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Multiple critical points of perturbed symmetric strongly indefinite functionals

Points critiques multiples de perturbations de fonctionnelles symétriques fortement indéfinies

Denis Bonheure [a,*,1], Miguel Ramos [b,2]

[a] *Université catholique de Louvain, Institut de Mathématique Pure et Appliquée, Chemin du Cyclotron, 2, B-1348 Louvain-la-Neuve, Belgium*

[b] *University of Lisbon, CMAF – Faculty of Science, Av. Prof. Gama Pinto 2, 1649-003 Lisboa, Portugal*

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Abstract

We prove that the elliptic system

$$-\Delta u = |v|^{q-2}v + k(x), \quad x \in \Omega, \tag{1}$$

$$-\Delta v = |u|^{p-2}u + h(x), \quad x \in \Omega, \tag{2}$$

where Ω is a regular bounded domain of $\mathbb{R}^N$, $N \geqslant 3$ and $h, k \in L^2(\Omega)$, admits an unbounded sequence of solutions $(u_k, v_k) \in H_0^1(\Omega) \times H_0^1(\Omega)$, provided $2 < p \leqslant q$ and $\frac{N}{2}(1 - \frac{1}{p} - \frac{1}{q}) < \frac{p-1}{p}$. We also prove a generic multiplicity result for exponents in the open region bounded by the lines $p = 2$, $q = 2$ and the critical hyperbola.

Résumé

Nous démontrons que le système elliptique ((1), (2)) où Ω est un domaine régulier de $\mathbb{R}^N$, $N \geqslant 3$ et $h, k \in L^2(\Omega)$, possède une suite non bornée de solutions $(u_k, v_k) \in H_0^1(\Omega) \times H_0^1(\Omega)$, pour autant que $2 < p \leqslant q$ et $\frac{N}{2}(1 - \frac{1}{p} - \frac{1}{q}) < \frac{p-1}{p}$. Nous démontrons également un résultat générique de mulplicité lorsque les exposants se situent dans l'ouvert délimité par les droites $p = 2$, $q = 2$ et l'hyperbole critique.

MSC: 35J50; 35J55; 58E05

Keywords: Elliptic system; Strongly indefinite functional; Perturbation from symmetry; Lyapunov–Schmidt reduction; Genericity

* Corresponding author.

E-mail addresses: denis.bonheure@uclouvain.be (D. Bonheure), mramos@ptmat.fc.ul.pt (M. Ramos).

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1. Introduction

Let Ω be a smooth bounded domain of $\mathbb{R}^N$, $N \geqslant 3$, and $h, k \in L^2(\Omega)$. We consider an elliptic system of the form

$$\begin{cases} -\Delta u = |v|^{q-2}v + k(x) & \text{in } \Omega, \\ -\Delta v = |u|^{p-2}u + h(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

with $p, q > 2$. Here q stands for the largest exponent appearing in (1.1), that is we assume without loss of generality that $p \leqslant q$. In case $h(x) = k(x)$ and $p = q > 2$, the system reduces to a single equation

$$-\Delta u = |u|^{p-2}u + h(x) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega. \tag{1.2}$$

This equation can be seen as a (large) perturbation of an equation possessing a natural $\mathbb{Z}_2$-symmetry and thus a large number of solutions are expected. One can indeed obtain infinitely many solutions, provided the growth range of the nonlinearity is suitably restricted. Namely, Bahri and Berestycki [3], Struwe [24], and, with a different approach, Rabinowitz [16,17] proved the existence of infinitely many solutions for problem (1.2) under the restriction

$$\frac{2}{p} + \frac{1}{p-1} > \frac{2N-2}{N}, \tag{1.3}$$

while, later on, Bahri and Lions [4] and Tanaka [26] (see also [14]) showed that it is sufficient to assume

$$p < \frac{2N-2}{N-2}. \tag{1.4}$$

Moreover, assuming the "natural" growth restriction $p < 2N/(N-2)$, Bahri [2] proved that there is an open dense set of functions $h \in H^{-1}(\Omega)$

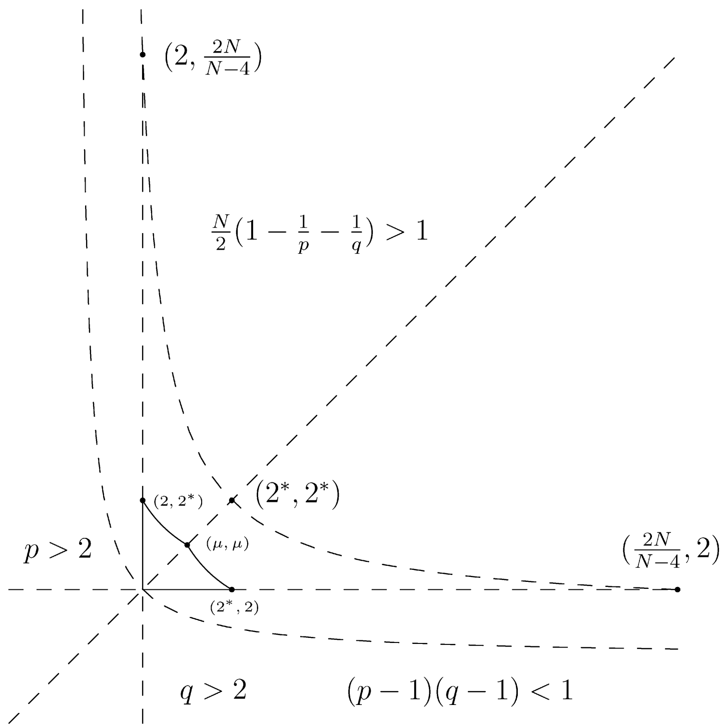

Fig. 1. The (p, q)-region covered by Theorem 1 is the one bounded by the continuous lines. When $p = q$, we recover the critical value $(\mu, \mu) = (\frac{2N-2}{N-2}, \frac{2N-2}{N-2})$. The upper hyperbola delimits the subcritical region while the lower hyperbola delimits the "superlinearity" region.

Theorem 1. *Let $h, k \in L^2(\Omega)$ and $2 < p \leqslant q$ be such that*

$$\frac{N}{2}\left(1 - \frac{1}{p} - \frac{1}{q}\right) < \frac{p-1}{p}. \tag{1.7}$$

Then the system (1.1) *admits an unbounded sequence of solutions $(u_k, v_k)_k \subset H_0^1(\Omega) \times H_0^1(\Omega)$.*

We stress that the condition (1.7) is sharp in the sense that it reduces to (1.4) in the case $p = q$, see Fig. 1. In particular, this condition is implied by that expressed in (1.6). On the other hand, (1.7) does force both p and q to be smaller than the Sobolev exponent $2N/(N-2)$. Observe also that we do assume both equations to be superlinear. To our knowledge, it is not known whether Theorem 1 extends to superlinear systems under the milder assumption $(p-1)(q-1) > 1$.

