Available online at www.sciencedirect.com
ScienceDirect
Ann. I. H. Poincaré – AN 26 (2009) 689–704
ANNALES DE L'INSTITUT HENRI POINCARÉ
ANALYSE NON LINÉAIRE
www.elsevier.com/locate/anihpc

Geometric expansion, Lyapunov exponents and foliations ☆

Radu Saghin [a,*], Zhihong Xia [b]

[a] *Centre de Recerca Matematica, Apartat 50, Bellaterra, 08193, Spain*
[b] *Department of Mathematics, Northwestern University, Evanston, IL 60208, USA*

Received 28 May 2007; received in revised form 2 June 2008; accepted 22 July 2008

Available online 26 July 2008

Abstract

We consider hyperbolic and partially hyperbolic diffeomorphisms on compact manifolds. Associated with invariant foliation of these systems, we define some topological invariants and show certain relationships between these topological invariants and the geometric and Lyapunov growths of these foliations. As an application, we show examples of systems with persistent non-absolute continuous center and weak unstable foliations. This generalizes the remarkable results of Shub and Wilkinson to cases where the center manifolds are not compact.

Keywords: Partially hyperbolic; Fubini's nightmare; Homology of foliations

1. Introduction

Let M be an n-dimensional compact Riemannian manifold. Let $f \in \mathrm{Diff}^r(M)$ be a C^r diffeomorphism on M, $r \geqslant 1$. We will also consider volume-preserving diffeomorphisms in our examples. Let W be a k-dimensional foliation of M with C^1 leaves. We say that the foliation is invariant under f, if f maps leaves of W to leaves. We first define volume growth of f on leaves. We will also assume that the leaves are orientable. For any $x \in M$, let $W(x)$ be the leaf through x and let $W_r(x)$ be the k-dimensional disk on $W(x)$ centered at x, with radius r.

For most of the paper we will assume that the leaves of W have uniform exponential growth under the iterates of f, i.e., there are constants $\lambda > 1$ and $C > 0$ such that

$$\left\| df_x^n v \right\| \geqslant C\lambda^n \|v\|$$

for all $x \in M$, all $v \in T_x W(x)$ and all $n \in \mathbb{N}$, where $W(x)$ is the leaf of W through the point x.

Examples of these expanding foliations can be found in hyperbolic and partially hyperbolic diffeomorphisms. A map $f \in \mathrm{Diff}^r(M)$ is said to be *partially hyperbolic* if there is an invariant splitting of the tangent bundle of M, $TM = E^s \oplus E^c \oplus E^u$, with at least two of them non-trivial, and there exist $\alpha > \alpha' > 1$, $\beta > \beta' > 1$ and $C > 0$, $D > 0$, $C' > 0$, $D' > 0$ such that

☆ Research supported in part by National Science Foundation.

* Corresponding author.
E-mail addresses: rsaghin@crm.cat (R. Saghin), xia@math.northwestern.edu (Z. Xia).

0294-1449/$ – see front matter © 2008 Elsevier Masson SAS. All rights reserved.
doi:10.1016/j.anihpc.2008.07.001

(1) E^u is uniformly expanding:

$$\|Df^k(v_u)\| \geqslant C\alpha^k \|v_u\|, \quad \forall v_u \in E^u,\ k \in \mathbb{N},$$

(2) E^s is uniformly contracting:

$$\|Df^k(v_s)\| \leqslant D\beta^{-k} \|v_s\|, \quad \forall v_s \in E^s,\ k \in \mathbb{N},$$

(3) E^u dominates E^c, and E^c dominates E^s:

$$D'(\beta')^{-k}\|v_c\| \leqslant \|Df^k(v_c)\| \leqslant C'(\alpha')^k \|v_c\|, \quad \forall v_c \in E^c,\ k \in \mathbb{N}.$$

We remark that in some papers it is allowed that the bounds for the expansion rates depend on the points.

The unstable distribution E^u is integrable and it integrates to the unstable foliation. The unstable foliation of a hyperbolic or partially hyperbolic diffeomorphism is certainly uniformly expanding. Likewise, the stable distribution E^s of a hyperbolic or partially hyperbolic diffeomorphism can be integrated to obtain the stable foliation which is uniformly expanding for f^{-1}.

The growth rate of an uniformly expanding foliation can be measured in several different ways. We first define the geometric growth rate. This is related to the volume growth used by Yomdin and Newhouse for the study of entropy of diffeomorphisms (see [12,4]). The difference is that we consider only k-dimensional disks on the leaves of the foliation. Let

$$\chi(x,r) = \limsup_{n\to\infty} \frac{1}{n} \log \mathrm{Vol}\big(f^n\big(W_r(x)\big)\big).$$

$\chi(x,r)$ measures the volume growth of the disc of radius r in $W(x)$ centered at x. Let

$$\chi = \chi(r) = \sup_{x\in M} \chi(x,r).$$

Then, χ is the maximum volume growth rate of the foliation W under f.

Lemma 1.1. *The volume growth χ of the foliation W is independent of r and of the Riemannian metric on M.*

Proof. Let $r, r' > 0$. For any $x \in M$, because $W_r(x)$ is compact, there exist $y_1, y_2, \ldots, y_l \in W_r(x)$ such that

$$W_r(x) \subset \bigcup_{i=1}^{l} W_{r'}(y_i).$$

Then

$$\mathrm{Vol}\big(f^n\big(W_r(x)\big)\big) \leqslant \sum_{i=1}^{l} \mathrm{Vol}\, f^n\big(\big(W_{r'}(y_i)\big)\big),$$

so

$$\chi(x,r) \leqslant \max_{1\leqslant i\leqslant l} \chi(y_i, r') \leqslant \chi(r').$$

This holds for every $x \in M$, so $\chi(r) \leqslant \chi(r')$. In the same way one can prove that $\chi(r') \leqslant \chi(r)$ so

$$\chi(r) = \chi(r'), \quad \forall r, r' > 0,$$

so χ is independent of r.

Now suppose that we have another Riemannian metric on M, the volume of a disk D with respect to this new metric is denoted $\mathrm{Vol}'(D)$, and the volume growth of $W_r(x)$ with respect to this new metric is denoted $\chi'(x,r)$. There exist a constant $C > 0$ such that

$$C^{-1}\,\mathrm{Vol}(D) \leqslant \mathrm{Vol}'(D) \leqslant C\,\mathrm{Vol}(D), \quad \forall D \subset M.$$

Then

$$\frac{\log \mathrm{Vol}(f^n(W_r(x)))}{n} - \frac{\log C}{n} \leqslant \frac{\log \mathrm{Vol}'(f^n(W_r(x)))}{n} \leqslant \frac{\log \mathrm{Vol}(f^n(W_r(x)))}{n} + \frac{\log C}{n}$$

so

$$\chi(x,r) = \chi'(x,r),$$

which means that the volume growth is independent of the Riemannian metric. □

The geometric growth is hard to compute and its dependence on the points and on the map itself is not very clear. We will define a topological growth rate for the foliation. This will depend on the homology that the invariant foliation carries and the action induced by f on the homology. Typically this topological growth will be much easier to compute and it is a local constant for maps in $\mathrm{Diff}^r(M)$. It turns out, as we will show, the geometric growth $\chi(x,r)$ and topological growth are the same for foliations carrying certain homological information. As a consequence, $\chi(x,r)$ is independent of x and r and remains the same under small perturbations. We will define this homological invariant using De Rahm currents.

The third type of growth rate for an invariant foliation is measured by the Lyapunov exponents in the tangent spaces of the leaves of the foliations. The Lyapunov exponents are positive in the leaves of an expanding foliation. We can integrate, over the manifold, the sum of all Lyapunov exponents in the leaves and we call this integral the Lyapunov growth. We will show that, if the foliation is absolutely continuous, the Lyapunov growth is smaller than the geometric growth. As a consequence, if the Lyapunov growth is larger than the geometric growth, then the foliation must be singular.

Shub and Wilkinson showed some remarkable examples where the center foliations, whose leaves are circles, persistently fail to be absolutely continuous in some partially hyperbolic volume preserving diffeomorphisms (see [10]). Moreover, every center leaf intersects a full measure set in a set of measure zero. They call these types of foliations *pathological*. Using our results, we will give examples of persistent pathological foliations with non-compact center leaves.

