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Second-order analysis for optimal control problems with pure state constraints and mixed control-state constraints

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Abstract

This paper deals with the optimal control problem of an ordinary differential equation with several pure state constraints, of arbitrary orders, as well as mixed control-state constraints. We assume (i) the control to be continuous and the strengthened Legendre–Clebsch condition to hold, and (ii) a linear independence condition of the active constraints at their respective order to hold. We give a complete analysis of the smoothness and junction conditions of the control and of the constraints multipliers. This allows us to obtain, when there are finitely many nontangential junction points, a theory of no-gap second-order optimality conditions and a characterization of the well-posedness of the shooting algorithm. These results generalize those obtained in the case of a scalar-valued state constraint and a scalar-valued control.

Résumé

Dans cet article on s'intéresse au problème de commande optimale d'une équation différentielle ordinaire avec plusieures contraintes pures sur l'état, d'ordres quelconques, et des contraintes mixtes sur la commande et sur l'état. On suppose que (i) la commande est continue et la condition forte de Legendre–Clebsch satisfaite, et (ii) une condition d'indépendance linéaire des contraintes actives est satisfaite. Des résultats de régularité des solutions et multiplicateurs et des conditions de jonction sont donnés. Lorsqu'il y a un nombre fini de points de jonction, on obtient des conditions d'optimalité du second-ordre nécessaires ou suffisantes, ainsi qu'une caractérisation du caractère bien posé de l'algorithme de tir. Ces résultats généralisent les résultats obtenus dans le cas d'une contrainte sur l'état et d'une commande scalaires.

MSC: 49K15; 49K40; 34K35

Keywords: Optimal control; State constraint; Higher order; Mixed control-state constraint; Junction conditions; Necessary or sufficient second-order optimality conditions; Shooting algorithm

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1. Introduction

This paper deals with optimal control problems with a vector-valued state constraint. Mixed control-state constraints (state constraints of order zero) are included in the analysis. It is assumed that the control is continuous and the strengthened Legendre–Clebsch condition holds, and that each component of the state constraint is of arbitrary (but finite) order q_i.

Second-order optimality conditions for state-constrained optimal control problems were recently studied in [22,34, 35,2]. The presence of pure state constraints introduces an additional curvature term in the second-order necessary condition, in contrast with mixed control-state constraints, see [32,30]. An analysis of the junction conditions may help to narrow the gap with the second-order sufficient condition. There are, to our knowledge, relatively few papers dealing with optimal control problems with several state constraints *of order greater than one*. One of them is an unpublished paper by Maurer [28]. In e.g. [15,24,11,12,25,27], several constraints of *first-order* were considered, but when dealing with constraints of higher order, then often only one constraint (and sometimes also a scalar control) is considered, see e.g. [18,16,26]. When there are several constraints of different orders, and more control variables than active constraints, then even the regularity of the control and of the state constraint multipliers on the interior of the arcs of the trajectory is not an obvious question. In [28, Lemma 4.1], it is shown that the control u is $C^{q_{\max}}$ (where $q_{\max}$ is the bigger order of the active constraints), under the assumption that there are as many active state constraints as control variables. In [28, Theorem 4.2], it is shown that the state constraints multipliers are smooth on the interior of arcs, but with the extra assumption that the control u is $C^{q_{\max}}$.

The motivation of this paper is to extend the no-gap second-order optimality conditions and the characterization of the well-posedness of the shooting algorithm, obtained in [3,1] and [4], respectively, for an optimal control problem with a scalar-valued state constraint and control, to the case of a vector-valued state constraint and control. The critical step is the extension of the junctions conditions obtained in the scalar case (i.e., with a scalar-valued state constraint and control) by Jacobson, Lele and Speyer [18]. This result says that some of the time derivatives of the control are continuous at a junction point until an order that depend on the *order* of the (scalar) state constraint, and on the nature of the junction point (entry/exit of boundary arcs versus touch points). This result has an important role when deriving the second-order necessary condition, since, with this regularity result and under suitable assumptions, it can be shown that boundary arcs have typically no contribution to the curvature term. This enables to derive a second-order sufficient condition as close as possible to the necessary one (no-gap), and to obtain a characterization of the well-posedness of the shooting algorithm. We show in particular that the shooting algorithm is ill-posed if a component of the state constraint of order $q_i \geqslant 3$ has a boundary arc.

In this paper, the focus is on the proofs that are not directly obtained from the scalar case, and in particular the (nontrivial) extension of the junction condition result of [18]. Our main assumption is the simplest one that the gradients w.r.t. the control variable of the time derivatives of the active constraints at their respective order are linearly independent. This enables to write locally the system under a "normal form", where the dynamics corresponding to the state constraints is linearized, and the different components of the constraints are decoupled.

The paper is organized as follows. In Section 2, we present the problem, notation, basic definitions and assumptions. In Section 3, we give sufficient conditions implying the continuity of the control, and we show local higher regularity of the control and constraints multipliers on the interior of arcs. In Section 4, we give some technical lemmas needed to put the system under a "normal form". This will be used in Section 5, where we give the junction conditions results. In Section 6, the no-gap second-order optimality conditions is stated. In Section 7, we recall the shooting formulation and state a characterization of the well-posedness of the shooting algorithm, under the additional assumption that the junction times of the different components of the state constraint do not coincide.

2. Framework

Let n, m, r, s be positive integers. If r and/or s is equal to zero, then the statements of this paper remain correct if the corresponding terms are removed. Denote by $\mathcal{U} := L^\infty(0,T;\mathbb{R}^m)$ (resp. $\mathcal{Y} := W^{1,\infty}(0,T;\mathbb{R}^n)$) the control (resp. state) space. We consider the following optimal control problem:

$$(\mathcal{P}) \qquad \min_{(u,y)\in\mathcal{U}\times\mathcal{Y}} \int_0^T \ell\big(u(t), y(t)\big)\,\mathrm{d}t + \phi\big(y(T)\big) \tag{2.1}$$

$$\text{subject to}\quad \dot y(t) = f\big(u(t), y(t)\big) \quad \text{for a.a. } t \in [0,T]; \quad y(0) = y_0, \tag{2.2}$$

$$g_i\big(y(t)\big) \leqslant 0 \quad \text{for all } t \in [0,T],\ i = 1,\dots,r, \tag{2.3}$$

$$c_i\big(u(t), y(t)\big) \leqslant 0 \quad \text{for a.a. } t \in [0,T],\ i = r+1,\dots,r+s. \tag{2.4}$$

The data of the problem are the distributed cost $\ell:\mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$, final cost $\phi:\mathbb{R}^n \to \mathbb{R}$, dynamics $f:\mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$, pure state constraint $g:\mathbb{R}^n \to \mathbb{R}^r$, mixed control-state constraint $c:\mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^s$, (fixed) final time $T > 0$, and (fixed) initial condition $y_0 \in \mathbb{R}^n$. We make the following assumptions on the data:

(A0) The mappings ℓ, ϕ, f, g and c are (at least) of class C^2 with locally Lipschitz continuous second-order derivatives, and the dynamics f is Lipschitz continuous.
(A1) The initial condition satisfies $g_i(y_0) < 0$ for all $i = 1,\dots,r$.

Throughout the paper it is assumed that assumption (A0) holds.

Notations. The space of row vectors is denoted by $\mathbb{R}^{n*}$. We denote by $A^\top$ the adjoint operator of a linear operator A or the transpose operator in $\mathbb{R}^{n\times m}$. Given a measurable set $\mathcal{I} \subset (0,T)$, we denote by $L^s(\mathcal{I})$ the Lebesgue space of measurable functions such that $\|u\|_s := (\int_{\mathcal{I}} |u(t)|^s\,\mathrm{d}t)^{1/s}$ (resp. $\|u\|_\infty := \sup\mathrm{ess}_{t\in\mathcal{I}} |u(t)|$) for $1 \leqslant s < +\infty$ (resp. $s = +\infty$) is finite. Given an open set $\mathcal{I} \subset (0,T)$, $k \in \mathbb{N}^*$ and $1 \leqslant s \leqslant +\infty$, the space $W^{k,s}(\mathcal{I})$ denotes the Sobolev space of functions having their weak derivatives until order k in $L^s(\mathcal{I})$. The standard norm of $W^{k,s}$ is denoted by $\|\cdot\|_{k,s}$. We say that a function is nonpositive, if it takes values in $\mathbb{R}_-$.

The Banach space of vector-valued continuous functions is denoted by $C([0,T];\mathbb{R}^r)$ and supplied with the product norm $\|x\|_\infty := \sum_{i=1}^r \|x_i\|_\infty$. The space of vector-valued Radon measures, dual space to $C([0,T];\mathbb{R}^r)$, is denoted by $\mathcal{M}([0,T];\mathbb{R}^{r*})$ and identified with vector-valued functions of bounded variation (BV) vanishing at T. The duality product between $C([0,T];\mathbb{R}^r)$ and $\mathcal{M}([0,T];\mathbb{R}^{r*})$ is denoted by $\langle \eta, x\rangle = \sum_{i=1}^r \int_0^T x_i\,\mathrm{d}\eta_i$. The cones of nonpositive continuous functions and nonnegative Radon measures over $[0,T]$ are denoted respectively by $K := C_-([0,T];\mathbb{R}^r)$ and $\mathcal{M}_+([0,T];\mathbb{R}^{r*})$.

The dual space to $L^\infty(0,T)$, denoted by $(L^\infty)^*(0,T)$, is the space of finitely additive set functions (see [14, p. 258]) letting invariant the sets of zero Lebesgue's measure. The duality product over $(L^\infty)^*$ and L^∞ is denoted by $\langle \lambda, x\rangle$, and when $\lambda \in L^1$, we have $\langle \lambda, x\rangle = \int_0^T \lambda(t)x(t)\,\mathrm{d}t$. The set of vector-valued essentially bounded functions $L^\infty(0,T;\mathbb{R}^s)$ is supplied with the product topology. The set of essentially bounded functions with value in $\mathbb{R}^s_-$ almost everywhere is denoted by $\mathcal{K} := L^\infty_-(0,T;\mathbb{R}^s)$, and the set of elements λ in $(L^\infty)^*(0,T;\mathbb{R}^s)$ such that $\langle \lambda, x\rangle$ is nonpositive for all $x \in L^\infty_-(0,T;\mathbb{R}^s)$ is denoted by $(L^\infty)^*_+(0,T;\mathbb{R}^s)$.

We denote by B_X the unit (open) ball of the Banach space X. By $\mathrm{cl}\,S$, $\mathrm{int}\,S$ and ∂S we denote respectively the closure, interior and boundary of the set S. The cardinal of a finite set J is denoted by $|J|$. The restriction of a function φ defined over $[0,T]$ to a set $A \subset [0,T]$ is denoted by $\varphi|_A$. The indicator function of a set A is denoted by $\mathbf{1}_A$. Given a Banach space X and $A \subset X^*$ the dual space to X, we denote by $A^\perp$ the space of $x \in X$ such that $\langle \xi, x\rangle = 0$ for all $\xi \in A$. If A is a singleton, then $\xi^\perp := \{\xi\}^\perp$. The left and right limits of a function of bounded variation φ over $[0,T]$ are denoted by $\varphi(\tau^\pm) := \lim_{t\to\tau^\pm} \varphi(t)$ and jumps are denoted by $[\varphi(\tau)] := \varphi(\tau^+) - \varphi(\tau^-)$. Fréchet derivatives of f, g_i, etc. w.r.t. arguments $u \in \mathbb{R}^m$, $y \in \mathbb{R}^n$, etc. are denoted by a subscript, for instance $f_u(u,y) = D_u f(u,y)$, $g_{i,y}(y) = D_y g_i(y)$. An exception to this rule is that given $u \in \mathcal{U}$, we denote by y_u the (unique) solution in $\mathcal{Y}$ of the state equation (2.2).

Abstract formulation. We denote by $J:\mathcal{U} \to \mathbb{R}$, $G:\mathcal{U} \to C([0,T];\mathbb{R}^r)$ and $\mathcal{G}:\mathcal{U} \to L^\infty(0,T;\mathbb{R}^s)$ the cost function $J(u) := \int_0^T \ell(u(t), y_u(t))\,\mathrm{d}t + \phi(y_u(T))$ and the constraints mappings defined by $G(u) := g(y_u)$ and $\mathcal{G}(u) := c(u, y_u)$. Recall that the constraints cones are defined by $K = C_-([0,T];\mathbb{R}^r)$ and $\mathcal{K} = L^\infty_-(0,T;\mathbb{R}^s)$. The abstract formulation of $(\mathcal{P})$ (used in Section 6 and in Appendix A) is the following:

$$(\mathcal{P}) \quad \min_{u\in\mathcal{U}} J(u), \quad \text{subject to } G(u) \in K,\ \mathcal{G}(u) \in \mathcal{K}. \tag{2.5}$$

The choice of the functional space for the pure state constraints (here, the space of continuous functions) is discussed later in Remark 2.4.

A *trajectory* (u,y) is an element of $\mathcal{U} \times \mathcal{Y}$ satisfying the state equation (2.2). A *feasible* trajectory is one that satisfies the constraints (2.3) and (2.4). We say that a feasible trajectory $(u,y) = (u,y_u)$ is a *local solution* (weak minimum) of $(\mathcal{P})$, if it minimizes (2.1) over the set of feasible trajectories $(\tilde{u}, \tilde{y})$ satisfying $\|\tilde{u} - u\|_\infty \leqslant \delta$, for some $\delta > 0$.

2.1. Constraint qualification condition

Given a measurable (nonpositive) function x, we denote the *contact set* by

$$\Delta(x) := \{t \in [0, T]:\ x(t) = 0\} \tag{2.6}$$

and, for $n \in \mathbb{N}^*$,

$$\Delta_n(x) := \left\{t \in [0, T]:\ x(t) \geqslant -\frac{1}{n}\right\}. \tag{2.7}$$

Given a feasible trajectory (u, y), define the sets of *active state constraints* and *active mixed constraints* at a.a. time $t \in [0, T]$ respectively by:

$$I^g(t) := \{i \in \{1, \dots, r\}:\ g_i(y(t)) = 0\}, \tag{2.8}$$

$$I^c(t) := \{i \in \{r + 1, \dots, r + s\}:\ t \in \Delta(c_i(u, y))\}, \tag{2.9}$$

and let

$$I(t) := I^g(t) \cup I^c(t). \tag{2.10}$$

An *arc* of the trajectory (u, y) is a maximal *open* interval of *positive measure* $\mathcal{I} = (\tau_1, \tau_2)$, such that $I(t)$ is constant, for all $t \in (\tau_1, \tau_2)$.

For $\varepsilon > 0$, $n \in \mathbb{N}^*$ and a.a. $t \in [0, T]$, define the set of *nearly active state constraints* and *nearly active mixed constraints* respectively by:

$$I^g_\varepsilon(t) := \bigcup\{I(\sigma);\ \sigma \in (t - \varepsilon, t + \varepsilon) \cap [0, T]\}, \tag{2.11}$$

$$I^c_n(t) := \{i \in \{r + 1, \dots, r + s\}:\ t \in \Delta_n(c_i(u, y))\} \tag{2.12}$$

and the set of nearly active constraints by

$$I_{\varepsilon,n}(t) := I^g_\varepsilon(t) \cup I^c_n(t). \tag{2.13}$$

The contact sets of the constraints are denoted by

$$\Delta_i := \Delta(g_i(y)) \quad \text{for } i = 1, \dots, r, \tag{2.14}$$

$$\Delta_i := \Delta(c_i(u, y)) \quad \text{for } i = r + 1, \dots, r + s, \tag{2.15}$$

and, for $\delta > 0$ and $n \in \mathbb{N}^*$,

$$\Delta^\delta_i := \{t \in (0, T):\ \mathrm{dist}\{t, \Delta(g_i(y))\} < \delta\}, \quad i = 1, \dots, r, \tag{2.16}$$

$$\Delta^n_i := \Delta_n(c_i(u, y)), \quad i = r + 1, \dots, r + s. \tag{2.17}$$

Orders of the state constraints. Let $i = 1, \dots, r$. If f and g_i are C^{q_i} mappings, we may define inductively the functions $\mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$, $g^{(j)}_i(u, y) := g^{(j-1)}_{i,y}(y) f(u, y)$ for $j = 1, \dots, q_i$, with $g^{(0)}_i := g_i$, if we have $g^{(j)}_{i,u} \equiv 0$ for all $j = 0, \dots, q_i - 1$, i.e. $g^{(j)}_{i,u}(u, y) = 0$ for all $(u, y) \in \mathbb{R}^m \times \mathbb{R}^n$. Then $\frac{\mathrm{d}^j}{\mathrm{d}t^j} g_i(y(t)) = g^{(j)}_i(u(t), y(t))$, and for all $j < q_i$, we have that $g^{(j)}_i(u, y) = g^{(j)}_i(y)$. Let q_i be the smallest number of derivations, so that a dependence w.r.t. u appears, i.e. such that $g^{(q_i)}_{i,u}$ is not identically zero over $\mathbb{R}^m \times \mathbb{R}^n$ (this intrinsic definition of the order does not depend on a given trajectory $(u, y) \in \mathcal{U} \times \mathcal{Y}$ nor on the time). If q_i is finite, we say that q_i is the *order* of the component g_i. If q_i is finite, for all i, we define the *highest order* $q_{\max} := \max^r_{i=1} q_i$, and the *orders vector* $q := (q_1, \dots, q_r) \in \mathbb{N}^r$ is the vector of orders of the constraint $g = (g_1, \dots, g_r)$. In all the paper, it is assumed in addition to (A0) that

(A0$_q$) Each component of the state constraint g_i, $i = 1, \dots, r$, is of finite order q_i, and f and g are (at least) $C^{q_{\max}+1}$.

Remark 2.1. When performing the analysis in the L^∞-vicinity of a given trajectory $(u, y) \in \mathcal{U} \times \mathcal{Y}$, it is sufficient, for the results of this paper, to restrict the variable $y \in \mathbb{R}^n$ in the above definition of the mappings $g_i^{(j)}$ and of the order q_i to an open neighborhood in $\mathbb{R}^n$ of $\{y(t);\ t \in \Delta_i\}$ for each $i = 1, \ldots, r$. Likewise, the order of the constraint q_i needs only to be defined in the neighborhood of each connected component of the contact set Δ_i and may differ over two distinct connected components.

Note that when the state constraint g_i is of order q_i, relations such as

$$g_{i,y}^{(j)}(u, y) = g_{i,yy}^{(j-1)}(y) f(u, y) + g_{i,y}^{(j-1)}(y) f_y(u, y), \tag{2.18}$$

are satisfied, for all $j = 1, \ldots, q_i$. This will be useful in some of the proofs.

We assume w.l.o.g. in this paper that $u \to c_{i,u}(u, y)$ is not identically zero, for all $i = r + 1, \ldots, r + s$, since otherwise $c_i(u, y)$ is a pure state constraint. We may interpret mixed control-state constraints as state constraint of order zero, setting

$$q_i := 0 \quad \text{and} \quad g_i^{(0)}(u, y) := c_i(u, y), \quad \text{for all } i = r + 1, \ldots, r + s. \tag{2.19}$$

Given a subset $J \subset \{1, \ldots, r + s\}$, say $J = \{i_1 < \cdots < i_k\}$, define the mapping $G_J^{(q)} : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^{|J|}$ by:

$$G_J^{(q)}(u, y) := \begin{pmatrix} g_{i_1}^{(q_{i_1})}(u, y) \\ \vdots \\ g_{i_k}^{(q_{i_k})}(u, y) \end{pmatrix}, \quad \text{for all } (u, y) \in \mathbb{R}^m \times \mathbb{R}^n. \tag{2.20}$$

By (2.19), mixed control-state constraints are taken into account in this definition. When $J = \{1, \ldots, r+s\}$, we denote just (2.20) by $G^{(q)}(u, y)$.