The proof of Theorem 1 is worked out in several steps in the next sections. It combines the perturbation argument from Rabinowitz [16] and Tanaka [26] for the single equation (1.2) with a Lyapunov–Schmidt type reduction used in Ramos and Tavares [18] (see also Ramos and Yang [19]). We provide a new estimate to the Morse index of solutions of the unperturbed system (1.1) (see Section 3.2) which can be seen as an extension of the one in [4,26] for the single equation.

It should be pointed out that contrarily to the above quoted papers [7,10,27], we do not rely on Galerkin type arguments; indeed, using our reduction method allows to get rid of the indefiniteness of the energy functional associated to the system, giving rise to critical points whose energy is controlled (from below) by their Morse indices (cf. Lemma 9). Concerning the unperturbed problem (1.1) (i.e. with $h = k = 0$), we obtain as a byproduct a short proof of the multiplicity result obtained in [1, Theorem 33]. We emphasize that in the unperturbed case, we can also deal with the natural growth condition $1/p + 1/q > (N-2)/N$, see Proposition 4.

We believe that our direct approach to the problem may turn to be useful to prove other results concerning the system (1.1). For instance, it becomes a simple task to adapt the argument of Bahri [2] to deduce that the multiplicity result is generic, in the sense that if $2 < p, q < 2N/(N-2)$, then, for (h, k) on a residual subset of $H^{-1}(\Omega) \times$

$H^{-1}(\Omega)$, the problem admits infinitely many weak solutions. We refer to Theorem 11 below for a more general statement concerning the case where $1/p + 1/q > (N-2)/N$.

For the sake of simplicity, we have restricted our attention in this paper to the model problem (1.1). It will be clear from the proofs that we could have dealt with more general nonlinearities, as done in [3,4,16,17].

Our paper is organized as follows. In Section 2, we introduce our functional settings and recall the basics of the reduction method borrowed from Ramos and Tavares [18]. Section 3 deals with technical lemmas used in the proof of Theorem 1 while Section 4 contains the proof in itself. Since we rely on the arguments in [16,17], we keep the proof short by merely emphasizing the difficulties which arise from the indefinite character of our problem. Section 5 deals with the adaptation of Bahri's genericity result to our framework.

We write throughout the paper $f(s) = |s|^{p-2}s$, $F(s) = |s|^p/p$, $g(s) = |s|^{q-2}s$ and $G(s) = |s|^q/q$ with $2 < p \leqslant q < 2^* = 2N/(N-2)$ and, if not explicitly stated, all integrals are taken over Ω. The notation $\|\cdot\|$ refers to the usual norm of $H_0^1(\Omega)$. Throughout the text, C denotes a positive constant that can change from line to line.

2. Functional settings

Let $E := H_0^1(\Omega) \times H_0^1(\Omega)$. The energy functional $I : E \to \mathbb{R}$ associated to the elliptic problem (1.1) writes

$$I(u, v) = \int \big(\langle \nabla u, \nabla v\rangle - F(u) - G(v) - h(x)u - k(x)v\big). \tag{2.1}$$

This is a C^2 functional whose derivative is given by

$$I'(u, v)(\varphi, \psi) = \int \big(\langle \nabla u, \nabla \psi\rangle + \langle \nabla v, \nabla \varphi\rangle - f(u)\varphi - g(v)\psi - h(x)\varphi - k(x)\psi\big),$$

and since both p and q are subcritical, it is easily seen that I satisfies the Palais–Smale condition (PS in short) over E, namely that every sequence $(u_n, v_n)_n \subset E$ such that $I'(u_n, v_n) \to 0$ and $I(u_n, v_n)$ is bounded admits a convergent subsequence (see e.g. [23, p. 1457]); here one makes use of the compact embedding $H_0^1(\Omega) \subset L^q(\Omega)$.

Next we consider the reduced functional $J : H_0^1(\Omega) \to \mathbb{R}$ defined by

$$J(\alpha) := I(\alpha + \psi_\alpha, \alpha - \psi_\alpha) := \max_{\psi \in H_0^1(\Omega)} I(\alpha + \psi, \alpha - \psi). \tag{2.2}$$

It follows from [18, Proposition 2.1] that the map

$$\Psi : H_0^1(\Omega) \to H_0^1(\Omega) : \alpha \mapsto \psi_\alpha$$

is well defined and of class C^1. We observe that, for every $\phi \in H_0^1(\Omega)$, ψ_α satisfies

$$I'(\alpha + \psi_\alpha, \alpha - \psi_\alpha)(\phi, -\phi) = 0, \tag{2.3}$$

that is ψ_α is the unique solution of the following equation in $H_0^1(\Omega)$,

$$-2\Delta\psi_\alpha = g(\alpha - \psi_\alpha) - f(\alpha + \psi_\alpha) + k(x) - h(x). \tag{2.4}$$

Moreover, using (2.3), we infer that for all $\alpha, \phi \in H_0^1(\Omega)$,

$$J'(\alpha)\phi = I'(\alpha + \psi_\alpha, \alpha - \psi_\alpha)(\phi, \phi). \tag{2.5}$$

Combining now (2.3) and (2.5), we deduce the following crucial proposition.

Proposition 2. *The map*

$$\eta : H_0^1(\Omega) \to E : \alpha \mapsto (\alpha + \psi_\alpha, \alpha - \psi_\alpha)$$

provides a homeomorphism between critical points of the reduced functional J and critical points of the functional I.

Proof. Observe that for any $(\zeta, \xi) \in H_0^1(\Omega) \times H_0^1(\Omega)$, we have

$$I'(\alpha + \psi_\alpha, \alpha - \psi_\alpha)(\zeta, \xi) = I'(\alpha + \psi_\alpha, \alpha - \psi_\alpha)\left(\frac{\zeta - \xi}{2}, -\frac{\zeta - \xi}{2}\right) + I'(\alpha + \psi_\alpha, \alpha - \psi_\alpha)\left(\frac{\zeta + \xi}{2}, \frac{\zeta + \xi}{2}\right)$$

and there you have it. $\square$

In particular, one can study the reduced functional J to find solutions of the system (1.1).

We now prove that the Palais–Smale condition holds for J.

