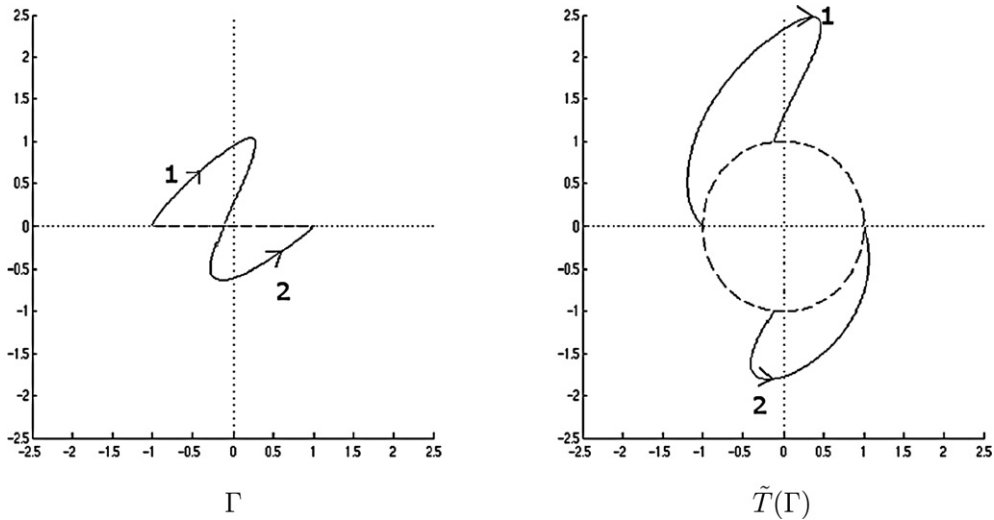
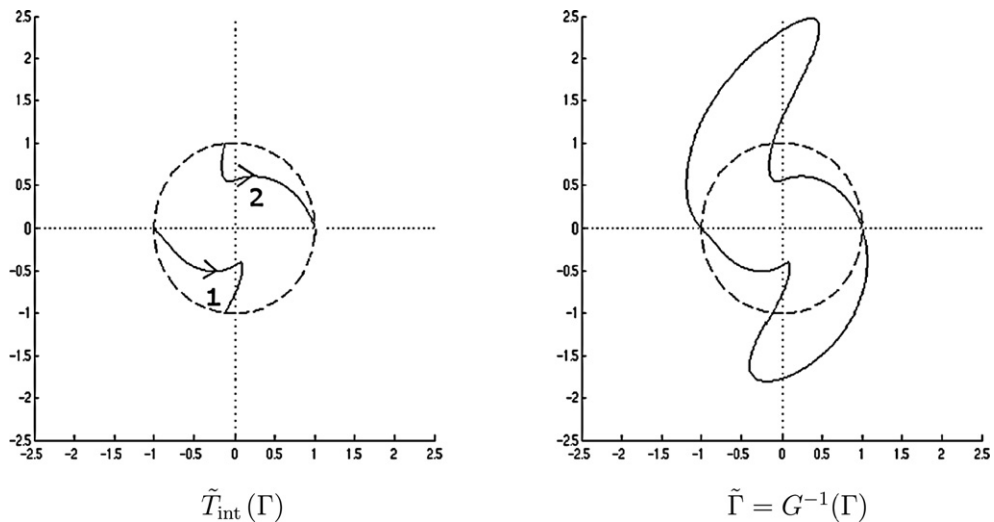


Fig. 2. Γ intersects $[-1, 1]$ at one point.



To show the continuity of \tilde{T}' on $\tilde{T}(\Gamma(t_0))$, we consider for example that $x \in (0, 1)$ and that $\Im(\Gamma(t_0^-)) > 0$ and $\Im(\Gamma(t_0^+)) < 0$. We can compute

$$\begin{aligned} \tilde{T}(\Gamma(t_0^-)) &= x + i\sqrt{1-x^2}, \\ \tilde{T}(\Gamma(t_0^+)) &= 1 + xi/\sqrt{1-x^2}, \quad \text{because } \tilde{T} = T_+ \text{ in a neighborhood of } x \\ \tilde{T}'(\Gamma(t_0^-)) &= 1 - xi/\sqrt{1-x^2}, \end{aligned}$$

to check that $-\tilde{T}(\Gamma(t_0^-))^2\tilde{T}'(\Gamma(t_0^+)) = \tilde{T}'(\Gamma(t_0^-))$, which allows us to conclude that

$$\tilde{T}'_{\text{int}}(\Gamma(t_0^+)) = -1/\tilde{T}(\Gamma(t_0^+))^2\tilde{T}'(\Gamma(t_0^+)) = \tilde{T}'(\Gamma(t_0^-)).$$

We leave to the reader the other cases. Let us do just another case: $x = 0$ then

$$\tilde{T}(\Gamma(t_0^-)) = \pm i,$$

$$\tilde{T}'(\Gamma(t_0^+)) = 1,$$

$$\tilde{T}'(\Gamma(t_0^-)) = 1,$$

and as $-(\pm i)^2 = 1$ we have the continuity of \tilde{T}' . As \tilde{T}'' is bounded, \tilde{T}' is Lipschitz, so the curve \tilde{T} is $C^{1,1}$ and closed. We have just studied the case of one or zero intersection of Γ with the segment $(-1, 1)$ but this argument works in the general case because we have a finite number of intersection. For example, if $\Gamma \subset [-1, 1]$ in a neighborhood of -1 , then $\tilde{G} \subset S$, so \tilde{G} is obviously $C^{1,1}$ in this neighborhood.

We denote by $\tilde{\Pi}$ the unbounded connected component of $\mathbb{R}^2 \setminus \tilde{\Gamma}$. We claim that we can construct T_2 , a biholomorphism between Π and $\tilde{\Pi}$, such that $T_2^{-1} = G$. Indeed, if we introduce $A = \tilde{\Pi} \cap \tilde{D}$ and $B = (\text{int } \tilde{\Pi}^c) \cap D^c$ (see Fig. 1), we observe that $B = 1/A$, because $\gamma_2 = 1/\gamma_1$ and $1/S = S$. As $G(1/z) = G(z)$, G is bijective on $\text{int } D^c$ and $1/(\partial D \cap \tilde{\Pi}) \subset \tilde{\Pi}^c$ then G is bijective on $\tilde{\Pi}$ and $G(\tilde{\Pi}) = \mathbb{R}^2 \setminus \Gamma$. Therefore, we have an function T_2 mapping the exterior of the Jordan arc to the exterior of the inner domain of a $C^{1,1}$ Jordan curve, such that $T_2^{-1}(z) = 1/2(z + 1/z)$.

Next, we just have to use the Riemann mapping Theorem and we find a conformal mapping F between $\tilde{\Pi}$ and D^c , such that $T \equiv F \circ T_2$ maps Π to D^c and $F(\infty) = \infty$. To finish the proof, we use the Kellogg–Warschawski Theorem (see Theorem 3.6 of [6], which can be applied for the exterior problems), to observe that F and F' have a continuous extension up to the boundary. Therefore, adding the fact that DF and DF^{-1} are bounded at infinity (see Remark 2.5), we find the same properties as in the segment case, in particular that DT blows up at the endpoints of the curve like the inverse of the square root of the distance (see (2.2)). \square

Remark 2.4. If Γ intersects the segment $[-1, 1]$ infinitely many times, the curve $\tilde{\Gamma}$ may not be even C^1 . For example a curve which starts as $t \mapsto (t - 1; e^{1/t^2} \sin(1/t))$, $t \in [0, 1/4]$, has two sequences $t_n \rightarrow 0$ and $\tilde{t}_n \rightarrow 0$ such $\tilde{T}'(\Gamma)/|\tilde{T}'(\Gamma)|$ tends to i and $-i$.

Remark 2.5. If we have a biholomorphism H between the exterior of an open connected and simply connected domain A and D^c , such that $H(\infty) = \infty$, then there exists a non-zero real number β and a holomorphic function $h : \Pi \rightarrow \mathbb{C}$ such that:

$$H(z) = \beta z + h(z)$$

with

$$h'(z) = O\left(\frac{1}{|z|^2}\right), \text{ as } |z| \rightarrow \infty.$$

This property can be applied for the F above to see that DF and DF^{-1} are bounded.

Proof. Indeed, after a translation we can suppose that $0 \in \text{int } A$, and we consider $W(z) = 1/H(1/z)$. The function W is holomorphic in a neighborhood of 0 and can be written as $W(z) = a_0 + a_1z + a_2z^2 + \dots$. We have $W(0) = 0$ so $a_0 = 0$. Now we want to prove that $a_1 \neq 0$ thanks to the univalence. Indeed, if $a_1 = 0$, we consider the first non-zero a_k , and we observe that there exists $R > 0$ such that $|W(z) - a_kz^k| \leq |a_k||z^k|$ in $B(0, R)$. Next, we denote by $g(z) = a_kz^k$. On $\partial B(0, R)$, $|W(z) - g(z)| \leq |a_k|R^k \leq |g(z)|$. Then we can apply the Rouché Theorem to conclude that W and g have the same number of zeros in $B(0, R)$, which is a contradiction with the fact that W is a biholomorphism and g not. Therefore $a_1 \neq 0$ and $H(z) = z/a_1 + b_0 + b_1/z + \dots$, which ends the proof. Multiplying H by $|a_1|/\bar{a}_1$, we can assume that $\beta = 1/a_1$ is real number. \square

2.2. The Biot–Savart law

Let Ω_0 be a bounded, open, connected, simply connected subset of the plane, the boundary of which, denoted by Γ_0 , is a C^∞ Jordan curve. We will denote by Π_0 the unbounded connected component of $\mathbb{R}^2 \setminus \Gamma_0$, so that $\Omega_0^c = \overline{\Pi_0}$.

We denote by $G_{\Pi_0} = G_{\Pi_0}(x, y)$ the Green’s function, whose the value is:

$$G_{\Pi_0}(x, y) = \frac{1}{2\pi} \log \frac{|T_0(x) - T_0(y)|}{|T_0(x) - T_0(y)^*||T_0(y)|} \tag{2.4}$$

writing $x^* = x/|x|^2$. The Green’s function is the unique function which verifies:

$$\begin{cases} \Delta_y G_{\Pi_0}(x, y) = \delta(y - x) & \text{for } x, y \in \Pi_0, \\ G_{\Pi_0}(x, y) = 0 & \text{for } y \in \Gamma_0, \\ G_{\Pi_0}(x, y) = G_{\Pi_0}(y, x). \end{cases} \tag{2.5}$$

Then the kernel of the Biot–Savart law is $K_{\Pi_0} = K_{\Pi_0}(x, y) \equiv \nabla_x^\perp G_{\Pi_0}(x, y)$. With $(x_1, x_2)^\perp = \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}$, the explicit formula of K_{Π_0} is given by

$$K_{\Pi_0}(x, y) = \frac{((T_0(x) - T_0(y))DT_0(x))^\perp}{2\pi|T_0(x) - T_0(y)|^2} - \frac{((T_0(x) - T_0(y)^*)DT_0(x))^\perp}{2\pi|T_0(x) - T_0(y)^*|^2}. \tag{2.6}$$

We require information on far-field behavior of K_{Π_0} . We will use several times the following general relation:

$$\left| \frac{a}{|a|^2} - \frac{b}{|b|^2} \right| = \frac{|a - b|}{|a||b|}, \tag{2.7}$$

which can be easily checked by squaring both sides.

We now find the following inequality:

$$|K_{\Pi_0}(x, y)| \leq C \frac{|T_0(y) - T_0(y)^*|}{|T_0(x) - T_0(y)||T_0(x) - T_0(y)^*|}.$$

For $f \in C_c^\infty(\Pi_0)$, we introduce the notation

$$K_{\Pi_0}[f] = K_{\Pi_0}[f](x) \equiv \int_{\Pi_0} K_{\Pi_0}(x, y) f(y) dy.$$

It is easy to see, for large $|x|$, that $|K_{\Pi_0}[f](x)| \leq C_1/|x|^2$ where C_1 depends on the size of the support of f . We used here the explicit formulas for the biholomorphism T_0 (Remark 2.5).

Lemma 2.6. *The vector field $u = K_{\Pi_0}[f]$ is a solution of the elliptic system:*

$$\begin{cases} \operatorname{div} u = 0 & \text{in } \Pi_0, \\ \operatorname{curl} u = f & \text{in } \Pi_0, \\ u \cdot \hat{n} = 0 & \text{on } \Gamma_0, \\ \lim_{|x| \rightarrow \infty} |u| = 0. \end{cases}$$

The proof of this lemma is straightforward.