In Section 2 we define the topological growth of a foliation, we discuss about some properties and we show how to relate it to the volume growth in some situations. In Section 3 we talk about the Lyapunov growth and relate it to the volume growth in the case of absolutely continuous foliations. Section 4 contains the examples of non-absolutely continuous foliations.

As another application of the homological invariants we defined here, Hua, Saghin and Xia proved certain continuity properties of topological entropy for partially hyperbolic diffeomorphisms, see [3].

2. Topological growth of a foliation

In this section, we will define the topological growth of a foliation. For this we will associate some homologies to the foliation and then we will analyze the action of f_* on these homologies. The natural objects used to define homologies of foliations are the closed currents supported on the foliation (see [5,11]). This approach was used for example in the study of entropy of axiom A diffeomorphisms (see [9,7]). Here we will restrict our attention to some specific currents supported on the foliation, which are related to the dynamics of f.

Let W be a k-dimensional foliation of M, invariant under f. For any positive integer n, any $x \in M$ and $r > 0$, we define the *dynamical currents* $C_n(x,r)$:

$$C_n(x,r)(\omega) = \frac{1}{\mathrm{Vol}(f^n(W_r(x)))} \int\limits_{f^n(W_r(x))} \omega,$$

for any k-form ω on M. These currents depend on x and r. Also they depend on the Riemannian metric on M. The currents $C_n(x,r)$ are uniformly bounded so there must be subsequences with weak limits. Let C be such a limit, i.e., we have a sequence $n_i \to \infty$ such that for any k-form ω we have $\lim_{i\to\infty} C_{n_i}(x,r)(\omega) = C(\omega)$.

A current C is said to be *closed* if for any exact k-form $\omega = d\alpha$, we have $C(\omega) = C(d\alpha) = 0$. If C is closed, it has a homology class $[C] = h_C \in H_k(M,\mathbb{R})$. This homology class is non-trivial if there exist a closed k-form ω such that $C(\omega) \neq 0$.

We would like to investigate the conditions under which the sub-sequential limits of the currents $C_n(x,r)$ are closed. In general, $C_n(x,r)$ itself is not closed. From Stokes' theorem, we have:

$$C_n(x,r)(\omega) = \frac{1}{\mathrm{Vol}(f^n(W_r(x)))} \int\limits_{f^n(W_r(x))} d\alpha$$
$$= \frac{1}{\mathrm{Vol}(f^n(W_r(x)))} \int\limits_{\partial f^n(W_r(x))} \alpha$$
$$= \frac{1}{\mathrm{Vol}(f^n(W_r(x)))} \int\limits_{\partial W_r(x)} (f^*)^n \alpha.$$

If the above sequence approaches zero as $n \to \infty$, then every sub-sequential limit of the currents $C_n(x,r)$ is closed. In many situations, the volume growth of $f^n(W_r(x))$ is larger than the lower dimensional volume growth of its boundary. One of them is when the foliation W is one-dimensional:

Proposition 2.1. *Suppose the one-dimensional foliation W is invariant under f and uniformly expanded by it. Then every limit of the sequence $C_n(x,r)$ is closed.*

Proof. Let $\omega = d\alpha$ be an exact 1-form on M. Then α is a 0-form, or a real valued function on M and hence $\int_{\partial W_r(x)} (f^*)^n \alpha$ is the difference of that function evaluated at the two end points of $f^n(W_r(x))$ and therefore it is uniformly bounded. On the other hand, W is uniformly expanded by f, so

$$\lim_{n\to\infty} \mathrm{Vol}\big(f^n\big(W_r(x)\big)\big) = \infty$$

thus $C_n(x,r)(\omega) \to 0$ as $n \to \infty$ and consequently all the limit currents are closed. □

If the foliation W is one-dimensional, there is also an intuitive way to see the limit currents of $C_n(x,r)$, related to the Schwartzmann asymptotic cycles (see [8]). In this case $f^n(W_r(x))$ is just a C^1 curve on the manifold, oriented with the same orientation of W. One can join the endpoints with another C^1 curve of length smaller or equal to the diameter of M, denoted l_n, to get a piecewise C^1 closed curve, denoted by d_n. We will use the notation $|\gamma|$ for the length of a piecewise C^1 curve γ. Now let $h_n = \frac{[d_n]}{|d_n|} \in H_1(M,\mathbb{R})$ be the homology class of d_n divided by the length of d_n. Here h_n will depend of course of the choice of the curve l_n and the Riemannian metric on M. However asymptotically $C_n(x,r)(\omega)$ is like $(h_n,[\omega])$ if the foliation W is uniformly expanding:

$$\big(h_n,[\omega]\big) = \frac{1}{|d_n|} \int\limits_{d_n} \omega$$
$$= \frac{1}{\mathrm{Vol}(f^n(W_r(x))) + |l_n|} \Bigg(\int\limits_{f^n(W_r(x))} \omega + \int\limits_{l_n} \omega \Bigg)$$
$$= \frac{\mathrm{Vol}(f^n(W_r(x)))}{\mathrm{Vol}(f^n(W_r(x))) + |l_n|} C_n(x,r)(\omega) + \frac{1}{\mathrm{Vol}(f^n(W_r(x))) + |l_n|} \int\limits_{l_n} \omega.$$

But because W is expanded by f we get that $\lim_{n\to\infty} \mathrm{Vol}(f^n(W_r(x))) = \infty$, and $|l_n|$ is uniformly bounded, so

$$\lim_{n\to\infty} \frac{\mathrm{Vol}(f^n(W_r(x)))}{\mathrm{Vol}(f^n(W_r(x))) + |l_n|} = 1,$$
$$\lim_{n\to\infty} \frac{1}{\mathrm{Vol}(f^n(W_r(x))) + |l_n|} \int\limits_{l_n} \omega = 0,$$

and from this we get:

$$\lim_{n\to\infty} \big(h_n,[\omega]\big) = \lim_{n\to\infty} C_n(x,r)(\omega).$$

This means that the homologies of the (closed) limit currents of $C_n(x,r)$ are exactly the limits of the homologies h_n.

Another situation when the limit currents are closed is when we have some convenient uniform bounds for the expansion rates on W:

Proposition 2.2. *Suppose the k-dimensional foliation W is invariant under f and the following condition holds*:

$$\sup_{v^{k-1}\in\Lambda^{k-1}TW}\frac{\|f_*^N v^{k-1}\|}{\|v^{k-1}\|}<\inf_{v^k\in\Lambda^kTW}\frac{\|f_*^N v^k\|}{\|v^k\|}$$

for some natural number $N>0$. Then every limit current of $C_n(x,r)$ is closed.

Proof. For $n=aN+b$, $n,a,b,\in\mathbb{N}$, we have the following inequalities:

$$\begin{aligned}\operatorname{Vol}\bigl(f^n\bigl(W_r(x)\bigr)\bigr)&=\int_{W_r(x)}\operatorname{Jac}f^n|_{W_r(x)}=\int_{W_r(x)}\operatorname{Jac}f^{aN+b}|_{W_r(x)}\\&\geqslant C_1\operatorname{Vol}\bigl(W_r(x)\bigr)\cdot\left(\inf_{v^k\in\Lambda^kTW}\frac{\|f_*^N v^k\|}{\|v^k\|}\right)^a\\&:=C_1\operatorname{Vol}\bigl(W_r(x)\bigr)\beta_1^a;\\\operatorname{Vol}\bigl(\partial f^n\bigl(W_r(x)\bigr)\bigr)&=\int_{\partial W_r(x)}\operatorname{Jac}f^n|_{\partial W_r(x)}=\int_{\partial W_r(x)}\operatorname{Jac}f^{aN+b}|_{\partial W_r(x)}\\&\leqslant C_2\operatorname{Vol}\bigl(\partial W_r(x)\bigr)\cdot\left(\sup_{v^{k-1}\in\Lambda^{k-1}TW}\frac{\|f_*^N v^{k-1}\|}{\|v^{k-1}\|}\right)^a\\&:=C_2\operatorname{Vol}\bigl(\partial W_r(x)\bigr)\beta_2^a,\end{aligned}$$

for some $C_1,C_2>0$, and then for the exact form $\omega=d\alpha$ we get:

$$\begin{aligned}\bigl|C_n(x,r)(\omega)\bigr|&=\left|\frac{1}{\operatorname{Vol}(f^n(W_r(x)))}\int_{\partial f^n(W_r(x))}\alpha\right|\\&\leqslant C\frac{\operatorname{Vol}(\partial f^n(W_r(x)))}{\operatorname{Vol}(f^n(W_r(x)))}\leqslant C'\left(\frac{\beta_2}{\beta_1}\right)^a\end{aligned}$$

for some $C,C'>0$. But from the hypothesis we have $\beta_1>\beta_2$, so

$$\lim_{n\to\infty}\bigl|C_n(x,r)(\omega)\bigr|=0,$$

and all the limit currents are closed. □

We remark that the condition in Proposition 2.2 is an open condition, it is verified also for small perturbations of f and W. This is for example the case when f is close to a linear map on the torus $\mathbb{T}^n$ and W is any of the expanding foliations close to the linear one. We will consider this case in more details later.