Lemma 3. *The reduced functional J satisfies the Palais–Smale condition in $H_0^1(\Omega)$ and, moreover, for any finite dimensional subspace $X \subset H_0^1(\Omega)$,*

$$J(\alpha) \to -\infty \quad \text{as } \|\alpha\| \to \infty,\ \alpha \in X. \tag{2.6}$$

Proof. Let $(\alpha_n)_n \subset H_0^1(\Omega)$ be a Palais–Smale sequence for J and write $\psi_n := \Psi(\alpha_n)$. Then, it is clear that the sequence $(\eta(\alpha_n))_n \subset E$ is a Palais–Smale sequence for I. Since PS holds for I, we deduce that, up to a subsequence, $\alpha_n + \psi_n \to u$ and $\alpha_n - \psi_n \to v$ for some $(u, v) \in E$. In particular, we have $\alpha_n \to (u + v)/2$ so that our first claim follows.

Now, take a finite dimensional subspace $X \subset H_0^1(\Omega)$. Assume by contradiction that there exists an unbounded sequence $(\alpha_n)_n \subset X$ such that

$$\liminf_{n\to\infty} J(\alpha_n) > -\infty.$$

Computing $J(\alpha_n)$, we easily see that the sequence $(\|\psi_n\|/\|\alpha_n\|)_n$ is bounded and

$$\lim_{n\to\infty} \int \left| \frac{\alpha_n}{\|\alpha_n\|} \pm \frac{\psi_n}{\|\alpha_n\|} \right|^p = 0.$$

It then follows that

$$\lim_{n\to\infty} \int \left| \frac{\alpha_n}{\|\alpha_n\|} \right|^p = 0,$$

which is impossible since X has finite dimension. This completes the proof. □

At this point, we are already able to prove an existence result in the unperturbed case. In this way, we recover with a direct proof the existence result in [1, Theorem 33] and [27, Section 3].

Proposition 4. *Assume that $2 < p \leqslant q$ are such that*

$$\frac{N}{2}\left(1 - \frac{1}{p} - \frac{1}{q}\right) < 1.$$

Then the system (1.1) *with $h \equiv k \equiv 0$ admits an unbounded sequence of solutions $(u_k, v_k)_k \subset H_0^1(\Omega) \times H_0^1(\Omega)$.*

Proof. Assume first that $p \leqslant q < 2^* = 2N/(N-2)$. If $h(x) = k(x) = 0$, then we deduce that

$$J(\alpha) \geqslant I(\alpha, \alpha) \geqslant c\|\alpha\|^2$$

provided $\|\alpha\| = \rho$ with $\rho > 0$ small enough. The results then follows straightforwardly from the $\mathbb{Z}_2$-version of the Mountain Pass Theorem (cf. e.g. [17, Theorem 9.12]).

Next, observe that assuming $p, q < 2^*$ is not restrictive. Indeed, if $q > 2^*$, define $g_n(s)$ as a smooth odd truncation of $g(s)$ such that $g_n(s) = g(s)$ if $|s| \leqslant n + 1$, $|g_n|$ is strictly convex and grows like $|s|^{p-1}$ at $\pm\infty$.

Since $p < 2^*$, extending the case of pure powers to our new settings, it is easily seen that for every $n \in \mathbb{N}$, the modified system

$$\begin{cases} -\Delta u = g_n(v) & \text{in } \Omega, \\ -\Delta v = |u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.7}$$

has an unbounded sequence of solutions $(u_k, v_k)_k \subset H_0^1(\Omega) \times H_0^1(\Omega)$.

Finally, arguing as in [18, Section 5], it comes out that those solutions are bounded independently of n. This means that for every $k \in \mathbb{N}$, the first k solutions of (2.7) are indeed solutions of the original system provided n is chosen large enough. Since this is true for every $k \in \mathbb{N}$, the conclusion follows. □

3. Preliminaries

Our proof of Theorem 1 mainly consists in adapting Rabinowitz's perturbation argument [16,17] to our framework, together with a new result on the Morse index of the solutions of (1.1). Some preliminary estimates are in order. For convenience, we write in the sequel

$$u_\alpha := \alpha + \psi_\alpha, \qquad v_\alpha := \alpha - \psi_\alpha. \tag{3.1}$$

3.1. The modified functional

Rabinowitz's approach [16,17] mainly relies on an estimate of the deviation from symmetry, see (3.14). Since the original functional does not enjoy this property, following [16,17], we next define a modified functional.

At first, we observe that, for any $\alpha \in H_0^1(\Omega)$,

$$J'(\alpha)\alpha = 2\|\alpha\|^2 - \int \big(g(v_\alpha)\alpha + f(u_\alpha)\alpha + k(x)\alpha + h(x)\alpha\big), \tag{3.2}$$

while (2.4) shows that

$$2\|\psi_\alpha\|^2 = \int \big(g(v_\alpha)\psi_\alpha - f(u_\alpha)\psi_\alpha + k(x)\psi_\alpha - h(x)\psi_\alpha\big). \tag{3.3}$$

Taking (3.3) into account, we infer that

$$J(\alpha) - \frac{1}{2}J'(\alpha)\alpha = \int \left(\frac{g(v_\alpha)v_\alpha}{2} - G(v_\alpha) - \frac{k(x)}{2}v_\alpha\right) + \int \left(\frac{f(u_\alpha)u_\alpha}{2} - F(u_\alpha) - \frac{h(x)}{2}u_\alpha\right).$$

Henceforth, there exist $A, B > 0$ such that, for every $\alpha \in H_0^1(\Omega)$,

$$\begin{aligned} 2A\left(\int \big(F(u_\alpha) + G(v_\alpha)\big) - 1\right) &\leqslant J(\alpha) - \frac{1}{2}J'(\alpha)\alpha \\ &\leqslant B\left(\int \big(F(u_\alpha) + G(v_\alpha)\big) + 1\right). \end{aligned} \tag{3.4}$$

Let $\chi \in \mathcal{D}(]-2,2[)$, $0 \leqslant \chi \leqslant 1$, with $\chi = 1$ in $[-1,1]$ and consider the C^1 map $\theta : H_0^1(\Omega) \to \mathbb{R}$,

$$\theta(\alpha) := \chi\left(\frac{A\int(F(u_\alpha) + G(v_\alpha))}{\sqrt{J^2(\alpha)+1}}\right),$$

where $A > 0$ was introduced in (3.4). Accordingly, we consider the functional $\tilde{I} : E \to \mathbb{R}$ defined by

$$\tilde{I}(u,v) := I(u,v) + \left(1 - \theta\left(\frac{u+v}{2}\right)\right)\int \big(h(x)u + k(x)v\big)$$

and, similarly to (2.2),

$$\tilde{J}(\alpha) := \tilde{I}(\alpha + \tilde{\psi}_\alpha, \alpha - \tilde{\psi}_\alpha) := \max_{\psi \in H_0^1(\Omega)} \tilde{I}(\alpha + \psi, \alpha - \psi). \tag{3.5}$$

Lemma 5. *There exists $C > 0$ such that, for any $\alpha \in H_0^1(\Omega)$,*

(i) $\|\psi_\alpha - \tilde{\psi}_\alpha\| \leqslant C$;
(ii) $\|\tilde{\psi}_\alpha + \tilde{\psi}_{-\alpha}\| \leqslant C$.