2.3. Harmonic vector fields

We will denote by \hat{n} the unit normal exterior to Π_0 at Γ_0 . In what follows all contour integrals are taken in the counter-clockwise sense, so that $\int_{\Gamma_0} F \cdot ds = - \int_{\Gamma_0} F \cdot \hat{n}^\perp ds$.

Proposition 2.7. *There exists a unique classical solution $H = H_{\Gamma_0}$ of the problem:*

$$\begin{cases} \operatorname{div} H = 0 & \text{in } \Pi_0, \\ \operatorname{curl} H = 0 & \text{in } \Pi_0, \\ H \cdot \hat{n} = 0 & \text{on } \Gamma_0, \\ \lim_{|x| \rightarrow \infty} |H| = 0, \\ \int_{\Gamma_0} H \cdot \mathbf{ds} = 1. \end{cases} \quad (2.8)$$

Moreover, $H_{\Gamma_0} = \mathcal{O}(1/|x|)$ as $|x| \rightarrow \infty$.

To prove this, one can check that $H_{\Gamma_0}(x) = \frac{1}{2\pi} \nabla^\perp \log|T_0(x)|$ is the unique solution. The details can be found in [2].

3. Flow in an exterior domain

Let us formulate precisely here the small obstacle limit.

3.1. The initial-boundary value problem

Let $u = u(x, t) = (u_1(x_1, x_2, t), u_2(x_1, x_2, t))$ be the velocity of an incompressible, ideal flow in Ω_0^c . We assume that u is tangent to Γ_0 and $u \rightarrow 0$ as $|x| \rightarrow \infty$. The evolution of such a flow is governed by the Euler equations:

$$\begin{cases} \partial_t u + u \cdot \nabla u = -\nabla p & \text{in } \Pi_0 \times (0, \infty), \\ \operatorname{div} u = 0 & \text{in } \Pi_0 \times [0, \infty), \\ u \cdot \hat{n} = 0 & \text{in } \Gamma_0 \times [0, \infty), \\ \lim_{|x| \rightarrow \infty} |u| = 0 & \text{for } t \in [0, \infty), \\ u(x, 0) = u_0(x) & \text{in } \Omega^c, \end{cases} \quad (3.1)$$

where $p = p(x, t)$ is the pressure. An important quantity for the study of this problem is the vorticity:

$$\omega = \operatorname{curl}(u) = \partial_1 u_2 - \partial_2 u_1.$$

The velocity and the vorticity are coupled by the elliptic system:

$$\begin{cases} \operatorname{div} u = 0 & \text{in } \Pi_0 \times [0, \infty) \\ \operatorname{curl} u = \omega & \text{in } \Pi_0 \times [0, \infty), \\ u \cdot \hat{n} = 0 & \text{in } \Gamma_0 \times [0, \infty), \\ \lim_{|x| \rightarrow \infty} |u| = 0 & \text{for } t \in [0, \infty). \end{cases}$$

Lemma 2.6 and Proposition 2.7 assure us that the general solution of this system is given by $u = u(x, t) = K_{\Pi_0}[\omega(\cdot, t)](x) + \alpha H_{\Pi_0}(x)$ for a function $\alpha = \alpha(t)$. However, using the fact that the circulation γ of u around Γ is conserved, we prove that α does not depend on the time, and $\alpha(t) = \gamma + \int_{\Pi_0} \operatorname{curl} u_0(x)$ (see [2]). Therefore, if we give the circulation, then we have the uniqueness of the solution of the previous system.

Finally, we can now write the vorticity formulation of this problem as:

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega = 0 & \text{in } \Pi_0 \times (0, \infty), \\ u = K_{\Pi_0}[\omega] + \alpha H_{\Pi_0} & \text{in } \Pi_0 \times [0, \infty), \\ \omega(x, 0) = \operatorname{curl} u_0(x) & \text{in } \Pi_0. \end{cases} \quad (3.2)$$

3.2. The evanescent obstacle

We will formulate in this subsection a family of problems, parametrized by the size of the obstacle. Therefore, we fix ω_0 such that its support is compact and does not intersect Γ .

We will consider a family of domain Ω_ε , containing Γ , with ε small enough, such that the support of ω_0 does not intersect Ω_ε . If we denote by T_ε the biholomorphism between $\Pi_\varepsilon \equiv \Omega_\varepsilon^c$ and D^c , then we suppose the following properties:

Assumption 3.1. The biholomorphism family $\{T_\varepsilon\}$ verifies

- (i) $\|T_\varepsilon - T\|_{L^\infty(B(0,R)\cap\Pi_\varepsilon)} \rightarrow 0$ for any $R > 0$,
- (ii) $\det(DT_\varepsilon^{-1})$ is bounded in Π_ε independently of ε ,
- (iii) $\|DT_\varepsilon - DT\|_{L^3(B(0,R)\cap\Pi_\varepsilon)} \rightarrow 0$ for any $R > 0$,
- (iv) there exist $R > 0$ and $C > 0$ such that $|DT_\varepsilon(x)| \leq C|x|$ on $B(0, R)^c$.

Remark 3.2. We can observe that point (iii) implies that for any R , DT_ε is bounded in $L^p(B(0, R) \cap \Pi_\varepsilon)$ independently of ε , for $p \leq 3$.

Just before going on, we give here one example of an obstacle family.

Example 3.3. We consider $\Omega_\varepsilon \equiv T^{-1}(B(0, 1 + \varepsilon) \setminus D)$. In this case, $T_\varepsilon = \frac{1}{1+\varepsilon}T$ verifies the previous assumption. If Γ is a segment, then Ω_ε is the interior of an ellipse around the segment.

The problem of this example is that the shape of the obstacle is the same of Γ .

We naturally denote by $\Gamma_\varepsilon = \partial\Omega_\varepsilon$ and $\Pi_\varepsilon = \text{int}\Omega_\varepsilon^c$. We denote also by $G_\varepsilon, K_\varepsilon$ and H_ε the previous functions corresponding at Π_ε .

Consider also the following problem:

$$\begin{cases} \partial_t \omega^\varepsilon + u^\varepsilon \cdot \nabla \omega^\varepsilon = 0 & \text{in } \Pi_\varepsilon \times (0, \infty), \\ u^\varepsilon = K^\varepsilon[\omega^\varepsilon] + \alpha H^\varepsilon & \text{in } \Pi_\varepsilon \times [0, \infty), \\ \omega^\varepsilon(x, 0) = \omega_0(x) & \text{in } \Pi_\varepsilon. \end{cases}$$

It follows from the work of Kikuchi [4] that, for any $\varepsilon > 0$, if ω_0 is sufficiently smooth then this system has a unique solution.

We now write the explicit formulas for K^ε and H^ε :

$$K^\varepsilon = \frac{1}{2\pi} DT_\varepsilon^t(x) \left(\frac{(T_\varepsilon(x) - T_\varepsilon(y))^\perp}{|T_\varepsilon(x) - T_\varepsilon(y)|^2} - \frac{(T_\varepsilon(x) - T_\varepsilon(y)^*)^\perp}{|T_\varepsilon(x) - T_\varepsilon(y)^*|^2} \right) \tag{3.3}$$

and

$$H^\varepsilon = \frac{1}{2\pi} DT_\varepsilon^t(x) \left(\frac{(T_\varepsilon(x))^\perp}{|T_\varepsilon(x)|^2} \right). \tag{3.4}$$

We introduce in the same way, K and H , replacing T_ε by T .

The regularity of T implies that H^ε is bounded in L^2_{loc} , which is really better than the punctual limit for the obstacle (see [2]) where H^ε is just L^1_{loc} . In our case, the limit is easier to see when $T_\varepsilon \rightarrow T$, and this extra regularity will allow us the passing to the limit.

4. A priori estimates

These estimates are important to conclude on the asymptotic behavior of the sequences (u^ε) and (ω^ε) . The transport nature of (3.3) gives us the classical estimates for the vorticity: $\|\omega^\varepsilon(\cdot, t)\|_{L^p(\Pi_\varepsilon)} = \|\omega_0\|_{L^p(\mathbb{R}^2)}$ and for $p \in [1, +\infty]$. In this article, we suppose that ω_0 is L^∞ and compactly supported. Moreover we choose ε small enough, so that the support of ω_0 does not intersect Π_ε .

4.1. Velocity estimate

We begin by recalling a result found in [1].

Lemma 4.1. Let $a \in (0, 2)$, $S \subset \mathbb{R}^2$ and $h : S \rightarrow \mathbb{R}_+$ be a function in $L^1(S) \cap L^\infty(S)$. Then

$$\int_S \frac{h(y)}{|x - y|^a} dy \leq C \|h\|_{L^1(S)}^{1-a/2} \|h\|_{L^\infty(S)}^{a/2}.$$

The goal of this subsection is to find a velocity estimate thanks to the explicit formula of u^ε in function of ω^ε and γ (Section 3.2):

$$u^\varepsilon(x, t) = \frac{1}{2\pi} DT_\varepsilon^t(x)(I_1 + I_2) + \alpha H^\varepsilon(x) \tag{4.1}$$

with

$$I_1 = \int_{\Pi_\varepsilon} \frac{(T_\varepsilon(x) - T_\varepsilon(y))^\perp}{|T_\varepsilon(x) - T_\varepsilon(y)|^2} \omega^\varepsilon(y, t) dy, \tag{4.2}$$

and

$$I_2 = - \int_{\Pi_\varepsilon} \frac{(T_\varepsilon(x) - T_\varepsilon(y)^*)^\perp}{|T_\varepsilon(x) - T_\varepsilon(y)^*|^2} \omega^\varepsilon(y, t) dy. \tag{4.3}$$

We begin by estimating I_1 and I_2 .

Lemma 4.2. *Let $a \in (0, 2)$ and $h : \Pi_\varepsilon \rightarrow \mathbb{R}_+$ be a function in $L^1(\Pi_\varepsilon) \cap L^\infty(\Pi_\varepsilon)$. We introduce*

$$I_{1,a} = \int_{\Pi_\varepsilon} \frac{|h(y)|}{|T_\varepsilon(x) - T_\varepsilon(y)|^a} dy, \tag{4.4}$$

$$\tilde{I}_2 = \int_{\Pi_\varepsilon} \frac{|h(y)|}{|T_\varepsilon(x) - T_\varepsilon(y)^*|} dy. \tag{4.5}$$

There exists a constant $C > 0$ depending only on the shape of Γ , such that

$$|I_{1,a}| \leq C \|h\|_{L^1}^{1-a/2} \|h\|_{L^\infty}^{a/2} \quad \text{and} \quad |\tilde{I}_2| \leq C (\|h\|_{L^1}^{1/2} \|h\|_{L^\infty}^{1/2} + \|h\|_{L^1}).$$

Remark 4.3. It will be clear from the proof that similar estimates hold true with T_ε replaced by T .

Proof. We start with the $I_{1,a}$ estimate. Let $J = J(\xi) \equiv |\det(DT_\varepsilon^{-1})(\xi)|$ and $z = T_\varepsilon(x)$. Making also the change of variables $\eta = T_\varepsilon(y)$, we find

$$|I_{1,a}| \leq \int_{|\eta| \geq 1} \frac{1}{|z - \eta|^a} |h(T_\varepsilon^{-1}(\eta))| J(\eta) d\eta. \tag{4.6}$$

Next, we introduce $f^\varepsilon(\eta) = |h(T_\varepsilon^{-1}(\eta))| J(\eta) \chi_{\{|\eta| \geq 1\}}$, with χ_E the characteristic function of the set E . Changing variables back, we get

$$\|f^\varepsilon\|_{L^1(\mathbb{R}^2)} = \|h\|_{L^1}.$$

The second point of Assumption 3.1 allows us to write

$$\|f^\varepsilon\|_{L^\infty(\mathbb{R}^2)} \leq C \|h\|_{L^\infty}.$$

So we apply the previous lemma for f and we finally find

$$|I_{1,a}| \leq \int_{\mathbb{R}^2} \frac{1}{|z - \eta|^a} f^\varepsilon(\eta) d\eta \leq C_1 \|f^\varepsilon\|_{L^1}^{1-a/2} \|f^\varepsilon\|_{L^\infty}^{a/2}. \tag{4.7}$$

This concludes the estimate for $I_{1,a}$.