The controllability lemma. *For $\kappa \in [1, +\infty]$, let*

$$\mathcal{V}_\kappa := L^\kappa\big(0, T; \mathbb{R}^m\big), \qquad \mathcal{Z}_\kappa := W^{1,\kappa}\big(0, T; \mathbb{R}^n\big). \tag{2.21}$$

Given a trajectory (u, y) and $v \in \mathcal{V}_\kappa$, we denote by z_v the (unique) solution in $\mathcal{Z}_\kappa$ of the linearized state equation

$$\dot z(t) = f_u\big(u(t), y(t)\big)v(t) + f_y\big(u(t), y(t)\big)z(t) \quad \text{a.e. on } [0, T], \quad z(0) = 0. \tag{2.22}$$

Lemma 2.2. *Let (u, y) be a trajectory, and let $\kappa \in [1, +\infty]$. For all $v \in \mathcal{V}_\kappa$ and all $i = 1, \ldots, r$, we have that $g_{i,y}(y(\cdot))z_v(\cdot) \in W^{q_i,\kappa}(0, T)$ and*:

$$\frac{\mathrm{d}^j}{\mathrm{d}t^j}\big(g_{i,y}\big(y(t)\big)z_v(t)\big) = g_{i,y}^{(j)}\big(y(t)\big)z_v(t), \quad \text{for all } j = 1, \ldots, q_i - 1, \tag{2.23}$$

$$\frac{\mathrm{d}^{q_i}}{\mathrm{d}t^{q_i}}\big(g_{i,y}\big(y(t)\big)z_v(t)\big) = g_{i,u}^{(q_i)}\big(u(t), y(t)\big)v(t) + g_{i,y}^{(q_i)}\big(u(t), y(t)\big)z_v(t). \tag{2.24}$$

Proof. It suffices to use the linearized state equation (2.22), the relation (2.18), and that $g_{i,y}^{(j-1)} f_u = g_{i,u}^{(j)} \equiv 0$ for all $j = 1, \ldots, q_i - 1$ to obtain (2.23)–(2.24) by induction on j. □

Consider the following constraint qualification condition:

$$\begin{aligned} &\text{there exist } \gamma, \varepsilon > 0 \text{ and } n \in \mathbb{N}^* \text{ such that} \\ &\gamma|\xi| \leqslant \big|G_{I_{\varepsilon,n}(t),u}^{(q)}\big(u(t), y(t)\big)^\top \xi\big|, \quad \text{for all } \xi \in \mathbb{R}^{|I_{\varepsilon,n}(t)|} \text{ and a.a. } t \in [0, T]. \end{aligned} \tag{2.25}$$

Lemma 2.3. *Let (u, y) be a trajectory satisfying* (A1) *and* (2.25)*. Then for all $\kappa \in [1, +\infty]$ and all $\delta \in (0, \varepsilon)$, where ε is given in* (2.25)*, the linear mapping*

$$\begin{aligned} &\mathcal{V}_\kappa \to \prod_{i=1}^{r} W^{q_i,\kappa}\big(\Delta_i^\delta\big) \times \prod_{i=r+1}^{r+s} L^\kappa\big(\Delta_i^n\big), \\ &v \mapsto \begin{pmatrix} ((g_{i,y}(y(\cdot))z_v(\cdot))|_{\Delta_i^\delta})_{1\leqslant i\leqslant r} \\ ((c_{i,u}(u(\cdot), y(\cdot))v(\cdot) + c_{i,y}(u(\cdot), y(\cdot))z_v(\cdot))|_{\Delta_i^n})_{r+1\leqslant i\leqslant r+s} \end{pmatrix} \end{aligned} \tag{2.26}$$

where z_v is the unique solution in $\mathcal{Z}_k$ of the linearized state equation (2.22)*, is onto, and hence has a bounded right inverse by the open mapping theorem.*

Recall that $\varphi|_{\mathcal{I}}$ denotes the restriction of the function φ to the set $\mathcal{I} \subset [0, T]$.

Proof. Let $\psi = (\psi_i)_{1 \leqslant i \leqslant r+s} \in \prod_{i=1}^{r} W^{q_i,\kappa}(\Delta_i^\delta) \times \prod_{i=r+1}^{r+s} L^\kappa(\Delta_i^n)$. In order to have $\psi_i = g_{i,y}(y)z_v$ on Δ_i^δ for all $i = 1, \dots, r$, it is necessary and sufficient by Lemma 2.2 that, a.e. on Δ_i^δ,

$$g_{i,u}^{(q_i)}(u, y)v + g_{i,y}^{(q_i)}(u, y)z_v = \psi_i^{(q_i)} \tag{2.27}$$

and that, for every point τ in the left boundary of Δ_i^δ (note that there exist finitely many such points),

$$g_{i,y}^{(j)}\big(y(\tau)\big)z_v(\tau) = \psi_i^{(j)}(\tau), \quad \text{for all } j = 0, \dots, q_i - 1. \tag{2.28}$$

The relation (2.27) with $q_i = 0$, $g_i^{(0)} = c_i$ and $\psi_i^{(0)} := \psi_i$ must be satisfied as well a.e. on Δ_i^n for all $i = r+1, \dots, r+s$. Set $M(t) := G_{I_{\varepsilon,n}(t),u}^{(q)}(u(t), y(t))$. By (2.25), the matrix $M(t)M(t)^\top$ is invertible at a.a. t, so we may take a.e., if $I_{\varepsilon,n}(t) \neq \emptyset$ (take $v(t) = 0$ if $I_{\varepsilon,n}(t) = \emptyset$):

$$v(t) = M(t)^\top \big(M(t)M(t)^\top\big)^{-1}\big\{\varphi(t) - G_{I_{\varepsilon,n}(t),y}^{(q)}\big(u(t), y(t)\big)z_v(t)\big\}, \tag{2.29}$$

where z_v is the solution of (2.22) with v given by (2.29), and the right-hand side $\varphi = (\varphi_i)_{i \in I_{\varepsilon,n}(t)}$ is as follows. We have $\varphi_i(t) = \psi_i(t)$ if $i = r+1, \dots, r+s$ and $t \in \Delta_i^n$, and $\varphi_i(t) = \psi_i^{(q_i)}(t)$ if $i = 1, \dots, r$ and $t \in \Delta_i^\delta$. On $\Delta_i^\varepsilon \setminus \Delta_i^\delta$, φ_i can be chosen equal e.g. to a polynomial function of order $2q_i - 1$, in order to match, in arbitrary small time $\varepsilon - \delta > 0$, the first $q_i - 1$ time derivatives of $g_{i,y}(y)z_v$ with those of ψ_i, i.e. so that (2.28) holds for all left endpoints τ of Δ_i^δ. $\square$

If the control u is continuous (see Proposition 3.1 and assumption (A2)), (2.25) is always satisfied if the *linear independence condition* below holds:

there exists $\gamma > 0$ such that

$$\gamma|\xi| \leqslant \big|G_{I(t),u}^{(q)}\big(u(t), y(t)\big)^\top \xi\big|, \quad \text{for all } \xi \in \mathbb{R}^{|I(t)|} \text{ and a.a. } t \in [0, T], \tag{2.30}$$

i.e. $G_{I(t),u}^{(q)}(u(t), y(t))$ is uniformly onto, for all $t \in [0, T]$. This assumption (without the mixed control-state constraints) was already used in [28].

For $J = \{i_1 < \cdots < i_k\} \subset \{r+1, \dots, r+s\}$, let us denote

$$c_J(u, y) := \big(c_{i_1}(u, y), \dots, c_{i_k}(u, y)\big)^\top.$$

We will also use in Proposition 3.1 the constraint qualification (2.31) below, weaker than (2.25), involving only the mixed control-state constraints:

there exist $n \in \mathbb{N}^*$ and $\gamma > 0$ such that

$$\gamma|\xi| \leqslant \big|c_{I_n^c(t),u}\big(u(t), y(t)\big)^\top \xi\big| \quad \text{for all } \xi \in \mathbb{R}^{|I_n^c(t)|} \text{ and a.a. } t \in [0, T]. \tag{2.31}$$

Remark 2.4. There are two possible natural choices for the functional space of the pure state constraints: either the space of continuous functions $C^0 := C([0, T]; \mathbb{R}^r)$, or the space $W^{q,\infty} := \prod_{i=1}^r W^{q_i,\infty}(0, T)$, where q_i denotes the order of the ith component of the constraint, in which the constraint is "onto" by Lemma 2.3. Considering the state constraints in C^0 instead of $W^{q,\infty}$, we have multipliers in $\mathcal{M}([0, T]; \mathbb{R}^{r*})$ rather than in the dual space of $W^{q,\infty}$. Existence of multipliers in $\mathcal{M}([0, T]; \mathbb{R}^{r*})$ is ensured under natural hypotheses (see below). Moreover, since the inclusion of $W^{q,\infty}$ in C^0 is dense and continuous, by surjectivity of the constraint in $W^{q,\infty}$ we obtain that the multipliers associated in both formulations are one-to-one, and we inherit nice properties such as uniqueness of the multiplier in $\mathcal{M}([0, T]; \mathbb{R}^{r*})$.

2.2. First-order optimality condition

Define the classical *Hamiltonian* and *Lagrangian* functions of $(\mathcal{P})$, $H:\mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^{n*} \to \mathbb{R}$ and $L:\mathcal{U} \times M([0,T];\mathbb{R}^{r*}) \times (L^\infty)^*(0,T;\mathbb{R}^{s*}) \to \mathbb{R}$ by:

$$H(u,y,p) := \ell(u,y) + pf(u,y), \tag{2.32}$$

$$L(u;\eta,\lambda) := J(u) + \langle \eta, G(u)\rangle + \langle \lambda, \mathcal{G}(u)\rangle, \tag{2.33}$$

for the duality products in the appropriate spaces.

Robinson's constraint qualification for the abstract problem (2.5) is as follows:

$$\exists \varepsilon > 0, \quad \varepsilon B_{C\times L^\infty} \subset \big(G(u), \mathcal{G}(u)\big) + \big(DG(u), D\mathcal{G}(u)\big)\mathcal{U} - K \times \mathcal{K}. \tag{2.34}$$

It is easy to see that under the assumptions of Lemma 2.3, (2.34) holds. Some elements of proof of the next theorem are recalled in Appendix A.2. The existence and uniqueness of the multipliers are a consequence of Lemma 2.3.

Theorem 2.5. *Let* $(u,y) \in \mathcal{U} \times \mathcal{Y}$ *be a local solution of* $(\mathcal{P})$*, satisfying* (A1), (2.34) *and* (2.31). *Then there exist* $p \in BV([0,T];\mathbb{R}^{n*})$, $\eta \in \mathcal{M}([0,T];\mathbb{R}^{r*})$ *and* $\lambda \in L^\infty(0,T;\mathbb{R}^{s*})$ *such that*

$$\dot y(t) = f\big(u(t),y(t)\big) \quad \text{for a.a. } t \in [0,T];\ y(0) = y_0, \tag{2.35}$$

$$p(t) = \int_t^T \big\{H_y\big(u(\sigma),y(\sigma),p(\sigma)\big) + \lambda(\sigma)c_y\big(u(\sigma),y(\sigma)\big)\big\}\,\mathrm{d}\sigma + \int_t^T \mathrm{d}\eta(\sigma)g_y\big(y(\sigma)\big) + \phi_y\big(y(T)\big), \tag{2.36}$$

$$0 = H_u\big(u(t),y(t),p(t)\big) + \lambda(t)c_u\big(u(t),y(t)\big) \quad \text{for a.a. } t \in [0,T], \tag{2.37}$$

$$0 \geqslant g_i\big(y(t)\big), \quad \mathrm{d}\eta_i \geqslant 0, \quad \int_0^T g_i\big(y(t)\big)\,\mathrm{d}\eta_i(t) = 0, \quad i = 1,\dots,r, \tag{2.38}$$

$$0 \geqslant c_i\big(u(t),y(t)\big), \quad \lambda_i(t) \geqslant 0 \ \text{ a.e.}, \int_0^T c_i\big(u(t),y(t)\big)\lambda_i(t)\,\mathrm{d}t = 0, \quad i = r+1,\dots,r+s. \tag{2.39}$$

We say that (u,y) is a *stationary point* of $(\mathcal{P})$, if there exist $p \in BV([0,T];\mathbb{R}^{n*})$, $\eta \in \mathcal{M}([0,T];\mathbb{R}^{r*})$ and $\lambda \in L^\infty(0,T;\mathbb{R}^{s*})$ such that (2.35)–(2.39) hold.

When the Hamiltonian and the mixed control-state constraints are convex w.r.t. the control variable (and in particular when assumption (2.43) below holds), then (2.37) and (2.39) are equivalent to

$$u(t) \in \operatorname*{argmin}_{w\in\mathbb{R}^m,\, c(w,y(t))\leqslant 0} H\big(w,y(t),p(t)\big) \quad \text{for a.a. } t \in [0,T]. \tag{2.40}$$

Here $\lambda(t)$ is the multiplier associated with the constraint (in $\mathbb{R}^m$) $c(w,y(t)) \leqslant 0$. We thus recover in this particular case Pontryagin's Minimum Principle, see [13,10,29].

Assumptions. Let the *augmented Hamiltonian of order zero* $H^0:\mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^{n*} \times \mathbb{R}^{s*} \to \mathbb{R}$ be defined by

$$H^0(u,y,p,\lambda) := H(u,y,p) + \lambda c(u,y). \tag{2.41}$$

Given (u,y) a stationary point of $(\mathcal{P})$, we will make the assumptions below:

(A2) The control u is *continuous* on $[0,T]$, and (strengthened Legendre–Clebsch condition)

$$\begin{gathered}\text{there exists } \alpha > 0 \text{ such that for all } t \in [0,T],\\ \alpha|\upsilon|^2 \leqslant H^0_{uu}\big(u(t),y(t),p(t),\lambda(t)\big)(\upsilon,\upsilon) \quad \text{for all } \upsilon \in \mathbb{R}^m.\end{gathered} \tag{2.42}$$

(A3) The data of the problem are (at least) $C^{2q_{\max}}$, and the linear independence condition (2.30) is satisfied.

Remark 2.6. The only condition (2.42) is not enough to ensure the continuity of the control, as shows the following example:

$$\min_{u\in L^\infty(0,T)} \int_0^2 \left\{u(t)^4 - 2u(t)^2 + \big(y(t)-1\big)u(t)\right\} \mathrm{d}t, \qquad \dot y(t) = 1,\ y(0)=0,$$

where the minimizer u jumps from the minimum close to 1 for $t = y(t) < 1$ to the minimum close to -1 for $t = y(t) > 1$, although (2.42) holds.

We will see in Proposition 3.1 that if (u, y) is a stationary point such that the Hamiltonian $H(\cdot, y(t), p(t))$ is uniformly strongly convex and the mixed control-state constraints are convex w.r.t. the control along the trajectory, which is equivalent to the condition below (stronger than (2.42))

$$\begin{aligned} &\text{there exists } \alpha > 0 \text{ such that for all } t \in [0,T] \text{ and all } (\hat u, \hat\lambda) \in \mathbb{R}^m \times \mathbb{R}^{s*}_+,\\ &\quad \alpha|\upsilon|^2 \leqslant H^0_{uu}\big(\hat u, y(t), p(t), \hat\lambda\big)(\upsilon,\upsilon) \quad \text{for all } \upsilon \in \mathbb{R}^m, \end{aligned} \tag{2.43}$$

and if (2.31) holds, then u is continuous on $[0,T]$. Therefore (2.43) and (2.31) imply that (A2) holds.

Remark 2.7. In some of the results of Sections 3 and 5, assumption (2.42) in (A2) can be weakened by assuming the uniform positivity of H^0_{uu} only on a subspace of $\mathbb{R}^m$ depending on the active constraints, namely

$$\begin{aligned} &\text{there exists } \alpha > 0 \text{ such that for a.a. } t \in [0,T],\\ &\quad \alpha|\upsilon|^2 \leqslant H^0_{uu}\big(u(t), y(t), p(t), \lambda(t)\big)(\upsilon,\upsilon) \quad \text{for all } \upsilon \in \mathbb{R}^m \text{ satisfying}\\ &\quad g^{(q_i)}_{i,u}\big(u(t), y(t)\big)\upsilon = 0 \quad \text{for all } i = 1,\dots,r+s \text{ such that } t \in \operatorname{int}\Delta_i. \end{aligned} \tag{2.44}$$

3. First regularity results

In the scalar case (when both the state constraint $g(y)$ and the control are scalar-valued, i.e. $m = r = 1$), and when there is no constraint on the control, the regularity of the control on the interior of arcs follows from the implicit function theorem, applied by (A2) to the relation $H_u(u(t), y(t), p(t)) = 0$ on the interior of unconstrained arcs (when $g(y(t)) < 0$), and by (A3) to $g^{(q)}(u(t), y(t)) = 0$ on the interior of boundary arcs (when $g(y(t)) = 0$). Knowing that u (and y) are smooth on boundary arcs, we can then differentiate w.r.t. t (in the measure sense) the relation $H_u(u(t), y(t), p(t))$ on boundary arcs, as many times as necessary, until we express, using (A3), the measure $\mathrm{d}\eta$ as $\eta_0(t)\,\mathrm{d}t$, with $\eta_0(t)$ a smooth function of $(u(t), y(t), p(t))$. Therefore we obtain that the state constraint multiplier η is continuously differentiable on the interior of boundary arcs.

Maurer in [28] extended this approach to the particular case when $r = m$ (and $s = 0$) (as many control as active state constraints), but this proof has no direct extension to the case $1 \leqslant r < m$.

In Section 3.1, we show that assumptions (2.43) and (2.31) imply the continuity of the control over $[0,T]$ (Proposition 3.1), and therefore also (A2) (no constraint regularity for the state constraint is needed). Moreover, (A2)–(A3) imply that the multipliers associated with mixed control-state constraints and with state constraints of first-order are continuous. In Section 3.2 we show higher regularity of the control and of the constraints multipliers on the interior of the arcs of the trajectory (Proposition 3.6). Our proof is based on the use of *alternative multipliers* (Definition 3.3).

3.1. Continuity of the control

Proposition 3.1. *Let (u, y) be a stationary point of $(\mathcal{P})$.*

(i) *Assume that* (2.43) *and* (2.31) *hold. Then the control u is continuous on* $[0,T]$.
(ii) *Assume that* (A2) *and* (2.30) *hold. Then the multiplier λ associated with the mixed control-state constraints and the multipliers η_i associated with components g_i of the state constraint of* first order ($q_i = 1$) *are continuous on* $[0,T]$.

In the absence of constraints of order greater than one, point (ii) is well known, see e.g. [15,16].

Proof of Proposition 3.1. Assumption (2.43) implies that for each $t \in [0,T]$, the problem (2.40) has a strongly convex cost function and convex constraints, therefore the control $u(t)$ is the unique solution of (2.40). In view of (2.31), $\lambda(t)$ is the unique associated multiplier. By (2.31) and (2.43), classical results on stability analysis in nonlinear programming (e.g. an easy application of Robinson's strong regularity theory [36], see also [19]) show that there exists a Lipschitz continuous function $\Upsilon : \mathbb{R}^n \times \mathbb{R}^{n*} \to \mathbb{R}^m \times \mathbb{R}^{s*}$ such that $(u(t), \lambda(t)) = \Upsilon(y(t), p(t))$, for a.a. $t \in [0,T]$. Since the composition of a Lipschitz continuous function with a function of bounded variation is a function of bounded variation, it follows that u and λ are of bounded variation, and hence have a right and a left limit everywhere.