Proof. By definition, $\tilde{\psi}_\alpha$ solves

$$-2\Delta\tilde{\psi}_\alpha = g(\alpha - \tilde{\psi}_\alpha) - f(\alpha + \tilde{\psi}_\alpha) + \theta(\alpha)\big(k(x) - h(x)\big). \tag{3.6}$$

By subtracting this equation from that in (2.4) and multiplying by $\psi_\alpha - \tilde{\psi}_\alpha$, we get

$$2\|\psi_\alpha - \tilde{\psi}_\alpha\|^2 = \int \big(g(v_\alpha) - g(\alpha - \tilde{\psi}_\alpha)\big)(\psi_\alpha - \tilde{\psi}_\alpha) + \int \big(f(\alpha + \tilde{\psi}_\alpha) - f(\alpha + \psi_\alpha)\big)(\psi_\alpha - \tilde{\psi}_\alpha)$$
$$+ \big(1 - \theta(\alpha)\big) \int \big(k(x) - h(x)\big)(\psi_\alpha - \tilde{\psi}_\alpha).$$

Writing

$$\int \big(g(v_\alpha) - g(\alpha - \tilde{\psi}_\alpha)\big)(\psi_\alpha - \tilde{\psi}_\alpha) = \int \int_{\psi_\alpha(x)}^{\tilde{\psi}_\alpha(x)} g'(\alpha - s)\, ds (\psi_\alpha - \tilde{\psi}_\alpha),$$

similarly for the second term and using the fact that $f' \geqslant 0$, $g' \geqslant 0$, we deduce the estimate

$$2\|\psi_\alpha - \tilde{\psi}_\alpha\|^2 \leqslant \big(1 - \theta(\alpha)\big) \int \big(k(x) - h(x)\big)(\psi_\alpha - \tilde{\psi}_\alpha),$$

so that (i) follows. The second statement can be proved in the same way, by comparing (3.6) with the identity

$$-2\Delta\tilde{\psi}_{-\alpha} = g(-\alpha - \tilde{\psi}_{-\alpha}) - f(-\alpha + \tilde{\psi}_{-\alpha}) + \theta(-\alpha)\big(k(x) - h(x)\big)$$
$$= -g(\alpha + \tilde{\psi}_{-\alpha}) + f(\alpha - \tilde{\psi}_{-\alpha}) + \theta(-\alpha)\big(k(x) - h(x)\big). \qquad \square$$

Large critical values of $\tilde{J}$ are in fact critical values of J. To establish this property, we need a further preliminary estimate. For any $\alpha \in H_0^1(\Omega)$, we define

$$\phi_\alpha := D\psi_\alpha \alpha = \frac{d}{dt}\bigg|_{t=0} \Psi(\alpha + t\alpha). \tag{3.7}$$

By differentiating either (2.3) or (2.4), we see that ϕ_α is the unique solution of the following equation in $H_0^1(\Omega)$:

$$-2\Delta\phi_\alpha = g'(v_\alpha)(\alpha - \phi_\alpha) - f'(u_\alpha)(\alpha + \phi_\alpha). \tag{3.8}$$

Lemma 6. *There exists $C > 0$ such that for every $\alpha \in H_0^1(\Omega)$,*

$$\int \big|f(u_\alpha)(\alpha + \phi_\alpha)\big| + \big|g(v_\alpha)(\alpha - \phi_\alpha)\big| \leqslant C\Big(\int \big(F(u_\alpha) + G(v_\alpha)\big) + 1\Big),$$

where u_α and v_α are defined by (3.1).

Proof. Subtracting the equation in (3.8) from that in (2.4) and taking $\phi_\alpha - \psi_\alpha$ as test function yields

$$2\|\phi_\alpha - \psi_\alpha\|^2 + \int \big(f'(u_\alpha) + g'(v_\alpha)\big)(\phi_\alpha - \psi_\alpha)^2$$
$$= \int \big(g'(v_\alpha)v_\alpha - g(v_\alpha) + f(u_\alpha) - f'(u_\alpha)u_\alpha\big)(\phi_\alpha - \psi_\alpha) + \int \big(h(x) - k(x)\big)(\phi_\alpha - \psi_\alpha). \tag{3.9}$$

The last term on the right-hand side can be estimated using Schwarz inequality. In order to deal with the first terms, observe that for any $\delta > 0$, we have

$$\int \big(g'(v_\alpha)v_\alpha - g(v_\alpha)\big)(\phi_\alpha - \psi_\alpha) \leqslant C\delta \int \big|g'(v_\alpha)\big||\phi_\alpha - \psi_\alpha|^2 + \frac{C}{\delta} \int \big|g'(v_\alpha)\big||v_\alpha|^2$$
$$\leqslant C\delta \int \big|g'(v_\alpha)\big||\phi_\alpha - \psi_\alpha|^2 + \frac{C}{\delta} \int G(v_\alpha),$$

where $C > 0$ only depends on q. Handling the term $(f(u_\alpha) - f'(u_\alpha)u_\alpha)(\phi_\alpha - \psi_\alpha)$ in the same way and taking $\delta < 1$, we now deduce the estimate

$$\int \big(f'(u_\alpha) + g'(v_\alpha)\big)(\phi_\alpha - \psi_\alpha)^2 \leqslant C\Big(\int \big(F(u_\alpha) + G(v_\alpha)\big) + 1\Big),$$

where $C > 0$ depends on p, q and δ.