Let us estimate \tilde{I}_2 :

$$|\tilde{I}_2| \leq \int_{\Pi_\varepsilon} \frac{1}{|T_\varepsilon(x) - T_\varepsilon(y)^*|} |h(y)| dy.$$

We use, as before, the notations J, z and the change of variables η

$$|\tilde{I}_2| \leq \int_{|\eta| \geq 1} \frac{1}{|z - \eta^*|} |h(T_\varepsilon^{-1}(\eta))| J(\eta) d\eta. \tag{4.8}$$

Next, we again change variables writing $\theta = \eta^*$, to obtain:

$$\begin{aligned} |\tilde{I}_2| &\leq \int_{|\theta| \leq 1} \frac{1}{|z - \theta|} |h(T_\varepsilon^{-1}(\theta^*))| J(\theta^*) \frac{d\theta}{|\theta|^4} \\ &= \left(\int_{|\theta| \leq 1/2} + \int_{1/2 \leq |\theta| \leq 1} \right) \equiv I_{21} + I_{22}. \end{aligned}$$

First we estimate I_{21} . As $z = T_\varepsilon(x)$, one has that $|z| \geq 1$, and if $|\theta| \leq 1/2$ then $|z - \theta| \geq 1/2$. Hence

$$|I_{21}| \leq \int_{|\theta| \leq 1/2} 2 |h(T_\varepsilon^{-1}(\theta^*))| J(\theta^*) \frac{d\theta}{|\theta|^4} \tag{4.9}$$

$$= 2 \int_{|\eta| \geq 2} |h(T_\varepsilon^{-1}(\eta))| J(\eta) d\eta \leq 2 \|h\|_{L^1}. \tag{4.10}$$

Finally, we estimate I_{22} . Let $g^\varepsilon(\theta) = |h(T_\varepsilon^{-1}(\theta^*))| J(\theta^*) / |\theta|^4$. We have

$$I_{22} = \int_{1/2 \leq |\theta| \leq 1} \frac{1}{|z - \theta|} g^\varepsilon(\theta) d\theta.$$

As above, we deduce by changing variables back that

$$\|g^\varepsilon\|_{L^1(1/2 \leq |\theta| \leq 1)} \leq \|h\|_{L^1}.$$

It is also trivial to see that

$$\|g^\varepsilon\|_{L^\infty(1/2 \leq |\theta| \leq 1)} \leq C \|h\|_{L^\infty}.$$

By Lemma 4.1

$$|I_{22}| = \int_{1/2 \leq |\theta| \leq 1} \frac{1}{|z - \theta|} g^\varepsilon(\theta) d\theta \tag{4.11}$$

$$\leq C \|g^\varepsilon\|_{L^1(1/2 \leq |\theta| \leq 1)}^{1/2} \|g^\varepsilon\|_{L^\infty(1/2 \leq |\theta| \leq 1)}^{1/2} \leq C \|h\|_{L^1}^{1/2} \|h\|_{L^\infty}^{1/2}. \quad \square \tag{4.12}$$

Since $|T_\varepsilon(x)| \geq 1$, one can easily see from (3.4) that $|H^\varepsilon(x)| \leq |DT_\varepsilon(x)|$. Moreover, applying the previous lemma with $a = 1$ and $h = \omega^\varepsilon \in L^1 \cap L^\infty$, we get that

$$|I_1| \leq C \|\omega^\varepsilon\|_{L^1}^{1/2} \|\omega^\varepsilon\|_{L^\infty}^{1/2} \quad \text{and} \quad |I_2| \leq C (\|\omega^\varepsilon\|_{L^1}^{1/2} \|\omega^\varepsilon\|_{L^\infty}^{1/2} + \|\omega^\varepsilon\|_{L^1}).$$

Thanks to the explicit formula (4.1), we can deduce directly the following theorem:

Theorem 4.4. u^ε is bounded in $L^\infty(\mathbb{R}^+, L^2_{\text{loc}}(\Pi_\varepsilon))$ independently of ε . More precisely, there exists a constant $C > 0$ depending only on the shape of Γ and the initial conditions $\|\omega_0\|_{L^1}, \|\omega_0\|_{L^\infty}$, such that

$$\|u^\varepsilon(\cdot, t)\|_{L^p(S)} \leq C \|DT_\varepsilon\|_{L^p(S)}, \quad \text{for all } p \in [1, \infty] \text{ and for any subset } S \text{ of } \Pi_\varepsilon.$$

The difference with [2] is that we have an estimate L^p_{loc} instead of L^∞ , but in our case, this estimate concerns all the velocity u^ε . It is one of the reason of the use of a different method to the velocity convergence.

4.2. Cutoff function

If we want to compare the different velocity and vorticity, the issue is that u^ε and ω^ε are defined on an ε -dependent domain. For this reason we extend the velocity and vorticity on \mathbb{R}^2 by multiplying by an ε -dependent cutoff function for a neighborhood of Ω_ε .

Let $\Phi \in C^\infty(\mathbb{R})$ be a non-decreasing function such that $0 \leq \Phi \leq 1$, $\Phi(s) = 1$ if $s \geq 2$ and $\Phi(s) = 0$ if $s \leq 1$. Then we introduce

$$\Phi^\varepsilon = \Phi^\varepsilon(x) = \Phi\left(\frac{|T_\varepsilon(x)| - 1}{\varepsilon}\right).$$

Clearly Φ^ε is $C^\infty(\mathbb{R}^2)$ vanishing in a neighborhood of $\overline{\Omega}^\varepsilon$.

We require some properties of $\nabla \Phi^\varepsilon$ which we collect in the following lemma.

Lemma 4.5. *The function Φ^ε defined above has the following properties:*

- (a) $H^\varepsilon \cdot \nabla \Phi^\varepsilon \equiv 0$ in Π_ε ,
- (b) there exists a constant $C > 0$ such that the Lebesgue measure of the support of $\Phi^\varepsilon - 1$ is bounded by $C\varepsilon$.

Proof. First, we remark that

$$H^\varepsilon(x) = \frac{1}{2\pi} \nabla^\perp \log |T_\varepsilon(x)| = \frac{1}{2\pi |T_\varepsilon(x)|} \nabla^\perp |T_\varepsilon(x)|,$$

and

$$\nabla \Phi^\varepsilon = \frac{1}{\varepsilon} \Phi' \left(\frac{|T_\varepsilon(x)| - 1}{\varepsilon} \right) \nabla |T_\varepsilon(x)| \tag{4.13}$$

what gives us the first point.

Finally, the support of $\Phi^\varepsilon - 1$ is contained in the subset $\{x \in \Pi_\varepsilon \mid 1 \leq |T_\varepsilon(x)| \leq 1 + 2\varepsilon\}$. The Lebesgue measure can be estimated as follows:

$$\int_{1 \leq |T_\varepsilon(x)| \leq 1 + 2\varepsilon} dx = \int_{1 \leq |z| \leq 1 + 2\varepsilon} |\det(DT_\varepsilon^{-1})|(z) dz \leq C_1 \varepsilon$$

for ε small enough. \square

We introduced the cutoff function Φ^ε in order to extend the velocity and the vorticity to \mathbb{R}^2 . One needs to make sure that the limit velocity and vorticity are not affected by the way the extension is constructed. We observe that our method of extension does not produce an error in the limit velocity and vorticity. Indeed, we denote by \tilde{u}^ε , respectively $\tilde{\omega}^\varepsilon$, the extension of u^ε , respectively ω^ε , by 0 inside the obstacle and we prove that $\lim_{\varepsilon \rightarrow 0} \tilde{u}^\varepsilon = \lim_{\varepsilon \rightarrow 0} \Phi^\varepsilon u^\varepsilon$ and $\lim_{\varepsilon \rightarrow 0} \tilde{\omega}^\varepsilon = \lim_{\varepsilon \rightarrow 0} \Phi^\varepsilon \omega^\varepsilon$ in $D'(\mathbb{R}^2)$. Indeed, using Theorem 4.4 and Remark 3.2, point (b) of the previous lemma allows us to state that

$$\|\Phi^\varepsilon u^\varepsilon - u^\varepsilon\|_{L^2(\Pi_\varepsilon)} \leq C \|DT_\varepsilon\|_{L^3(\text{supp}(\Phi^\varepsilon - 1))} |C\varepsilon|^{1/6}$$

and

$$\|\Phi^\varepsilon \omega^\varepsilon - \omega^\varepsilon\|_{L^p(\Pi_\varepsilon)} \leq C \|\omega_0\|_{L^\infty} |C\varepsilon|^{1/p}$$

for all $p \in [1, \infty)$.

In the case where the limit is a point [2], the Lebesgue measure of the support of $\nabla \Phi^\varepsilon$ is bounded by $C\varepsilon^2$, which implies that the norm L^1 of this gradient tends to 0. Moreover, the authors use a part of velocity v^ε bounded independently of ε , so they can compute the limit of $v^\varepsilon \cdot \nabla \Phi^\varepsilon$ and $v^\varepsilon \cdot \nabla^\perp \Phi^\varepsilon$ which is necessary for the calculation of the curl and div. Finally they conclude thanks to the Div–Curl Lemma.

In our case, for $1 \leq p < 4$ we have $\|\nabla \Phi^\varepsilon\|_{L^p} \leq C_p/\varepsilon$, and we cannot compute the limit of $u^\varepsilon \cdot \nabla^\perp \Phi^\varepsilon$. For this reason we cannot use a similar proof as in [2].

However, the following lemma gives us a piece of information about the limit behavior.

Lemma 4.6. $u^\varepsilon \cdot \nabla \Phi^\varepsilon \rightarrow 0$ strongly in $L^1(\mathbb{R}^2)$ and uniformly in time, when $\varepsilon \rightarrow 0$.

Proof. Using the explicit formulas (4.1)–(4.3) and (4.13), we write

$$u^\varepsilon(x) \cdot \nabla \Phi^\varepsilon(x) = u^{\varepsilon\perp}(x) \cdot \nabla^\perp \Phi^\varepsilon(x) = -\frac{1}{2\pi\varepsilon} \Phi' \left(\frac{|T_\varepsilon(x)| - 1}{\varepsilon} \right) \times \int_{\Pi_\varepsilon} \left(\frac{T_\varepsilon(x) - T_\varepsilon(y)}{|T_\varepsilon(x) - T_\varepsilon(y)|^2} - \frac{T_\varepsilon(x) - T_\varepsilon(y)^*}{|T_\varepsilon(x) - T_\varepsilon(y)^*|^2} \right) \omega^\varepsilon(y, t) dy DT_\varepsilon(x) DT_\varepsilon'(x) \frac{T_\varepsilon(x)^\perp}{|T_\varepsilon(x)|}.$$

But T_ε is holomorphic, so DT_ε is of the form $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ and we can check that $DT_\varepsilon(x)DT_\varepsilon'(x) = (a^2 + b^2)\text{Id} = |\det(DT_\varepsilon)(x)|\text{Id}$, so

$$u^\varepsilon(x) \cdot \nabla \Phi^\varepsilon(x) = \frac{\Phi'((|T_\varepsilon(x)| - 1)/\varepsilon)|\det(DT_\varepsilon)(x)|}{2\pi\varepsilon|T_\varepsilon(x)|} \int_{\Pi_\varepsilon} \left(\frac{T_\varepsilon(y) \cdot T_\varepsilon(x)^\perp}{|T_\varepsilon(x) - T_\varepsilon(y)|^2} - \frac{T_\varepsilon(y)^* \cdot T_\varepsilon(x)^\perp}{|T_\varepsilon(x) - T_\varepsilon(y)^*|^2} \right) \omega^\varepsilon(y, t) dy.$$

We compute the L^1 norm, next we change variables twice $\eta = T_\varepsilon(y)$ and $z = T_\varepsilon(x)$, to have

$$\|u^\varepsilon \cdot \nabla \Phi^\varepsilon\|_{L^1} = \frac{1}{2\pi\varepsilon} \int_{|z| \geq 1} \left| \Phi' \left(\frac{|z| - 1}{\varepsilon} \right) \right| \left| \int_{|\eta| \geq 1} \left(\frac{\eta \cdot z^\perp / |z|}{|z - \eta|^2} - \frac{\eta^* \cdot z^\perp / |z|}{|z - \eta^*|^2} \right) J(\eta) \omega^\varepsilon(T_\varepsilon^{-1}(\eta), t) d\eta \right| dz,$$

where $J(\eta) = |\det(DT_\varepsilon^{-1})(\eta)|$.