Fix $t \in [0,T]$. We sometimes omit the time argument t. Denote respectively by u^+ and u^- the right and left limits of u at time t. Set $[u] := u^+ - u^-$ and for $\sigma \in [0,1]$, $u^\sigma := \sigma u^+ + (1-\sigma)u^-$. We use similar notations for λ and p. By the costate equation (2.36), p has at most countably many jumps, of type

$$[p] = p^+ - p^- = -\sum_{i=1}^{r} \nu_i g_{i,y}\big(y(t)\big), \quad \text{with } \nu_i := \big[\eta_i(t)\big] \geqslant 0. \tag{3.1}$$

Recall that H^0 denotes the augmented Hamiltonian of order zero (2.41). It follows from (2.37) that

$$\begin{aligned} 0 &= H^0_u\big(u^+, y, p^+, \lambda^+\big) - H^0_u\big(u^-, y, p^-, \lambda^-\big) \\ &= \int_0^1 \big\{H^0_{uu}\big(u^\sigma, y, p^\sigma, \lambda^\sigma\big)[u] + [p]f_u\big(u^\sigma, y\big) + [\lambda]c_u\big(u^\sigma, y\big)\big\}\,\mathrm{d}\sigma. \end{aligned}$$

Using (3.1) and observing that, by definition of the order of the state constraint, $g_{i,y}f_u = g^{(1)}_{i,u}$ equals zero if $q_i > 1$, we obtain that

$$\int_0^1 H^0_{uu}\big(u^\sigma, y, p^\sigma, \lambda^\sigma\big)[u]\,\mathrm{d}\sigma = \int_0^1 \sum_{i:\, q_i=1} \nu_i g^{(1)}_{i,u}\big(u^\sigma, y\big)\,\mathrm{d}\sigma - \int_0^1 [\lambda]c_u\big(u^\sigma, y\big)\,\mathrm{d}\sigma. \tag{3.2}$$

Noticing that $H^0_{uu}(u^\sigma, y, p^\sigma, \lambda^\sigma) = \sigma H^0_{uu}(u^\sigma, y, p^+, \lambda^+) + (1-\sigma)H^0_{uu}(u^\sigma, y, p^-, \lambda^-)$ and taking the scalar product of both sides of (3.2) by $[u]$, we get using hypothesis (2.43) that

$$\alpha\big|[u]\big|^2 \leqslant \sum_{i:\, q_i=1} \nu_i\big[g^{(1)}_i(u,y)\big] - [\lambda]\big[c(u,y)\big]. \tag{3.3}$$

If $\nu_i > 0$, then $g_i(y(t)) = 0$, and hence $[g^{(1)}_i(u,y)] \leqslant 0$ since t is a local maximum of $g_i(y)$. By (2.39), $\lambda^\pm(t)$ belongs to the normal cone to $\mathbb{R}^s_-$ at point $c(u^\pm(t), y(t))$. By monotonicity of the normal cone, we obtain that $[\lambda][c(u,y)] \geqslant 0$. Therefore, the right-hand side in (3.3) is nonpositive, implying that $[u] = 0$, i.e. u is continuous at t. This shows (i).

Since $[u] = 0$, the right-hand side of (3.2) equals zero. By (2.30), the vectors $(g^{(1)}_{i,u}(u,y))$ for $i \in I^g(t) \cap \{i:\ q_i = 1\}$ and $c_{i,u}(u,y)$ for $i \in I^c(t)$ are jointly linearly independent. It follows that $[\lambda] = 0$ and $\nu_i = 0$, for all i corresponding to first-order state-constraint components. This achieves the proof of (ii). □

Remark 3.2. For point (ii) in Proposition 3.1, it is sufficient to have the linear independence condition (2.30) for mixed control-state constraints and *first-order* components of the state constraint only.

3.2. Higher regularity on interior of arcs

We recall that an arc of the trajectory (u, y) is a maximal open interval of positive measure with a constant set of active constraints (2.10), and that mixed control-state constraints are considered as state constraint of order zero by (2.19).

Definition 3.3. Let (u, y) be a stationary point of $(\mathcal{P})$, and (τ_1, τ_2) an arc of the trajectory, with constant set of active constraints $I(t) = J \subset \{1, \dots, r+s\}$, for all $t \in (\tau_1, \tau_2)$. The *alternative multipliers* on (τ_1, τ_2) are as follows. Define the functions η^j_i for $i = 1, \dots, r+s$ and $j = 1, \dots, q_i$ if $i \leqslant r$, $j = 0$ if $i > r$, by

$$\begin{aligned}
&\eta_i^1(t) := -\int \mathrm{d}\eta_i(\sigma) = Cst - \eta_i(t), \quad i \in J,\ i \leqslant r,\\
&\eta_i^j(t) := -\int \eta_i^{j-1}(\sigma)\,\mathrm{d}\sigma, \quad j = 2,\dots,q_i,\ i \in J,\ i \leqslant r,\\
&\eta_i^j(t) := 0, \quad j = 1,\dots,q_i,\ i \in \{1,\dots,r\} \setminus J,\\
&\eta_i^0(t) := \lambda_i(t), \quad i \in J,\ i > r.
\end{aligned} \tag{3.4}$$

We denote here by Cst an arbitrary integration constant. The alternative multipliers (p^q, η^q) are defined by $\eta^q := (\eta_1^{q_1}, \dots, \eta_{r+s}^{q_{r+s}})$ and

$$p^q(t) := p(t) - \sum_{i=1}^{r} \sum_{j=1}^{q_i} \eta_i^j(t) g_{i,y}^{(j-1)}\big(y(t)\big). \tag{3.5}$$

The *alternative Hamiltonian of order q* $H^q : \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^{n*} \times \mathbb{R}^{(r+s)*} \to \mathbb{R}$ is defined by:

$$H^q\big(u, y, p^q, \eta^q\big) := H\big(u, y, p^q\big) + \eta^q G^{(q)}(u, y) = H\big(u, y, p^q\big) + \sum_{i=1}^{r+s} \eta_i^{q_i} g_i^{(q_i)}(u, y), \tag{3.6}$$

with H the classical Hamiltonian (2.32).

Lemma 3.4. *Let (u, y) be a stationary point of $(\mathcal{P})$, with multipliers (p, η, λ). Then on the interior of each arc (τ_1, τ_2) of the trajectory, with a constant set of active constraints $I(t) = J \subset \{1, \dots, r+s\}$ on (τ_1, τ_2), the following holds, with the alternative multipliers of Definition* 3.3*, for all $t \in (\tau_1, \tau_2)$: p_q is absolutely continuous on (τ_1, τ_2) and*

$$-\dot{p}^q(t) = H_y^q\big(u(t), y(t), p^q(t), \eta^q(t)\big), \tag{3.7}$$

$$H^q\big(\cdot, y(t), p^q(t), \eta^q(t)\big) = H^0\big(\cdot, y(t), p(t), \lambda(t)\big), \tag{3.8}$$

and for all $i = 1, \dots, r+s$:

$$g_i^{(q_i)}\big(u(t), y(t)\big) = 0, \quad i \in J, \tag{3.9}$$

$$\eta_i^{q_i}(t) = 0, \quad i \notin J. \tag{3.10}$$

Remark 3.5. An obvious consequence of (3.8) is that u minimizes $H^0(\cdot, y(t), p(t), \lambda(t))$ iff it minimizes $H^q(\cdot, y(t), p^q(t), \eta^q(t))$, and in particular, by (2.37), a stationary point satisfies

$$0 = H_u^q\big(u(t), y(t), p^q(t), \eta^q(t)\big). \tag{3.11}$$

Proof. For the sake of completeness of the paper, let us recall the proof, due to Maurer in [28] when there are no mixed control-state constraints. Relation (3.9) follows from differentiation w.r.t. $t \in (\tau_1, \tau_2)$ of the relation $g_i(y(t)) = 0$, for $i \in J$, $i \leqslant r$ and (3.10) follows from definition (3.4). By definition of the constraint order q_i, the function $g_i^{(j)}(u, y)$ does not depend on u, for all $j = 1, \dots, q_i - 1$ and $i = 1, \dots, r$, and hence, for all $\hat{u} \in \mathbb{R}^m$, we have:

$$\begin{aligned}
H^0(\hat{u}, y, p, \lambda) &= H^0\big(\hat{u}, y, p^q, \lambda\big) + \big(p - p^q\big) f(\hat{u}, y)\\
&= H^0\big(\hat{u}, y, p^q, \lambda\big) + \sum_{i=1}^{r} \sum_{j=1}^{q_i} \eta_i^j g_i^{(j)}(\hat{u}, y)\\
&= H^q\big(\hat{u}, y, p^q, \eta^q\big) + F(t),
\end{aligned}$$

where

$$F(t) := \sum_{i=1}^{r} \sum_{j=1}^{q_i - 1} \eta_i^j(t) g_i^{(j)}\big(y(t)\big)$$

does not depend on $\hat{u}$. For all $i = 1, \dots, r$, if $i \in J$, then $g_i^{(j)}(y(t)) = 0$, and if $i \notin J$, then $\eta_i^j(t) = 0$ by (3.4). Consequently, $F(t) = 0$, which proves (3.8).

We show now (3.7). Using (3.5) and that $\dot{\eta}_i^j = -\eta_i^{j-1}$, for $j = 2, \dots, q_i$, $i \leqslant r$, we have:

$$-\mathrm{d}p^q = -\mathrm{d}p + \sum_{i=1}^{r} \left\{ \sum_{j=1}^{q_i} \eta_i^j g_{i,yy}^{(j-1)}(y) f(u, y)\,\mathrm{d}t - \sum_{j=2}^{q_i} \eta_i^{j-1} g_{i,y}^{(j-1)}(y)\,\mathrm{d}t - \mathrm{d}\eta_i g_{i,y}(y) \right\}. \tag{3.12}$$

Since

$$-\mathrm{d}p = H_y\big(u, y, p^q\big)\,\mathrm{d}t + \big(p - p^q\big) f_y(u, y)\,\mathrm{d}t + \sum_{i=1}^{r} \mathrm{d}\eta_i g_{i,y}(y) + \sum_{i=r+1}^{r+s} \lambda_i c_{i,y}(u, y)\,\mathrm{d}t,$$

substituting $p - p^q$ into (3.12) using (3.5), we obtain:

$$\begin{aligned} -\mathrm{d}p^q = {} & H_y\big(u, y, p^q\big)\,\mathrm{d}t + \sum_{i=r+1}^{r+s} \eta_i^0 g_{i,y}^{(0)}(u, y)\,\mathrm{d}t \\ & + \sum_{i=1}^{r} \left\{ \sum_{j=1}^{q_i} \eta_i^j \big(g_{i,y}^{(j-1)}(y) f_y(u, y) + g_{i,yy}^{(j-1)}(y) f(u, y)\big) - \sum_{j=2}^{q_i} \eta_i^{j-1} g_{i,y}^{(j-1)}(y) \right\} \mathrm{d}t. \end{aligned}$$

Using (2.18), it follows that

$$-\mathrm{d}p^q = H_y\big(u, y, p^q\big)\,\mathrm{d}t + \sum_{i=1}^{r+s} \eta_i^{q_i} g_{i,y}^{(q_i)}(u, y)\,\mathrm{d}t,$$

which shows (3.7) and achieves the proof. □

Proposition 3.6. *Assume that the data are (at least) $C^{2q_{\max}}$. Let (u, y) be a stationary point of $(\mathcal{P})$, with multipliers (p, η, λ), and let $(\tau_1, \tau_2) \subset [0, T]$ be such that $I(t)$ is constant on (τ_1, τ_2), u is continuous on (τ_1, τ_2), and* (2.44) *and* (2.30) *are satisfied on (τ_1, τ_2). Then on (τ_1, τ_2), u is $C^{q_{\max}}$, y is $C^{q_{\max}+1}$, p is C^1, λ is $C^{q_{\max}}$ and the state constraint multiplier η_i is $C^{q_{\max}-q_i+1}$, for all $i = 1, \dots, r$.*

Proof. Denote by $J \subset \{1, \dots, r + s\}$ the constant set of active constraints $I(t)$ for $t \in (\tau_1, \tau_2)$. The Jacobian w.r.t. u and $(\eta_i^{q_i})_{i \in J}$ of Eqs. (3.11) and (3.9), the latter being rewritten as $G_J^{(q)}(u(t), y(t)) = 0$, is given by

$$\begin{pmatrix} H_{uu}(u, y, p^q) + \sum_{i \in J} \eta_i^{q_i} g_{i,uu}^{(q_i)}(u, y) & G_{J,u}^{(q)}(u, y)^\top \\ G_{J,u}^{(q)}(u, y) & 0 \end{pmatrix}. \tag{3.13}$$

By (3.8),

$$H_{uu}\big(u, y, p^q\big) + \sum_{i \in J} \eta_i^{q_i} g_{i,uu}^{(q_i)}(u, y) = H_{uu}^q\big(u, y, p^q, \eta^q\big) = H_{uu}^0(u, y, p, \lambda)$$

is positive definite on $\operatorname{Ker} G_{J,u}^{(q)}(u, y)$ by (2.44), and by (2.30), $G_{J,u}^{(q)}(u, y)$ is onto. Since by assumption u is continuous, by (2.30) and (3.11), we deduce that $(\eta_i^{q_i})_{i \in J}$ is also continuous. Thus we can apply the implicit function theorem to express u and $(\eta_i^{q_i})_{i \in J}$ as $C^{q_{\max}}$ implicit functions of (y, p^q). Since (y, p^q) is solution of a $C^{q_{\max}-1}$ differential equation system (2.2) and (3.7), we deduce that $(y, p^q, u, \eta_i^{q_i})$, $i \in J$, are $C^{q_{\max}}$ on (τ_1, τ_2). By (3.10), the components $\eta_i^{q_i}$ for $i \notin J$ being equal to zero on (τ_1, τ_2) are also trivially $C^{q_{\max}}$ on (τ_1, τ_2). Finally, recall that the classical multipliers η_i and p are related to the alternative ones by (3.4), i.e. $\eta_i(t) = (-1)^{q_i} \frac{\mathrm{d}^{q_i-1}}{\mathrm{d}t^{q_i-1}} \eta_i^{q_i}(t)$, and (3.5). It follows that each component η_i is $C^{q_{\max}-q_i+1}$ for $i \leqslant r$, $\lambda_i = \eta_i^0$ is $C^{q_{\max}}$, for all $i = r + 1, \dots, r + s$, and p is C^1, locally on (τ_1, τ_2). □

4. Local exact linearization of the "constraint dynamics"

We first give in Section 4.1 a result of "local invariance" of stationary points by a local change of coordinates and nonlinear feedback (Lemma 4.2). We use this result in Section 4.3 to show that, assuming (A3) and the continuity of u, we can locally "linearize the constraints dynamics" (Lemma 4.6), and we will use this "normal form" of the system in the proof of the junctions conditions results in Proposition 5.3. For that, a technical lemma (Lemma 4.4) given in Section 4.2 is needed, which will also be used in the proofs of Proposition 7.2 and Theorem 7.6.

4.1. Local invariance of stationary points by change of coordinates

Definition 4.1. Let (u, y) be a trajectory, and $t_0 \in (0, T)$. A couple of mappings (ϕ, ψ) is a C^k *local change of state variables and nonlinear feedback* at time t_0, $k \geqslant 1$, if there exist $\delta > 0$ and an open neighborhood $V_u \times V_y$ in $\mathbb{R}^m \times \mathbb{R}^n$ of $\{(u(t), y(t));\ t \in (t_0 - \delta, t_0 + \delta)\}$, such that $\phi : V_y \to \phi(V_y) =: V_z$, $\psi : V_u \times V_y \to \psi(V_u \times V_y) =: V_v$ and there exist $\bar\phi : V_z \to V_y$ and $\bar\psi : V_v \times V_z \to V_u$ such that for all $(u, y, v, z) \in V_u \times V_y \times V_v \times V_z$, we have

$$z = \phi(y) \quad \Leftrightarrow \quad y = \bar\phi(z); \qquad v = \psi(u, y) \quad \Leftrightarrow \quad u = \bar\psi(v, z)$$

and the inverse mappings $\bar\phi$ and $\bar\psi$ are C^k over V_z and $V_v \times V_z$, respectively.

Lemma 4.2 *(Invariance of stationarity equations). Let (u, y) be a trajectory, and $t_0 \in (0, T)$. Let (ϕ, ψ) be a local change of state variable and nonlinear feedback at time t_0, with $\delta > 0$ as in Definition* 4.1*. Then (u, y) satisfies with multipliers (p, η, λ) the stationarity equations* (2.35)–(2.36) *and* (2.37)–(2.39) *locally on $(t_0 - \delta, t_0 + \delta)$, iff (v, z, π) defined on $(t_0 - \delta, t_0 + \delta)$ by*

$$z(t) := \phi\big(y(t)\big); \qquad v(t) := \psi\big(u(t), y(t)\big); \qquad \pi(t) := p(t)\phi_y^{-1}\big(y(t)\big) \tag{4.1}$$

satisfies on $(t_0 - \delta, t_0 + \delta)$:

$$\dot z(t) = \hat f\big(v(t), z(t)\big), \tag{4.2}$$

$$-\mathrm{d}\pi(t) = \hat H_z\big(v(t), z(t), \pi(t)\big)\,\mathrm{d}t + \mathrm{d}\eta(t)\hat g_z\big(z(t)\big) + \lambda(t)\hat c_z\big(v(t), z(t)\big)\,\mathrm{d}t, \tag{4.3}$$

$$0 = \hat H_v\big(v(t), z(t), \pi(t)\big) + \lambda(t)\hat c_v\big(v(t), z(t)\big) \quad \text{a.e.} \tag{4.4}$$

$$\hat g\big(z(t)\big) \leqslant 0;\ \mathrm{d}\eta \geqslant 0; \quad \int_{t_0-\delta}^{t_0+\delta} \mathrm{d}\eta(t)\hat g\big(z(t)\big) = 0; \tag{4.5}$$

$$\hat c\big(v(t), z(t)\big) \leqslant 0;\ \lambda(t) \geqslant 0 \text{ a.e.}; \quad \int_{t_0-\delta}^{t_0+\delta} \lambda(t)\hat c\big(v(t), z(t)\big)\,\mathrm{d}t = 0; \tag{4.6}$$

with the new dynamics, integral cost function, Hamiltonian, and state and mixed constraints given by

$$\hat f(v, z) := \phi_y\big(\bar\phi(z)\big) f\big(\bar\psi(v, z), \bar\phi(z)\big), \tag{4.7}$$

$$\hat\ell(v, z) := \ell\big(\bar\psi(v, z), \bar\phi(z)\big), \tag{4.8}$$

$$\hat H(v, z, \pi) := \hat\ell(v, z) + \pi \hat f(v, z), \tag{4.9}$$

$$\hat g(z) := g\big(\bar\phi(z)\big), \tag{4.10}$$

$$\hat c(v, z) := c\big(\bar\psi(v, z), \bar\phi(z)\big). \tag{4.11}$$

In addition, the augmented Hamiltonian of order 0 *and the time derivatives of the state constraint (all components supposed to be of finite order q_i, $i = 1, \dots, r$), are invariant, i.e., on $V_z \times V_v$:*

$$\hat H^0(v, z, \pi, \lambda) := \hat H(v, z, \pi) + \lambda \hat c(v, z) \ = \ H^0\big(\bar\psi(v, z), \bar\phi(z), \pi\phi_y\big(\bar\phi(z)\big), \lambda\big); \tag{4.12}$$

$$\hat g_i^{(j)}(z) = g_i^{(j)}\big(\bar\phi(z)\big), \quad \text{for all } j = 1, \dots, q_i - 1,\ i = 1, \dots, r; \tag{4.13}$$

$$\hat g_i^{(q_i)}(v, z) = g_i^{(q_i)}\big(\bar\psi(v, z), \bar\phi(z)\big), \quad i = 1, \dots, r. \tag{4.14}$$

Proof. Assume that (u, y, p, η, λ) satisfies (2.35)–(2.36) and (2.37)–(2.39) for $t \in (t_0 - \delta, t_0 + \delta)$, and let us show that $(v, z, \pi, \eta, \lambda)$ satisfies (4.2)–(4.6) on $(t_0 - \delta, t_0 + \delta)$. The converse is proved similarly by symmetry. By (4.1), (4.7) and (4.10)–(4.11), it is obvious that (4.2), (4.5) and (4.6) follow from (2.35) and (2.38)–(2.39). Moreover, we have

$$\begin{aligned}\hat{H}^0_v(v, z, \pi, \lambda) &= D_v\big\{\ell\big(\bar\psi(v, z), \bar\phi(z)\big) + \pi\phi_y\big(\bar\phi(z)\big)f\big(\bar\psi(v, z), \bar\phi(z)\big) + \lambda c\big(\bar\psi(v, z), \bar\phi(z)\big)\big\}\\ &= H^0_u\big(\bar\psi(v, z), \bar\phi(z), p, \lambda\big)\bar\psi_v(v, z).\end{aligned}$$