By writing $f(u_\alpha)(\alpha+\phi_\alpha) = f(u_\alpha)u_\alpha + \frac{f(u_\alpha)}{u_\alpha}u_\alpha(\phi_\alpha - \psi_\alpha)$, we now infer that

$$\int \left|f(u_\alpha)(\alpha+\phi_\alpha)\right| \leqslant \int \left(CF(u_\alpha) + \left|\frac{f(u_\alpha)}{u_\alpha}\right| \left(u_\alpha^2 + (\phi_\alpha - \psi_\alpha)^2\right)\right) \leqslant C\int \left(F(u_\alpha) + f'(u_\alpha)(\phi_\alpha - \psi_\alpha)^2\right).$$

Arguing similarly to treat the term $g(v_\alpha)(\alpha - \phi_\alpha)$, the conclusion easily follows. □

Lemma 7. *If* $\theta(\alpha) \neq 0$ *then*

$$J(\alpha) - \tilde J(\alpha) = \mathrm{o}(1)J(\alpha) \tag{3.10}$$

and

$$J'(\alpha)\alpha - \tilde J'(\alpha)\alpha = \mathrm{o}(1)J(\alpha) + \mathrm{o}(1)J'(\alpha)\alpha \tag{3.11}$$

where $\mathrm{o}(1) \to 0$ *as* $J(\alpha) \to \infty$. *In particular, if* $\alpha \in H_0^1(\Omega)$ *is such that* $\tilde J'(\alpha) = 0$ *and* $\tilde J(\alpha)$ *is sufficiently large then* $\theta(\alpha) = 1$ *and* $J'(\alpha) = 0$.

Proof. By assumption,

$$A\int \left(F(u_\alpha) + G(v_\alpha)\right) \leqslant 2\sqrt{J^2(\alpha)+1}, \tag{3.12}$$

where as before, $u_\alpha = \alpha + \psi_\alpha$ and $v_\alpha = \alpha - \psi_\alpha$. Now, by (3.3), Hölder inequality and Sobolev embeddings, we see that

$$\|\psi_\alpha\| \leqslant C + C\left(\int |u_\alpha|^p\right)^{\frac{p-1}{p}} + C\left(\int |v_\alpha|^q\right)^{\frac{q-1}{q}}. \tag{3.13}$$

Thanks to Lemma 5, a similar estimate holds for $\int(F(\tilde u_\alpha) - F(u_\alpha))$ and for $\int(G(\tilde v_\alpha) - G(v_\alpha))$, where $\tilde u_\alpha := \alpha + \tilde\psi_\alpha$ and $\tilde v_\alpha := \alpha - \tilde\psi_\alpha$. Indeed, using the inequality

$$F(x) - F(y) \leqslant |x-y|\left(\left|f(x)\right| + \left|f(y)\right|\right),$$

we infer that

$$\left|\int \left(F(\tilde u_\alpha) - F(u_\alpha)\right)\right| \leqslant C\|\psi_\alpha - \tilde\psi_\alpha\|\left[\left(\int |u_\alpha|^p\right)^{\frac{p-1}{p}} + \left(\int |\tilde u_\alpha|^p\right)^{\frac{p-1}{p}}\right].$$

Arguing in the same way to estimate $\int(G(\tilde v_\alpha) - G(v_\alpha))$ and taking the first statement of Lemma 5 into account, we finally deduce that

$$\left|\int \left(F(\tilde u_\alpha) - F(u_\alpha)\right) + \int \left(G(\tilde v_\alpha) - G(v_\alpha)\right)\right| \leqslant C + C\left(\int |u_\alpha|^p\right)^{\frac{p-1}{p}} + C\left(\int |v_\alpha|^q\right)^{\frac{q-1}{q}}.$$

From this last estimate and (3.12), we readily conclude that

$$J(\alpha) - \tilde J(\alpha) = \mathrm{o}(1)J(\alpha), \quad \text{as } J(\alpha) \to \infty.$$

Similar estimates are used to deduce (3.11). Compute

$$J'(\alpha)\alpha - \tilde J'(\alpha)\alpha = \int \left(f(\tilde u_\alpha) - f(u_\alpha)\right)\alpha + \int \left(g(v_\alpha) - g(\tilde v_\alpha)\right)\alpha + \left(\theta(\alpha) - 1\right)\int \left(h(x) + k(x)\right)\alpha + \theta'(\alpha)\alpha \int \left(h(x)\tilde u_\alpha + k(x)\tilde v_\alpha\right).$$

The first terms can be estimated using by now familiar arguments. To deal with the extra term

$$\theta'(\alpha)\alpha \int \left(h(x)\tilde u_\alpha + k(x)\tilde v_\alpha\right),$$

we make use of the estimate derived in Lemma 6. Indeed, this lemma implies that

$$\int \big(f(u_\alpha)(\alpha+\phi_\alpha)+g(v_\alpha)(\alpha-\phi_\alpha)\big)=\mathrm{O}(1)J(\alpha)$$

and since we have

$$\theta'(\alpha)\alpha=\chi'\bigg(\frac{A\int(F(u_\alpha)+G(v_\alpha))}{(J^2(\alpha)+1)^{1/2}}\bigg) \times\bigg[\frac{A\int(f(u_\alpha)(\alpha+\phi_\alpha)+g(v_\alpha)(\alpha-\phi_\alpha))}{(J^2(\alpha)+1)^{1/2}}-\frac{J'(\alpha)\alpha\, A\int(F(u_\alpha)+G(v_\alpha))}{(J^2(\alpha)+1)^{3/2}}\bigg],$$

a straightforward computation leads to (3.11).

At last, suppose that $\tilde J'(\alpha)=0$ and $\tilde J(\alpha)$ is large. Arguing by contradiction, it is easily seen that we must have $\theta(\alpha)\neq 0$. Indeed, it follows from Lemma 5 and a computation similar to (3.2)–(3.4) (with $h=k=0$) that having $\theta(\alpha)=0$ is impossible. Then, we infer that $J(\alpha)$ is large as well and therefore (3.11) shows that $J'(\alpha)\alpha=\mathrm{o}(1)J(\alpha)$, as $J(\alpha)\to+\infty$. Hence, we deduce that

$$\big(1+\mathrm{o}(1)\big)J(\alpha)=J(\alpha)-\frac{1}{2}J'(\alpha)\alpha,\quad \text{as } J(\alpha)\to+\infty.$$

Combining this with the first inequality in (3.4) yields $\theta(\alpha)=1$ (in fact, θ takes the value 1 near α). Clearly, in this case, we have $\tilde\psi_\alpha=\psi_\alpha$ and $\tilde J'(\alpha)=J'(\alpha)$, as claimed. □

It is now an easy task to prove that if $(\alpha_n)_n\subset H_0^1(\Omega)$ is a Palais–Smale sequence for $\tilde J$ at a sufficiently large level then $\theta(\alpha_n)=1$ and therefore $(\alpha_n)_n$ is a Palais–Smale sequence for J as well. In particular (cf. Lemma 3), $\tilde J$ satisfies the Palais–Smale condition at large energy levels. Of course, the property displayed in (2.6) also holds for $\tilde J$.

Next we analyze the deviation from symmetry enjoyed by $\tilde J$. As in [16,17], this estimate is crucial in the proof of our main result.