Definition 2.3. We say that a k-dimensional invariant foliation W carries a non-trivial homology $h_C\in H_k(M,\mathbb{R})$ if for some $x\in M$, $r>0$ the currents $C_n(x,r)$ defined above have a closed sub-sequential limit C and $h_C=[C]\neq 0$.

We say that a k-dimensional invariant foliation W carries a unique non-trivial homology (up to rescale) if for all $x\in M$, $r>0$, all sub-sequential limits of the currents $C_n(x,r)$ are closed and their homologies are unique up to scalar multiplication and are uniformly bounded away from zero.

We remark that there is no natural size for the currents supported on a foliation or for the homologies of the closed ones. So the homology can be unique only up to rescale, because its size depends on the Riemannian metric.

A closed current is non-trivial if there is a closed k-form ω such that $C(\omega)\neq 0$. The homology class of a non-trivial closed current is non-trivial. One way to show that the closed current C is non-trivial is to show that there is a closed k-form ω such that ω is non-degenerate on W, i.e. ω is non-degenerate on $T_xW(x)$ for any $x\in M$.

Proposition 2.4. *Suppose C is a limit current of $C_n(x,r)$ which is also closed, and there exist a closed k-form ω which is non-degenerate on the k-dimensional foliation W. Then C is non-trivial, i.e. $[C]\neq 0$.*

Proof. When we have a non-degenerate k-form on the leaves of W, by compactness of the manifold, there exists a constant $c > 0$ such that

$$\left|\int_D \omega\right| \geqslant c\,\mathrm{Vol}(D)$$

for any disk D on the leaves of W, and therefore

$$\left|C_n(x,r)(\omega)\right| = \frac{1}{\mathrm{Vol}(f^n(W_r(x)))}\left|\int_{f^n(W_r(x))} \omega\right| \geqslant c.$$

This implies that $C(\omega) \geqslant c > 0$, or C is non-trivial. □

We remark that actually one can see from the proof that if there is a closed form non-degenerate on the foliation, then the set of homologies carried by the foliation must be bounded away from zero. The next proposition discusses the homologies carried by an invariant expanding foliation.

Proposition 2.5. *Let W be an expanding invariant foliation, let $H_{x,r} \subset H_k(M,\mathbb{R})$ be the set of homologies of the closed limit currents of $C_n(x,r)$, and let*

$$H = \bigcup_{x\in M,\, r>0} H_{x,r} \subset H_k(M,\mathbb{R})$$

be the set of homologies carried by W. Then

(1) *$H_{x,r}$ is closed and bounded and $\mathbb{R}H_{x,r}$ is invariant under*

$$f_*: H_k(M,\mathbb{R}) \to H_k(M,\mathbb{R});$$

if $H_{x,r}$ is bounded away from zero, then $\mathbb{R}H_{x,r}$ is closed invariant.

(2) *H spans a linear space, invariant under $f_*: H_k(M,\mathbb{R}) \to H_k(M,\mathbb{R})$.*

Proof. First we observe that the map f naturally induces an action on the currents, defined by:

$$f_*C(\omega) = C(f^*\omega)$$

for any k current C and k form ω. Obviously, if C is closed, then f_*C is closed too and

$$[f_*C] = f_*[C] \in H_k(M,\mathbb{R}).$$

Obviously the set of limit currents of $C_n(x,r)$ is closed, so the subset of closed limit currents is also closed, and then $H_{x,r}$ must be closed. If $H_{x,r}$ is bounded away from zero, obviously $\mathbb{R}H_{x,r}$ is also closed. $H_{x,r}$ is bounded because the sequence $C_n(x,r)$ is bounded.

Now we want to prove the invariance. If $H_{x,r} = \emptyset$ then there is nothing to prove. Let a current C be a closed sub-sequential limit of $C_n(x,r)$, so $[C] \in H_{x,r}$. Then

$$C(\omega) = \lim_{i\to\infty} \frac{1}{\mathrm{Vol}(f^{n_i}(W_r(x)))} \int_{f^{n_i}(W_r(x))} \omega,$$

for any k-form on M. Therefore

$$\begin{aligned}(f_*C)(\omega) &= \lim_{i\to\infty} \frac{1}{\mathrm{Vol}(f^{n_i}(W_r(x)))} \int_{f^{n_i}(W_r(x))} f^*\omega \\ &= \lim_{i\to\infty} \frac{1}{\mathrm{Vol}(f^{n_i}(W_r(x)))} \int_{f^{(n_i+1)}(W_r(x))} \omega \\ &= \lim_{i\to\infty} \frac{\mathrm{Vol}(f^{(n_i+1)}(W_r(x)))}{\mathrm{Vol}(f^{n_i}(W_r(x)))} \cdot \frac{1}{\mathrm{Vol}(f^{(n_i+1)}(W_r(x)))} \int_{f^{(n_i+1)}(W_r(x))} \omega.\end{aligned}$$

Since the ratio $\mathrm{Vol}(f^{(n_i+1)}(W_r(x)))/\mathrm{Vol}(f^{n_i}(W_r(x)))$ is uniformly bounded, both from above and away from zero, there is a convergent subsequence. Without loss of generality, we may assume that the sequence actually converges and there is a constant $\lambda > 0$ such that

$$\lim_{i\to\infty} \frac{\mathrm{Vol}(f^{(n_i+1)}(W_r(x)))}{\mathrm{Vol}(f^{n_i}(W_r(x)))} = \lambda.$$

This implies that

$$\lim_{i\to\infty} C_{n_{i+1}}(x,r)(\omega) = \frac{1}{\lambda}(f_*C)(\omega),$$

for any k-form ω, so f_*C/λ is also a sub-sequential limit of the current $C_n(x,r)$, then

$$[\lambda^{-1} f_*C] = \lambda^{-1} f_*[C] \in H_{x,r}$$

is also a homology of a closed limit current of $C_n(x,r)$, so $\mathbb{R}H_{x,r}$ is invariant.

The second statement follows immediately from the first one. This proves the proposition. □

Now suppose that the foliation W carries a unique non-trivial homology. Let $h_C = [C] \in H_k(M,\mathbb{R})$, where C is a limit current as defined above. The next proposition shows that h_C is actually an eigenvector of the induced linear map by f on the homology of M.

Proposition 2.6. *Let W be a k-dimensional expanding invariant foliation that carries a unique non-trivial homology h_C. Then h_C is an eigenvector of the induced linear map*:

$$f_*: H_k(M,\mathbb{R}) \to H_k(M,\mathbb{R}).$$

Proof. From Proposition 2.5 we know that H spans a subspace of $H_k(M,\mathbb{R})$ invariant under f_*. Because W has unique non-trivial homology, this invariant subspace must be $\mathbb{R}h_C$. Then obviously h_C must be an eigenvector of the induced linear map:

$$f_*: H_k(M,\mathbb{R}) \to H_k(M,\mathbb{R}). \qquad \square$$

Keeping the assumption that W has unique non-trivial homology, let λ_W be the eigenvalue of f_* corresponding to an eigenvector h_C associated to W, as in Proposition 2.6. We call λ_W the *topological growth* of the foliation W. We will see below that the topological growth and the volume growth are the same for a foliation that carries a unique non-trivial homology, except that the volume growth we defined here is an exponent, while the topological growth is a multiplier.