Since $\bar\psi_v$ is invertible, this gives (4.4). It remains to check the costate equation. We have

$$\begin{aligned}\hat{H}^0_z(v, z, \pi, \lambda) = {} & H^0_u\big(\bar\psi(v, z), \bar\phi(z), p, \lambda\big)\bar\psi_z(v, z) + H^0_y\big(\bar\psi(v, z), \bar\phi(z), p, \lambda\big)\bar\phi_z(z)\\ & + \pi\phi_{yy}\big(\bar\phi(z)\big)\big(\bar\phi_z(z), f\big(\bar\psi(v, z), \bar\phi(z)\big)\big).\end{aligned} \tag{4.15}$$

By definition of π in (4.1), we have

$$\begin{aligned}\mathrm{d}p(t) &= \mathrm{d}\big\{\pi(t)\phi_y\big(\bar\phi\big(z(t)\big)\big)\big\}\\ &= \mathrm{d}\pi(t)\phi_y\big(\bar\phi\big(z(t)\big)\big) + \pi(t)\phi_{yy}\big(\bar\phi(z)\big)f\big(\bar\psi(v, z), \bar\phi(z)\big)\,\mathrm{d}t.\end{aligned}$$

Since $\phi_y(\bar\phi(z))\bar\phi_z(z) \equiv I_d$, using (2.36), (4.15) and (2.37) on $(t_0 - \delta, t_0 + \delta)$, we obtain

$$\begin{aligned}-\mathrm{d}\pi(t) &= -\mathrm{d}p(t)\bar\phi_z(z) + \pi(t)\phi_{yy}\big(\bar\phi(z)\big)\big(f\big(\bar\psi(v, z), \bar\phi(z)\big), \bar\phi_z(z)\big)\,\mathrm{d}t\\ &= \hat{H}^0_z(v, z, \pi, \lambda)\,\mathrm{d}t + \mathrm{d}\eta g_y\big(\bar\phi(z)\big)\bar\phi_z(z) = \hat{H}^0_z(v, z, \pi, \lambda)\,\mathrm{d}t + \mathrm{d}\eta\hat{g}_z(z),\end{aligned}$$

which gives (4.3). From (4.7) and (4.10), by induction for $j = 1, \dots, q_i$, we obtain

$$\begin{aligned}\hat{g}^{(j)}_i(v, z) &= \hat{g}^{(j-1)}_{i,z}(z)\hat{f}(v, z)\\ &= g^{(j-1)}_{i,y}\big(\bar\phi(z)\big)\bar\phi_z(z)\phi_y\big(\bar\phi(z)\big)f\big(\bar\psi(v, z), \bar\phi(z)\big)\\ &= g^{(j-1)}_{i,y}\big(\bar\phi(z)\big)f\big(\bar\psi(v, z), \bar\phi(z)\big) = g^{(j)}_i\big(\bar\psi(v, z), \bar\phi(z)\big),\end{aligned}$$

which shows (4.13)–(4.14) and achieves the proof. □

Remark 4.3. With the notations and assumptions of Lemma 4.2, we have

$$\hat{H}^0_{vv}(v, z, \pi, \lambda) = H^0_{uu}(u, y, p, \lambda)\big(\bar\psi_v(v, z), \bar\psi_v(v, z)\big) + H^0_u(u, y, p, \lambda)\bar\psi_{vv}(v, z) \tag{4.16}$$

and, for $J \subset \{1, \dots, r + s\}$, defining $\hat{G}^{(q)}_J(v, z) := (\hat{g}^{(q_i)}_i(v, z))_{i \in J}$, with still $q_i := 0$ and $\hat{g}^{(0)}_i := \hat{c}_i$ for $i = r + 1, \dots, r + s$, we obtain by (4.14) and (4.11):

$$\hat{G}^{(q)}_{J,v}\big(v(t), z(t)\big) = G^{(q_i)}_{J,u}\big(u(t), y(t)\big)\bar\psi_v\big(v(t), z(t)\big).$$

Since $H^0_u(u, y, p, \lambda) = 0$ at a stationary point, and $\bar\psi_v(v, z)$ is invertible over $V_v \times V_z$, we obtain that if (u, y) is a stationary point, then assumptions (2.42) (or (2.44)) and (2.30) are locally invariant by local change of coordinate and nonlinear feedback (but of course, with possibly different positive constants α and γ).

4.2. The linear independence lemma

Given $J \subset \{1, \dots, r\}$, we denote by $|q_J| := \sum_{i \in J} q_i$ and $|q| := \sum_{i=1}^r q_i$. Define the mapping $\Gamma_J : \mathbb{R}^n \to \mathbb{R}^{|q_J|}$ that with y associates the "J" state constraints and their time derivatives depending on y only, by:

$$\Gamma_J(y) := \begin{pmatrix} g_{i_1}(y)\\ \vdots\\ g^{(q_{i_1}-1)}_{i_1}(y)\\ \vdots\\ g_{i_s}(y)\\ \vdots\\ g^{(q_{i_s}-1)}_{i_s}(y)\end{pmatrix}, \quad J = \{i_1, \dots, i_s\},\ i_1 < \dots < i_s. \tag{4.17}$$

Lemma 4.4. *Let $\hat{y} \in \mathbb{R}^n$ and $J \subset \{1, \dots, r\}$. Assume that there exists $\hat{w} \in \mathbb{R}^m$ such that $G^{(q)}_{J,u}(\hat{w}, \hat{y})$ has full rank $|J|$. Then the matrix $\Gamma_{J,y}(\hat{y})$ has full rank, equals to $|q_J|$.*

The above result is well known in the case when the dynamics and the constraints are linear, but since we were not able to find a reference for it in the general nonlinear case, we give a proof below, which uses the relations (4.19) established in [28].

Proof. For $\tau \in (0, T)$ and small $\delta > 0$, consider the solution y of the state equation $\dot{y}(t) = f(u(t), y(t))$ over $(\tau - \delta, \tau + \delta)$, with $y(\tau) = \hat{y}$ and $u : (\tau - \delta, \tau + \delta) \to \mathbb{R}^m$ is here any $C^{q_{\max}}$ function such that $u(\tau) = \hat{w}$. For $k = 1, \dots, q_{\max} - 1$, define the mappings $A_k : (\tau - \delta, \tau + \delta) \to \mathbb{R}^{n \times m}$ by:

$$\begin{cases} A_0(t) := f_u(u(t), y(t)), \\ A_k(t) := f_y(u(t), y(t))A_{k-1}(t) - \dot{A}_{k-1}(t), \quad 1 \leqslant k \leqslant q_{\max} - 1. \end{cases} \tag{4.18}$$

The proof of the lemma is based on the following relations, due to [28]. For all $t \in (\tau - \delta, \tau + \delta)$ and $i = 1, \dots, r$, we have:

$$\begin{cases} g^{(j)}_{i,y}(y(t))A_k(t) = 0 \quad \text{for } k, j \geqslant 0,\ k + j \leqslant q_i - 2, \\ g^{(j)}_{i,y}(y(t))A_{q_i - j - 1}(t) = g^{(q_i)}_{i,u}(u(t), y(t)) \quad \text{for } 0 \leqslant j \leqslant q_i - 1. \end{cases} \tag{4.19}$$

For the sake of completeness of the paper, let us recall how to prove (4.19). We first show that for all $j = 0, \dots, q_i - 1$, the following assertion

$$g^{(j)}_{i,y}\big(y(t)\big)A_k(t) = 0 \quad \forall t \in (\tau - \delta, \tau + \delta) \tag{4.20}$$

implies that

$$g^{(j+1)}_{i,y}\big(u(t), y(t)\big)A_k(t) = g^{(j)}_{i,y}\big(y(t)\big)A_{k+1}(t) \quad \forall t \in (\tau - \delta, \tau + \delta). \tag{4.21}$$

Indeed, by derivation of (4.20) w.r.t. time, we get using (2.18)

$$\begin{aligned} 0 &= g^{(j)}_{i,yy}(y)f(u, y)A_k + g^{(j)}_{i,y}(y)\dot{A}_k \\ &= g^{(j)}_{i,yy}(y)f(u, y)A_k + g^{(j)}_{i,y}\big(f_y(u, y)A_k - A_{k+1}\big) \\ &= g^{(j+1)}_{i,y}(u, y)A_k - g^{(j)}_{i,y}(y)A_{k+1}. \end{aligned}$$

This gives (4.21). We also have that $g^{(j)}_{i,u}(u, y) = g^{(j-1)}_{i,y}(y)f_u(u, y) = g^{(j-1)}_{i,y}(y)A_0$ for $j = 1, \dots, q_i$. Since $g^{(j)}_{i,u} = 0$ for $j \leqslant q_i - 1$, it follows that $g^{(j)}_{i,y}A_0 = 0$ for $j = 0, \dots, q_i - 2$. By (4.21), we deduce that $g^{(j)}_{i,y}A_1 = 0$ for $j = 0, \dots, q_i - 3$. By induction, this proves the first equation in (4.19). Since $g^{(q_i-2)}_{i,y}A_0 = 0 = g^{(q_i-3)}_{i,y}A_1 = \cdots = g_{i,y}A_{q_i-2}$, by (4.21) we obtain $g^{(q_i)}_{i,u} = g^{(q_i-1)}_{i,y}A_0 = g^{(q_i-2)}_{i,y}A_1 = \cdots = g_{i,y}A_{q_i-1}$, which proves the second equation in (4.19).

Assume w.l.o.g. that $J = \{1, \dots, r'\}$, with $r' \leqslant r$, and that $q_1 \geqslant q_2 \geqslant \cdots \geqslant q_{r'} \geqslant 1$. Consider the matrix

$$K(t) := \big(A_{q_1-1}(t) \quad \dots \quad A_1(t) \quad A_0(t)\big) \in \mathbb{R}^{n \times mq_1}, \tag{4.22}$$

and form the product matrix

$$P(t) := \Gamma_{J,y}\big(y(t)\big)K(t) \in \mathbb{R}^{|q_J| \times mq_1}. \tag{4.23}$$

Let $\tilde{q}_i := \sum_{l=1}^{i} q_l$, and for $i = 1, \dots, r'$, denote by $P_i(t) \in \mathbb{R}^{q_i \times mq_1}$ the submatrix formed by the rows $\tilde{q}_{i-1} + 1$ to $\tilde{q}_i$ of $P(t)$. By (4.19), we have

$$P_i(t) = \begin{pmatrix} * & g^{(q_i)}_{i,u}(u(t), y(t)) & \dots & 0 \\ * & \vdots & \ddots & \vdots \\ \underbrace{*}_{m(q_1 - q_i)} & * & \cdots & g^{(q_i)}_{i,u}(u(t), y(t)) \end{pmatrix}. \tag{4.24}$$

Let us show that $P(\tau)$ has full rank $|q_J|$. For that consider a linear combination of the rows ℓ_j of $P(\tau)$, $\sum_{j=1}^{|q_J|}\beta_j\ell_j = 0$. By (4.24), only the rows of $P(\tau)$ for $j = \tilde{q}_i$, $i = 1, \dots, r'$, have a contribution to the last m components of $\sum_{j=1}^{|q_J|}\beta_j\ell_j$. It is easily seen that these last m components are a linear combination of the rows of $G^{(q)}_{J,u}(u(\tau), y(\tau))$, with coefficients $\beta_{\tilde{q}_i}$. Since $u(\tau) = \hat{w}$ and $G^{(q)}_{J,u}(\hat{w}, y(\tau))$ has full rank by hypothesis, it follows that $\beta_{\tilde{q}_i} = 0$ for all $i = 1, \dots, r'$. Repeating the same argument, we obtain that $\beta_j = 0$ for all $j = 1, \dots, |q_J|$, i.e. the product matrix $P(t)$ has rank $|q_J|$. Therefore, the matrix $\Gamma_{J,y}(y(\tau))$ has rank $|q_J|$. □

Corollary 4.5. *Let a trajectory (u, y) satisfy* (2.30). *Then the matrix $\Gamma_{I^g(t),y}(y(t))$ has full rank, equals to $|q_{I^g(t)}|$, for all $t \in [0, T]$ (and consequently, $\sum_{i\in I^g(t)} q_i \leqslant n$).*

4.3. Locally normal form of the state equation

Lemma 4.6. *Let (u, y) be a trajectory and $t_0 \in (0, T)$ such that u is continuous at t_0. Assume that f, g are (at least) $C^{2q_{\max}}$, that* (2.30) *holds at $t = t_0$, and w.l.o.g. that $I(t_0) = \{1, \dots, r'\} \cup \{r + 1, \dots, r + s'\} =: J$. Then there exists a $C^{q_{\max}}$ local change of variable and nonlinear feedback (ϕ, ψ), defined over a neighborhood of $(u(t_0), y(t_0))$, such that, with the notations of Lemma* 4.2, *the new dynamics $\hat{f}$ writes on $(t_0 - \delta, t_0 + \delta)$, with $\tilde{q}_i := \sum_{l=1}^{i} q_l$ (and $\tilde{q}_0 = 0$):*

$$\begin{cases} \dot{z}_{\tilde{q}_{i-1}+1}(t) = z_{\tilde{q}_{i-1}+2}(t), \\ \quad\vdots \\ \dot{z}_{\tilde{q}_i-1}(t) = z_{\tilde{q}_i}(t), \\ \dot{z}_{\tilde{q}_i}(t) = v_i(t), \end{cases} \qquad i = 1, \dots, r', \tag{4.25}$$

$$\dot{z}_N(t) = \hat{f}_N\big(v(t), z(t)\big),$$

where z_N and $\hat{f}_N$ denote components $|q_J| + 1, \dots, n$ of z and $\hat{f}$, and the state and mixed constraints $\hat{g}$ and $\hat{c}$ are given by:

$$\hat{g}_i\big(z(t)\big) = z_{\tilde{q}_{i-1}+1}(t) \leqslant 0, \quad i = 1, \dots, r', \tag{4.26}$$

$$\hat{c}_i\big(v(t), z(t)\big) = v_{i-r+r'}(t) \leqslant 0, \quad i = r + 1, \dots, r + s'. \tag{4.27}$$

Under this change of coordinates, the active state constraints $\hat{g}_i$ and their time derivatives until order q_i are linear, and the active mixed control-state constraints $\hat{c}_i$ are linear as well, and depend only on the control.

Proof. By Corollary 4.5, the Jacobian $\Gamma_{J,y}(y(t_0))$ has full-rank, equal to $|q_J|$, and since y is continuous at t_0, there exist $\delta > 0$ and a diffeomorphism ϕ defined over an open neighborhood V_y in $\mathbb{R}^n$ of $\{y(t);\ t \in (t_0 - \delta, t_0 + \delta)\}$, such that $\phi_k(y) = \Gamma_J(y)|_k$, for all $k = 1, \dots, |q_J|$.

By (2.30), there exists then an open neighborhood V_u of $u(t_0)$ in $\mathbb{R}^m$, such that all $u \in V_u$ can be partitioned in $u = (u_G, u_N) \in \mathbb{R}^{r'+s'} \times \mathbb{R}^{m-r'-s'}$, and $G^{(q)}_{J,u_G}(u(t_0), y(t_0))$ is invertible (note that $|J| = r' + s'$). Consequently, reducing V_u and V_y if necessary, the mapping

$$\psi(\cdot, y) : u \mapsto \begin{pmatrix} g_1^{(q_1)}(u, y) \\ \vdots \\ g_{r'}^{(q_{r'})}(u, y) \\ c_{r+1}(u, y) \\ \vdots \\ c_{r+s'}(u, y) \\ u_N \end{pmatrix} \tag{4.28}$$

has an invertible Jacobian $\psi_u(u, y)$, for all $(u, y) \in V_u \times V_y$. Since by assumption, u is continuous at t_0, reducing δ if necessary, V_u is a neighborhood of $\{u(t);\ t \in (t_0 - \delta, t_0 + \delta)\}$.

Therefore, (ϕ, ψ) is a $C^{q_{\max}}$ local change of state variables and nonlinear feedback, so Lemma 4.2 applies, and formulae (4.7) and (4.10)–(4.11) give the expressions (4.25) and (4.26)–(4.27). □

5. Junctions conditions analysis

In Proposition 3.6, it was shown that when assumptions (A2) and (A3) hold, the control and multipliers are smooth on the interior of the arcs of the trajectory. In this section we study the regularity of the control and multipliers at the junction between two arcs. The main result of this section is Proposition 5.3 which generalizes the result obtained by Jacobson, Lele and Speyer [18] in the particular case of a scalar control and scalar state constraint.

5.1. Junction points

The set of *junction points* (or junction times) of constraint $i = 1, \dots, r+s$, is defined as the endpoints in $(0, T)$ of the contact set Δ_i and is denoted by $\mathcal{T}^i := \partial \Delta_i$.

A *boundary* (resp. *interior*) *arc* of component g_i is a maximal *open* interval of positive measure $\mathcal{I}_i \subset [0, T]$, such that $g_i(y(t)) = 0$ (resp. $g_i(y(t)) < 0$) for all $t \in \mathcal{I}_i$. If $(\tau_{\text{en}}^i, \tau_{\text{ex}}^i)$ is a boundary arc of g_i, then τ_{en}^i and τ_{ex}^i are called respectively *entry* and *exit* point (or time) of the constraint g_i. A *touch* point τ_{to}^i in $(0, T)$ is an isolated contact point for constraint g_i (endpoint of two interior arcs). Similar definitions of boundary and interior arcs, entry, exit and touch points for the mixed control-state constraints c_i, $i = r+1, \dots, r+s$, hold. Thus entry, exit and touch points are by definition junction points.

Definition 5.1. We say that a junction point τ is *regular*, if it is endpoint of two arcs.

By the above definition, a cluster point of junction times is not a regular junction time. The (disjoint and possibly empty) sets of *regular* entry, exit and touch points of constraint g_i and c_i will be respectively denoted by $\mathcal{T}_{\text{en}}^i$, $\mathcal{T}_{\text{ex}}^i$, and $\mathcal{T}_{\text{to}}^i$. Thus $\mathcal{T}^i \supset \mathcal{T}_{\text{en}}^i \cup \mathcal{T}_{\text{ex}}^i \cup \mathcal{T}_{\text{to}}^i$ with equality for all $i = 1, \dots, r+s$, iff all the junction points are regular (equivalently, iff $\mathcal{T}^i$ is *finite* for all $i = 1, \dots, r+s$). The set of all junctions times of the trajectory (u, y) will be denoted by $\mathcal{T}$, with

$$\mathcal{T} := \bigcup_{i=1}^{r+s} \mathcal{T}^i. \tag{5.1}$$

Definition 5.2. A touch point $\tau_{\text{to}}^i \in \mathcal{T}_{\text{to}}^i$ of the state constraint g_i, for $i = 1, \dots, r$, is said to be *essential*, if it belongs to the support of the multiplier η_i, that is if $[\eta_i(\tau_{\text{to}}^i)] > 0$.

In other words, a touch point is essential, if strict complementarity locally holds at that touch point. Otherwise, it is said *nonessential*. The set of essential (resp. nonessential) touch points for constraint i will be denoted by $\mathcal{T}_{\text{to}}^{i,\text{ess}}$ (resp. $\mathcal{T}_{\text{to}}^{i,\text{nes}}$). For mixed control-state constraints, since $\lambda \in L^\infty$, we will say by extension that touch points of mixed control-state constraints are always nonessential. The regularity of u, η, λ given in Proposition 3.6 is not affected by the presence of nonessential touch points.