Thanks to Lemma 4.5, we know that $\|\frac{1}{\varepsilon}\Phi'(\frac{|z|-1}{\varepsilon})\|_{L^1} \leq C$. So it is sufficient to prove that

$$\left\| \int_{|\eta| \geq 1} \left(\frac{\eta \cdot z^\perp / |z|}{|z - \eta|^2} - \frac{\eta^* \cdot z^\perp / |z|}{|z - \eta^*|^2} \right) J(\eta) \omega^\varepsilon(T_\varepsilon^{-1}(\eta), t) d\eta \right\|_{L^\infty(1+\varepsilon \leq |z| \leq 1+2\varepsilon)} \rightarrow 0 \tag{4.14}$$

as $\varepsilon \rightarrow 0$, uniformly in time.

Let

$$A = \frac{\eta \cdot z^\perp / |z|}{|z - \eta|^2} - \frac{\eta^* \cdot z^\perp / |z|}{|z - \eta^*|^2}.$$

We compute

$$A = \left(\frac{(|z|^2 - 2z \cdot \eta / |\eta|^2 + 1/|\eta|^2) - 1/|\eta|^2(|z|^2 - 2z \cdot \eta + |\eta|^2)}{|z - \eta|^2 |z - \eta^*|^2} \right) \eta \cdot \frac{z^\perp}{|z|} = \frac{(|z|^2 - 1)(1 - 1/|\eta|^2)}{|z - \eta|^2 |z - \eta^*|^2} \eta \cdot \frac{z^\perp}{|z|}.$$

We now use that $|z| \geq 1$, to write

$$|z - \eta^*| \geq 1 - \frac{1}{|\eta|}.$$

Moreover, $|\eta^*| \leq 1$ allows to have

$$|z - \eta^*| \geq |z| - 1.$$

We can now estimate A by:

$$|A| \leq \frac{(|z| + 1)(1 + 1/|\eta|)(|z| - 1)^b}{|z - \eta|^2 |z - \eta^*|^b} \left| \eta \cdot \frac{z^\perp}{|z|} \right|$$

with $0 \leq b \leq 1$, to be chosen later. We remark also that $\eta \cdot \frac{z^\perp}{|z|} = (\eta - z) \cdot \frac{z^\perp}{|z|}$ and the Cauchy–Schwarz inequality gives

$$\left| \eta \cdot \frac{z^\perp}{|z|} \right| \leq |\eta - z|.$$

We now use the fact that $|z| - 1 \leq 2\varepsilon$, to estimate (4.14):

$$\left| \int_{|\eta| \geq 1} AJ(\eta)\omega^\varepsilon(T_\varepsilon^{-1}(\eta), t) d\eta \right| \leq (2 + 2\varepsilon) \cdot 2 \cdot (2\varepsilon)^b \int_{|\eta| \geq 1} \frac{J(\eta)|\omega^\varepsilon(T_\varepsilon^{-1}(\eta), t)|}{|z - \eta||z - \eta^*|^b} d\eta.$$

In the same way we passed from (4.6) to (4.7), we obtain for $p < 2$:

$$\left\| \frac{J(\eta)^{1/p} |\omega^\varepsilon(T_\varepsilon^{-1}(\eta), t)|^{1/p}}{|z - \eta|} \right\|_{L^p} = \left(\int_{|\eta| \geq 1} \frac{J(\eta)|\omega^\varepsilon(T_\varepsilon^{-1}(\eta), t)|}{|z - \eta|^p} \right)^{1/p} \leq C_p.$$

Moreover, as we passed from (4.8) to (4.10) and (4.12), we obtain for $bq = 1$:

$$\left\| \frac{J(\eta)^{1/q} |\omega^\varepsilon(T_\varepsilon^{-1}(\eta), t)|^{1/q}}{|z - \eta^*|^b} \right\|_{L^q} = \left(\int_{|\eta| \geq 1} \frac{J(\eta)|\omega^\varepsilon(T_\varepsilon^{-1}(\eta), t)|}{|z - \eta^*|} \right)^{1/q} \leq C_q.$$

We choose $b > 0$, $1/p + 1/q = 1$, and using the Hölder inequality we finish the proof. For example if we fix $b = 1/4$, $q = 4$ and $p = 4/3$, we obtain

$$\|u^\varepsilon \cdot \nabla \Phi^\varepsilon\|_{L^1} \leq C(2 + 2\varepsilon) \cdot 2 \cdot (2\varepsilon)^{1/4} C_{4/3} C_4$$

which tends to zero when ε tends to zero. \square

If the proof is a little bit technical, the idea is natural. On the boundary, the velocity u^ε is tangent to Γ_ε , whereas $\nabla \Phi^\varepsilon$ is normal. To see that, we can check that $A = 0$ when $x \in \Gamma_\varepsilon$ (which means that $|z| = |T_\varepsilon(x)| = 1$).

Before going to the last section, we derive directly from the PDE a temporal estimate for the vorticity.

4.3. Temporal estimate

If we fix $\mathbf{T} > 0$, we remark that there exists $R_1 > 0$ such that the support of $\omega^\varepsilon(\cdot, t)$ is contained in $B(0; R_1)$ for all $0 \leq t \leq \mathbf{T}$.

To see that, let R_0 be such that $B(0; R_0)$ contains the support of ω_0 . Eq. (3.2) means that ω^ε is transported by the velocity field u^ε and the trajectory of a particle moving with the flow verifies

$$\partial_t X = u^\varepsilon(X, t).$$

Moreover, Theorem 4.4 states $|u^\varepsilon(x)| \leq C|DT_\varepsilon(x)|$ and the last point of Assumption 3.1 states that there exist $R > 0$ and $C_1 > 0$ such that DT_ε is bounded by $C_1|x|$ outside $B(0, R)$. If a material particle reaches the region $B(0, \max(R_0, R))^c$, its velocity is uniformly bounded by $CC_1|x|$, and we obtain the following inequality:

$$\partial_t |X|^2 = X \cdot \partial_t X \leq CC_1|X|^2,$$

that holds true in such a region. Applying Gronwall Lemma, we observe that the trajectory of a material particle is bounded independently of ε (up to the fixed time \mathbf{T}).

Lemma 4.7. *There exists a constant C , which does not depend on $t \in [0, \mathbf{T}]$ and ε such that*

$$\|\Phi^\varepsilon \partial_t \omega^\varepsilon\|_{H^{-2}} \leq C.$$

Proof. We write the equation verified by $\Phi^\varepsilon \omega^\varepsilon$:

$$\Phi^\varepsilon \partial_t \omega^\varepsilon = -\Phi^\varepsilon u^\varepsilon \cdot \nabla \omega^\varepsilon = -\operatorname{div}(\Phi^\varepsilon u^\varepsilon \omega^\varepsilon) + \omega^\varepsilon u^\varepsilon \cdot \nabla \Phi^\varepsilon$$

which is bounded in H^{-2} for the following reason. Note that Φ^ε and ω^ε are uniformly bounded, u^ε is bounded in $L^2(B(0, R_1))$ thanks to Theorem 4.4, Remark 3.2 and the previous remark. Moreover $\nabla \Phi^\varepsilon \cdot u^\varepsilon \rightarrow 0$ in L^1 according to the previous Lemma. We finally conclude, bearing in mind that L^1 and H^{-1} are embedded into H^{-2} . \square

5. Passing to the limit

5.1. Strong compactness in velocity

Fix $\mathbf{T} > 0$. We will need the following lemmas to the passing to the strong limit $L^2_{\text{loc}}([0, \mathbf{T}] \times \mathbb{R}^2)$ of the sequence $\Phi^\varepsilon u^\varepsilon$.

As in the previous subsection, let $R_1 > 0$ be such that the support of $\omega^\varepsilon(\cdot, t)$ is contained in $B(0; R_1)$ for all $0 \leq t \leq \mathbf{T}$ and $0 < \varepsilon < \varepsilon_0$.

Lemma 5.1. *For all fixed $x \in \Pi$, there exists $\varepsilon_x > 0$ such that $x \in \Pi_\varepsilon$ for all $\varepsilon \leq \varepsilon_x$. The two following functions*

$$f_{x,\varepsilon}(y) = \frac{T_\varepsilon(x) - T_\varepsilon(y)}{|T_\varepsilon(x) - T_\varepsilon(y)|^2} \quad \text{and} \quad g_{x,\varepsilon}(y) = \frac{T_\varepsilon(x) - T_\varepsilon(y)^*}{|T_\varepsilon(x) - T_\varepsilon(y)^*|^2}$$

are bounded in $L^{4/3}(B(0, R_1) \cap \Pi_\varepsilon)$ independently of $\varepsilon \leq \varepsilon_x$ (but not necessarily independent of x).

Moreover,

$$f_x(y) = \frac{T(x) - T(y)}{|T(x) - T(y)|^2} \quad \text{and} \quad g_x(y) = \frac{T(x) - T(y)^*}{|T(x) - T(y)^*|^2}$$

are bounded in $L^{4/3}(B(0, R_1))$.

Proof. Bearing in mind the properties of T and that $T_\varepsilon \rightarrow T$ uniformly in $B(0, R_1)$, we know that $T_\varepsilon(B(0, R_1)) \subset B(0, \tilde{R})$, for some $\tilde{R} > 0$ independent of ε .

To bound $f_{x,\varepsilon}$, we change variables $\eta = T_\varepsilon(y)$ and denote by $z = T_\varepsilon(x)$:

$$\begin{aligned} \int_{B(0, R_1) \cap \Pi_\varepsilon} \frac{1}{|T_\varepsilon(x) - T_\varepsilon(y)|^{4/3}} dy &\leq \int_{1 \leq |\eta| \leq \tilde{R}} \frac{|\det(DT_\varepsilon^{-1})|(\eta)}{|z - \eta|^{4/3}} d\eta \\ &\leq 2\pi C \int_0^{|\tilde{R}|} \frac{1}{r^{1/3}} dr \\ &\leq C_1(x). \end{aligned}$$

For the second function, we begin in the same way, next we change again variables with $\theta = \eta^*$:

$$\begin{aligned} \int_{B(0, R_1) \cap \Pi_\varepsilon} \frac{1}{|T_\varepsilon(x) - T_\varepsilon(y)^*|^{4/3}} dy &\leq \int_{1 \leq |\eta| \leq \tilde{R}} \frac{|\det(DT_\varepsilon^{-1})|(\eta)}{|z - \eta^*|^{4/3}} d\eta \\ &\leq C \int_{1/\tilde{R} \leq |\theta| \leq 1} \frac{1}{|z - \theta|^{4/3} |\theta|^4} d\theta \\ &\leq \tilde{C} \int_{|\theta| \leq 1} \frac{1}{|z - \theta|^{4/3}} d\theta \\ &\leq C_2(x) \end{aligned}$$

with C_2 not depending on ε .

Replacing T by T_ε , one can obtain the bounds for f_x and g_x . \square

We now consider the limit of $\Phi^\varepsilon \omega^\varepsilon$. As $\Phi^\varepsilon \omega^\varepsilon \in L^\infty([0, \infty), L^1 \cap L^\infty(\mathbb{R}^2))$ and $\Phi^\varepsilon \partial_t \omega^\varepsilon \in L^\infty_{\text{loc}}([0, \infty), H^{-2}(\mathbb{R}^2))$, we can extract a subsequence such that $\Phi^\varepsilon \omega^\varepsilon \rightarrow \omega$ weak-* in $L^\infty([0, \infty), L^4(\mathbb{R}^2))$ with $\omega \in L^\infty([0, \infty), L^1 \cap L^\infty(\mathbb{R}^2))$ and $\partial_t \omega \in L^\infty([0, \infty), H^{-2}(\mathbb{R}^2))$. Moreover, we want that $\Phi^\varepsilon \omega^\varepsilon(\cdot, t) \rightarrow \omega(\cdot, t)$ weak L^4 for all t . As $\|\Phi^\varepsilon \omega^\varepsilon(\cdot, t)\|_{L^4(\mathbb{R}^2)} \leq \|\omega_0\|_{L^4} < \infty$, and thanks to Alaoglu's Theorem, for all t we can extract a subsequence which

verifies $\Phi^\varepsilon \omega^\varepsilon(\cdot, t) \rightarrow \omega(\cdot, t)$ weak L^4 . The problem is that the subsequence depends on the time. Let us look for a common subsequence for all t . We observe that $\omega(t)$ is defined for all $t \geq 0$. Indeed, $\partial_t \omega \in L^\infty([0, \infty), H^{-2}(\mathbb{R}^2))$, so $\omega \in C^0([0, \infty), H^{-2}(\mathbb{R}^2))$. Moreover, since $\omega \in L^\infty([0, \infty), L^1 \cap L^\infty(\mathbb{R}^2))$ we also have that $\omega(t) \in L^p(\mathbb{R}^2)$ for all $t \geq 0$, and $\sup_{t \geq 0} \|\omega(t)\|_{L^p(\mathbb{R}^2)} < \infty$ for all $p > 1$.