Theorem 2.7. *Let W be an expanding invariant foliation that carries a unique non-trivial homology h_W. Let λ_W be the topological growth of the foliation. Then the volume growth defined before,*

$$\chi(x,r) = \limsup_{i\to\infty} \frac{1}{i}\log\bigl(\mathrm{Vol}\bigl(f^i\bigl(W_r(x)\bigr)\bigr)\bigr)$$
$$= \lim_{i\to\infty} \frac{1}{i}\log\mathrm{Vol}\bigl(f^i\bigl(W_r(x)\bigr)\bigr) = \ln\lambda_W,$$

for any $x \in M$ and any $r > 0$.

Proof. The volume of a piece of leaf in a foliation depends on the Riemannian metric defined on M. So in general, the volume does not grow uniformly with each iteration. We will rescale the volume at each step so that there will be uniform growth. Let $h_W \in H_k(M,\mathbb{R})$ be a homology carried by W. Let ω_W be a closed k-form such that the pairing between h_W and $[\omega_W]$ is nonzero. For any $x \in M$ and $r > 0$, we choose a sequence of numbers d_i, $i \in \mathbb{N}$ such that

$$\lim_{i\to\infty} d_i C_i(\omega_W) = \bigl(h_W, [\omega_W]\bigr).$$

Moreover, there are numbers $0 < c_1 \leqslant c_2$ such that d_i can be chosen with $c_2^{-1} \leqslant d_i \leqslant c_1^{-1}$. Then, because of the uniqueness of homologies carried by the foliation, every limit current of $\{d_i C_i\}_{i\in\mathbb{N}}$ must have the homology h_W. This implies that the relation

$$\lim_{i\to\infty} d_i C_i(\omega) = \big(h_W, [\omega]\big)$$

holds for every closed form ω, so also for $f^*\omega$. Therefore,

$$\begin{aligned}
\big(h_W, [f^*\omega]\big) &= \lim_{i\to\infty} \frac{d_i}{\mathrm{Vol}(f^i(W_r(x)))} \int\limits_{f^i(W_r(x))} f^*\omega \\
&= \lim_{i\to\infty} \frac{d_i}{\mathrm{Vol}(f^i(W_r(x)))} \int\limits_{f^{i+1}(W_r(x))} \omega \\
&= \lim_{i\to\infty} \frac{d_{i+1}^{-1}\,\mathrm{Vol}(f^{i+1}(W_r(x)))}{d_i^{-1}\,\mathrm{Vol}(f^i(W_r(x)))} \cdot \frac{d_{i+1}}{\mathrm{Vol}(f^{i+1}(W_r(x)))} \int\limits_{f^{i+1}(W_r(x))} \omega \\
&= \lim_{i\to\infty} \frac{d_{i+1}^{-1}\,\mathrm{Vol}(f^{i+1}(W_r(x)))}{d_i^{-1}\,\mathrm{Vol}(f^i(W_r(x)))} \cdot \big(h_W, [\omega]\big).
\end{aligned}$$

Therefore

$$\begin{aligned}
\lim_{i\to\infty} \frac{d_{i+1}^{-1}\,\mathrm{Vol}(f^{i+1}(W_r(x)))}{d_i^{-1}\,\mathrm{Vol}(f^i(W_r(x)))} &= \big(h_W, [f^*\omega]\big)/\big(h_W, [\omega]\big) \\
&= \big(f_* h_W, [\omega]\big)/\big(h_W, [\omega]\big) = \lambda_W.
\end{aligned}$$

This implies that

$$\begin{aligned}
\chi(x,r) &= \limsup_{i\to\infty} \frac{1}{i} \log\Big(\mathrm{Vol}\big(f^i\big(W_r(x)\big)\big)\Big) \\
&= \lim_{i\to\infty} \frac{1}{i} \log \mathrm{Vol}\big(f^i\big(W_r(x)\big)\big) \\
&= \lim_{i\to\infty} \frac{1}{i} \log\Big(d_i^{-1}\,\mathrm{Vol}\big(f^i\big(W_r(x)\big)\big)\Big) \\
&= \lim_{i\to\infty} \frac{1}{i} \log\left(d_0^{-1}\,\mathrm{Vol}\big(W_r(x)\big) \cdot \left(\prod_{j=1}^{i} \frac{d_i^{-1}\,\mathrm{Vol}(f^i(W_r(x)))}{d_{i-1}^{-1}\,\mathrm{Vol}(f^{i-1}(W_r(x)))}\right)\right) \\
&= \lim_{i\to\infty} \frac{1}{i} \sum_{j=1}^{i} \left(\log \frac{d_i^{-1}\,\mathrm{Vol}(f^i(W_r(x)))}{d_{i-1}^{-1}\,\mathrm{Vol}(f^{i-1}(W_r(x)))}\right) \\
&= \log \lambda_W.
\end{aligned}$$

Here we used the elementary fact that if $\lim_{i\to\infty} a_i = a$, then

$$\lim_{i\to\infty} \frac{1}{i} \sum_{j=1}^{i} a_i = a.$$

This proves the theorem. □

Now we will discuss some situations where the above results apply and remind some related results.

The closed currents supported on a foliation (Ruelle–Sullivan currents) appear in [11,7]. They are equivalent to transverse measures to the foliation. They may not exist always, for example in the case of the center-unstable foliation of an Anosov flow. One of the most famous existence results is due to Plante, who proves that there are closed currents (or transversal measures) if the foliation is oriented and there are leaves with sub-exponential growth (see [5]). An uniformly expanding foliation has all the leaves with sub-exponential growth, so it has transversal measures. The topological growth was studied for the case of uniformly hyperbolic transitive diffeomorphisms, where the logarithm of it must be equal to the entropy (see [7,9]). In this case the unstable and stable foliations have unique non-trivial homology. It is still an open question weather there are non-transitive uniformly hyperbolic diffeomorphisms, so actually all these results apply for all the known Anosov diffeomorphisms.

For the stable and unstable foliations of partially hyperbolic diffeomorphisms, the closed currents exist, but they are not always non-trivial. For example, for the time one map of an Anosov flow, every closed current supported on the stable or the unstable foliation has zero homology. This is because the map is homotopic to the identity, so the action on the homology is the identity, and this makes impossible to have non-trivial closed currents on the unstable foliation, because they would have to be expanded. However, in other known situations, the stable and unstable foliations of partially hyperbolic diffeomorphisms have unique non-trivial homology. This is the case for skew products over Anosov systems, some derived from Anosov maps (skew products over), linear partially hyperbolic maps of the torus and small perturbations.

We will present in more detail the case of maps on the n-torus $\mathbb{T}^n$ close to a linear map. Consider an $n \times n$ matrix A with determinant one and with integer entries. The matrix A induces a toral automorphism: $T_A : \mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n \to \mathbb{T}^n$ defined by $T_A x = Ax \bmod \mathbb{Z}^n$. We will consider the standard Riemannian metric on the torus $\mathbb{T}^n$. If all eigenvalues are away from the unit circle, then T_A is a hyperbolic toral automorphism. If the eigenvalues of A are mixed, with some on the unit circle and some away from unit circle, then T_A is partially hyperbolic. Let λ_i, $1 \leqslant i \leqslant n$, be the eigenvalues of A counted with their multiplicity, ordered such that

$$|\lambda_1| \leqslant |\lambda_2| \leqslant \cdots \leqslant |\lambda_{n-k}| \leqslant 1 < |\lambda_{n-k+1}| \leqslant \cdots \leqslant |\lambda_n|.$$