Recall now the alternative multipliers in Section 3.2. Let τ be a *regular* junction time, i.e. τ is the right and left endpoint of two arcs, (τ_1, τ) and (τ, τ_2), with constant set of active constraints J_1 and J_2, respectively. Note that $J_1 \cup J_2 \subset I(\tau)$, the inclusion being strict iff τ is a touch point for at least one of the constraint. The multipliers η_i^j for $j = 1, \dots, q_i$ and $i = 1, \dots, r$, being defined in (3.4) up to a polynomial function of order $j-1$ on each arc (τ_1, τ) and (τ, τ_2), their jump at τ are well-defined. According to (3.5) and (2.36), it holds, with $\nu_\tau^i := [\eta_i(\tau)] \geqslant 0$:

$$\begin{aligned}
[p^q(\tau)] &= [p(\tau)] - \sum_{i \in I(\tau)} \sum_{j=1}^{q_i} [\eta_i^j(\tau)] g_{i,y}^{(j-1)}(y(\tau)) \\
&= -\sum_{i \in I(\tau)} \left\{ \left(\nu_\tau^i + [\eta_i^1(\tau)]\right) g_{i,y}(y(\tau)) + \sum_{j=2}^{q_i} [\eta_i^j(\tau)] g_{i,y}^{(j-1)}(y(\tau)) \right\}.
\end{aligned} \tag{5.2}$$

<table>
<tr><th>q</th><th>entry/exit points</th><th>ess. touch points</th></tr>
<tr><td>0</td><td rowspan="3">0</td><td rowspan="2"></td></tr>
<tr><td>1</td></tr>
<tr><td>2</td><td></td></tr>
<tr><td>3</td><td rowspan="2">2</td><td>1</td></tr>
<tr><td>4</td><td></td></tr>
<tr><td>5</td><td rowspan="2">4</td><td>3</td></tr>
<tr><td>6</td><td></td></tr>
</table>

Fig. 1. Order of continuity of the control at a regular junction point, in function of the order of the constraint q and the nature of the junction point (in the scalar case).

5.2. Junction conditions

We say that a function $u \in L^\infty(0, T; \mathbb{R}^m)$ is *continuous until order* $k \geqslant 0$ at point $\tau \in (0, T)$, if u and its time derivatives $\dot{u}, \dots, u^{(k)}$ are continuous at τ. We say that u is *discontinuous at order* $k' \geqslant 1$ at point τ, if u is continuous until order $k' - 1$ and if the time derivative $u^{(k')}$ of order k' is discontinuous at τ. This integer k' will be called the *order of discontinuity* of the control. If u is not continuous at τ (resp. if u is C^∞ at τ), we say that u has order of discontinuity 0 (resp. ∞).

The next theorem is an extension of the junction conditions results of Jacobson, Lele and Speyer [18] to the case of a vector-valued state constraint and control. Let us recall their result. Given an optimal control problem with a scalar control $u(t) \in \mathbb{R}$ and a scalar state constraint $g(y(t)) \leqslant 0$, if (u, y) is a stationary point satisfying assumptions (A2)–(A3), then the time derivatives of u are continuous at a regular junction point until an order that depends on the *order* q of the (scalar) state constraint, and on the *nature* of the junction point (regular entry/exit points versus essential touch points). More precisely, for constraints of first order, u is continuous at entry/exit points, and essential touch points cannot occur (see Proposition 3.1(ii)). For constraints of *even* order $q \geqslant 2$, u is continuous until order $q - 2$ at regular entry/exit points and essential touch points. For constraints of *odd* order $q \geqslant 3$, u is continuous until order $q - 1$ at regular entry/exit points and until order $q - 2$ at essential touch points. The result is illustrated in Fig. 1. The junction condition results for mixed control-constraints ($q = 0$) were added.

When studying the second-order necessary condition (see Section 6), we have to compute the expression (6.11) at junction points τ. To this end, we use Taylor expansions of the nominator and denominator in the neighborhood of τ, and for this we need to know the order of discontinuity of the function $g_i(y(t))$ at regular entry/exit points. Since $\frac{\mathrm{d}^{q_i}}{\mathrm{d}t^{q_i}} g_i(y(t)) = g_i^{(q_i)}(u(t), y(t))$, we see that the order of discontinuity of $g_i(y(t))$ is at least q_i plus the order of discontinuity of the control.

Proposition 5.3. *Assume that the data are* (*at least*) $C^{2q_{\max}}$. *Let* (u, y) *be a stationary point of* $(\mathcal{P})$, *and let* $\tau \in (0, T)$ *be a regular junction point. Assume that* u *is continuous at* τ *and that* (2.44) *and* (2.30) *are satisfied at* $t = \tau$. *Let*

$$q_\tau := \min\{q_i;\ \tau \in \mathcal{T}_{\mathrm{en}}^i \cup \mathcal{T}_{\mathrm{ex}}^i \cup \mathcal{T}_{\mathrm{to}}^{i,\mathrm{ess}},\ i \in I(\tau)\}. \tag{5.3}$$

(i) *If* $q_\tau \geqslant 3$, *then the control is continuous at* τ *until order* $q_\tau - 2$.
(ii) *If in addition, the following holds*:

$$\begin{aligned} &q_\tau \text{ is odd, and for all } i \text{ such that } q_i = q_\tau \text{ and } \tau \in \mathcal{T}^i \setminus \mathcal{T}_{\mathrm{to}}^{i,\mathrm{nes}}, \\ &\tau \text{ is an entry or exit point, i.e. } \tau \in \mathcal{T}_{\mathrm{en}}^i \cup \mathcal{T}_{\mathrm{ex}}^i, \end{aligned} \tag{5.4}$$

then the control is continuous at τ *until order* $q_\tau - 1$.

The alternative multipliers $\eta_i^{q_i}$ *for all* $i = 1, \dots, r + s$ *such that* $\tau \in \operatorname{int} \Delta_i$ *are continuous at* τ *until the same order as the control. In particular,*

$$\text{(i}'\text{)}\quad \textit{If } q_\tau \geqslant 3,\qquad \nu_\tau^i = [\eta_i(\tau)] = 0\quad \textit{for all } i \in I(\tau) \textit{ such that } q_i < q_\tau, \tag{5.5}$$

$$\text{(ii}'\text{)}\quad \textit{If (5.4) holds},\quad \nu_\tau^i = [\eta_i(\tau)] = 0\quad \textit{for all } i \in I(\tau) \textit{ such that } q_i \leqslant q_\tau. \tag{5.6}$$

Remark 5.4. If $q_\tau = 1$, then (5.4) always holds since components of first order of the state constraint have no essential touch points by Proposition 3.1(ii). It follows then from Proposition 3.1 that point (i′) (resp. (ii′)) of Proposition 5.3 holds true when $q_\tau = 2$ (resp. $q_\tau = 1$).

Proof. Let $\tau \in \mathcal{T}$ be such that $q_\tau > 2$. Assume w.l.o.g. that

$$I(\tau) = \{1, \dots, r'\} \cup \{r+1, \dots, r+s'\} =: J,\quad 1 \leqslant q_1 \leqslant \cdots \leqslant q_{r'}. \tag{5.7}$$

We will use the local invariance of stationary points of Lemma 4.2 for the particular choice of (ϕ, ψ) given in Lemma 4.6, and write the optimality conditions in these variables (v, z). Since $u(t) = \bar\psi(v(t), z(t))$, $\bar\psi$ is $C^{q_{\max}}$, and $\bar\psi_v(v(t), z(t))$ is invertible in the neighborhood of τ, the continuity of $u, \dots, u^{(j)}$ for $j \leqslant q_{\max}$ is equivalent to the continuity of $v, \dots, v^{(j)}$. Assume w.l.o.g. that $\delta > 0$ is so small that $\mathcal{T} \cap (\tau - \delta, \tau + \delta) = \{\tau\}$. Define

$$r_k := \operatorname{Card}\{i \in I(\tau);\ 1 \leqslant q_i \leqslant k\},\quad 0 \leqslant k \leqslant q_{\max},\quad r_0 := 0.$$

Then $r_{q_{\max}} = r'$, and the useful relation below holds, for all $1 \leqslant i \leqslant r'$ and $1 \leqslant k \leqslant q_{\max}$:

$$r_{k-1} < i \leqslant r_k \quad \text{iff} \quad q_i = k. \tag{5.8}$$

Denote the nonlinear part of the Hamiltonian by:

$$\hat L(v, z, \pi_N) := \hat\ell(v, z) + \sum_{k=|q_J|+1}^{n} \pi_k \hat f_k(v, z) = \hat\ell(v, z) + \pi_N \hat f_N(v, z),$$

where, similarly to y_N and $\hat f_N$, we denote by π_N the last $n - |q_J|$ components of π, and still denote $\tilde q_i := \sum_{l=1}^i q_l$ for $i = 0, \dots, r'$. Then (v, z) is solution on $(\tau - \delta, \tau + \delta)$ of the state equation (4.25), and, since

$$\hat G_J^{(q)}(v, z) = (v_1, \dots, v_{r'}, v_{r'+1}, \dots, v_{r'+s'})^\top,$$

the *alternative* costate and control equations (recall Lemma 3.4 and Remark 3.5) satisfied on $(\tau - \delta, \tau) \cup (\tau, \tau + \delta)$ are respectively given by:

$$\begin{cases} -\dot\pi^q_{\tilde q_{i-1}+1}(t) = \hat L_{z_{\tilde q_{i-1}+1}}(v(t), z(t), \pi_N^q(t)), \\ -\dot\pi^q_{\tilde q_{i-1}+2}(t) = \hat L_{z_{\tilde q_{i-1}+2}}(v(t), z(t), \pi_N^q(t)) + \pi^q_{\tilde q_{i-1}+1}(t), \\ \quad\vdots \\ -\dot\pi^q_{\tilde q_i}(t) = \hat L_{z_{\tilde q_i}}(v(t), z(t), \pi_N^q(t)) + \pi^q_{\tilde q_i - 1}(t), \end{cases} \quad i = 1, \dots, r', \tag{5.9}$$

$$-\dot\pi_N^q(t) = \hat L_{z_N}\big(v(t), z(t), \pi_N^q(t)\big); \tag{5.10}$$

$$0 = \hat L_{v_i}\big(v(t), z(t), \pi_N^q(t)\big) + \pi^q_{\tilde q_i}(t) + \eta_i^{q_i}(t),\quad i = 1, \dots, r', \tag{5.11}$$

$$0 = \hat L_{v_i}\big(v(t), z(t), \pi_N^q(t)\big) + \eta^0_{i-r'+r}(t),\quad i = r'+1, \dots, r'+s', \tag{5.12}$$

$$0 = \hat L_{v_N}\big(v(t), z(t), \pi_N^q(t)\big), \tag{5.13}$$

where v_N denotes the remaining $m - r' - s'$ components of the control. Since $\hat g_{i,y}^{(j-1)}(z)$ is the $(\tilde q_{i-1} + j)$th basis vector, by (5.2), the jump of each component of π^q satisfies, using that $\tilde q_{i-1} + 1 = i$ if $i \leqslant r_1$ (recall that here, $\nu_\tau^i = [\eta_i(\tau)] \geqslant 0$ and by Proposition 3.1(ii), $\nu_\tau^i = 0$ if $q_i = 1$, i.e. if $i \leqslant r_1$ by (5.8)):

$$\begin{aligned} &[\pi_i^q(\tau)] + [\eta_i^1(\tau)] = -\nu_\tau^i = 0,\quad i = 1, \dots, r_1, \\ &[\pi^q_{\tilde q_{i-1}+1}(\tau)] + [\eta_i^1(\tau)] = -\nu_\tau^i \leqslant 0,\quad i = r_1+1, \dots, r', \\ &[\pi^q_{\tilde q_{i-1}+j}(\tau)] + [\eta_i^j(\tau)] = 0,\quad j = 2, \dots, q_i,\ i = r_1+1, \dots, r', \\ &[\pi_N^q(\tau)] = 0. \end{aligned} \tag{5.14}$$

For future reference, we rewrite the above relations as

$$\begin{aligned}
&\big[\pi^q_{\tilde q_i}(\tau)\big]+\big[\eta_i^{q_i}(\tau)\big]=-\nu^i_\tau=0,\quad i=1,\dots,r_1,\\
&\big[\pi^q_{\tilde q_i-q_i+1}(\tau)\big]+\big[\eta_i^{1}(\tau)\big]=-\nu^i_\tau\leqslant 0,\quad i=r_1+1,\dots,r',\\
&\big[\pi^q_{\tilde q_i-j}(\tau)\big]+\big[\eta_i^{q_i-j}(\tau)\big]=0,\quad j=0,\dots,q_i-2,\ i=r_1+1,\dots,r',\\
&\big[\pi^q_N(\tau)\big]=0.
\end{aligned}\tag{5.15}$$

By Proposition 3.6, the control and state constraint alternative multiplier η^q are $C^{q_{\max}}$ on interiors of arcs, therefore we may define over $(\tau-\delta,\tau)\cup(\tau,\tau+\delta)$ the functions a_i^j for $i=1,\dots,r'+s'$ and $j=0,\dots,q_{\max}$ by:

$$a_i^0(t):=\hat L_{v_i}\big(v(t),z(t),\pi^q_N(t)\big),$$

$$\begin{cases} a_i^{j+1}(t):=-\frac{\mathrm d}{\mathrm dt}a_i^j(t)+\hat L_{z_{\tilde q_i-j}}(v(t),z(t),\pi_N^q(t)), & 0\leqslant j\leqslant q_i-1,\\ a_i^{j+1}(t):=-\frac{\mathrm d}{\mathrm dt}a_i^j(t), & q_i\leqslant j\leqslant q_{\max}.\end{cases}$$

After j derivations of row i of (5.11) and (5.12), $1\leqslant j\leqslant q_{\max}$, we obtain using (5.9) that the following holds, on $(\tau-\delta,\tau)\cup(\tau,\tau+\delta)$:

$$0=a_i^j(t)+\pi^q_{\tilde q_i-j}(t)+\eta_i^{q_i-j}(t),\quad 1\leqslant j\leqslant q_i-1,\ i=1,\dots,r',\tag{5.16}$$

$$0=a_i^j(t)+(-1)^{q_i-j}\eta_i^{q_i-j}(t),\quad q_i\leqslant j\leqslant q_{\max},\ i=1,\dots,r',\tag{5.17}$$

$$0=a_i^j(t)+(-1)^{-j}\eta^{-j}_{i-r'+r}(t),\quad 1\leqslant j\leqslant q_{\max},\ i=r'+1,\dots,r'+s'.\tag{5.18}$$

Here, for all $i\in J$, we define for $q_i-j\leqslant 0$, $\eta_i^{q_i-j}:=(-1)^{q_i}\frac{\mathrm d^j}{\mathrm dt^j}\eta_i^{q_i}(t)$. We have, by definition of the functions a_i^j, for all $1\leqslant j\leqslant q_{\max}$ and $i=1,\dots,r'+s'$, with (5.9)–(5.10),

$$\begin{aligned}a_i^j(t)={}&(-1)^j\hat L_{v_iv}\big(v(t),z(t),\pi_N^q(t)\big)v^{(j)}(t)\\&+\text{a continuous function of }\big(v^{(j-1)}(t),\dots,v(t),z(t),\pi^q_N(t)\big).\end{aligned}\tag{5.19}$$

This implies in particular that if $v,\dots,v^{(j-1)}$ are continuous at τ, then the jump of a_i^j at time τ is given by

$$\big[a_i^j(\tau)\big]=(-1)^j\hat L_{v_iv}\big(v(t),z(t),\pi^q_N(\tau)\big)\big[v^{(j)}(\tau)\big].$$

Similarly, by derivations of (5.13), we obtain, for all $1\leqslant j\leqslant q_{\max}$:

$$\begin{aligned}0={}&(-1)^j\hat L_{v_Nv}\big(v(t),z(t),\pi_N^q(t)\big)v^{(j)}(t)\\&+\text{a continuous function of }\big(v^{(j-1)}(t),\dots,v(t),z(t),\pi^q_N(t)\big).\end{aligned}\tag{5.20}$$

Let us show now that the time derivatives of the control v are continuous until order $q_\tau-2$. By assumption, v is continuous at τ. By induction, assume that $v,\dots,v^{(j-1)}$ are continuous at τ, for $j<q_\tau-2$. Taking the jump at τ in (5.16)–(5.17) and (5.20), we obtain, for $i=1,\dots,r'+s'$ (recall that by (5.8), $i\leqslant r_j$ iff $1\leqslant q_i\leqslant j$):

$$\begin{aligned}
&0=(-1)^j\hat L_{v_iv}\big(v(\tau),z(\tau),\pi^q_N(\tau)\big)\big[v^{(j)}(\tau)\big]+(-1)^{q_i-j}\big[\eta_i^{q_i-j}(\tau)\big],\quad i\leqslant r_j,\\
&0=(-1)^j\hat L_{v_iv}\big(v(\tau),z(\tau),\pi^q_N(\tau)\big)\big[v^{(j)}(\tau)\big]+\big[\pi^q_{\tilde q_i-j}(\tau)\big]+\big[\eta_i^{q_i-j}(\tau)\big],\quad r_j<i\leqslant r',\\
&0=(-1)^j\hat L_{v_iv}\big(v(\tau),z(\tau),\pi^q_N(\tau)\big)\big[v^{(j)}(\tau)\big]+(-1)^{-j}\big[\eta^{-j}_{i-r'+r}(\tau)\big],\quad i>r',\\
&0=(-1)^j\hat L_{v_Nv}\big(v(\tau),z(\tau),\pi^q_N(\tau)\big)\big[v^{(j)}(\tau)\big].
\end{aligned}\tag{5.21}$$

We denote in the sequel by $v_{k+1:l}$ the subvector of components $k+1,\dots,l$ of v. Similarly, $\nu_\tau^{k+1:l}$ denotes the column vector of components ν^i_τ for $i=k+1,\dots,l$. Recall that by (5.8), $q_i-j=1$ iff $r_j<i\leqslant r_{j+1}$, and $q_i-j>1$

iff $i > r_{j+1}$. Since $\hat{H}^0_{vv} = \hat{L}_{vv}$ depends only on $(v, z, \pi^q_N = \pi_N)$, we write in what follows $\hat{H}^0_{vv}(v, z, \pi^q_N)$ instead of $\hat{H}^0_{vv}(v, z, \pi, \lambda)$, and using (5.15), Eqs. (5.21) become:

$$\hat{H}^0_{vv}\big(v(\tau), z(\tau), \pi^q_N(\tau)\big) \begin{pmatrix} [v^{(j)}_{1:r_j}(\tau)] \\ [v^{(j)}_{r_j+1:r_{j+1}}(\tau)] \\ [v^{(j)}_{r_{j+1}+1:r'}(\tau)] \\ [v^{(j)}_{r'+1:r'+s'}(\tau)] \\ [v^{(j)}_{r'+s'+1:m}(\tau)] \end{pmatrix} = \begin{pmatrix} (-1)^{q_i+1}[\eta^{q_i-j}_i(\tau)] \\ (-1)^j \nu_\tau^{r_j+1:r_{j+1}} \\ 0 \\ -[\eta^{-j}_{i-r'+r}(\tau)] \\ 0 \end{pmatrix}. \tag{5.22}$$

By Remark 4.3, $\hat{H}^0_{vv}(v(\tau), z(\tau), \pi^q_N(\tau))$ satisfies (2.44) for some positive constant α'. Since $[v^{(j)}(\tau)]$ is such that $\hat{g}^{(q_i)}_{i,v}(v(\tau), z(\tau))[v^{(j)}(\tau)] = [v^{(j)}_i(\tau)] = 0$ for all $i = 1, \dots, r'$ such that $\tau \in \operatorname{int} \Delta_i$, and $\hat{g}^{(q_i)}_{i,v}(v(\tau), z(\tau))[v^{(j)}(\tau)] = [v^{(j)}_{i+r'-r}(\tau)] = 0$ for all $i = r+1, \dots, r+s'$ such that $\tau \in \operatorname{int} \Delta_i$, it follows that

$$\alpha' \big|\big[v^{(j)}(\tau)\big]\big|^2 \leqslant \big[v^{(j)}(\tau)\big]^\top \hat{H}^0_{vv}\big(v(\tau), z(\tau), \pi^q_N(\tau)\big)\big[v^{(j)}(\tau)\big]. \tag{5.23}$$

For all $j \leqslant q_\tau - 1$, by definition of q_τ, we have $\tau \in \operatorname{int} \Delta_i$, for all $i = 1, \dots, r_j$ and hence, $[v^{(j)}_i(\tau)] = 0$ for all $i = 1, \dots, r_j$. Since $q_\tau > 0$, we have for the same reason $[v^{(j)}_i(\tau)] = 0$ for all $i = r'+1, \dots, r'+s'$. Therefore, (5.22) writes

$$\hat{H}^0_{vv}\big(v(\tau), z(\tau), \pi^q_N(\tau)\big) \begin{pmatrix} 0 \\ [v^{(j)}_{r_j+1:r_{j+1}}(\tau)] \\ [v^{(j)}_{r_{j+1}+1:r'}(\tau)] \\ 0 \\ [v^{(j)}_{r'+s'+1:m}(\tau)] \end{pmatrix} = \begin{pmatrix} (-1)^{q_i+1}[\eta^{q_i-j}_i(\tau)] \\ (-1)^j \nu_\tau^{r_j+1:r_{j+1}} \\ 0 \\ -[\eta^{-j}_{i-r'+r}(\tau)] \\ 0 \end{pmatrix}. \tag{5.24}$$

For $j \leqslant q_\tau - 2$, we also have $\tau \in \operatorname{int} \Delta_i$, for all $i \leqslant r_{j+1}$, and hence $[v^{(j)}_{r_j+1:r_{j+1}}(\tau)] = 0$. Multiplying on the left (5.24) by $[v^{(j)}(\tau)]^\top$, we obtain that the product with the right-hand side is zero, and therefore $[v^{(j)}(\tau)]^\top \hat{H}^0_{vv}(v(\tau), z(\tau), \pi^q_N(\tau))[v^{(j)}(\tau)] = 0$. From (5.23) it follows that $v^{(j)}$ is continuous at τ, and the right-hand side in (5.24) is equal to zero. This implies that the alternative multipliers $\eta^{q_i}_i$ are C^j at τ, and the second row of (5.14) is satisfied with equality, that is $\nu^i_\tau = 0$, for all $i = 1, \dots, r_{j+1}$, i.e. such that $q_i \leqslant j+1 \leqslant q_\tau - 1$ and $\tau \in \operatorname{int} \Delta_i$. By induction, we proved that $v, \dots, v^{(q_\tau-2)}$ are continuous. This shows (i) and (i′).