Proposition 5.2. *There exists a subsequence of $\Phi^\varepsilon \omega^\varepsilon$ (again denoted by $\Phi^\varepsilon \omega^\varepsilon$) such that $\Phi^\varepsilon \omega^\varepsilon(\cdot, t) \rightarrow \omega(\cdot, t)$ weak $L^4(\mathbb{R}^2)$ for all t .*

Proof. We can choose a common subsequence for all rational times, by the diagonal extraction, because \mathbb{Q} is countable. We now prove that this subsequence converges for any t .

Let $\varphi \in C_c^\infty(\mathbb{R}^2)$, and

$$f_\varepsilon(t) = \int_{\mathbb{R}^2} \varphi(\Phi^\varepsilon \omega^\varepsilon - \omega) dx.$$

Then $f_\varepsilon \rightarrow 0$ for all $t \in \mathbb{Q}$. Moreover

$$f'_\varepsilon(t) = \int_{\mathbb{R}^2} \varphi(\Phi^\varepsilon \partial_t \omega^\varepsilon - \partial_t \omega) dx,$$

which allows us to state that the family $\{f_\varepsilon\}$ is equicontinuous, using the temporal estimate (Lemma 4.7) and that $\varphi \in H^2$.

Therefore, we have an equicontinuous family which tends to 0 on a dense subset, so it tends to 0 for all t .

To finish, let $\varphi \in L^{4/3}(\mathbb{R}^2)$. The set C_c^∞ being dense into $L^{4/3}$, there exists a sequence $\varphi_n \in C_c^\infty(\mathbb{R}^2)$ which converges to φ in $L^{4/3}$. We introduce $f_{n,\varepsilon}$ in the same way as f_ε , replacing φ by φ_n . Let t be fixed, we have by the first part

$$\text{for all } n, \quad f_{n,\varepsilon} \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0. \tag{5.1}$$

Moreover

$$f_\varepsilon - f_{n,\varepsilon} = \int_{\mathbb{R}^2} (\varphi - \varphi_n)(\Phi^\varepsilon \omega^\varepsilon - \omega) dx,$$

$$|f_\varepsilon - f_{n,\varepsilon}| \leq \left(\|\omega_0\|_{L^4} + \sup_{t \geq 0} \|\omega(t)\|_{L^4} \right) \|\varphi - \varphi_n\|_{L^{4/3}}.$$

Therefore $f_\varepsilon - f_{n,\varepsilon}$ tends to 0 uniformly in ε , which according to (5.1) allows to conclude that $f_\varepsilon \rightarrow 0$. This completes the proof. \square

Lemma 5.3. *The two following functions:*

$$f_\varepsilon(x) = \int_{\mathbb{R}^2} \frac{(T(x) - T(y))^\perp}{|T(x) - T(y)|^2} (\Phi^\varepsilon(y) \omega^\varepsilon(y) - \omega(y)) dy$$

and

$$g_\varepsilon(x) = \int_{\mathbb{R}^2} \frac{(T(x) - T(y)^*)^\perp}{|T(x) - T(y)^*|^2} (\Phi^\varepsilon(y) \omega^\varepsilon(y) - \omega(y)) dy$$

tend to 0 as $\varepsilon \rightarrow 0$ in $L^2([0, \mathbf{T}], L^6_{\text{loc}}(\mathbb{R}^2))$.

Proof. Let K be a compact set of \mathbb{R}^2 . Firstly, we fix $t \in [0, \mathbf{T}]$ and we prove that the norm $L^6(K)$ of f_ε tends to 0.

For all $x \in K \setminus \Gamma$,

$$f_\varepsilon(x) = \int_{\mathbb{R}^2} f_x(y)^\perp (\Phi^\varepsilon(y) \omega^\varepsilon(y) - \omega(y)) dy,$$

with f_x given in Lemma 5.1. Furthermore, Lemma 5.1 states that f_x is bounded in $L^{4/3}$ and as $\Phi^\varepsilon \omega^\varepsilon \rightarrow \omega$ weak L^4 , we obtain that for fixed x ,

$$f_\varepsilon(x) \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0.$$

Moreover, we can apply Lemma 4.2 (estimate of $I_{1,a}$) to f_ε , with $h(y) = \Phi^\varepsilon(y)\omega^\varepsilon(y) - \omega(y)$ and we obtain a bound for f_ε independently of x, t and ε .

Then, $f_\varepsilon^6 \rightarrow 0$ almost everywhere as $\varepsilon \rightarrow 0$, and $|f_\varepsilon^6|$ is uniformly bounded. We can apply the dominated convergence theorem to conclude that for fixed t and K a bounded set

$$\int_K |f_\varepsilon(x)|^6 dx \rightarrow 0.$$

We now let t vary and we apply again the dominated convergence theorem to obtain the result on f_ε . Using the estimate of \tilde{I}_2 in Lemma 4.2, we proceed in the same manner to prove the result for g_ε . \square

Moreover, we need two last lemmas which are a consequence of the convergence of T_ε to T (Assumption 3.1).

Lemma 5.4. For all fixed $x \in \Pi$,

$$\frac{(T_\varepsilon(x) - T_\varepsilon(y)^*)^\perp}{|T_\varepsilon(x) - T_\varepsilon(y)^*|^2} - \frac{(T(x) - T(y)^*)^\perp}{|T(x) - T(y)^*|^2} \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0$$

in $L^{4/3}(B(0, R_1) \cap \Pi_\varepsilon)$ (where the norm is taken with respect to y).

Proof. Let x be fixed. Using the relation (2.7) we have

$$\int_{B(0, R_1) \cap \Pi_\varepsilon} \left(\frac{|(T(x) - T_\varepsilon(x)) - (T(y)^* - T_\varepsilon(y)^*)|}{|T(x) - T(y)^*| |T_\varepsilon(x) - T_\varepsilon(y)^*|} \right)^{4/3} dy \equiv \int_{B(0, R_1) \cap \Pi_\varepsilon} h_{x,\varepsilon}(y) dy.$$

By Assumption 3.1, we know that $h_{x,\varepsilon}(y) \rightarrow 0$ pointwise as $\varepsilon \rightarrow 0$. Moreover, if $x \notin \Gamma$ then $|T(x)| > 1$, and as $|T_\varepsilon(x)| \rightarrow |T(x)| \neq 1$ we can write

$$\begin{aligned} |T(x) - T(y)^*| &\geq |T(x)| - 1 > 0, \\ |T_\varepsilon(x) - T_\varepsilon(y)^*| &\geq |T_\varepsilon(x)| - 1 \geq 1/2(|T(x)| - 1) > 0, \end{aligned}$$

for ε small enough (depending on x). Then $h_{x,\varepsilon}$ can be bounded by a constant which does not depend on y and ε , which allows us to apply the dominated convergence theorem to deduce that $\int_{B(0, R_1) \cap \Pi_\varepsilon} h_{x,\varepsilon}(y) dy \rightarrow 0$ as $\varepsilon \rightarrow 0$. \square

Lemma 5.5. One has that

$$\Phi^\varepsilon(x) \int_{\mathbb{R}^2} \left(\frac{(T_\varepsilon(x) - T_\varepsilon(y))^\perp}{|T_\varepsilon(x) - T_\varepsilon(y)|^2} - \frac{(T(x) - T(y))^\perp}{|T(x) - T(y)|^2} \right) \Phi^\varepsilon(y) \omega^\varepsilon(y) dy \rightarrow 0$$

in $L^\infty_{\text{loc}}([0, \infty) \times \mathbb{R}^2)$ as $\varepsilon \rightarrow 0$.

Proof. Let $x \in B(0, R)$ and $y \in \text{supp } \omega^\varepsilon$. Using the relation (2.7) we can introduce and bound

$$\begin{aligned} \tilde{h}_{x,\varepsilon} &\equiv \frac{|(T_\varepsilon(x) - T(x)) - (T_\varepsilon(y) - T(y))|}{|T_\varepsilon(x) - T_\varepsilon(y)| |T(x) - T(y)|} \\ &\leq \sqrt{2} \sup_{B(0, R_2) \cap \Pi_\varepsilon} (|T_\varepsilon - T|) \left(\frac{1}{\sqrt{|T_\varepsilon(x) - T_\varepsilon(y)| |T(x) - T(y)|}} + \frac{1}{|T_\varepsilon(x) - T_\varepsilon(y)| \sqrt{|T(x) - T(y)|}} \right) \end{aligned}$$

where $R_2 = \max(R, R_1)$.

Using Lemma 4.2 with $a = 5/4$ and $a = 5/3$ we conclude by the Hölder inequality:

$$\begin{aligned} \int_{\mathbb{R}^2} \tilde{h}_{x,\varepsilon}(y) \Phi^\varepsilon(y) |\omega^\varepsilon(y)| dy &\leq \sqrt{2 \sup_{B(0,R_2) \cap \Pi_\varepsilon} (|T_\varepsilon - T|)} \left(\left\| \frac{(\omega^\varepsilon)^{2/5}}{\sqrt{|T_\varepsilon(x) - T_\varepsilon(y)|}} \right\|_{L^{5/2}} \left\| \frac{(\omega^\varepsilon)^{3/5}}{|T(x) - T(y)|} \right\|_{L^{5/3}} \right. \\ &\quad \left. + \left\| \frac{(\omega^\varepsilon)^{3/5}}{|T_\varepsilon(x) - T_\varepsilon(y)|} \right\|_{L^{5/3}} \left\| \frac{(\omega^\varepsilon)^{2/5}}{\sqrt{|T(x) - T(y)|}} \right\|_{L^{5/2}} \right) \\ &\leq C \sqrt{\sup_{B(0,R_2) \cap \Pi_\varepsilon} (|T_\varepsilon - T|)}. \end{aligned}$$

Therefore, the uniform convergence of T_ε (Assumption 3.1) allows us to conclude. \square

Theorem 5.6. *One has that $\Phi^\varepsilon u^\varepsilon \rightarrow u$ strongly in $L^2_{\text{loc}}([0, \infty) \times \mathbb{R}^2)$, with*

$$u(x) = \frac{1}{2\pi} DT^t(x) \int_{\mathbb{R}^2} \left(\frac{(T(x) - T(y))^\perp}{|T(x) - T(y)|^2} - \frac{(T(x) - T(y))^*^\perp}{|T(x) - T(y)^*|^2} \right) \omega(y, t) dy + \alpha H(x). \tag{5.2}$$

Proof. We recall the explicit formula for $\Phi^\varepsilon u^\varepsilon$:

$$\Phi^\varepsilon u^\varepsilon = \frac{1}{2\pi} \Phi^\varepsilon(x) DT_\varepsilon^t(x) \int_{\Pi_\varepsilon} \left(\frac{(T_\varepsilon(x) - T_\varepsilon(y))^\perp}{|T_\varepsilon(x) - T_\varepsilon(y)|^2} - \frac{(T_\varepsilon(x) - T_\varepsilon(y))^*^\perp}{|T_\varepsilon(x) - T_\varepsilon(y)^*|^2} \right) \omega^\varepsilon(y, t) dy + \alpha \Phi^\varepsilon(x) H_\varepsilon(x).$$