In both hyperbolic or partially hyperbolic cases, let E^u be the k-dimensional unstable distribution of T_A on $\mathbb{T}^n$. At each point $x \in \mathbb{T}^n$, $E^u(x) \subset T_x\mathbb{T}^n$ is the unstable subspace for $dT_A : T_x\mathbb{T}^n \to T_x\mathbb{T}^n$, meaning that it is the space spanned by the (generalized) eigenspaces corresponding to the eigenvalues greater than one in absolute value—λ_n, λ_{n-1}, λ_{n-k+1}. Let W^u be the unstable foliation generated by E^u. Then it is easy to see that the condition in Proposition 2.2 is satisfied for $N = 1$, because the k-dimensional expansion in W^u, or the unstable Jacobian J_u, is constant

$$\frac{\|T_{A*}v^k\|}{\|v^k\|} = \left|\prod_{i=n-k+1}^{n} \lambda_i\right| := J_u, \quad \forall v^k \in \Lambda^k T W^u,$$

and the maximal $k-1$-dimensional expansion is at most the product of the greatest (in absolute value) $k-1$ eigenvalues, which is strictly smaller than J_u:

$$\frac{\|T_{A*}v^{k-1}\|}{\|v^{k-1}\|} \leqslant \left|\prod_{i=n-k+2}^{n} \lambda_i\right| = \frac{J_u}{|\lambda_{n-k+1}|} < J_u, \quad \forall v^{k-1} \in \Lambda^{k-1} T W^u.$$

This means that for any $x \in M$ and $r > 0$, all the limit currents of $C_n(x,r)$ are closed. Furthermore, there exist a closed differential form

$$\omega = dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}$$

which is non-trivial on W^u, and by Proposition 2.4 this means that all the limit currents of $C_n(x,r)$ are also non-trivial. Moreover, for any $i_1, i_2, \ldots, i_k \in \{1, 2, \ldots, n\}$, we remark that

$$C_n(x,r)(dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}) = C_{i_1,i_2,\ldots,i_k}$$

is independent of n, x, r (this is because T_A is linear, and $W^u(x)$ is just a plane with the same direction for every x), so all the limit currents will have the same homology, denoted by h_u. This of course will be an eigenvector of $T_{A*} : H_k(\mathbb{T}^n, \mathbb{R}) \to H_k(\mathbb{T}^n, \mathbb{R})$, corresponding to the unstable subspace E^u for A, and the eigenvalue will be exactly J_u and it will be a simple eigenvalue. So the topological growth of W^u is J_u and the volume growth is $\log J_u$.

Now suppose that f is a map which is C^1 close to T_A. Then f is also partially hyperbolic, and the unstable distribution $\tilde{E}^u$ will be C^0 close to E^u (this is because a dominated splitting depends continuously on the point and the map, there is a simple proof using invariant cone fields) and can be integrated to obtain a new unstable foliation $\tilde{W}^u$. We claim that the unstable foliation of f has also unique non-trivial homology, the same as the one of T_A, and the same topological and volume growth.

Proposition 2.8. *If f is sufficiently C^1 close to T_A, then the unstable foliation $\tilde{W}^u$ of f has unique non-trivial homology, the topological growth is equal to J_u, and consequently the volume growth of every unstable disk is* $\log J_u$.

Proof. We remind that the condition on the rates of growth along W^u needed for Proposition 2.2 is an open condition, if it holds for T_A and W^u it also holds for f C^1 close to T_A and $\tilde{W}^u$ such that $T\tilde{W}^u = \tilde{E}^u$ is C^0 close to $TW^u = E^u$, so all the sub-sequential limits of the currents $C_n(x,r)$ are closed for any $x \in M$, $r > 0$. Also, the existence of a non-degenerate closed form on a foliation is an open condition, if ω is non-degenerate on W^u and $\tilde{W}^u$ is such that $T\tilde{W}^u$ is C^0 close to TW^u, then ω is also non-degenerate on $\tilde{W}^u$. By Proposition 2.4 we get that all the limit currents of $C_n(x,r)$ are non-trivial. This implies that $H_{x,r}$ is bounded away from zero for any $x \in M$, $r > 0$.

Furthermore we observe that the map f is homotopic to the linear map T_A and hence the induced map on the homology is exactly the same as that of the linear map. Denote by $\tilde{J}_u(x)$ the new unstable Jacobian for f which now will depend of course on the point $x \in M$. For a C^1 small perturbation $\tilde{J}_u$ is close to J_u. By Proposition 2.5, for any $x \in M$ and $r > 0$, the set $\mathbb{R}H_{x,r}$ is a non-trivial closed invariant subset of $H_k(\mathbb{T}^n, \mathbb{R})$. Suppose $\mathbb{R}H_{x,r} \neq \mathbb{R}h_u$. Because $\mathbb{R}H_{x,r}$ is non-trivial closed invariant, $\mathbb{R}H_{x,r}$ must contain some homology $\tilde{h}$ which is not in $\mathbb{R}h_u$. Because J_u is a simple eigenvalue, $\tilde{h}$ cannot be a (generalized) eigenvector for J_u, so if we write $\tilde{h}$ in a basis formed of generalized eigenvectors we will find at least one non-trivial component in a direction of an eigenvector corresponding to some eigenvalue $\tilde{\lambda} \neq J_u$. Then

$$|\tilde{\lambda}| \leqslant |\lambda_n \lambda_{n-1} \cdots \lambda_{n-k+2} \lambda_{n-k}| < J_u,$$

and

$$\lim_{j\to\infty} \left| f_*^{-j} \tilde{h} \right|^{\frac{1}{j}} \geqslant \frac{1}{\tilde{\lambda}}.$$

We can assume without loss of generality that $\tilde{h} \in H_{x,r}$, so there is a limit current

$$C = \lim_{i\to\infty} C_{n_i}(x,r)$$

such that $[C] = \tilde{h}$. Following the proof of Proposition 2.5, with $f_* C$ replaced by $f_*^{-j} C$, we get that eventually for a subsequence we have

$$\lambda(j) f_*^{-j} C = \lim_{i\to\infty} C_{n_i - j}(x,r),$$

where

$$\lambda(j) = \lim_{i\to\infty} \frac{\operatorname{Vol}(f^{n_i}(W_r(x)))}{\operatorname{Vol}(f^{n_i - j}(W_r(x)))} = \frac{1}{\operatorname{Vol}(f^{n_i - j}(W_r(x)))} \int\limits_{f^{n_i - j}(W_r(x))} \tilde{J}_u(x) \tilde{J}_u\big(f(x)\big) \cdots \tilde{J}_u\big(f^{j-1}(x)\big).$$

By choosing f sufficiently close to T_A, we may assume that $\tilde{J}_u(x)$ is close enough to J_u so for some $\alpha > 1$ we have $\tilde{J}_u(x) > \alpha |\tilde{\lambda}|$ for all $x \in M$. From this we get:

$$\lambda(j) > \big(\alpha |\tilde{\lambda}|\big)^j.$$

Then $\lambda(j) f_*^{-j} C$ is also a limit of $C_n(x,r)$ and

$$\big[\lambda(j) f_*^{-j} C\big] = \lambda(j) f_*^{-j} \tilde{h} \in H_{x,r}.$$

But, taking the limit as j tends to infinity we get:

$$\lim_{j\to\infty} \left| \lambda(j) f_*^{-j} \tilde{h} \right| \geqslant \lim_{j\to\infty} \frac{\lambda(j)}{|\tilde{\lambda}|^j} \geqslant \lim_{j\to\infty} \alpha^j = \infty,$$

but this is a contradiction because $H_{x,r}$ is bounded.

This proves that $\tilde{W}^u$ has unique non-trivial homology, h_u, so the topological growth is J_u and the volume growth of every disk $\tilde{W}_r(x)$ must be $\log J_u$. □

3. Lyapunov exponents

The expansion of an invariant foliation W can also be described by the Lyapunov exponents. In this section, we will consider this analytical description and show its relations with the geometric expansion we described in the first section.

Let f be a diffeomorphism of M with an invariant probability measure μ. then for μ-a.e. $x \in M$, there exist real numbers $\lambda_1(x) > \cdots > \lambda_l(x)$ $(l \leqslant n)$; positive integers $n_1(x), \ldots, n_l(x)$ such that $n_1(x) + \cdots + n_l(x) = n$; and a measurable invariant splitting $T_xM = E_x^1 \oplus \cdots \oplus E_x^l$, with dimension $\dim(E_x^i) = n_i(x)$ such that

$$\lim_{j\to\infty} \frac{1}{j} \log\left\| D_x f^j(v_i)\right\| = \lambda_i(x),$$

whenever $v_i \in E_x^i$, $v_i \neq 0$.