Let now $j = q_\tau - 1$. Assume that (5.4) holds, i.e. q_τ is odd, and attained at entry/exit points. Then we have, near the boundary arc, due to the continuity of $v_i, \dots, v^{(q_\tau-2)}_i$ vanishing at entry/exit of boundary arc, for all $i = r_{q_\tau-1}+1, \dots, r_{q_\tau}$ (and hence $q_i = q_\tau$):

$$z_{\tilde{q}_{i-1}+1}(t) = \frac{(t-\tau)^{(2q_\tau-1)}}{(2q_\tau-1)!} v^{(q_\tau-1)}_i\big(\tau^\pm\big) + \mathcal{O}\big((t-\tau)^{2q_\tau}\big) \leqslant 0,$$

from which we deduce that $[v^{(q_\tau-1)}_i(\tau)] \leqslant 0$ at both entry and exit times. We still have $[v^{(q_\tau-1)}_i(\tau)] = 0$ for $i \leqslant r_{q_\tau-1}$ and for $i = r'+1, \dots, r'+s'$, since $q_i \leqslant q_\tau - 1$ implies that we are on the interior of a boundary arc for constraint i. Since $v, \dots, v^{(q_\tau-2)}$ are continuous, (5.24) holds for $j = q_\tau - 1$, hence we obtain by (5.23) and (5.14), since $\nu_\tau^{r_{q_\tau-1}+1:r_{q_\tau}} \geqslant 0$:

$$\begin{aligned} \alpha' \big|\big[v^{(q_\tau-1)}(\tau)\big]\big|^2 &\leqslant \big[v^{(q_\tau-1)}(\tau)\big]^\top \hat{H}^0_{vv}\big(v(\tau), z(\tau), \pi^q_N(\tau)\big)\big[v^{(q_\tau-1)}(\tau)\big] \\ &= (-1)^{q_\tau-1}\big[v^{(q_\tau-1)}_{r_{q_\tau-1}+1:r_{q_\tau}}(\tau)\big]^\top \nu_\tau^{r_{q_\tau-1}+1:r_{q_\tau}} \leqslant 0, \end{aligned}$$

which implies that $v^{(q_\tau-1)}$ is also continuous, and $\nu^i_\tau = 0$ for all $i \in I(\tau)$ such that $q_i = q_\tau$. This shows (ii) and (ii′) and achieves the proof. □

6. No-gap second-order optimality conditions

In this section, we extend the no-gap second-order optimality conditions of [1] given in the scalar case, to several state constraints, and include mixed control-state constraints. The main results of the section are Theorem 6.1 and Corollary 6.2.

6.1. Abstract optimization framework and main result

We consider here the abstract formulation (2.5) of $(\mathcal{P})$. We say that a local solution u of (2.5) satisfies the *quadratic growth condition*, if there exist $c, \rho > 0$ such that

$$J(u') \geqslant J(u) + c\|u' - u\|_2^2, \quad \text{for all } u'\text{: } \|u' - u\|_\infty < \rho,\ G(u') \in K,\ \mathcal{G}(u') \in \mathcal{K}. \tag{6.1}$$

Recall that the Lagrangian is given by (2.33). Let $(u, y = y_u)$ be a local solution of $(\mathcal{P})$ satisfying the assumptions of Theorem 2.5, with (unique) multipliers p, η and λ. A second-order necessary condition for (2.5) due to Kawasaki [20] is as follows:

$$D^2_{uu}L(u;\eta,\lambda)(v,v) - \sigma\big(\eta, T_K^{2,i}\big(G(u), DG(u)v\big)\big) - \sigma\big(\lambda, T_{\mathcal{K}}^{2,i}\big(\mathcal{G}(u), D\mathcal{G}(u)v\big)\big) \geqslant 0, \tag{6.2}$$

for all directions v in the critical cone $C(u)$ defined by

$$C(u) := \big\{v \in \mathcal{U}\text{: } DJ(u)v \leqslant 0,\ DG(u)v \in T_K\big(G(u)\big),\ D\mathcal{G}(u)v \in T_{\mathcal{K}}\big(\mathcal{G}(u)\big)\big\}. \tag{6.3}$$

Here $T_P(x)$ (for $P = K$ or $\mathcal{K}$) denotes the tangent cone (in the sense of convex analysis) to the set P at point $x \in P$, $T_P^{2,i}(x,h)$ is the *inner second-order tangent set* to P at $x \in P$ in direction h,

$$T_P^{2,i}(x,h) := \left\{w\text{: dist}\left(x + \varepsilon h + \frac{\varepsilon^2}{2}w, P\right) = \mathrm{o}\big(\varepsilon^2\big),\ \forall \varepsilon > 0\right\},$$

and $\sigma(\cdot, S)$ denotes the *support function* of the set S, defined for $\xi \in X^*$ by $\sigma(\xi, S) = \sup_{x\in S}\langle \xi, x\rangle$. The critical cone can be characterized as follows:

$$C(u) = \big\{v \in \mathcal{U}\text{: } DG(u)v \in T_K\big(G(u)\big) \cap \eta^\perp,\ D\mathcal{G}(u)v \in T_{\mathcal{K}}\big(\mathcal{G}(u)\big) \cap \lambda^\perp\big\}. \tag{6.4}$$

The term

$$\Sigma(u,v) := \sigma\big(\eta, T_K^{2,i}\big(G(u), DG(u)v\big)\big) + \sigma\big(\lambda, T_{\mathcal{K}}^{2,i}\big(\mathcal{G}(u), D\mathcal{G}(u)v\big)\big) \tag{6.5}$$

in (6.2) is called the *curvature term*. It is nonpositive, for all $v \in C(u)$. Note that the component i of $DG(u)v$ (resp. $D\mathcal{G}(u)v$) is the function $g_{i,y}(y(\cdot))z_v(\cdot)$ (resp. $c_{i,u}(u(\cdot), y(\cdot))v(\cdot) + c_{i,y}(u(\cdot), y(\cdot))z_v(\cdot)$), where z_v is the solution of the linearized state equation (2.22).

When there are only mixed control-state constraints, it is known that the latter have no contribution in the curvature term (6.5). This follows from the extended polyhedricity framework, see [5, Propositions 3.53 and 3.54] (the cone $\mathcal{K}$ is a polyhedric subset of L^∞ and $D\mathcal{G}(u)$ is "onto" by (2.31)). On the contrary, pure state constraints may have a nonzero contribution in the curvature term (6.5).

Since K has a product form, $K \equiv (K_0)^r$ with $K_0 := C_-[0,T]$, the *inner* second-order tangent set is also given under a product expression. This would be false, however, for the *outer* second-order tangent-set, see e.g. [5, p. 168]. Therefore we have, for $x = (x_i)_{1\leqslant i\leqslant r} \in K$ and $h = (h_i)_{1\leqslant i\leqslant r} \in T_K(x)$:

$$T_K^{2,i}(x,h) = \prod_{i=1}^r T_{K_0}^{2,i}(x_i, h_i). \tag{6.6}$$

Since the support function of a Cartesian product of sets is the sum of the support function for each set, the expression of pure state constraints in the curvature term can be deduced from the result by Kawasaki [21] for $K_0 = C_-[0,T]$. Recall that Δ_i is given by (2.14), and the *second-order contact set* is defined, for $v \in \mathcal{V}$, by

$$\Delta_i^2 := \big\{t \in \Delta_i;\ g_{i,y}\big(y(t)\big)z_v(t) = 0\big\}, \quad i = 1, \dots, r. \tag{6.7}$$

Then, by [21], we have

$$\sigma\big(\eta, T_K^{2,i}\big(G(u), DG(u)v\big)\big) = \sum_{i=1}^{r} \sigma\big(\eta_i, T_{K_0}^{2,i}\big(g_i(y), g_{i,y}(y)z_v\big)\big) = \sum_{i=1}^{r} \int_0^T \varsigma_i(t)\,\mathrm{d}\eta_i(t), \tag{6.8}$$

where, for all $i = 1, \dots, r$:

$$\varsigma_i(t) = \begin{cases} 0 & \text{if } t \in (\operatorname{int} \Delta_i) \cap \Delta_i^2, \\ \liminf\limits_{t' \to t;\, g_i(y(t'))<0} \dfrac{(\{g_{i,y}(y(t'))z_v(t')\}_+)^2}{2g_i(y(t'))} & \text{if } t \in (\partial \Delta_i) \cap \Delta_i^2, \\ +\infty & \text{otherwise} \end{cases} \tag{6.9}$$

where $h_+(t) := \max(0, h(t))$. We denote in the sequel by $\operatorname{supp}(\mathrm{d}\eta_i)$ the *support* of the measure η_i. We make the following assumption:

(A4) (i) Each component of the state constraint g_i, $i = 1, \dots, r$, has *finitely many* junctions times, and the state constraint is not active at final time, $g_i(y(T)) < 0$, $i = 1, \dots, r$.

This assumption implies that all entry and exit times of state constraints are regular. Using (6.8), and the fact that $\operatorname{supp}(\mathrm{d}\eta_i) \subset \Delta_i^2$ for all critical directions v, the curvature term has the expression below, for $v \in C(u)$ (see [21]), with $\nu_\tau^i = [\eta_i(\tau)]$

$$\sigma\big(\eta, T_K^{2,i}\big(G(u), DG(u)v\big)\big) = \sum_{i=1}^{r} \sum_{\tau \in \mathcal{T}_i \cap \Delta_i^2} \nu_\tau^i \varsigma_i(\tau). \tag{6.10}$$

We thus need to compute, for junction times $\tau \in \mathcal{T}_i \cap \Delta_i^2$,

$$\varsigma_i(\tau) = \liminf_{t \to \tau;\, g_i(y(t))<0} \frac{(\{g_{i,y}(y(t))z_v(t)\}_+)^2}{2g_i(y(t))}. \tag{6.11}$$

The tangentiality conditions (see assumption (A5)(i) below), under which boundary arcs with regular entry/exit points of state constraints have no contribution to the curvature term, are more delicate to state than in the scalar case, due to the possibility of having coinciding junction times of different components of the state constraints. Let $i = 1, \dots, r$ and $\tau \in \mathcal{T}_{\mathrm{en}}^i \cup \mathcal{T}_{\mathrm{ex}}^i$. Denote by k_i^τ the order of discontinuity at point τ of the function (of time) $g_i^{(q_i)}(u(t), y(t))$. By Proposition 5.3, we necessarily have $k_i^\tau \geqslant q_\tau - 1$. A Taylor expansion of the denominator in (6.11) gives then, in the neighborhood of τ on the interior arc-side

$$g_i\big(y(t)\big) = g_i^{(q_i+k_i^\tau)}\big(\tau^\pm\big)\frac{(t-\tau)^{q_i+k_i^\tau}}{(q_i+k_i^\tau)!} + \mathrm{o}\big((t-\tau)^{q_i+k_i^\tau}\big), \tag{6.12}$$

with $\tau^\pm = \tau^-$ (resp. τ^+) if $\tau \in \mathcal{T}_{\mathrm{en}}^i$ (resp. $\tau \in \mathcal{T}_{\mathrm{ex}}^i$), and $g_i^{(q_i+k_i^\tau)}(\tau^\pm) := \frac{\mathrm{d}^{q_i+k_i^\tau}}{\mathrm{d}t^{q_i+k_i^\tau}} g_i(y(t))|_{t=\tau^\pm}$ is nonzero by definition of k_i^τ.

Assume now that strict complementarity holds near τ on the boundary arc, in the sense that there exists $\varepsilon > 0$ small such that

$$[\tau, \tau+\varepsilon] \subset \operatorname{supp}(\mathrm{d}\eta_i) \quad \text{if } \tau \in \mathcal{T}_{\mathrm{en}}^i \quad \big(\text{resp. } [\tau-\varepsilon, \tau] \subset \operatorname{supp}(\mathrm{d}\eta_i) \text{ if } \tau \in \mathcal{T}_{\mathrm{ex}}^i\big). \tag{6.13}$$

Since $g_{i,y}(y)z_v \in W^{q_i,\infty}(0,T)$ by Lemma 2.2, for all critical directions $v \in C(u)$, the first $q_i - 1$ time derivatives of $g_{i,y}(y)z_v$ being continuous vanish at entry/exit of boundary arcs, and hence the following expansion holds, for t in the neighborhood of τ on the side of the interior arc of g_i:

$$g_{i,y}\big(y(t)\big)z_v(t) = \mathcal{O}\big((t-\tau)^{q_i}\big). \tag{6.14}$$

We thus obtain with (6.12) and (6.14) that there exists a constant $C > 0$ such that

$$\big|\varsigma_i(\tau)\big| \leqslant \lim_{t \to \tau} C|t-\tau|^{q_i - k_i^\tau}. \tag{6.15}$$

It follows that

$$\varsigma_i(\tau) > -\infty \quad \text{if } k_i^\tau \leqslant q_i \quad \text{and} \quad \varsigma_i(\tau) = 0 \quad \text{if } k_i^\tau < q_i. \tag{6.16}$$

Since $k_i^\tau \geqslant q_\tau - 1$ by Proposition 5.3, and $q_i \geqslant q_\tau$ whenever τ is an entry or exit point of constraint g_i, it makes sense to assume that $q_\tau - 1 \leqslant k_i^\tau \leqslant q_i$. In addition, the continuity of u implies that $k_i^\tau \geqslant 1$. By (6.16), we see that whenever

$$\max(1, q_\tau - 1) \leqslant k_i^\tau < q_i \tag{6.17}$$

then $\varsigma_i(\tau) = 0$, and hence $\nu_\tau^i \varsigma_i(\tau) = 0$.

Clearly, (6.17) requires that $q_i > 1$. In addition, when (5.4) holds and $q_i = q_\tau$, then it is necessary by Proposition 5.3(ii) that $k_i^\tau \geqslant q_\tau = q_i$, which is incompatible with (6.17). Therefore, we cannot assume that (6.17) holds when either $q_i = 1$ or (5.4) holds and $q_i = q_\tau$, and will rather assume in that case that

$$k_i^\tau = q_i. \tag{6.18}$$

By (6.16), assumption (6.18) ensures that $\varsigma_i(\tau)$ is finite. Moreover, if $q_i = 1$, then $\nu_\tau^i = 0$ by Proposition 3.1(ii), implying that $\nu_\tau^i \varsigma_i(\tau) = 0$. If (5.4) holds and $q_i = q_\tau$, then by Proposition 5.3(ii′), we have $\nu_\tau^i = 0$, i.e. $\nu_\tau^i \varsigma_i(\tau) = 0$ again. This shows that boundary arcs have no contribution to the curvature term (6.10) when assumptions (6.13) and (A5)(i) below hold:

(A5) (i) For all junction point $\tau \in \mathcal{T}_i$, $i = 1, \ldots, r$, if τ is an *entry or exit time* of constraint g_i, the function of time $g_i(y(t))$ has order of discontinuity $q_i + k_i^\tau$, and k_i^τ satisfies

$$\begin{cases} (6.18) & \text{if } q_i = 1 \text{ or if (5.4) holds and } q_i = q_\tau, \\ (6.17) & \text{otherwise.} \end{cases}$$

In the case when the junction times of the different components of the state constraints do not coincide (see assumption (A7) in Section 7), then assumption (A5)(i) has the simpler form (7.57) (see Remark 7.5).

The contribution of touch points to the curvature term (6.10) is classical, when the touch points are reducible, in the following sense. A touch point τ of a component g_i of the state constraint of order $q_i \geqslant 2$ is said to be *reducible*, if $t \mapsto \frac{\mathrm{d}^2}{\mathrm{d}t^2} g_i(y(t))$ is continuous at τ, and if

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} g_i\big(y(t)\big)\bigg|_{t=\tau} < 0. \tag{6.19}$$

We will make the assumption that

(A5) (ii) All *essential* touch points of constraint g_i, for all $i = 1, \ldots, r$, are *reducible*, i.e. satisfy (6.19).

Finally, we will also need the following assumption, implying (6.13):

(A6) (i) (Strict complementarity on interior of boundary arcs)

$$\frac{\mathrm{d}\eta_i}{\mathrm{d}t}(t) > 0, \quad \text{for a.a. } t \in \operatorname{int} \Delta_i, \text{ for all } i = 1, \ldots, r. \tag{6.20}$$

Let $\mathcal{V} := \mathcal{V}_2 = L^2(0, T; \mathbb{R}^m)$ and $\mathcal{Z} := \mathcal{Z}_2 = H^1(0, T; \mathbb{R}^n)$. Let

$$\hat{T}_{\mathcal{K}}\big(\mathcal{G}(u)\big) := \big\{\omega \in L^2\big(0, T; \mathbb{R}^s\big)\colon \omega_i \leqslant 0 \text{ a.e. on } \Delta_i,\ i = r+1, \ldots, r+s\big\}. \tag{6.21}$$

This is the extension of the tangent cone $T_{\mathcal{K}}(\mathcal{G}(u))$ over L^2. Since $\lambda \in L^\infty(0, T; \mathbb{R}^{r*})$, λ can be extended to a continuous linear form over $L^2(0, T; \mathbb{R}^r)$. We may then consider the extension of the critical cone over L^2 as follows:

$$\hat{C}_{L^2}(u) := \big\{v \in \mathcal{V}\colon DG(u)v \in T_K\big(G(u)\big) \cap \eta^\perp,\ D\mathcal{G}(u)v \in \hat{T}_{\mathcal{K}}\big(\mathcal{G}(u)\big) \cap \lambda^\perp\big\}. \tag{6.22}$$

We can now state the no-gap second-order conditions, that do not assume strict complementarity at touch points for the state constraints, and make no additional assumptions for the mixed control-state constraints.

Theorem 6.1.