Next we decompose:

$$\begin{aligned} (\Phi^\varepsilon u^\varepsilon - u)(x) &= \frac{1}{2\pi} (DT_\varepsilon^t(x) - DT^t(x)) \Phi^\varepsilon(x) \int_{\Pi_\varepsilon} \left(\frac{(T_\varepsilon(x) - T_\varepsilon(y))^\perp}{|T_\varepsilon(x) - T_\varepsilon(y)|^2} - \frac{(T_\varepsilon(x) - T_\varepsilon(y))^*^\perp}{|T_\varepsilon(x) - T_\varepsilon(y)^*|^2} \right) \omega^\varepsilon(y) dy \\ &\quad + \frac{1}{2\pi} DT^t(x) \Phi^\varepsilon(x) \int_{\Pi_\varepsilon} \left(\frac{(T_\varepsilon(x) - T_\varepsilon(y))^\perp}{|T_\varepsilon(x) - T_\varepsilon(y)|^2} - \frac{(T_\varepsilon(x) - T_\varepsilon(y))^*^\perp}{|T_\varepsilon(x) - T_\varepsilon(y)^*|^2} \right) \omega^\varepsilon(y) (1 - \Phi^\varepsilon(y)) dy \\ &\quad + \frac{1}{2\pi} DT^t(x) \Phi^\varepsilon(x) \int_{\mathbb{R}^2} \left(\frac{(T_\varepsilon(x) - T_\varepsilon(y))^\perp}{|T_\varepsilon(x) - T_\varepsilon(y)|^2} - \frac{(T(x) - T(y))^\perp}{|T(x) - T(y)|^2} \right) \Phi^\varepsilon(y) \omega^\varepsilon(y) dy \\ &\quad + \frac{1}{2\pi} DT^t(x) (\Phi^\varepsilon(x) - 1) \int_{\mathbb{R}^2} \frac{(T(x) - T(y))^\perp}{|T(x) - T(y)|^2} \Phi^\varepsilon(y) \omega^\varepsilon(y) dy \\ &\quad + \frac{1}{2\pi} DT^t(x) \int_{\mathbb{R}^2} \frac{(T(x) - T(y))^\perp}{|T(x) - T(y)|^2} (\Phi^\varepsilon(y) \omega^\varepsilon(y) - \omega(y)) dy \\ &\quad - \frac{1}{2\pi} DT^t(x) \Phi^\varepsilon(x) \int_{\mathbb{R}^2} \left(\frac{(T_\varepsilon(x) - T_\varepsilon(y))^*^\perp}{|T_\varepsilon(x) - T_\varepsilon(y)^*|^2} - \frac{(T(x) - T(y))^*^\perp}{|T(x) - T(y)^*|^2} \right) \Phi^\varepsilon(y) \omega^\varepsilon(y) dy \\ &\quad - \frac{1}{2\pi} DT^t(x) (\Phi^\varepsilon(x) - 1) \int_{\mathbb{R}^2} \frac{(T(x) - T(y))^*^\perp}{|T(x) - T(y)^*|^2} \Phi^\varepsilon(y) \omega^\varepsilon(y) dy \\ &\quad - \frac{1}{2\pi} DT^t(x) \int_{\mathbb{R}^2} \frac{(T(x) - T(y))^*^\perp}{|T(x) - T(y)^*|^2} (\Phi^\varepsilon(y) \omega^\varepsilon(y) - \omega(y)) dy \\ &\quad + \alpha \frac{1}{2\pi} \left(\Phi^\varepsilon(x) DT_\varepsilon^t(x) \frac{T_\varepsilon(x)^\perp}{|T_\varepsilon(x)|^2} - DT^t(x) \frac{T(x)^\perp}{|T(x)|^2} \right) \\ &\equiv J_1 + \dots + J_9. \end{aligned}$$

In every J_i , we use the fact that DT is bounded in L^3_{loc} (see Proposition 2.2). We also use the estimates of the integrals I_1 and I_2 independently of x, ε, t (see Lemma 4.2).

For J_4 and J_7 , we remark that $(\Phi^\varepsilon(x) - 1) \rightarrow 0$ in L^6 , DT is bounded in L^3_{loc} and the integral is bounded independently of x, ε and t (see Remark 4.3), which is sufficient to conclude that J_4 and J_7 converge to zero in $L^2_{loc}([0, \infty) \times \mathbb{R}^2)$.

A similar argument holds true for J_1 , since $DT_\varepsilon \rightarrow DT$ in L^3_{loc} by Assumption 3.1.

For J_2 , Lemma 5.1 states that for fixed x , the fractions are bounded in $L^{4/3}(B(0, R_1))$ independently of ε . Moreover ω^ε is bounded independently of t, ε and $1 - \Phi^\varepsilon(y) \rightarrow 0$ in L^4 . Therefore, for fixed $x \notin \Gamma$, the integral tends pointwise to 0. Moreover this integral is bounded (see Lemma 4.2) and using the dominated convergence theorem, we can observe that it tends to 0 in L^6_{loc} . So, we have the convergence of J_2 to zero because DT is bounded in L^3_{loc} .

The convergence of J_3 to 0 is a direct consequence of Lemma 5.5. Next, DT belongs to $L^3_{loc}(\mathbb{R}^2)$, and thanks to Lemma 5.3, we know that the integrals in J_5 and J_8 tend to zero in $L^2_{loc}([0, \infty), L^6_{loc}(\mathbb{R}^2))$. So J_5 and J_8 tend to zero in L^2_{loc} .

We now go to J_6 . Applying Lemma 5.4 and reasoning as we did for the second term: for fixed x and t , the integral tends pointwise to 0 because ω^ε is bounded in L^4 independently of t . Moreover, it is uniformly bounded by Lemma 4.2, and we can apply twice the dominated convergence theorem to obtain that the integral in J_6 converges to 0 in $L^2_{loc}([0, \infty), L^6_{loc}(\mathbb{R}^2))$. Using again the boundness of DT in L^3_{loc} we get the desired conclusion for J_6 .

The convergence of J_9 can be done more easily, because $1/|T| \leq 1$. Indeed we can decompose

$$J_9 = \frac{\alpha}{2\pi} (DT_\varepsilon^t(x) - DT^t(x)) \Phi^\varepsilon(x) \frac{T_\varepsilon(x)^\perp}{|T_\varepsilon(x)|^2} + \frac{\alpha}{2\pi} \Phi^\varepsilon(x) DT^t(x) \left(\frac{T_\varepsilon(x)^\perp}{|T_\varepsilon(x)|^2} - \frac{T(x)^\perp}{|T(x)|^2} \right) + \frac{\alpha}{2\pi} (\Phi^\varepsilon(x) - 1) DT^t(x) \frac{T(x)^\perp}{|T(x)|^2},$$

and the convergence to zero of J_9 is a direct consequence of points (i) and (iii) of Assumption 3.1. \square

The previous theorem provides an explicit formula expressing the limit velocity in terms of the limit vorticity. From this formula, we can deduce a few properties of the limit velocity u .

Proposition 5.7. *Let u be given as in Theorem 5.6. For fixed t , the velocity*

- (i) *is continuous on $\mathbb{R}^2 \setminus \Gamma$;*
- (ii) *is continuous up to $\Gamma \setminus \{-1; 1\}$, with different values on each side of Γ ;*
- (iii) *blows up at the endpoints of the curve like $C/\sqrt{|x-1||x+1|}$, which belongs to L^p_{loc} for $p < 4$;*
- (iv) *is tangent to the curve.*

Proof. To show that, we now prove that

$$A(x) \equiv \int_{\mathbb{R}^2} \frac{T(x) - T(y)}{|T(x) - T(y)|^2} \omega(y) dy \quad \text{and} \quad B(x) \equiv \int_{\mathbb{R}^2} \frac{T(x) - T(y)^*}{|T(x) - T(y)^*|^2} \omega(y) dy$$

are continuous on $\mathbb{R}^2 \setminus \Gamma$ as $\omega \in L^1 \cap L^\infty$. As in the proof of Lemma 4.2, we change variables, we introduce $f(\eta, t) = \omega(T^{-1}(\eta), t) J(\eta) \chi_{\{|\eta| \geq 1\}}$ and $z = T(x)$. Then we have $\tilde{A}(z) \equiv A(x) = \int_{|\eta| \geq 1} \frac{z - \eta}{|z - \eta|^2} f(\eta, t) d\eta$, which is continuous for $f \in L^1 \cap L^\infty$. In the same way we estimated I_2 in the proof of Lemma 4.2, we write

$$B(x) = \int_{|\eta| \geq 2} \frac{z - \eta^*}{|z - \eta^*|^2} f(\eta, t) d\eta + \int_{1/2 \leq |\theta| \leq 1} \frac{z - \theta}{|z - \theta|^2} g(\theta, t) d\theta \equiv B_1(z) + B_2(z),$$

with $g(\theta, t) = \omega(T^{-1}(\theta^*), t) J(\theta^*) / |\theta|^4$. As for A , we observe that B_2 is continuous. For B_1 , taking a sequence $f_n \in C_c^\infty$ such that $f_n \rightarrow f(\cdot, t)$ strongly in L^1 , we see that $B_{1,n} \equiv \int_{|\eta| \geq 2} \frac{z - \eta^*}{|z - \eta^*|^2} f_n(\eta, t) d\eta$ is continuous. As $|z - \eta^*| \geq 1/2$, we can conclude, after remarking that $\|B_n - B_1\|_{L^\infty} \leq 2\|f_n - f\|_{L^1}$, that B_1 is continuous.

Moreover, \tilde{A} , B_1 and B_2 are continuous up to the boundary. As $A(x) = \tilde{A}(T(x))$ and $B(x) = B_1(T(x)) + B_2(T(x))$, with T continuous up to $\Gamma \setminus \{-1; 1\}$, with different values on each side of Γ (see Proposition 2.2), we proved (i) and (ii).

The blowing up at the endpoints is the consequence of the expression of DT (see (2.2)), and the fact that $A(x)$ and $B(x)$ is bounded by Lemma 4.2.

Finally, to show that the velocity is tangent to the curve, we do a simplified, but similar calculation to the one in Lemma 4.7. Indeed, as $|T(x)| = 1$ on the curve Γ , $\nabla|T(x)|$ is orthogonal to the curve. According to Proposition 2.2, $\nabla|T(x)|$ is continuous up to the curve Γ with different values on each side. Let $x \in \Gamma \setminus \{-1; 1\}$, then for a sequence $x_n \in \Pi$ which tends to x , we can make the same calculation than in the beginning of the proof of Lemma 4.6, to get:

$$\begin{aligned} u(x_n) \cdot \nabla|T(x_n)| &= u(x_n)^\perp \cdot \nabla^\perp|T(x_n)| \\ &= -\frac{1}{2\pi} \int_{\mathbb{R}^2} \left(\frac{T(x_n) - T(y)}{|T(x_n) - T(y)|^2} - \frac{T(x_n) - T(y)^*}{|T(x_n) - T(y)^*|^2} \right) \omega(y, t) dy DT(x_n) DT^t(x_n) \frac{T(x_n)^\perp}{|T(x_n)|} \\ &\quad + \alpha \frac{\nabla|T(x_n)|}{|T(x_n)|} \cdot \nabla^\perp|T(x_n)| \\ &= \frac{1}{2\pi|T(x_n)|} \det(DT)(x_n) \int_{\mathbb{R}^2} A_n(y) \omega(y, t) dy \end{aligned}$$

with

$$\begin{aligned} A_n(y) &= \frac{T(y) \cdot T(x_n)^\perp}{|T(x_n) - T(y)|^2} - \frac{T(y)^* \cdot T(x_n)^\perp}{|T(x_n) - T(y)^*|^2} \\ &= \frac{|T(x_n) - T(y)^*|^2 - |T(x_n) - T(y)|^2 / |T(y)|^2}{|T(x_n) - T(y)|^2 |T(x_n) - T(y)^*|^2} T(y) \cdot T(x_n)^\perp \\ &= \frac{(1 - 1/|T(y)|^2)(|T(x_n)|^2 - 1)}{|T(x_n) - T(y)|^2 |T(x_n) - T(y)^*|^2} T(y) \cdot T(x_n)^\perp. \end{aligned}$$

If x_n tends to x on one side of the curve, then $|T(x_n)| \rightarrow 1$ and $A_n(y) \rightarrow 0$. So $A_n \omega(\cdot, t)$ tends pointwise to zero, and as the integral is bounded by Remark 4.3, we can conclude by the dominated convergence theorem that $u(x) \cdot \nabla|T(x)| = 0$ and that the velocity, on each side, is tangent to the curve. \square

Therefore we have a weak-* limit for the vorticity, and a strong limit for the velocity. We now study the relation between $\text{curl } u$ and ω .

5.2. Calculation of curl and div of the velocity

We first remark that $\text{div } u = 0$, which is obvious since the velocity is the orthogonal gradient of a function. Indeed $u = \nabla^\perp \psi$ with $\psi(x) = \int G_\pi(x, y) \omega(y) dy + \frac{1}{2\pi} \log|T(x)|$.

We now compute the curl of the limit velocity.

Recall that the curve Γ goes from -1 to 1 . Let $\vec{\tau} = \Gamma' / |\Gamma'|$ the tangent vector of Γ , u_{up} the limit of $u(\Gamma(s) + \rho \vec{\tau}^\perp)$ as $\rho \rightarrow 0^+$ and u_{down} the limit as $\rho \rightarrow 0^-$.

Lemma 5.8. *There exists a function g_ω which depends on Γ and ω such that*

$$\text{curl } u = \omega + g_\omega(s) \delta_\Gamma,$$

in the sense of distributions.