These numbers $\lambda_1(x), \ldots, \lambda_l(x)$ are called the Lyapunov exponents of $x \in M$. If the probability measure μ is ergodic, then these exponents are constant for a.e. (μ) $x \in M$. The existence of these Lyapunov exponents is the result of Oseledec's Multiplicative Ergodic Theorem.

Let E be an invariant sub-bundle of TM. For example it can be TW, the tangent spaces of leaves are preserved under the map. i.e., for any $x \in M$, $D_xf(T_xW(x)) = T_{f(x)}W(f(x))$. For any invariant probability measure μ and for a.e. (μ) $x \in M$, a subset of the Lyapunov splitting E_x^i, $i = 1, \ldots, l$, spans E_x. Let $\Lambda_E(x)$ be the sum (counting multiplicity $n_i(x)$) of the Lyapunov exponents $\lambda_i(x)$ corresponding to the sub-bundles E_x^i which are inside E_x. $\Lambda_E(x)$ is defined a.e. (μ) and it is also given by the formula:

$$\Lambda_E(x) = \lim_{j\to\infty} \frac{1}{j} \log\left\| \Lambda^k D_x f^j|_{\Lambda^k E_x}\right\|.$$

We also define the integrated Lyapunov exponent of E to be

$$\Lambda_E = \int_M \Lambda_E(x)\, d\mu.$$

Because of the Birkhoff Ergodic Theorem applied to the additive real-valued cocycle $x \mapsto \log\|\Lambda^k D_x f^j|_{\Lambda^k E_x}\|$, the integrated Lyapunov exponent of E is also equal to the integral over the manifold M of the logarithm of the Jacobian of f restricted to the sub-bundle E:

$$\Lambda_E = \int_M \log\left(\left\| \Lambda^k D_x f|_{\Lambda^k E_x}\right\|\right) d\mu.$$

When μ is ergodic, $\Lambda_E(x) = \Lambda_E$ a.e. (μ). If $E = TW$ we will denote $\Lambda_W(x) = \Lambda_{TW}(x)$ and $\Lambda_W = \Lambda_{TW}$.

For the following result, we need to define the concept of absolute continuity. For simplicity, we use a stronger version of absolute continuity. For any $x \in M$, let D_1 and D_2 be sufficiently small $(n-k)$-dimensional smooth disks transverse to $W_r(x)$ for some $r > 0$. One can locally define a map, called holonomy map for the foliation, from D_1 to D_2, $y_1 \mapsto y_2$ with $y_1 \in D_1$ and $y_2 = D_2 \cap W_r(y_1)$. The holonomy map is said to be absolutely continuous if it maps sets of measure zero in D_1 to sets of measure zero in D_2. The foliation is said to be absolutely continuous if the holonomy maps are absolutely continuous. If a foliation is absolutely continuous, a full measure set for a smooth measure intersects almost all leaves in full measure. Here the measure on the leaves is the Riemannian volume restricted to W, and almost all leaves is with respect to Riemannian volume on transversals.

The next result is a standard way to prove non-absolute continuity of foliations:

Lemma 3.1. *Let $f \in \mathrm{Diff}_\mu^r(M)$ be a diffeomorphism on M, preserving a smooth volume μ. Let W be a k-dimensional foliation of M, invariant under f and*

$$\chi(x, r) = \limsup_{i\to\infty} \frac{1}{i} \ln \mathrm{Vol}\big(f^i\big(W_r(x)\big)\big),$$

and let

$$\chi = \chi(r) = \sup_{x\in M} \chi(x, r).$$

Finally, let Λ_W be the integrated Lyapunov exponent of the foliation W for the invariant measure μ. If the foliation W is absolutely continuous, then

$$\Lambda_W \leqslant \chi.$$

Proof. Let $A \subset M$ be the set of Lyapunov generic points. i.e., for any $x \in A$, there exist the sum of the Lyapunov exponents for x on $T_x W$ and is equal to $\Lambda_W(x)$. This is a full measure set with respect to μ. We have that

$$\Lambda_W = \int_M \Lambda_W(x)\, d\mu,$$

so there exists a positive measure set $B \subset A \subset M$ such that for any $x \in B$ we have $\Lambda_W(x) \geqslant \Lambda_W$. The absolute continuity of W implies that there exists at least a leaf $W(x)$ for some $x \in M$ such that $W(x)$ intersects B in a set of positive measure (actually there is a positive set of such leaves). Denote by m_W the Riemannian volume on $W(x)$ and fix a disk $W_r(x)$ such that $m_W(W_r(x) \cap B) > 0$.

Let $\mathrm{Jac}_y(f^i) = \|\Lambda^k D_y f^i|_{\Lambda^k T_y W}\|$ be the Jacobian at $y \in M$ of the function f restricted to $W(y)$. If y is a Lyapunov regular point ($y \in A$), we have

$$\Lambda_W(y) = \lim_{i\to\infty} \frac{1}{i} \log\bigl(\mathrm{Jac}_y\bigl(f^i\bigr)\bigr).$$

Then for any small $\epsilon > 0$, for any $y \in W_r(x) \cap B \subset A$ there exist $N_y \in \mathbb{N}$ such that for all $i \geqslant N_y$ we have

$$\frac{1}{i} \log\bigl(\mathrm{Jac}_y\bigl(f^i\bigr)\bigr) \geqslant \Lambda_W(y) - \epsilon,$$

or

$$\mathrm{Jac}_y\bigl(f^i\bigr) \geqslant e^{i(\Lambda_W(y)-\epsilon)}.$$

Let $B_N \subset W_r(x) \cap B$ be the set of points y such that $N_y \leqslant N$. Then B_N is an increasing sequence of sets and the union is $W_r(x) \cap B$ which has positive measure, so there is an $N \in \mathbb{N}$ such that $m_W(B_N) > 0$. It follows that for any $y \in B_N$ and any $i > N$ we have

$$\mathrm{Jac}_y\bigl(f^i\bigr) \geqslant e^{i(\Lambda_W(y)-\epsilon)} \geqslant e^{i(\Lambda_W-\epsilon)}.$$

We will use this to estimate the volume of $f^i(W_r(x))$ for $i > N$:

$$\begin{aligned}
\mathrm{Vol}\bigl(f^i\bigl(W_r(x)\bigr)\bigr) &= \int_{f^i(W_r(x))} dm_W \\
&= \int_{W_r(x)} \mathrm{Jac}_y\bigl(f^i\bigr)\, dm_W \\
&\geqslant \int_{B_N} \mathrm{Jac}_y\bigl(f^i\bigr)\, dm_W \\
&\geqslant m_W(B_N) e^{i(\Lambda_W-\epsilon)}.
\end{aligned}$$

Therefore,

$$\chi \geqslant \chi(x,r) = \limsup_{i\to\infty} \frac{1}{i} \log \mathrm{Vol}\bigl(f^i\bigl(W_r(x)\bigr)\bigr) \geqslant \Lambda_W - \epsilon.$$

Since $\epsilon > 0$ is arbitrary, we have $\chi \geqslant \Lambda_W$. □

A simple corollary of the above lemma and its proof is the following:

Corollary 3.2. *Let $f \in \mathrm{Diff}^r_\mu(M)$ be a diffeomorphism on M, preserving a smooth volume μ. Let W be a k-dimensional foliation of M, invariant under f and let Λ_W be the integrated Lyapunov exponent of the foliation W for the measure μ.*

If $\chi < \Lambda_W$, then the foliation W is not absolutely continuous. Moreover, if μ is ergodic, then there is a full measure set $A \in M$ such that every leaf $W(x)$ of the foliation W intersect A in a zero measure set,

$$\mu_W\big(W(x) \cap A\big) = 0,$$

for all $x \in M$, where μ_W is the conditional measure of μ on the leaves of W.

Proof. The first statement of the corollary is just a consequence of Lemma 3.1. In the second statement of the corollary let A be the set of Lyapunov regular points x such that $\Lambda_W(x) = \Lambda_W$. If a leaf of W intersects A in a positive measure set then the same argument from the proof of the lemma will show that the leaf expands under iterates of f at an exponential rate greater or equal to Λ_W, so strictly greater than the volume growth χ, which gives a contradiction. □

4. Perturbations and examples

In this section we show how to perturb a linear map of the torus in order to make an intermediate foliation non-absolute continuous in a persistent way. The main tool used here is a result of A. Baraviera and C. Bonatti (see [1]). Before we state it, we have to define dominated splittings.