Lemma 8. *There exists $C>0$ such that*

$$\big|\tilde J(\alpha)-\tilde J(-\alpha)\big|\leqslant C\big(\big|\tilde J(\alpha)\big|^{1/p}+1\big),\quad \forall\alpha\in H_0^1(\Omega). \tag{3.14}$$

Proof. The estimate in (3.13) is not accurate enough for our purposes. Instead, we start with the observation that, according to the definition in (3.5),

$$\tilde J(\alpha)\geqslant\tilde I(\alpha-\tilde\psi_{-\alpha},\alpha+\tilde\psi_{-\alpha})=\tilde I(-\tilde u_{-\alpha},-\tilde v_{-\alpha})$$

and

$$\tilde J(-\alpha)\geqslant\tilde I(-\alpha-\tilde\psi_\alpha,-\alpha+\tilde\psi_\alpha)=\tilde I(-\tilde u_\alpha,-\tilde v_\alpha),$$

where as usual, we use the notations $\tilde u_\alpha=\alpha+\tilde\psi_\alpha$, $\tilde v_\alpha=\alpha-\tilde\psi_\alpha$, $\tilde u_{-\alpha}=-\alpha+\tilde\psi_{-\alpha}$ and $\tilde v_{-\alpha}=-\alpha-\tilde\psi_{-\alpha}$. We then compute

$$\big|\tilde J(\alpha)-\tilde J(-\alpha)\big|\leqslant\big(\theta(\alpha)+\theta(-\alpha)\big)\int\big(\big|h(x)\tilde u_\alpha\big|+\big|h(x)\tilde u_{-\alpha}\big|+\big|k(x)\tilde v_\alpha\big|+\big|k(x)\tilde v_{-\alpha}\big|\big).$$

Using Lemma 5, this leads to the estimate

$$\big|\tilde J(\alpha)-\tilde J(-\alpha)\big|\leqslant C\Big(1+\int\big(\big|h(x)u_\alpha\big|+\big|k(x)v_\alpha\big|\big)\Big).$$

The conclusion now follows from Hölder inequality and the estimate (3.10). □

3.2. Morse index

We now focus on the Morse index of the solutions of the unperturbed system (1.1) with $h(x) = k(x) = 0$. Let $I^* : E \to \mathbb{R}$ be the functional associated to the unperturbed problem

$$\begin{cases} -\Delta u = |v|^{q-2}v & \text{in } \Omega, \\ -\Delta v = |u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.15}$$

Consider the associated reduced functional

$$J^*(\alpha) := I^*\big(\alpha + \psi_\alpha^*, \alpha - \psi_\alpha^*\big) := \max_{\psi \in H_0^1(\Omega)} I^*(\alpha + \psi, \alpha - \psi). \tag{3.16}$$

Recall that if α is a critical point of J^* then

$$-2\Delta\alpha = f\big(u_\alpha^*\big) + g\big(v_\alpha^*\big), \tag{3.17}$$

where $u_\alpha^* := \alpha + \psi_\alpha^*$, $v_\alpha^* := \alpha - \psi_\alpha^*$ and ψ_α^* is the unique solution of the following equation in $H_0^1(\Omega)$:

$$-2\Delta\psi_\alpha^* = g\big(v_\alpha^*\big) - f\big(u_\alpha^*\big). \tag{3.18}$$

Denote by $m^*(\alpha)$ the *augmented Morse index* of the critical point α with respect to J^*, i.e. the number of nonpositive eigenvalues of the quadratic form $(J^*)''(\alpha)$. We next derive a bound on $m^*(\alpha)$.

Proposition 9. *There exists $C > 0$ such that for every critical point $\alpha \in H_0^1(\Omega)$ of J^*,*

$$m^*(\alpha) \leqslant C J^*(\alpha)^{(1-\frac{1}{p}-\frac{1}{q})\frac{N}{2}}.$$

Proof. According to (3.17) and (3.18), $m^*(\alpha)$ is the number of eigenvalues $\mu \leqslant 1$ of the problem

$$-2\Delta\varphi = \mu\big(f'\big(u_\alpha^*\big)(\varphi + \phi) + g'\big(v_\alpha^*\big)(\varphi - \phi)\big), \quad \varphi \in H_0^1(\Omega), \tag{3.19}$$

where $\phi \in H_0^1(\Omega)$ solves

$$-2\Delta\phi = g'\big(v_\alpha^*\big)(\varphi - \phi) - f'\big(u_\alpha^*\big)(\varphi + \phi). \tag{3.20}$$

By denoting $V = (f'(u_\alpha^*) + g'(v_\alpha^*))/2$ and $W = (f'(u_\alpha^*) - g'(v_\alpha^*))/2$, we can rephrase (3.19)–(3.20) by

$$-\Delta\varphi = \mu(V\varphi + W\phi) \quad \text{and} \quad (-\Delta + V)\phi = W\varphi.$$

Hence, $m^*(\alpha)$ is the number of eigenvalues $\mu \leqslant 1$ of the problem

$$-\Delta\varphi = \mu T\varphi, \quad \varphi \in H_0^1(\Omega),$$

where T is the compact operator

$$T := V - W(-\Delta + V)^{-1}W.$$

Now, let

$$m(x) := \min\big\{f'\big(u_\alpha^*(x)\big), g'\big(v_\alpha^*(x)\big)\big\}.$$

Observe that, since $|W| \leqslant V - m \leqslant V$, we have

$$\begin{aligned} \langle T\varphi, \varphi\rangle - \int m\varphi^2 &= \int V\varphi^2 - \int W\varphi\phi - \int m\varphi^2 \\ &\geqslant \int |W|\varphi^2 - \int W\varphi\phi \\ &\geqslant \int |W|\varphi^2 - \frac{1}{2}\int |W|\varphi^2 - \frac{1}{2}\int |W|\phi^2 \\ &= \frac{1}{2}\int |W|\big(\varphi^2 - \phi^2\big). \end{aligned}$$

Multiplying the identity $-\Delta\phi + V\phi = W\varphi$ by ϕ and integrating, we get that

$$\int |W|\phi^2 \leqslant \int V\phi^2 \leqslant \int |W|\varphi^2. \tag{3.21}$$

Hence, we deduce that

$$\langle T\varphi, \varphi\rangle \geqslant \langle S\varphi, \varphi\rangle := \int m\varphi^2, \quad \forall \varphi \in H_0^1(\Omega). \tag{3.22}$$