Moreover $g_\omega = (u_{\text{down}} - u_{\text{up}}) \cdot \vec{\tau}$ which corresponds to the jump of the velocity on the curve.

Proof. This proof is divided in two part. The first step consists to show that $\text{curl } u - \omega$ is concentrated on the curve Γ , and we find in the second step the expression of g_ω .

We begin with $\text{curl } H$. We remember that $H = 1/2\pi \nabla^\perp \log(T(x))$. Let $\varphi \in C_0^\infty(\mathbb{R}^2)$, we write:

$$\begin{aligned} \int_{\Pi} \operatorname{curl}(H)\varphi &= - \int_{\Pi} H \cdot \nabla^{\perp} \varphi = - \frac{1}{2\pi} \int_{\Pi} \frac{T(x)}{|T(x)|^2} DT(x) \nabla \varphi(x) dx \\ &= - \frac{1}{2\pi} \int_{D^c} \frac{z}{|z|^2} DT(T^{-1}(z)) \nabla \varphi(T^{-1}(z)) |\det T^{-1}|(z) dz \end{aligned}$$

where we changed variables $z = T(x)$. Since T is holomorphic, we remark that

$$\nabla(\varphi \circ T^{-1})(z) = DT^{-1}(z) \nabla \varphi(T^{-1}(z)) = |\det T^{-1}|(z) DT(T^{-1}(z)) \nabla \varphi(T^{-1}(z)).$$

We use the polar variables ($z = re^{i\theta}$) to find

$$\begin{aligned} \int_{\Pi} \operatorname{curl}(H)\varphi &= - \frac{1}{2\pi} \int_0^{2\pi} \int_1^{\infty} \nabla(\varphi \circ T^{-1})(z) \cdot \frac{z}{r} dr d\theta = - \frac{1}{2\pi} \int_0^{2\pi} \int_1^{\infty} \frac{d}{dr} (\varphi \circ T^{-1}(re^{i\theta})) dr d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} \varphi \circ T^{-1}(\cos(\theta), \sin(\theta)) d\theta. \end{aligned}$$

The last integral can be written as an integral of φ on the curve Γ with a certain weight.

We make the same calculation with the explicit formula of the velocity, and if we consider the translations $\tau_1 : z \mapsto z + T(y)$ and $\tau_2 : z \mapsto z + T(y)^*$, we obtain:

$$\begin{aligned} \int \operatorname{curl}(u) \cdot \varphi &= - \frac{1}{2\pi} \int_{\mathbb{R}^2} \left(\int_{D^c - T(y)} \nabla(\varphi \circ T^{-1} \circ \tau_1)(z) \cdot \frac{z}{|z|^2} dz \right) \omega(y) dy \\ &\quad + \frac{1}{2\pi} \int_{\mathbb{R}^2} \left(\int_{D^c - T(y)^*} \nabla(\varphi \circ T^{-1} \circ \tau_2)(z) \cdot \frac{z}{|z|^2} dz \right) \omega(y) dy + \alpha \int \operatorname{curl}(H) \cdot \varphi \\ &= \int_{\mathbb{R}^2} \varphi(y) \omega(y) dy + \frac{1}{2\pi} \int_{\mathbb{R}^2} \int_{\theta_0(y)}^{\theta_1(y)} [\varphi \circ T^{-1}(A_{1,y}(\theta)) - \varphi \circ T^{-1}(A_{2,y}(\theta))] d\theta \omega(y) dy \\ &\quad - \frac{1}{2\pi} \int_{\mathbb{R}^2} \int_0^{2\pi} \varphi \circ T^{-1}(A_{3,y}(\theta)) \omega(y) dy + \frac{\alpha}{2\pi} \int_0^{2\pi} \varphi \circ T^{-1}(\cos(\theta), \sin(\theta)) d\theta, \end{aligned}$$

with A_1, A_2 and A_3 constructed like this: we consider the half-line starting at $T(y)$ having an angle θ with the abscissa axis. There exist two angles $\theta_0 < \theta_1$ such that the half-line is tangent to the unit circle. If we choose $\theta \in (\theta_0, \theta_1)$, then the half-line intersects the circle in two points A_2 and further A_1 . For A_3 we do the same thing with the half-line starting at $T(y)^*$. In this case, we obtain each time an intersection with the unit circle.

Now, the difficulty is the change of variables. Indeed, as $T^{-1}(A_{i,y}(\theta)) \in \Gamma$, we should change the variable $s = T^{-1}(A_{i,y}(\theta))$ and we would obtain $\int_{\mathbb{R}^2} \int_{\Gamma} \varphi(s) f_{i,y}(s) ds$, but this calculation is too complicated. In fact, we have just proved that $\operatorname{curl} u = \omega + g_{\omega}(s) \delta_{\Gamma}$, and we will directly find the expression of g_{ω} . For that, we consider the solution v of the Green problem without obstacle. That is,

$$\operatorname{div} v = 0 \quad \text{and} \quad \operatorname{curl} v = \omega \quad \text{in } \mathbb{R}^2.$$

The explicit formula is $v(x) = \int_{\mathbb{R}^2} \frac{(x-y)^{\perp}}{|x-y|^2} \omega(y) dy$. We denote by $w = u - v$. Then $\operatorname{curl} w = g_{\omega} \delta_{\Gamma}$.

We now prove that $g_{\omega} = (w_{\text{down}} - w_{\text{up}}) \cdot \vec{\tau}$, with w_{up} and w_{down} defined in the same manner as u_{up} and u_{down} .

So, for $x \in \Gamma \setminus \{-1; 1\}$, there exists a small neighborhood O of x , such $O \setminus \Gamma$ is the union of two connected domains: O_{up} and O_{down} . On the one side,

$$\int_O \operatorname{curl} w \varphi = \int_{\Gamma} \varphi(s) g_{\omega}(s) ds.$$

On the other side,

$$\begin{aligned} \int_O \operatorname{curl} w \varphi &= - \int_O w \cdot \nabla^\perp \varphi = - \int_{O_{\text{up}}} w \cdot \nabla^\perp \varphi - \int_{O_{\text{down}}} w \cdot \nabla^\perp \varphi \\ &= \int_{O_{\text{up}}} \operatorname{curl} w \varphi - \int_{\partial O_{\text{up}}} \varphi w \cdot \vec{\tau} + \int_{O_{\text{down}}} \operatorname{curl} w \varphi - \int_{\partial O_{\text{down}}} \varphi w \cdot \vec{\tau} \\ &= - \int_\Gamma \varphi w_{\text{up}} \cdot \vec{\tau} + \int_\Gamma \varphi w_{\text{down}} \cdot \vec{\tau} \end{aligned}$$

because $w = u - v$ is continuous on $\mathbb{R}^2 \setminus \Gamma$.

As we want, we have $g_\omega = (w_{\text{down}} - w_{\text{up}}) \cdot \vec{\tau}$. Moreover, adding the regular part v , we have

$$g_\omega = \operatorname{curl} w = (u_{\text{down}} - u_{\text{up}}) \cdot \vec{\tau}. \tag{5.3}$$

Therefore, we obtain:

$$\int \varphi \operatorname{curl}(u) = \int_{\mathbb{R}^2} \varphi(y) \omega(y) dy + \int_\Gamma \varphi(s) g_\omega(s) \cdot ds,$$

with g_ω , bounded outside the endpoints, and equivalent at the endpoints to

$$\frac{1}{\pi} \frac{A(\pm 1)}{\sqrt{|s-1||s+1|}}$$

with

$$A(\pm 1) = C_{\pm 1} \left| \int_{\mathbb{R}^2} \left(\frac{(T(\pm 1) - T(y))^\perp}{|T(\pm 1) - T(y)|^2} - \frac{(T(\pm 1) - T(y)^*)^\perp}{|T(\pm 1) - T(y)^*|^2} \right) \omega(y, t) dy + \alpha T(\pm 1)^\perp \right|$$

which is bounded. Indeed, we can prove that g_ω is continuous as we prove that u is continuous in Proposition 5.7. \square

Getting a simplification is really hard, and we remark that we cannot obtain a result like $\operatorname{curl} u = \omega + g(s)\delta_\Gamma$. Even in the simpler case of the segment, the calculation of g_ω does not give a good result. However, we can explicit the calculation of $\operatorname{curl}(H)$ in the case where the curve Γ is the segment $[-1, 1]$: $T^{-1}(\cos(\theta), \sin(\theta)) = (\cos(\theta), 0)$ and we change the variable $\eta = \cos \theta$ to have

$$\int \operatorname{curl}(H)\varphi = 2 \cdot \frac{1}{2\pi} \int_0^\pi \varphi(\cos(\theta), 0) d\theta = \frac{1}{\pi} \int_{-1}^1 \frac{\varphi(\eta, 0)}{\sqrt{1-\eta^2}} d\eta.$$

Moreover, we remark that g_ω , like the velocity, blows up at the endpoints of the curve Γ as the inverse of the square root of the distance.

5.3. The asymptotic vorticity equation in \mathbb{R}^2

We begin by observing that the sequence $\{\Phi^\varepsilon \omega^\varepsilon\}$ is bounded in $L^\infty([0, \mathbf{T}], L^4)$, then, passing to a subsequence if necessary, we have

$$\Phi^\varepsilon \omega^\varepsilon \rightharpoonup \omega, \quad \text{weak-* in } L^\infty([0, \mathbf{T}], L^4).$$

We already have a limit velocity.

The purpose of this section is to prove that u and ω verify, in an appropriate sense, the system:

$$\begin{cases} \omega_t + u \cdot \nabla \omega = 0, & \text{in } \mathbb{R}^2 \times (0, \infty), \\ \operatorname{div} u = 0 \text{ and } \operatorname{curl} u = \omega + g_\omega(s)\delta_\Gamma, & \text{in } \mathbb{R}^2 \times [0, \infty), \\ |u| \rightarrow 0, & \text{as } |x| \rightarrow \infty, \\ \omega(x, 0) = \omega_0(x), & \text{in } \mathbb{R}^2, \end{cases} \tag{5.4}$$

where δ_Γ is the Dirac function along the curve and g_ω is given in (5.3).

Definition 5.9. The pair (u, ω) is a weak solution of the previous system if

(a) for any test function $\varphi \in C_c^\infty([0, \infty) \times \mathbb{R}^2)$ we have

$$\int_0^\infty \int_{\mathbb{R}^2} \varphi_t \omega \, dx \, dt + \int_0^\infty \int_{\mathbb{R}^2} \nabla \varphi \cdot u \omega \, dx \, dt + \int_{\mathbb{R}^2} \varphi(x, 0) \omega_0(x) \, dx = 0,$$

(b) we have $\operatorname{div} u = 0$ and $\operatorname{curl} u = \omega + g_\omega \delta_\Gamma$ in the sense of distributions of \mathbb{R}^2 , with $|u| \rightarrow 0$ at infinity.

Theorem 5.10. The pair (u, ω) obtained at the beginning of this subsection is a weak solution of the previous system.

Proof. The second point of the definition is directly verified by the previous section and by the estimate of Section 2.2 about the far-field behavior. Indeed, the velocity u verifies $|u| \rightarrow 0$ at infinity, thanks to the explicit expressions for $K[\omega]$ and H , using the uniform compact support of ω .

Next, we introduce an operator I_ε , which for a function $\varphi \in C_0^\infty([0, \infty) \times \mathbb{R}^2)$ gives:

$$I_\varepsilon[\varphi] \equiv \int_0^\infty \int_{\mathbb{R}^2} \varphi_t (\Phi^\varepsilon)^2 \omega^\varepsilon \, dx \, dt + \int_0^\infty \int_{\mathbb{R}^2} \nabla \varphi \cdot (\Phi^\varepsilon u^\varepsilon) (\Phi^\varepsilon \omega^\varepsilon) \, dx \, dt.$$

To prove that (u, ω) is a weak solution, we will show that

- (i) $I_\varepsilon[\varphi] + \int_{\mathbb{R}^2} \varphi(x, 0) \omega_0(x) \, dx \rightarrow 0$ as $\varepsilon \rightarrow 0$,
- (ii) $I_\varepsilon[\varphi] \rightarrow \int_0^\infty \int_{\mathbb{R}^2} \varphi_t \omega \, dx \, dt + \int_0^\infty \int_{\mathbb{R}^2} \nabla \varphi \cdot u \omega \, dx \, dt$ as $\varepsilon \rightarrow 0$.

Clearly these two steps complete the proof.