(i) (*Necessary condition*) *Let* (u, y) *be a local solution of* $(\mathcal{P})$ *and* (p, η, λ) *its* (*unique*) *associated multipliers, satisfying* (A1)–(A3), (A4)(i), (A5)(i)–(ii) *and* (A6)(i), *and* $\nu_\tau^i = [\eta_i(\tau)]$. *Then*

$$D^2_{uu}L(u;\eta,\lambda)(v,v) - \sum_{i=1}^{r} \sum_{\tau \in \mathcal{T}_{\mathrm{to}}^{i,\mathrm{ess}}} \nu_\tau^i \frac{(g_{i,y}^{(1)}(y(t))z_v(t))^2}{\frac{\mathrm{d}^2}{\mathrm{d}t^2} g_i(y(t))|_{t=\tau}} \geqslant 0 \quad \forall v \in \hat{C}_{L^2}(u). \tag{6.23}$$

(ii) (*Sufficient condition*) *Let* (u, y) *be a stationary point of* $(\mathcal{P})$ *with multipliers* (p, η, λ), *satisfying* (2.42), *and* $\nu_\tau^i = [\eta_i(\tau)]$. *For* $i = 1, \dots, r$ *such that* $q_i \geqslant 2$, *let* $\mathcal{T}_{\mathrm{red}}^i$ *denote a* finite *set* (*possibly empty*) *of reducible touch points of constraint* g_i. *If*

$$D^2_{uu}L(u;\eta,\lambda)(v,v) - \sum_{i:\, q_i \geqslant 2} \sum_{\tau \in \mathcal{T}_{\mathrm{red}}^{i}} \nu_\tau^i \frac{(g_{i,y}^{(1)}(y(t))z_v(t))^2}{\frac{\mathrm{d}^2}{\mathrm{d}t^2} g_i(y(t))|_{t=\tau}} > 0 \quad \forall v \in \hat{C}_{L^2}(u) \setminus \{0\}, \tag{6.24}$$

then (u, y) *is a local solution of* $(\mathcal{P})$ *satisfying the quadratic growth condition* (6.1).

Note that under (A2)–(A3), $\mathcal{T}_{\mathrm{to}}^{i,\mathrm{ess}} = \emptyset$ if $q_i \leqslant 1$. It is easy to obtain from the above theorem a characterization of the quadratic growth.

Corollary 6.2. *Let* (u, y) *be a stationary point of* $(\mathcal{P})$ *with multipliers* (p, η, λ), *satisfying* (A1)–(A3), (A4)(i), (A5)(i)–(ii) *and* (A6)(i), *and* $\nu_\tau^i = [\eta_i(\tau)]$. *Then* (u, y) *is a local solution of* $(\mathcal{P})$ *satisfying the quadratic growth condition* (6.1) *iff*

$$D^2_{uu}L(u;\eta,\lambda)(v,v) - \sum_{i=1}^{r} \sum_{\tau \in \mathcal{T}_{\mathrm{to}}^{i,\mathrm{ess}}} \nu_\tau^i \frac{(g_{i,y}^{(1)}(y(t))z_v(t))^2}{\frac{\mathrm{d}^2}{\mathrm{d}t^2} g_i(y(t))|_{t=\tau}} > 0 \quad \forall v \in \hat{C}_{L^2}(u) \setminus \{0\}. \tag{6.25}$$

Denote by $Q(v)$ the left-hand side of (6.23) and (6.25). An explicit computation of the Hessian of the Lagrangian $D^2_{uu}L(u;\eta,\lambda)(v,v)$ shows that

$$\begin{aligned} Q(v) = {} & \int_0^T H^0_{(u,y),(u,y)}(u,y,p,\lambda)\big((v,z_v),(v,z_v)\big)\,\mathrm{d}t + \phi_{yy}\big(y(T)\big)\big(z_v(T),z_v(T)\big) \\ & + \sum_{i=1}^{r} \Bigg(\int_0^T g_{i,yy}\big(y(t)\big)\big(z_v(t),z_v(t)\big)\,\mathrm{d}\eta_i(t) - \sum_{\tau \in \mathcal{T}_{\mathrm{to}}^{i,\mathrm{ess}}} \nu_\tau^i \frac{(g_{i,y}^{(1)}(y(t))z_v(t))^2}{\frac{\mathrm{d}^2}{\mathrm{d}t^2} g_i(y(t))|_{t=\tau}} \Bigg). \end{aligned} \tag{6.26}$$

Let us recall that a *Legendre form* Q (see [17]) is a weakly lower semi-continuous quadratic form defined over an Hilbert space, that satisfies the following property: for all weakly convergent sequences (v_n), $(v_n) \rightharpoonup \bar{v}$, we have that $v_n \to \bar{v}$ strongly if $Q(v_n) \to Q(\bar{v})$. An example of a Legendre form is $v \mapsto \|v\|^2$, with $\|\cdot\|$ the norm of the Hilbert space. Under assumption (2.42), it is not difficult to show that (6.26) is a Legendre form (see e.g. [1, Lemma 21]). This is no more true if (2.42) is replaced by the weaker hypothesis (2.44).

6.2. Proof of Theorem 6.1

Denote the *radial cone* to $\mathcal{K}$ at point $x \in \mathcal{K}$ by:

$$\mathcal{R}_{\mathcal{K}}(x) = \big\{h \in L^\infty;\ \exists \varepsilon_0 > 0,\ x + \varepsilon h \in \mathcal{K},\ \text{for all } 0 < \varepsilon < \varepsilon_0\big\}. \tag{6.27}$$

Since $\mathcal{K}$ is a closed convex set, $T_{\mathcal{K}}(x) = \mathrm{cl}(\mathcal{R}_{\mathcal{K}}(x))$. Let

$$C_0(u) := \big\{v \in C(u),\ DG(u)v|_i(\tau) < 0,\ \text{for all } \tau \in \mathcal{T}_{\mathrm{to}}^{\mathrm{nes},i},\ i = 1,\dots,r,\ D\mathcal{G}(u)v \in \mathcal{R}_{\mathcal{K}}\big(\mathcal{G}(u)\big)\big\}. \tag{6.28}$$

This subset of the critical cone contains the critical directions that "avoid" nonessential touch points of the state constraint, and such that the derivatives of the mixed constraints belong to the radial cone $\mathcal{R}_{\mathcal{K}}(\mathcal{G}(u))$.

Lemma 6.3. *Under the assumptions of Theorem* 6.1(i), *for all* $v \in C_0(u)$, *the term* (6.5) *has the expression*

$$\Sigma(u,v) = \sum_{i=1}^{r} \sum_{\tau \in \mathcal{T}_{\mathrm{to}}^{i,\mathrm{ess}}} \nu_\tau^i \frac{(g_{i,y}^{(1)}(y(t))z_v(t))^2}{\frac{\mathrm{d}^2}{\mathrm{d}t^2} g_i(y(t))|_{t=\tau}}. \tag{6.29}$$

Proof. It is easy to see that if $D\mathcal{G}(u)v \in \mathcal{R}_{\mathcal{K}}(\mathcal{G}(u))$, then $0 \in T_{\mathcal{K}}^{2,i}(\mathcal{G}(u), D\mathcal{G}(u)v)$. Hence $\sigma(\lambda, T_{\mathcal{K}}^{2,i}(\mathcal{G}(u), D\mathcal{G}(u)v)) = 0$. It remains then in (6.5) the contribution of state constraints. As shown in the previous subsection, when assumptions (A5)(i) and (A6)(i) hold, entry and exit points of boundary arcs of the state constraints have a zero contribution to the curvature term. The term (6.11) for the contribution of essential touch points satisfying (6.19) is computed explicitly, in the same manner as in the scalar case (see [1, Proposition 14]). Finally, nonessential touch points do not belong to Δ_i^2 for $v \in C_0(u)$, and hence have no contribution in the sum (6.10). The results follows. □

Lemma 6.4. *Under the assumptions of Theorem* 6.1(i):

(i) *The set* $C_0(u)$ *is dense in* $C(u)$.
(ii) *The set* $C(u)$ *is dense in the set* $\hat{C}_{L^2}(u)$.

The key point in the proof below is the controllability Lemma 2.3, that enables to handle separately the arguments for the state constraints and for the mixed control-state constraints, in the following way. Under the assumptions of Lemma 2.3, with n_0 the n of (2.25), for all $\kappa \in [1, +\infty]$, there exists a constant $C = C(\kappa) > 0$ such that for all $(w, \omega) \in \mathcal{W}_\kappa \times L^\kappa(0, T; \mathbb{R}^s)$, with

$$\mathcal{W}_\kappa := \prod_{i=1}^{r} W^{q_i,\kappa}(0,T), \tag{6.30}$$

there exists $v \in \mathcal{V}_\kappa$ such that

$$g_{i,y}(y)z_v = w_i \quad \text{on } \Delta_i,\ \forall i = 1, \dots, r, \tag{6.31}$$

$$c_{i,u}(u,y)v + c_{i,y}(u,y)z_v = \omega_i \quad \text{a.e. on } \Delta_i^{n_0},\ \forall i = r+1, \dots, r+s, \tag{6.32}$$

$$\|v\|_\kappa \leqslant C\big(\|w\|_{\mathcal{W}_\kappa} + \|\omega\|_\kappa\big). \tag{6.33}$$

Proof. (i) Let $v \in C(u)$, and set $w := DG(u)v$ and $\omega := D\mathcal{G}(u)v$. Let φ be a C^∞ function with support in $[-1, 1]$ and which is positive on $(-1, 1)$. Set $w_{n,i} := w_i - \sum_{\tau \in \mathcal{T}_{\mathrm{to}}^{i,\mathrm{nes}}} \frac{1}{n^{q_i+1}} \varphi(n(\cdot - \tau))$ for $i = 1, \dots, r$. Then, for n large enough, $w_{n,i}(\tau) < 0$ for all $\tau \in \mathcal{T}_{\mathrm{to}}^{i,\mathrm{nes}}$, $w_{n,i} = w_i$ outside a neighborhood of $\mathcal{T}_{\mathrm{to}}^{i,\mathrm{nes}}$, and $\|w_{n,i} - w_i\|_{q_i,\infty} \to 0$ when $n \to +\infty$. Further, since $\mathcal{R}_{\mathcal{K}}(\mathcal{G}(u)) \cap \lambda^\perp$ in dense in $T_{\mathcal{K}}(\mathcal{G}(u)) \cap \lambda^\perp$ (see Lemma A.2 in Appendix A), there exists a sequence $(\omega_n) \subset \mathcal{R}_{\mathcal{K}}(\mathcal{G}(u)) \cap \lambda^\perp$ such that $\|\omega_n - \omega\|_\infty \to 0$. By the controllability Lemma 2.3, there exists $v_n \in \mathcal{U}$ that satisfies (6.31)–(6.32) with (w_n, ω_n), and $\|v_n - v\|_\infty \leqslant C(\|w_n - w\|_{\mathcal{W}_\infty} + \|\omega_n - \omega\|_\infty)$. By construction it follows that $v_n \in C_0(u)$, and $v_n \to v$ in L^∞.

(ii) Let $v \in \hat{C}_{L^2}(u)$, and again let $w := DG(u)v$ and $\omega := D\mathcal{G}(u)v$. By Lemmas 16–17 in [1] (this is where assumption (A6)(i) is used), we can construct a sequence $(w_n) \subset \prod_{i=1}^{r} W^{q_i,\infty}(0,T)$ such that $w_{n,i} = 0 = w_i$ on each boundary arc of g_i, $i = 1, \dots, r$, $w_{n,i}(\tau) = w_i(\tau)$ at each touch point $\tau \in \mathcal{T}_i$, and $\|w_{n,i} - w_i\|_{q_i,2} \to 0$. So $w_n \in T_K(G(u)) \cap \eta^\perp$. Now by Lemma A.3 in Appendix A, there exists a sequence $(\omega_n) \subset T_{\mathcal{K}}(\mathcal{G}(u)) \cap \lambda^\perp$ such that $\|\omega_n - \omega\|_2 \to 0$. By Lemma 2.3 again, there exists $v_n \in \mathcal{U}$ that satisfies (6.31)–(6.32) with (w_n, ω_n) and $\|v_n - v\|_2 \leqslant C(\|w_n - w\|_{\mathcal{W}_2} + \|\omega_n - \omega\|_2)$. By construction we have $v_n \in C(u)$, and $v_n \to v$ in L^2. □

Proof of Theorem 6.1. For the necessary condition, we use the abstract condition (6.2) and compute the curvature term (6.5). By Lemma 6.3, we have the expression of the curvature term for all $v \in C_0(u)$. Since the right-hand side of (6.29) is continuous for the norm of L^2, we obtain the result by a density argument in view of Lemma 6.4.

For the sufficient condition, we follow [1, Theorems 18 and 27]. The idea is to use a reduction approach, i.e. to reformulate the state constraint around finitely many reducible touch points of the components g_i of the state constraint of order $q_i \geqslant 2$. More precisely, for $\mathcal{T}^i_{\text{red}} := \{\tau^i_1, \dots, \tau^i_{N_i}\}$, $\varepsilon, \delta > 0$ small enough, and

$$\Omega_i := [0,T] \setminus \bigcup_{k=1}^{N_i} \left(\tau^i_k - \varepsilon, \tau^i_k + \varepsilon\right),$$

the constraint $G(u') \in K$ in (2.5) can be equivalently replaced, for all $\|u' - u\|_\infty \leqslant \delta$, by

$$g_i\left(y_{u'}(t)\right) \leqslant 0 \quad \text{for all } t \in \Omega_i \quad \text{and} \quad g_i\left(y_{u'}\left(t^i_k(u')\right)\right) \leqslant 0,\ k = 1, \dots, N_i,\ \forall i\colon q_i \geqslant 2, \tag{6.34}$$

where $t^i_k(u')$ is the unique point of maximum of the function $g_i(y_{u'}(\cdot))$ over the set $(\tau^i_k - \varepsilon, \tau^i_k + \varepsilon)$. The Hessian of the Lagrangian of the reduced problem is equal to the quadratic form $Q(v)$, i.e. has an additional term that matches the curvature term. Now assume that (6.1) does not hold. Then there exists a sequence (u_n), $u_n \to u$ in L^∞, satisfying the constraints (6.34) and $\mathcal{G}(u_n) \in \mathcal{K}$, and such that

$$J(u_n) \leqslant J(u) + \mathrm{o}\left(\|u_n - u\|_2^2\right). \tag{6.35}$$

Set $\varepsilon_n := \|u_n - u\|_2$ and $v_n := \varepsilon_n^{-1}(u_n - u)$. Being bounded in L^2, assume that $v_n \rightharpoonup v$ weakly in L^2. By (6.35), a second-order expansion of the Lagrangian of the reduced problem shows that

$$Q(v_n) \leqslant \mathrm{o}(1). \tag{6.36}$$

Moreover, since

$$\mathcal{K} \ni \mathcal{G}(u_n) = \mathcal{G}(u) + \varepsilon_n D\mathcal{G}(u) v_n + \varepsilon_n r_n$$

with $\|r_n\|_2 \to 0$, we deduce that $D\mathcal{G}(u)v_n + r_n \in \hat{T}_{\mathcal{K}}(\mathcal{G}(u))$. Taking the weak limit in L^2, we obtain that $D\mathcal{G}(u)v \in \hat{T}_{\mathcal{K}}(\mathcal{G}(u))$. Proceeding similarly for the state constraints, and since as a consequence of (6.35), we have $DJ(u)v \leqslant 0$, we deduce that $v \in \hat{C}_{L^2}(u)$. It follows then from (6.24) and (6.36), since Q is weakly lower semi-continuous, that $Q(v) = 0$, and hence, $Q(v_n) \to Q(v)$. Since Q is a Legendre form by hypothesis (2.42), this implies that $v_n \to v$ strongly, contradicting that $\|v_n\|_2 = 1$ for all n. This completes the proof. □

7. The shooting algorithm

In presence of state constraints, a reformulation of the optimality conditions is needed to apply so-called shooting methods. For an overview of the different formulations of optimality conditions existing in the literature, see the survey by Hartl et al. [16]. The shooting algorithm takes only into account a part of the optimality conditions, and the remainder conditions, referred as “additional conditions”, have to be checked afterwards. In this section, we first recall the alternative formulation used in the shooting algorithm (Definition 7.1). Additional conditions are given, under which the alternative formulation is equivalent to the first-order optimality condition of $(\mathcal{P})$ (Proposition 7.2). It is shown that some of those additional conditions are automatically satisfied (Lemma 7.3). Finally we give a characterization of the well-posedness of the shooting algorithm (Theorem 7.6), which is the main result of this section.

Given a finite subset $\mathcal{S}$ of $(0,T)$, we denote by $PC^k_{\mathcal{S}}[0,T]$ the set of functions over $[0,T]$ that are of class C^k outside $\mathcal{S}$ and have, as well as their first k derivatives, a left and a right limit over $\mathcal{S}$ and a left (respectively right) limit at T (respectively 0).

7.1. Shooting formulation

The formulation for the shooting algorithm presented in this section was introduced by Bryson et al. [7]. The presence of additional conditions was first underlined by Jacobson, Lele and Speyer [18], see also Kreindler [23]. See an example of implementation in e.g. [31] and numerical applications in e.g. [8,6].

Recall that H^q denotes the alternative Hamiltonian (3.6). We assume in the sequel that assumptions (A2)–(A4)(i) hold, and that first-order components of the state constraint do not have touch points (which is typically satisfied in view of Proposition 3.1(ii), since first-order components of the state constraint only have nonessential touch points). We assume in addition that

(A4) (ii) Each component of the mixed control-state constraint $c_i(u,y)$, $i=r+1,\dots,r+s$, has *finitely many* boundary arcs, and no touch points.

Under (A4) (which stands for (A4)(i)–(ii)), we denote by $\mathcal{I}_b^i$ the closure of the union of boundary arcs of each constraint $i=1,\dots,r+s$, i.e. $\mathcal{I}_b^i := \bigcup_{k=1}^{N_b^i}[\tau_{\text{en}}^{i,k},\tau_{\text{ex}}^{i,k}]$ for $\mathcal{T}_{\text{en}}^i := \{\tau_{\text{en}}^{i,1} < \cdots < \tau_{\text{en}}^{i,N_b^i}\}$ and a similar definition of $\mathcal{T}_{\text{ex}}^i$.

In the alternative formulation presented in Definition 3.3, the integration constants in (3.4) on a boundary arc of g_i are arbitrary. In the sequel, we will choose like in [28] these constants, on each boundary arc $(\tau_{\text{en}}^i,\tau_{\text{ex}}^i)$ of g_i, such that the functions η_i^j for $i=1,\dots,r$ and $j=1,\dots,q_i$ satisfy, for $t\in(\tau_{\text{en}}^i,\tau_{\text{ex}}^i)$,

$$\eta_i^1(t) := \eta_i\big(\tau_{\text{ex}}^{i+}\big) - \eta_i(t), \quad \eta_i^j(t) := \int_t^{\tau_{\text{ex}}^i} \eta_i^{j-1}(\sigma)\,\mathrm{d}\sigma, \quad j=2,\dots,q_i,$$

and we still have $\eta_i^j = 0$ outside boundary arcs of g_i and $\eta_i^0 = \lambda_i$ for $i=r+1,\dots,r+s$. With this formulation, the alternative costate p_q is continuous at exit points and discontinuous at entry and touch points, which allows to take the jump parameters $\nu_\tau^{i,j}$ and ν_τ^i involved in the jump condition (7.9) as *shooting parameters* in the shooting algorithm.