It follows from (3.22) that $m^*(\alpha) \leqslant m_S^*(\alpha)$, where the latter quantity denotes the number of eigenvalues $\mu \leqslant 1$ of the problem

$$-\Delta\varphi = \mu m(x)\varphi, \quad \varphi \in H_0^1(\Omega).$$

According to a well-known estimate obtained in [9,15,20] (see e.g. [22] for a proof), we have that

$$m_S^*(\alpha) \leqslant C \int \left|m(x)\right|^{N/2}$$

for some universal constant $C > 0$. Now, since $|m(x)| \leqslant f'(u_\alpha^*(x))^{1/2} g'(v_\alpha^*(x))^{1/2}$ and since, by assumption, $\frac{p-2}{p}\frac{N}{4} + \frac{q-2}{q}\frac{N}{4} = (1 - \frac{1}{p} - \frac{1}{q})\frac{N}{2} < 1$, we conclude, using Hölder inequality, that

$$m_S^*(\alpha) \leqslant C\left(\int f'\left(u_\alpha^*\right)^{p/(p-2)}\right)^{N(p-2)/4p}\left(\int g'\left(v_\alpha^*\right)^{q/(q-2)}\right)^{N(q-2)/4q},$$

that is

$$m_S^*(\alpha) \leqslant C\left(\int \left|u_\alpha^*\right|^p\right)^{N(p-2)/4p}\left(\int \left|v_\alpha^*\right|^q\right)^{N(q-2)/4q}.$$

Going back to the original system $-\Delta u = |v|^{q-2}v$, $-\Delta v = |u|^{p-2}u$, we observe that $\int |u_\alpha^*|^p = \int |v_\alpha^*|^q$ and

$$\begin{aligned} J^*(\alpha) = I^*\left(u_\alpha^*, v_\alpha^*\right) &= \left(\frac{1}{2} - \frac{1}{p}\right)\int \left|u_\alpha^*\right|^p + \left(\frac{1}{2} - \frac{1}{q}\right)\int \left|v_\alpha^*\right|^q \\ &= \left(1 - \frac{1}{p} - \frac{1}{q}\right)\int \left|u_\alpha^*\right|^p \\ &= \left(1 - \frac{1}{p} - \frac{1}{q}\right)\int \left|v_\alpha^*\right|^q, \end{aligned}$$

so that the conclusion follows. □

4. Proof of Theorem 1

We have now all the ingredients we need to complete the proof of our main result. Let us write

$$H_0^1(\Omega) = E_k \oplus E_k^\perp,$$

where, for each $k\mathbb{N}_0$, E_k is spanned by the first k eigenfunctions of the Laplacian operator in $H_0^1(\Omega)$. Arguing as in Lemma 3, we can provide a large constant $R_k > 0$ such that $\tilde{J}(\alpha) < 0$ for every $\alpha \in E_k$ satisfying $\|\alpha\| > R_k$. Let

$$G_k := \left\{\gamma \in C\left(B_{R_k}(0) \cap E_k; H_0^1(\Omega)\right) \mid \gamma(-u) = -\gamma(u),\ \gamma_{|\partial B_{R_k}(0)\cap E_k} = \mathrm{Id}\right\},$$

and define the minimax levels

$$\tilde{b}_k := \inf_{\gamma\in G_k} \max\left\{\tilde{J}\left(\gamma(\alpha)\right):\ \alpha \in B_{R_k}(0) \cap E_k\right\}. \tag{4.1}$$

Following Rabinowitz's idea [17], we will exploit these levels to deduce the statement of Theorem 1 by an indirect argument.

Proof of Theorem 1. Assume by contradiction that $\tilde{J}$ does not admit an unbounded sequence of critical values. Let $(\tilde{b}_k)_k$ be the sequence of minimax levels of $\tilde{J}$ defined by (4.1).

Claim 1. *There exist $C, k_0 > 0$ such that for all $k \geqslant k_0$,*

$$\tilde{b}_k \leqslant Ck^{p/(p-1)}. \tag{4.2}$$

Thanks to the estimate (3.14) of Lemma 8, the claim follows exactly as in [17, Proposition 10.46].

Now, similarly to [4,26], we use the information on the Morse index to obtain a lower bound on the growth of the sequence $\tilde{b}_k$.

Claim 2. *There exist $C' > 0$ and $k_0' > 0$ such that for all $k \geqslant k_0'$,*

$$\tilde{b}_k \geqslant C' k^{2pq/N(pq-p-q)}. \tag{4.3}$$

Let us fix a small $c > 0$ in such a way that the functional

$$\hat{I}(u, v) = \int \big(\langle \nabla u, \nabla v\rangle - cF(u) - cG(v)\big)$$

is such that $\tilde{I} - \hat{I}$ is bounded from below in $H_0^1(\Omega) \times H_0^1(\Omega)$. We also consider the associated reduced functional $\hat{J}$ defined by

$$\hat{J}(\alpha) := \hat{I}(\alpha + \hat{\psi}_\alpha, \alpha - \hat{\psi}_\alpha) := \max_{\psi \in H_0^1(\Omega)} \hat{I}(\alpha + \psi, \alpha - \psi)$$

and the corresponding minimax numbers

$$\hat{b}_k := \inf_{\gamma \in G_k} \max\big\{\hat{J}\big(\gamma(\alpha)\big)\colon \alpha \in B_{R_k}(0) \cap E_k\big\},$$

where taking R_k larger if necessary, we can assume that $\hat{J}(\alpha) < 0$ for every $\alpha \in E_k$ satisfying $\|\alpha\| > R_k$. Clearly, the sequence $\tilde{b}_k - \hat{b}_k$ is bounded from below.

According to [4] and [26, Theorem B], applied to $\hat{J}$, there exists a sequence $(\hat{\alpha}_k)_k$ of critical points of $\hat{J}$ such that

$$\hat{J}(\hat{\alpha}_k) \leqslant \hat{b}_k \quad \text{and} \quad \hat{m}(\hat{\alpha}_k) \geqslant k,$$

where $\hat{m}(\hat{\alpha}_k)$ denotes the augmented Morse index of the critical point $\hat{\alpha}_k$ with respect to $\hat{J}$. Then, by Proposition 9, we infer that

$$k^{2pq/N(pq-p-q)} \leqslant \hat{m}(\hat{\alpha}_k)^{2pq/N(pq-p-q)} \leqslant C\hat{J}(\hat{\alpha}_k) \leqslant C\hat{b}_k,$$

so that the claim follows.

Conclusion. In view of our assumption (1.7), by comparing (4.2)–(4.3), we reach a contradiction. Therefore, $\tilde{J}$ does have an unbounded sequence of critical values which, by Lemma 7, means that J does as well. Finally, the conclusion follows from Proposition 2.

This completes the proof of Theorem 1. □

5. Genericity

In this section, we focus on the genericity of our multiplicity result. The next theorem is the counterpart of Bahri's result [2] for a single equation. We first state it in a simple form and subsequently extend it in Theorem 11 below.