We say that $TM = E \oplus F$ is a dominated splitting for the diffeomorphism f if the sub-bundles E and F are invariant under Df and there is an $l \in \mathbb{N}$ such that for each $x \in M$ and each nonzero vectors $u \in E_x$, $v \in F_x$ we have

$$\frac{\|D_x f^l(u)\|}{\|u\|} < \frac{1}{2}\frac{\|D_x f^l(v)\|}{\|v\|}.$$

A dominated splitting is continuous and it persists after perturbations, meaning that any g which is C^1 close to f will also have a dominated splitting $TM = E' \oplus F'$ close to the dominated splitting $TM = E \oplus F$ for f. The definition can be of course extended for splittings with more than two sub-bundles.

Theorem 4.1. *Let M be a compact Riemannian manifold and μ a smooth volume form on M. Let f be a C^1 diffeomorphism of M preserving μ and admitting a dominated splitting $TM = E^1 \oplus E^2 \oplus E^3$. Then there are arbitrarily small volume preserving C^1 perturbations g of f such that, if $TM = \tilde{E}^1 \oplus \tilde{E}^2 \oplus \tilde{E}^3$ is the new dominated splitting for g, then the integrated Lyapunov exponent of $\tilde{E}^2$ with respect to g is strictly larger than the integrated Lyapunov exponent of E^2 with respect to f:*

$$\Lambda_{\tilde{E}^2}(g) > \Lambda_{E^2}(f).$$

The idea of the proof is the following. One has to make a small perturbation to 'mix' the direction of E^2 with the direction of E^3, while keeping the coordinates corresponding to E^1 almost unchanged. This mixing almost does not change the direction of $E^2 \oplus E^3$ and the Jacobian restricted to it, so the integrated Lyapunov exponent of $\tilde{E}^2 \oplus \tilde{E}^3$ is very close to the one of $E^2 \oplus E^3$. The perturbation will be local, supported on a small ball with very large returning time. This perturbation will change the direction of E^3 toward E^2 at the image of the ball, but then the dynamics of the map will tend to correct this perturbation, and if the return time is large enough then this perturbation becomes negligible for estimating the Jacobian along $\tilde{E}^3$ for the further iterates. Then, analyzing the change on the small ball where the perturbation is supported, one can prove that the integrated exponent corresponding to the new $\tilde{E}^3$ is 'significantly' smaller than the one of E^3. As a consequence, the integrated exponent corresponding to the new $\tilde{E}^2$ becomes larger than the one of E^2. For the details of the proof we send the reader to [1].

A. Baraviera and C. Bonatti show that a consequence of this result is the fact that a C^1 generic small perturbation of the time one map of an volume preserving Anosov flow has a non-absolutely continuous central foliation. Previously M. Shub and A. Wilkinson gave some examples of perturbations of skew products where the central foliation is again non-absolutely continuous in a persistent way. In their situation the central foliation consists of circles (see [10]). Recently M. Hirayama and Y. Pesin proved that C^1 generically a partially hyperbolic map with compact center leaves has the central foliation non-absolutely continuous (see [2]). All this results lead to the following conjecture:

Conjecture 1. *Generically the central foliation* (*if it exists*) *of a volume preserving partially hyperbolic diffeomorphism is non-absolutely continuous.*

We want to give another example of persistent non-absolutely continuous central and intermediate foliations of volume preserving partially hyperbolic diffeomorphisms that supports this conjecture.

Consider again a linear automorphism T_A of the torus $\mathbb{T}^n$ and suppose that this time the tangent bundle has a dominated invariant splitting $T\mathbb{T}^n = E^1 \oplus E^2 \oplus E^3$ with the dimension of E_i equal to k_i. We remark that T_A preserves the Lebesgue measure on $\mathbb{T}^n$. We will also denote by J_1, J_2, J_3 the Jacobians of T_A on E^1, E^2, E^3 and Λ_1, Λ_2, Λ_3 the integrated Lyapunov exponents corresponding to the invariant bundles E^1, E^2, E^3 (w.r.t. the Lebesgue invariant measure). We have

$$\Lambda_i = \log J_i, \quad i \in \{1, 2, 3\}.$$

We can integrate the invariant distributions E^1, E^2, E^3 to obtain invariant foliations W^1, W^2, W^3. This is not true for a general dominating splitting of a map. However in our case T_A is linear, and all these foliations exist. To see this consider $A : \mathbb{R}^n \to \mathbb{R}^n$ be the lift of T_A. We will make an abuse of notation and denote the lifted bundles also by E^i, and using the exponential map we identify $\mathbb{R}^n \equiv T_0\mathbb{R}^n$, $A \equiv D_0 A$ and $E_0^i \equiv W_0^i$, for $i \in \{1, 2, 3\}$, where 0 is the origin. Then the planes parallel to $E_0^i = W_0^i$ will form a foliation of $\mathbb{R}^n$ which projected to the torus $\mathbb{T}^n$ will give the foliation W^i tangent to E^i, for all $i \in \{1, 2, 3\}$.

We assume that W^2 and W^3 are uniformly expanding, or W^3 is a strong unstable foliation, W^2 is a weak unstable foliation and W^1 is a stable or a center-stable foliation. We can also integrate $E^2 \oplus E^3$ to get W^{23}, which is an unstable foliation, and $E^1 \oplus E^2$ to obtain W^{12}, which can be seen as a center-stable foliation. All this foliations have unique non-trivial homology which is an eigenvector of the map induced by T_A in the corresponding homology group. We denote this eigenvectors h_1, h_2, h_3. By taking f^2 if necessary, we can assume that f preserves the orientation of the invariant foliations. The topological growth of W^1, W^2, W^3 will be exactly the corresponding eigenvalues J_1, J_2, J_3. Because the map is linear, for all these foliations the volume growth, the Lyapunov growth and the logarithm of the topological growth coincide.

For any f a C^1 small perturbation of T_A we will have an f-invariant dominated splitting $T\mathbb{T}^n = \tilde{E}^1 \oplus \tilde{E}^2 \oplus \tilde{E}^3$ which is close to the invariant splitting for T_A. Also the foliations will persist, one can integrate $\tilde{E}^1$, $\tilde{E}^2$, $\tilde{E}^3$ to obtain the corresponding f-invariant foliations $\tilde{W}^1$, $\tilde{W}^2$, $\tilde{W}^3$. The new foliations exist because $\tilde{E}^3$ and $\tilde{E}^{23}$ are (strong) unstable distributions, so they can always be integrated to obtain strong unstable foliations $\tilde{W}^3$ and $\tilde{W}^{23}$, while the existence of $\tilde{W}^1$ and $\tilde{W}^{12}$ follows from the persistence under small perturbations of the center foliations for T_A, W^1 and W^{12}, because they are C^1, so they are plaque expansive (see [6]). Then $\tilde{W}^2$ will be just the intersection of $\tilde{W}^{12}$ and $\tilde{W}^{23}$.

To simplify the proof we will make the following technical assumption:

(H) J_2 is a simple eigenvalue of $T_{A*} : H_{k_2}(\mathbb{T}^n, \mathbb{R}) \to H_{k_2}(\mathbb{T}^n, \mathbb{R})$ and there is no other eigenvalue of absolute value J_2.

This is used just to obtain a simple proof of the fact that $\tilde{W}^2$ has unique non-trivial homology. The next result is also true without this assumption.

Theorem 4.2. *Suppose T_A is a linear automorphism of the torus $\mathbb{T}^n$ with an invariant dominated splitting $T\mathbb{T}^n = E^1 \oplus E^2 \oplus E^3$, with E^2 uniformly expanding, and satisfying the hypothesis* (H). *Then there exist an open set of volume preserving diffeomorphisms U, C^1 arbitrarily close to T_A, such that, for any $f \in U$, the weak unstable foliation of f, $\tilde{W}^2$, is non-absolutely continuous.*

Proof. By the previous theorem we can make an arbitrarily C^1 small perturbation of T_A to obtain a volume preserving diffeomorphism f such that

$$\Lambda_{\tilde{E}^2}(f) > \Lambda_2.$$

We remark that this property is true also for small C^1 perturbations of f, because the integrated Lyapunov exponent of a sub-bundle of a dominated splitting depends continuous on the map (it is just the integral of the corresponding Jacobian).