We begin by showing (i). As u^ε and ω^ε verify (3.2), it can be easily seen that

$$\int_0^\infty \int_{\mathbb{R}^2} \varphi_t (\Phi^\varepsilon)^2 \omega^\varepsilon \, dx \, dt = - \int_0^\infty \int_{\mathbb{R}^2} \nabla(\varphi (\Phi^\varepsilon)^2) \cdot u^\varepsilon \omega^\varepsilon \, dx \, dt - \int_{\mathbb{R}^2} \varphi(x, 0) (\Phi^\varepsilon)^2(x) \omega_0(x) \, dx.$$

Thus we compute

$$I_\varepsilon[\varphi] = -2 \int_0^\infty \int_{\mathbb{R}^2} \varphi \nabla \Phi^\varepsilon \cdot u^\varepsilon (\Phi^\varepsilon \omega^\varepsilon) \, dx \, dt - \int_{\mathbb{R}^2} \varphi(x, 0) (\Phi^\varepsilon)^2(x) \omega_0(x) \, dx.$$

We have:

$$\left| I_\varepsilon[\varphi] + \int_{\mathbb{R}^2} \varphi(x, 0) (\Phi^\varepsilon)^2(x) \omega_0(x) \, dx \right| \leq 2 \|\Phi^\varepsilon \omega^\varepsilon\|_{L^\infty(L^\infty)} \|\varphi\|_{L^1(L^\infty)} \|u^\varepsilon \cdot \nabla \Phi^\varepsilon\|_{L^\infty(L^1)} \rightarrow 0,$$

as $\varepsilon \rightarrow 0$ by Lemma 4.6. This shows (i) for all ε sufficiently small such that $(\Phi^\varepsilon)^2(x) \omega_0 = \omega_0$ since the support of ω_0 does not intersect the curve.

For (ii), the linear term presents no difficulty. The second term consists of the weak-strong pair vorticity-velocity:

$$\left| \iint \nabla \varphi \cdot (\Phi^\varepsilon u^\varepsilon) (\Phi^\varepsilon \omega^\varepsilon) - \iint \nabla \varphi \cdot u \omega \right| \leq \left| \iint \nabla \varphi \cdot (\Phi^\varepsilon u^\varepsilon - u) (\Phi^\varepsilon \omega^\varepsilon) \right| + \left| \iint \nabla \varphi \cdot u (\Phi^\varepsilon \omega^\varepsilon - \omega) \right|.$$

$\Phi^\varepsilon u^\varepsilon \rightarrow u$ strongly in $L^2([0, \mathbf{T}], L^2_{\text{loc}}(\mathbb{R}^2))$ thanks to Theorem 5.6. So the first term tends to zero because $\Phi^\varepsilon \omega^\varepsilon$ is bounded in $L^\infty([0, \infty), L^2(\mathbb{R}^2))$. In the same way, the second term tends to zero because $\Phi^\varepsilon \omega^\varepsilon \rightharpoonup \omega$ weak-* in $L^\infty([0, \mathbf{T}], L^4(\mathbb{R}^2))$ and $u \in L^\infty([0, \mathbf{T}], L^2_{\text{loc}}(\mathbb{R}^2))$. \square

5.4. The asymptotic velocity equation in \mathbb{R}^2

As the function u is bounded in L^2 , we can write the vorticity equation more simply than in [2]. The main calculation of this subsection can be found in [2].

We begin by introducing $v(x) = \int K(x - y)\omega(y) dy$ with $K(x) = \frac{1}{2\pi} \frac{x^\perp}{|x|^2}$, the solution without obstacle of

$$\begin{cases} \operatorname{div} v = 0 & \text{on } \mathbb{R}^2, \\ \operatorname{curl} v = \omega & \text{on } \mathbb{R}^2, \\ \lim_{|x| \rightarrow \infty} |v| = 0. \end{cases}$$

This velocity is bounded, and we denote the perturbation by $w = u - v$, which is bounded in L^p_{loc} for $p < 4$, and it verifies

$$\begin{cases} \operatorname{div} w = 0 & \text{on } \mathbb{R}^2, \\ \operatorname{curl} w = g_\omega(s)\delta_\Gamma & \text{on } \mathbb{R}^2, \\ \lim_{|x| \rightarrow \infty} |w| = 0. \end{cases}$$

We now prove that v verifies the following equation:

$$\begin{cases} v_t + v \cdot \nabla v + v \cdot \nabla w + w \cdot \nabla v - v(s)^\perp \tilde{g}_v(s)\delta_\Gamma = -\nabla p, & \text{in } \mathbb{R}^2 \times (0, \infty), \\ \operatorname{div} v = 0, & \text{in } \mathbb{R}^2 \times (0, \infty), \\ w(x) = \frac{1}{2\pi} \int_\Gamma \frac{(x-s)^\perp}{|x-s|^2} \tilde{g}_v(s) \cdot ds, & \text{in } \mathbb{R}^2 \times (0, \infty), \\ v(x, 0) = K[\omega_0], & \text{in } \mathbb{R}^2 \end{cases} \tag{5.5}$$

with $\tilde{g}_v = g_{\operatorname{curl} v}$.

In order to prove the equivalence of (5.4) and (5.5) it is sufficient to show that

$$\operatorname{curl}[v \cdot \nabla w + w \cdot \nabla v - v(s)^\perp \tilde{g}_v(s)\delta_\Gamma] = \operatorname{div}(\omega w) \tag{5.6}$$

for all divergence free fields $v \in W^{1,p}_{\text{loc}}$, with some $p > 2$. Indeed, if (5.6) holds, then we get for $\omega = \operatorname{curl} v$

$$\begin{aligned} 0 &= -\operatorname{curl} \nabla p = \operatorname{curl}[v_t + v \cdot \nabla v + v \cdot \nabla w + w \cdot \nabla v - v(s)^\perp \tilde{g}_v(s)\delta_\Gamma] \\ &= \omega_t + v \cdot \nabla \omega + w \cdot \nabla \omega = \omega_t + u \cdot \nabla \omega = 0 \end{aligned}$$

so relation (5.4) holds true. And vice versa, if (5.4) holds then we deduce that the left hand side of (5.5) has zero curl so it must be a gradient.

We now prove (5.6). As $W^{1,p}_{\text{loc}} \subset C^0$, $v(s)$ is well defined. Next, it suffices to prove the equality for smooth v , since we can pass to the limit on a subsequence of smooth approximations of v which converges strongly in $W^{1,p}_{\text{loc}}$ and C^0 . Now, it is trivial to check that, for a 2×2 matrix A with distribution coefficients, we have

$$\operatorname{curl} \operatorname{div} A = \operatorname{div} \begin{pmatrix} \operatorname{curl} C_1 \\ \operatorname{curl} C_2 \end{pmatrix}$$

where C_i denotes the i th column of A . For smooth v , we deduce

$$\begin{aligned} \operatorname{curl}[v \cdot \nabla w + w \cdot \nabla v] &= \operatorname{curl} \operatorname{div}(v \otimes w + w \otimes v) \\ &= \operatorname{div} \begin{pmatrix} \operatorname{curl}(vw_1) + \operatorname{curl}(wv_1) \\ \operatorname{curl}(vw_2) + \operatorname{curl}(wv_2) \end{pmatrix} \\ &= \operatorname{div}(w \operatorname{curl} v + v \cdot \nabla^\perp w + v \operatorname{curl} w + w \cdot \nabla^\perp v). \end{aligned}$$

It is a simple computation to check that

$$\operatorname{div}(v \cdot \nabla^\perp w + w \cdot \nabla^\perp v) = v \cdot \nabla^\perp \operatorname{div} w + w \cdot \nabla^\perp \operatorname{div} v + \operatorname{curl} v \operatorname{div} w + \operatorname{curl} w \operatorname{div} v.$$

Taking into account that we have free divergence fields, we can finish by writing

$$\begin{aligned} \operatorname{curl}[v \cdot \nabla w + w \cdot \nabla v] &= \operatorname{div}(w \operatorname{curl} v + v \tilde{g}_v(s)\delta_\Gamma) \\ &= \operatorname{div}(w \operatorname{curl} v) + \operatorname{curl}[v(s)^\perp \tilde{g}_v(s)\delta_\Gamma] \end{aligned}$$

which proves (5.6).

Now, we write a formulation for the velocity u , by replacing v by $u - w$ to obtain in \mathbb{R}^2

$$u_t + u \cdot \nabla u = -\nabla p + w_t + w \cdot \nabla w + v(s)^\perp \tilde{g}_v(s) \delta_\Gamma.$$

However, since $\text{curl } w = 0$, we can remark that $\text{curl}[w_t + w \cdot \nabla w + v(s)^\perp \tilde{g}_v(s) \delta_\Gamma] = 0$ in $\mathbb{R}^2 \setminus \Gamma$.

5.5. Formulation on $\mathbb{R}^2 \setminus \Gamma$

We can obtain directly an equation for u on $\mathbb{R}^2 \setminus \Gamma$ by passing to the limit $\varepsilon \rightarrow 0$. We multiply the velocity equation (3.2) by some divergence-free test vector field $\varphi \in C_c^\infty(\mathbb{R}^+ \times (\mathbb{R}^2 \setminus \Gamma))$ and assume that ε is small enough such that the support of φ is contained in Π_ε and do not intersected the support of $\nabla \Phi^\varepsilon$. After integration,

$$\int_0^\infty \int u^\varepsilon \cdot \partial_t \varphi + \int_{\mathbb{R}^2 \setminus \Gamma} u^\varepsilon(0, \cdot) \cdot \varphi(0, \cdot) + \int_0^\infty \int (u^\varepsilon \otimes u^\varepsilon) \cdot \nabla \varphi = 0,$$

which easily pass to the limit since $u^\varepsilon \rightarrow u$ strongly in L^2_{loc} by Theorem 5.6. Indeed, we can prove easily that $u_0^\varepsilon \rightarrow u_0$ in $L^2_{\text{loc}}(\mathbb{R}^2)$ thanks to the proof of Theorem 5.6. Therefore, the above relation holds true with u instead of u^ε and this is the formulation in the sense of distributions of the Euler equation in $\mathbb{R}^2 \setminus \Gamma$:

$$\begin{cases} u_t + u \cdot \nabla u = -\nabla p & \text{in } \mathbb{R}^2 \setminus \Gamma \times (0, \infty), \\ \text{div } u = 0 & \text{in } \mathbb{R}^2 \setminus \Gamma \times [0, \infty), \\ u \cdot \hat{n} = 0 & \text{on } \Gamma \times [0, \infty), \\ |u| \rightarrow 0, & \text{as } |x| \rightarrow \infty, \\ u(x, 0) = F(\omega_0) & \text{in } \mathbb{R}^2 \setminus \Gamma, \end{cases} \tag{5.7}$$

where F is the formula from Theorem 5.6 expressing explicitly the velocity in terms of vorticity and circulation.

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Appendix A. List of notations

Domains

$D \equiv B(0, 1)$ the unit disk,

$S \equiv \partial D$,

Γ is a Jordan arc (see Proposition 2.2),

$\Pi \equiv \mathbb{R}^2 \setminus \Gamma$,

Ω_0 is a bounded, open, connected, simply connected subset of the plane,

$\Gamma_0 \equiv \partial \Omega_0$ is a C^∞ Jordan curve,

$\Pi_0 \equiv \mathbb{R}^2 \setminus \Omega_0$,

Ω_ε is a family of a domain, verifying the same properties of Ω_0 , such as $\Omega_\varepsilon \rightarrow \Gamma$ as $\varepsilon \rightarrow 0$,

$\Gamma_\varepsilon \equiv \partial \Omega_\varepsilon$ and $\Pi_\varepsilon \equiv \mathbb{R}^2 \setminus \Omega_\varepsilon$.

Functions

ω_0 is the initial vorticity ($C_c^\infty(\Pi)$),

γ is the circulation of u_0^ε on Γ_ε (see introduction),

$(u^\varepsilon, \omega^\varepsilon)$ is the solution of the Euler equations on Π_ε ,
 T is a biholomorphism between Π and $\text{int } D^c$ (see Proposition 2.2),
 T_0 is a biholomorphism between Π_0 and $\text{int } D^c$,
 T_ε is a biholomorphism between Π_ε and $\text{int } D^c$ (see Assumption 3.1),
 K^ε and H^ε are given in (3.3) and (3.4),
 $K^\varepsilon[\omega^\varepsilon](x) \equiv \int_{\Pi_\varepsilon} K^\varepsilon(x, y)\omega^\varepsilon(y) dy$,
 Φ^ε is a cutoff function in a small ε -neighborhood of Ω_ε (see Subsection 4.2).

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