Theorem 10. *Let $2 < p \leqslant q < 2N/(N-2)$ and $n \in \mathbb{N}$. Then there exists an open dense subset $H_n \subset H^{-1}(\Omega) \times H^{-1}(\Omega)$ such that the system* (1.1) *admits at least n solutions for every $(h, k) \in H_n$. In particular, there exists a residual set $H = \bigcap_{n\in\mathbb{N}} H_n$ such that* (1.1) *has infinitely many solutions for every $(h, k) \in H$.*

As the proof essentially follows the lines of Bahri's paper, we only stress the special care our settings require.

Let $S = \{u \in H_0^1(\Omega) \mid \|u\| = 1\}$. One first observes that the functional $Q : S \to \mathbb{R}$ defined by

$$Q(\alpha) := \sup_{\lambda \in [0,+\infty[} J(\lambda\alpha)$$

is such that there exists $\lambda_0 > 0$ such that if $\lambda \geqslant \lambda_0$ and

$$\frac{d}{d\lambda} Q(\lambda\alpha) = 0,$$

then

$$\frac{d^2}{d\lambda^2} Q(\lambda\alpha) < 0.$$

This claim can be proved along the lines of [18, Lemma 2.2]. As a consequence, we can fix a large positive constant A such that the functional $Q - A$ belongs to the class (C), see [2, Definition 1].

Next, for α such that $Q(\alpha) > A$, define $\lambda(\alpha)$ as the unique positive solution of $Q(\alpha) = J(\lambda\alpha)$. The second inequality in statement (i) of Lemma 2 in [2], namely the fact that, for some $C > 0$, $\lambda(\alpha) \leqslant C(Q(\alpha)^{1/2} + 1)$ $\forall \alpha \in S$, assumes in our setting the weaker form $\lambda(\alpha) \leqslant C(Q(\alpha)^r + 1)$ $\forall \alpha \in S$, $r = \max\{(p-1)/p, (q-1)/q\}$; in particular, it follows that if $Q(\alpha)$ is bounded then $\lambda(\alpha)$ is bounded as well, and this is all that matters in the proof given in [2].

The remaining of Bahri's arguments can be deduced with obvious differences so that the proof of Theorem 10 can be completed arguing as in [2].

Next we deal with the more general situation where $p, q > 2$, $1/p + 1/q > (N-2)/N$. We use the functional framework introduced in [11]. Let us fix positive numbers s, t such that $s + t = 2$, $p < 2N/(N-2s)$ and $q < 2N/(N-2t)$. We denote by $(\mu_j)_j$ the nondecreasing sequence of eigenvalues of $(-\Delta, H_0^1(\Omega))$ and by $(\phi_j)_j$ the corresponding sequence of eigenvalues normalized in $L^2(\Omega)$. The Hilbert space $E^s(\Omega)$ is defined as

$$E^s(\Omega) := \left\{ u = \sum_{j=1}^{\infty} u_j \phi_j \in L^2(\Omega),\ \|u\|_{E^s(\Omega)}^2 := \sum_{j=1}^{\infty} \mu_j^s u_j^2 < \infty \right\},$$

and we denote by $A^s : E^s(\Omega) \to L^2(\Omega)$ the isometric isomorphism

$$A^s u := \sum_{j=1}^{\infty} \mu_j^{s/2} u_j \phi_j.$$

We point out that $E^1(\Omega) = H_0^1(\Omega)$ and $A^1 = (-\Delta)^{1/2}$. It can be proved that there is a compact inclusion $E^s(\Omega) \subset L^p(\Omega)$ and that solutions (u, v) of the system (1.1) with $h \in L^{p/(p-1)}(\Omega)$, $k \in L^{q/(q-1)}(\Omega)$ can be seen as critical points of the functional defined on $E^s(\Omega) \times E^t(\Omega)$ by

$$\int_\Omega A^s u A^t v - \int_\Omega F(u) - \int_\Omega G(v) - \int_\Omega hu - \int_\Omega kv.$$

In fact, as shown in [11, Theorem 1.2], any critical point $(u, v) \in E^s(\Omega) \times E^t(\Omega)$ turns out to satisfy $u \in W^{2,q/(q-1)}(\Omega) \cap W_0^{1,q/(q-1)}(\Omega)$, $v \in W^{2,p/(p-1)}(\Omega) \cap W_0^{1,p/(p-1)}(\Omega)$ and is a strong solution of (1.1).

For our purposes it is convenient to use a slightly different formulation. Let us denote $B_s := (A^s)^{-1} \circ A^1$, so that $B_s : H_0^1(\Omega) \to E^s(\Omega)$ is an isometric isomorphism with corresponding dual operator $B_s^* : (E^s(\Omega))' \to H^{-1}(\Omega)$. We then seek for critical points of the functional $I_{s,t} : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ defined by

$$I_{s,t}(u, v) = \int_\Omega \langle \nabla u, \nabla v \rangle - \int_\Omega F(B_s u) - \int_\Omega G(B_t v) - \int_\Omega h B_s u - \int_\Omega k B_t v.$$

Such a critical point will satisfy

$$-\Delta u = B_t^*\big(g(B_t v) + k\big) \quad \text{in } H^{-1}(\Omega), \qquad -\Delta v = B_s^*\big(f(B_s u) + h\big) \quad \text{in } H^{-1}(\Omega),$$

and so the pair $(\tilde{u}, \tilde{v}) = (B_s u, B_t v) \in E^s(\Omega) \times E^t(\Omega)$ will be a strong solution of the original system (1.1).

Next we can define the associated reduced functional $J_{s,t} : H_0^1(\Omega) \to \mathbb{R}$ as in (2.2). Starting from this point, there are no substantial changes with respect to our previous considerations. In particular, we can deduce the following theorem, which can be seen as an extension of Theorem 10.

Theorem 11. *Let $p, q > 2$, $1/p + 1/q > (N-2)/N$ and $n \in \mathbb{N}$. Then there exists an open dense subset $H_n \subset H^{-1}(\Omega) \times H^{-1}(\Omega)$ such that, for every $(h,k) \in H_n$, the system*

$$-\Delta u = B_t^*\big(g(B_t v) + k\big), \qquad -\Delta v = B_s^*\big(f(B_s u) + h\big) \tag{5.1}$$

admits at least n solutions in $H^{-1}(\Omega) \times H^{-1}(\Omega)$. In particular, there exists a residual set $H = \bigcap_{n\in\mathbb{N}} H_n$ such that (5.1) *has infinitely many solutions for every $(h,k) \in H$.*

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