We will use the following lemma:

Lemma 4.3. *Suppose T_A is a linear automorphism of the torus $\mathbb{T}^n$ with an invariant dominated splitting $T\mathbb{T}^n = E^1 \oplus E^2 \oplus E^3$, with E^2 uniformly expanding, and satisfying the hypothesis* (H). *Then for any f sufficiently C^1 close to T_A, the weak unstable foliation of f, $\tilde{W}^2$, has unique non-trivial homology, equal to the one of W^2 (h_2).*

Proof. The proof is similar to the proof of Proposition 2.8, the only difference is that now the eigenvalue J_2 of $f_*: H_{k_2}(\mathbb{T}^n, \mathbb{R}) \to H_{k_2}(\mathbb{T}^n \mathbb{R})$ is not strictly greater than the other eigenvalues in absolute value, it is just different.

We start by making the remark that we can choose f sufficiently close to A so that the Jacobian of f on $\tilde{W}^2$, denoted $\tilde{J}_2(x)$, is inside a small neighborhood of J_2 ($|J_2 - \tilde{J}_2(x)| < \epsilon$) which does not contain any other eigenvalue in absolute value of the map $T_{A*}: H_k(\mathbb{T}^n, \mathbb{R}) \to H_k(\mathbb{T}^n, \mathbb{R})$.

Because f is close to a linear map on the torus, one can apply Propositions 2.2 and 2.4 to conclude that every limit current on $\tilde{W}^2$ is closed and non-trivial (see also the proof of Proposition 2.8), and for every $x \in M$, $r > 0$, $H_{x,r}$ is bounded away from zero and infinity. Suppose that $\tilde{W}^2$ does not have the unique non-trivial homology h_2. Then there exist a disk $\tilde{W}^2_r(x)$ in $\tilde{W}^2$ and a subsequence of corresponding currents $C_{n_i}(x, r)$ such that $\lim_{i\to\infty} C_{n_i}(x, r) = C$ and $[C] = \tilde{h} \notin \mathbb{R}h_2$.

The condition (H) can be reformulated in the following way:

(H′) Suppose $h \in H_{k_2}(\mathbb{T}^n, \mathbb{R})$. Then $h \in \mathbb{R}h_2$ if and only if

$$\lim_{j\to\infty} \left|f_*^j h\right|^{\frac{1}{j}} = \lim_{j\to\infty} \left|f_*^{-j} h\right|^{-\frac{1}{j}} = J_2.$$

This implies that at least one of $\lim_{j\to\infty} |f_*^j h|^{\frac{1}{j}}$ and $\lim_{j\to\infty} |f_*^{-j} h|^{-\frac{1}{j}}$ must be different from J_2. We will assume that $\lim_{j\to\infty} |f_*^j h|^{\frac{1}{j}} = \tilde{\lambda} < J_2$, all the other cases are treated similarly.

As in Section 2 we get

$$f_*^j C = \lim_{i\to\infty} \frac{\mathrm{Vol}(f^{n_i+j}(W_r(x)))}{\mathrm{Vol}(f^{n_i}(W_r(x)))} \cdot C_{n_i+j}(x, r),$$

and eventually for a subsequence

$$\lim_{i\to\infty} \frac{\mathrm{Vol}(f^{n_i+j}(W_r(x)))}{\mathrm{Vol}(f^{n_i}(W_r(x)))} = \lambda(j),$$

so

$$\lim_{i\to\infty} C_{n_i+j}(x, r) = \frac{f_*^j C}{\lambda(j)}$$

is another closed limit current of $C_n(x, r)$, or

$$\frac{f_*^j \tilde{h}}{\lambda(j)} \in H_{x,r}, \quad \forall j > 0.$$

On the other hand $\lambda(j) \geqslant (J_2 - \epsilon)^j > (\alpha\tilde{\lambda})^j$ for some $\alpha > 1$, so

$$\lim_{j\to\infty} \left|\frac{f_*^j \tilde{h}}{\lambda(j)}\right| \leqslant \lim_{j\to\infty} \frac{\tilde{\lambda}^j}{\alpha\tilde{\lambda}^j} = \lim_{j\to\infty} \alpha^{-j} = 0,$$

which contradicts the fact that $H_{x,r}$ is bounded away from zero and infinity. □

Now we go back to the proof of the theorem. We know from the previous lemma that $\tilde{W}^2$ has unique non-trivial homology which is h_2. Then the volume growth of $\tilde{W}^2$ will have to be

$$\chi(f, \tilde{W}^2) = \log J_2 = \Lambda_2 < \Lambda_{\tilde{E}^2}(f) = \Lambda_{\tilde{W}^2}(f).$$

But then the volume growth of $\tilde{W}^2$ is strictly smaller then the integrated Lyapunov exponent $\Lambda_{\tilde{W}^2}$ of $\tilde{W}^2$, so from Corollary 3.2 from the previous sections follows that the foliation $\tilde{W}^2$ is non-absolutely continuous. The same is true for all the diffeomorphisms sufficiently C^1 close to f. □

We remark that this result can be generalized to dominated splittings with a larger number of sub-bundles. For example if $T\mathbb{T}^n = E^1 \oplus E^2 \oplus \cdots \oplus E^l$ is a dominated splitting for the linear map on the n-torus T_A, with E^2 uniformly expanding, then there are arbitrarily small C^1 perturbations f of T_A such that for any $2 \leqslant i \leqslant j < l$, there is an intermediate f-invariant sub-bundle $\tilde{E}^i \oplus \tilde{E}^{i+1} \oplus \cdots \oplus \tilde{E}^j$ close to $E^i \oplus E^{i+1} \oplus \cdots \oplus E^j$ which can be integrated to obtain a weak unstable foliation $\tilde{W}^{i,i+1,\ldots,j}$ which is not absolutely continuous in a persistent way. However if f is $C^{1+\epsilon}$ then the strong unstable foliations $\tilde{W}^{j,j+1,\ldots,l}$ are always absolutely continuous, for every $2 \leqslant j \leqslant l$.

Acknowledgements

We would like to thank Michael Shub, Charles Pugh and Amie Wilkinson for useful conversations. We would also like to thank the referee for pointing out some typos and for useful suggestions for improving the presentation. The first author would like to thank the Fields Institute and CRM Barcelona for their hospitality.

References

[1] A. Baraviera, C. Bonatti, Removing zero Lyapunov exponents, Ergodic Theory Dynam. Systems 23 (6) (2003) 1655–1670.
[2] M. Hirayama, Y. Pesin, Non-absolutely continuous foliations, Preprint.
[3] Y. Hua, R. Saghin, Z. Xia, Topological entropy and partially hyperbolic diffeomorphisms, Preprint math.DS/0608720.
[4] S.E. Newhouse, Continuity properties of entropy, Ann. of Math. (2) 129 (2) (1989) 215–235.
[5] J.F. Plante, Foliations with measure preserving holonomy, Ann. of Math. (2) 102 (2) (1975) 327–361.
[6] C. Pugh, M. Shub, Stably ergodic dynamical systems and partial hyperbolicity, J. Complexity 13 (1997) 125–179.
[7] D. Ruelle, D. Sullivan, Currents, flows and diffeomorphisms, Topology 14 (4) (1975) 319–327.
[8] S. Schwartzmann, Asymptotic cycles, Ann. of Math. 66 (1957) 270–284.
[9] M. Shub, R. Williams, Entropy and stability, Topology 14 (4) (1975) 329–338.
[10] M. Shub, A. Wilkinson, Pathological foliations and removable zero exponents, Invent. Math. 139 (3) (2000) 495–508.
[11] D. Sullivan, Cycles for the dynamical study of foliated manifolds and complex manifolds, Invent. Math. 36 (1976) 225–255.
[12] Y. Yomdin, Volume growth and entropy, Israel J. Math. 57 (3) (1987) 285–300.