Definition 7.1. A trajectory (u,y) having a finite set of junction times $\mathcal{T} = \bigcup_{i=1}^{r+s}\mathcal{T}_i$ satisfies the *alternative formulation*, if there exist $p^q \in PC_{\mathcal{T}}^{q_{\max}}([0,T];\mathbb{R}^{n*})$, $\eta^q \in PC_{\mathcal{T}}^{q_{\max}}([0,T];\mathbb{R}^{(r+s)*})$, and, for each $i=1,\dots,r$, for each entry time τ of g_i, there exist q_i jump parameters $(\nu_\tau^{i,j})_{1\leqslant j\leqslant q_i}$ and for each touch point τ of g_i with $q_i \geqslant 2$, there exists a jump parameter ν_τ^i, such that the following relations are satisfied (dependence in time is omitted):

$$\dot y = f(u,y) \quad \text{on } [0,T]; \qquad y(0)=y_0, \tag{7.1}$$
$$-\dot p^q = H_y^q\big(u,y,p^q,\eta^q\big) \quad \text{on } [0,T]\setminus\mathcal{T}, \tag{7.2}$$
$$0 = H_u^q\big(u,y,p^q,\eta^q\big) \quad \text{on } [0,T]\setminus\mathcal{T}, \tag{7.3}$$
$$g_i^{(q_i)}\big(u(t),y(t)\big) = 0 \quad \text{on } \mathcal{I}_b^i,\ i=1,\dots,r+s, \tag{7.4}$$
$$\eta_i^{q_i}(t) = 0 \quad \text{on } [0,T]\setminus\mathcal{I}_b^i,\ i=1,\dots,r+s, \tag{7.5}$$
$$p^q(T) = \phi_y\big(y(T)\big), \tag{7.6}$$

and, for all $i=1,\dots,r$ and each junction point $\tau\in\mathcal{T}^i$ of g_i:

$$g_i^{(j)}\big(y(\tau)\big) = 0 \quad \text{if } \tau\in\mathcal{T}_{\text{en}}^i,\ j=0,\dots,q_i-1, \tag{7.7}$$
$$g_i\big(y(\tau)\big) = 0 \quad \text{if } \tau\in\mathcal{T}_{\text{to}}^i, \tag{7.8}$$

and for each junction time $\tau\in\mathcal{T}$:

$$\big[p^q(\tau)\big] = -\sum_{i\leqslant r:\ \tau\in\mathcal{T}_{\text{en}}^i}\sum_{j=1}^{q_i}\nu_\tau^{i,j} g_{i,y}^{(j-1)}\big(y(\tau)\big) - \sum_{i\leqslant r:\ \tau\in\mathcal{T}_{\text{to}}^i}\nu_\tau^i g_{i,y}\big(y(\tau)\big). \tag{7.9}$$

The shooting algorithm consists in finding a zero of a finite-dimensional shooting mapping, using e.g. a Newton method. The *structure* of active constraints of the optimal trajectory, i.e. the number and order of boundary arcs and touch points of each component of the constraint, is assumed to be known (or guessed). The arguments of the shooting mapping are called the *shooting parameters*, and are composed of the initial value of costate $p^0\in\mathbb{R}^{n*}$, all the junction times (with the exception of nonessential touch points) of the pure state constraints and mixed control-state constraints, and all the jump parameters $\nu_\tau^{i,j}$ at entry times τ of g_i, $i=1,\dots,r$, $j=1,\dots,q_i$, and ν_τ^i at touch points τ of g_i, $i=1,\dots,r$, $q_i\geqslant 2$, that are involved in the jump condition of the costate (7.9).

By assumptions (A2)–(A3), the algebraic variable $(u(t),\eta^q(t))\in\mathbb{R}^m\times\mathbb{R}^{(r+s)*}$ satisfying (7.3)–(7.5) can be expressed as implicit function of the differential variables $(y(t),p^q(t))\in\mathbb{R}^n\times\mathbb{R}^{n*}$ on the interior of each arc of the trajectory (see the proof of Proposition 3.6). With a given set of shooting parameters is therefore associated at most a unique solution (u,y,p^q,η^q) of the Cauchy problem (7.1)–(7.2) with initial condition of the costate $p^q(0)=p^0$, the algebraic variable (u, η^q) satisfying (7.3)–(7.5) and the jump of p^q at junction times of pure state constraints being given by (7.9).

The shooting mapping is then defined as follows. With a given set of shooting parameters are associated the following conditions: the final condition (7.6), the interior point conditions (7.7)–(7.8), and the optimality conditions for junction times below, for all $\tau \in \mathcal{T}$ and all $i = 1, \dots, r+s$:

$$g_i^{(q_i)}\big(u(\tau^-), y(\tau)\big) = 0, \quad \text{if } \tau \in \mathcal{T}_{\text{en}}^i, \tag{7.10}$$

$$g_i^{(q_i)}\big(u(\tau^+), y(\tau)\big) = 0, \quad \text{if } \tau \in \mathcal{T}_{\text{ex}}^i, \tag{7.11}$$

$$g_i^{(1)}\big(y(\tau)\big) = 0, \quad \text{if } \tau \in \mathcal{T}_{\text{to}}^i \text{ and if } q_i \geqslant 2. \tag{7.12}$$

This is a mapping defined on a subset of $\mathbb{R}^{\bar{N}}$ to $\mathbb{R}^{\bar{N}}$, where $\bar{N}$ the dimension of the shooting mapping is as follows. Let N_{ba}^i be the total number of boundary arcs of constraints g_i for $i = 1, \dots, r$, and c_i for $i = r+1, \dots, r+s$, and N_{to} the total number of touch points of state constraints of order $q_i \geqslant 2$. Then

$$\bar{N} = n + \sum_{i=1}^{r+s} (q_i + 2) N_{ba}^i + 2N_{\text{to}}. \tag{7.13}$$

7.2. Additional conditions

It is of importance to check whether solutions of the shooting algorithm (i.e. trajectory associated with a zero of the shooting function) are stationary points of $(\mathcal{P})$. For this, we need to make explicit the relation between the multipliers in the alternative formulation (Definition 7.1) and in Theorem 2.5.

Given alternative multipliers (p^q, η^q) and jump parameters $(\nu_\tau^{i,j})$ at entry times and (ν_τ^i) at touch times, the related multipliers (p, η, λ) in Theorem 2.5 are given by the following relations. Define first

$$\eta_i^j(t) = (-1)^{q_i - j} \frac{\mathrm{d}^{q_i - j}}{\mathrm{d}t^{q_i - j}} \eta_i^{q_i}(t), \quad j = 0, \dots, q_i - 1,\ i = 1, \dots, r,\ t \notin \mathcal{T}, \tag{7.14}$$

then

$$\lambda_i(t) = \eta_i^0(t), \quad i = r+1, \dots, r+s,\ t \notin \mathcal{T}, \tag{7.15}$$

$$p(t) = p^q(t) + \sum_{i=1}^{r} \sum_{j=1}^{q_i} \eta_i^j(t) g_{i,y}^{(j-1)}\big(y(t)\big), \quad t \notin \mathcal{T}. \tag{7.16}$$

Finally, let

$$\mathrm{d}\eta_i(t) = \eta_i^0(t)\,\mathrm{d}t + \sum_{\tau \in \mathcal{T}} \nu_\tau^i \delta_\tau(t), \quad i = 1, \dots, r, \tag{7.17}$$

where $\delta_\tau(t)$ denotes the Dirac measure at time τ, and the jumps parameters ν_τ^i at junction points $\tau \in \mathcal{T}$, for all $i = 1, \dots, r$, are the ones in the alternative formulation if $\tau \in \mathcal{T}_{\text{to}}^i$, $\nu_\tau^i = 0$ if $i \notin I(\tau)$, and, if $\tau \in \mathcal{I}_b^i$, they are given by, in view of (5.2) and (7.9),

$$\nu_\tau^i = \nu_\tau^{i,1} - \eta_i^1(\tau^+) \quad \text{if } \tau \in \mathcal{T}_{\text{en}}^i, \tag{7.18}$$

$$\nu_\tau^i = \eta_i^1(\tau^-) \quad \text{if } \tau \in \mathcal{T}_{\text{ex}}^i, \tag{7.19}$$

$$\nu_\tau^i = -\big[\eta_i^1(\tau)\big] \quad \text{if } \tau \in \operatorname{int} \mathcal{I}_b^i. \tag{7.20}$$

Conversely, Proposition 3.6 ensures, whenever assumptions (A2)–(A4) are satisfied, that each component η_i of η admits a (unique) decomposition under the form (7.17). Therefore, classical multipliers (p, η, λ) of Theorem 2.5 uniquely determine the alternative multipliers and alternative jump parameters so that (7.14)–(7.20) as well as (7.25), (7.27), (7.29) below hold, these three last conditions being needed in order to fix the integration constants in (3.4) and the jumps parameters at entry times $(\nu_\tau^{i,j})$ for $j \geqslant 2$.

The *additional conditions* needed to obtain the equivalence between the alternative formulation (7.1)–(7.9) and the first-order optimality condition (2.35)–(2.39) are the following:

$$g_i\big(y(t)\big) < 0 \quad \text{on } [0,T] \setminus \big(\mathcal{I}_b^i \cup \mathcal{T}_{\text{to}}^i\big), \ \text{for all } i = 1,\dots,r, \tag{7.21}$$

$$c_i\big(u(t), y(t)\big) < 0 \quad \text{a.e. on } [0,T] \setminus \mathcal{I}_b^i, \ \text{for all } i = r+1,\dots,r+s, \tag{7.22}$$

$$(-1)^{q_i} \frac{\mathrm{d}^{q_i}}{\mathrm{d}t^{q_i}} \eta_i^{q_i}(t) \geqslant 0 \quad \text{on } \operatorname{int} \mathcal{I}_b^i, \ \text{for all } i = 1,\dots,r+s, \tag{7.23}$$

and, for all $\tau \in \mathcal{T}$ and all $i = 1,\dots,r$:

$$\nu_\tau^{i,1} - \eta_i^1(\tau^+) \geqslant 0, \quad \text{if } \tau \in \mathcal{T}_{\text{en}}^i, \tag{7.24}$$

$$\nu_\tau^{i,j} - \eta_i^j(\tau^+) = 0, \quad \text{if } \tau \in \mathcal{T}_{\text{en}}^i, \ j = 2,\dots,q_i, \tag{7.25}$$

$$\eta_i^1(\tau^-) \geqslant 0, \quad \text{if } \tau \in \mathcal{T}_{\text{ex}}^i, \tag{7.26}$$

$$\eta_i^j(\tau^-) = 0, \quad \text{if } \tau \in \mathcal{T}_{\text{ex}}^i, \ j = 2,\dots,q_i, \tag{7.27}$$

$$\big[\eta_i^1(\tau)\big] \leqslant 0, \quad \text{if } \tau \in \operatorname{int} \mathcal{I}_b^i, \tag{7.28}$$

$$\big[\eta_i^j(\tau)\big] = 0, \quad \text{if } \tau \in \operatorname{int} \mathcal{I}_b^i, \ j = 2,\dots,q_i, \tag{7.29}$$

$$\nu_\tau^i \geqslant 0, \quad \text{if } \tau \in \mathcal{T}_{\text{to}}^i, \tag{7.30}$$

For all i such that $q_i = 1$, the inequalities (7.24), (7.26), (7.28) and (7.30) are equalities. (7.31)

Proposition 7.2. *Let* (u, y) *be a trajectory satisfying* (A2)–(A4). *Then* (u, y) *is a stationary point, with multipliers* (p, η, λ), *iff* (u, y) *satisfies both the alternative formulation* (*Definition* 7.1) *and the additional conditions* (7.21)–(7.31). *Relations* (7.14)–(7.20) *and* (7.25), (7.27), (7.29) *are a one-to-one mapping between the multipliers* (p, η, λ) *involved in the first-order optimality condition of Theorem* 2.5, *and the alternative multipliers* (p^q, η^q) *and alternative jumps parameters* $(\nu_\tau^{i,j})$ *and* (ν_τ^i) *at respectively entry and touch points in the alternative formulation and additional conditions.*

The higher the order q_i of the constraint is, the more additional conditions have to be checked at regular entry/exit points of boundary arcs. Those conditions are analogous to the known conditions in the scalar case, with in addition the conditions (7.28)–(7.29), that were not apparent in the scalar case, and to our knowledge not known in the literature. Thus, when assumptions (A2)–(A3) hold, we are led to think that, like in the scalar case, boundary arcs with regular entry/exit times for components of the state constraint of order $q_i \geqslant 3$ may occur only in degenerate situations. We underline that this was not, however, an immediate result, since now we allow more control variables (more than one) and hence, more degrees of freedom.

Proof of Proposition 7.2. Let us show the equivalence between, on the one hand, the first-order optimality system of $(\mathcal{P})$ (2.35)–(2.39), and on the other hand, the alternative formulation (7.1)–(7.9) and the additional conditions (7.21)–(7.31).

First, $g_i(y(t)) \leqslant 0$ in (2.38) is equivalent to $g_i(y(t)) = 0$ on $\mathcal{I}_b^i$, (7.8) at touch points and (7.21) outside the contact set, and then $g_i(y(t)) = 0$ on $\mathcal{I}_b^i$ is equivalent to (7.4) for $i = 1,\dots,r$, with the q_i entry-point conditions (7.7). By Proposition 3.6, the state constraint multipliers η_i, $i = 1,\dots,r$, are regular on interiors of arcs, therefore, each component η_i can be put into the form (7.17), where jumps can occur only at junctions points, and the density of each component η_i^0 is continuous on the interior of arcs. It follows that η_i is a nonnegative measure ($\mathrm{d}\eta_i \geqslant 0$ in (2.38)), iff its density $\frac{\mathrm{d}\eta_i}{\mathrm{d}t}(t) = \eta_i^0(t) = (-1)^{q_i} \frac{\mathrm{d}^{q_i}}{\mathrm{d}t^{q_i}} \eta_i^{q_i}(t)$ is nonnegative, i.e. iff (7.23) holds for $i = 1,\dots,r$, and the jumps at junction times are nonnegative, i.e.

$$\nu_\tau^i = \big[\eta_i(\tau)\big] \geqslant 0, \quad \text{for all } i = 1,\dots,r \text{ and all } \tau \in \mathcal{T} = \bigcup_{i=1}^{r+s} \mathcal{T}^i. \tag{7.32}$$

The complementarity condition $\int_0^T g_i(y(t))\,\mathrm{d}\eta_i(t) = 0$ in (2.38) is then equivalent to (7.5) for $i = 1,\dots,r$ (the measure $\mathrm{d}\eta_i$ has support on the contact set of $g_i(y)$). Similarly, for mixed control-state constraints, since $\lambda \in L^\infty$, (2.39) is equivalent to (7.4)–(7.5) and (7.22)–(7.23) for $i = r+1,\dots,r+s$.

The state equations (2.35) and (7.1) are of course identical, and so are the final conditions of the costate (2.36) and (7.6) in view of (A4)(i). By Lemma 3.4, the costate and control equations (7.2) and (7.3) are equivalent, respectively, to the costate and control equations (2.36) and (2.37) *on the interior of arcs*. Now let us show the equivalence, at junction times, between on the one hand the costate equation (2.36) and (7.32), and on the other hand the jump condition (7.9) and the additional conditions (7.24)–(7.30). By (5.2) (recall that $[p(\tau)] = -\sum_{i\in I(\tau)} \nu_\tau^i g_{i,y}(y(\tau))$ with $\nu_\tau^i = [\eta_i(\tau)]$) and by (7.9), it holds respectively

$$\left[p^q(\tau)\right] = -\sum_{i\in I(\tau)} \left\{ \left(\nu_\tau^i + \left[\eta_i^1(\tau)\right]\right) g_{i,y}\left(y(\tau)\right) + \sum_{j=2}^{q_i} \left[\eta_i^j(\tau)\right] g_{i,y}^{(j-1)}\left(y(\tau)\right) \right\}, \tag{7.33}$$

$$\left[p^q(\tau)\right] = -\sum_{i\leqslant r:\ \tau\in\mathcal{T}_{\rm en}^i} \sum_{j=1}^{q_i} \nu_\tau^{i,j} g_{i,y}^{(j-1)}\left(y(\tau)\right) - \sum_{i\leqslant r:\ \tau\in\mathcal{T}_{\rm to}^i} \nu_\tau^i g_{i,y}\left(y(\tau)\right). \tag{7.34}$$

By Corollary 4.5, the vectors $g_{i,y}^{(j-1)}(y(\tau))$ are linearly independent, for all $i \in I(\tau)$ and $j = 1,\dots,q_i$, hence the relations (7.33)–(7.34) are equal, iff the coefficients of $g_{i,y}^{(j-1)}(y(\tau))$ are equal. We thus obtain, for all $\tau \in \mathcal{T}$ and $i \in I(\tau)$, if $\tau \in \mathcal{T}_{\rm en}^i$:

$$\nu_\tau^i + \left[\eta_i^1(\tau)\right] = \nu_\tau^{i,1} \quad \text{and} \quad \left[\eta_i^j(\tau)\right] = \nu_\tau^{i,j}, \quad j = 2,\dots,q_i,$$

which, with (7.32), is equivalent to (7.24)–(7.25), using that $\eta_i^j(\tau^-) = 0$ at entry point. If now $\tau \in \mathcal{T}_{\rm to}^i$, we obtain, since the multipliers η_i^j are equal to zero in the neighborhood of τ:

$$\left[\eta_i(\tau)\right] = \nu_\tau^i,$$

which, with (7.32), is equivalent to (7.30). Finally, if $\tau \in \operatorname{int}\mathcal{I}_b^i$ or if $\tau \in \mathcal{T}_{\rm ex}^i$, then we have

$$\left[\eta_i(\tau)\right] + \left[\eta_i^1(\tau)\right] = 0 \quad \text{and} \quad \left[\eta_i^j(\tau)\right] = 0, \quad j = 2,\dots,q_i,$$

which, with (7.32) again, is equivalent to (7.28)–(7.29) on interior of boundary arcs and to (7.26)–(7.27) at exit points, since $\eta_i^j(\tau^+) = 0$. Finally, whenever $q_i = 1$, then we know by Proposition 3.1 that η_i is continuous, i.e. $[\eta_i(\tau)] = 0$, and therefore all inequalities in (7.24)–(7.30) are in fact equalities. □

Like in the scalar case, the conditions (7.10)–(7.11) imposed in the shooting algorithm, related to the continuity of u, imply that some of the additional conditions are automatically satisfied by a solution of the shooting algorithm.

Lemma 7.3. *Let* (u, y) *satisfy the alternative formulation* (7.1)–(7.9)*, the strong assumption* (2.43) *and* (A3)–(A4)*, and assume that* $\mathcal{T}_{\rm to}^i = \emptyset$*, for all* i *such that* $q_i = 1$*. Then the following assertions are equivalent*:

(i) *For all* $i = 1,\dots,r$ *and all junction point* $\tau \in \mathcal{T}$*, if* $q_i = 1$ *the additional conditions* (7.24), (7.26) *and* (7.28) *are satisfied with equality and if* $q_i \geqslant 2$*, the additional conditions in* (7.25), (7.27) *and* (7.29) *are satisfied for* $j = q_i$*, i.e.*

$$\nu_\tau^{i,q_i} = \eta_i^{q_i}(\tau^+), \quad \text{if } \tau \in \mathcal{T}_{\rm en}^i, \tag{7.35}$$

$$\eta_i^{q_i}(\tau^-) = 0, \quad \text{if } \tau \in \mathcal{T}_{\rm ex}^i, \tag{7.36}$$

$$\left[\eta_i^{q_i}(\tau)\right] = 0, \quad \text{if } \tau \in \operatorname{int}\mathcal{I}_b^i, \tag{7.37}$$

and for all $i = r+1,\dots,r+s$, $\eta_i^{q_i} = \lambda_i$ *is continuous over* $[0, T]$.

(ii) *The conditions* (7.10)–(7.11) *are satisfied, for all* $\tau \in \mathcal{T}$ *and all* $i = 1,\dots,r+s$.

(iii) *The control* u *is continuous over* $[0, T]$.

Proof. Let $\tau \in \mathcal{T}$, and let $J := I(\tau) \setminus \{i = 1,\dots,r;\ \tau \in \mathcal{T}_{\rm to}^i\}$. Set $u^\pm := u(\tau^\pm)$, $[u] := u^+ - u^-$, and, for $\sigma \in [0, 1]$, $u^\sigma := u^- + \sigma(u^+ - u^-)$. Similar notations for p^q, η^q are used. Denote by $\tilde\nu^q = (\tilde\nu_i^{q_i})_{i\in J}$ the augmented (row) vector of jump parameters, satisfying $\tilde{\nu}_i^{q_i} = \nu_\tau^{i,q_i}$ for all $i \in J$ such that $\tau \in \mathcal{T}_{\text{en}}^i$ and $q_i \geqslant 1$, and $\tilde{\nu}_i^{q_i} = 0$ for all $i \in J$ such that $\tau \in \operatorname{int} \mathcal{I}_b^i \cup \mathcal{T}_{\text{ex}}^i$ or $q_i = 0$. By (7.3),

$$H_u^q\big(u^+, y(\tau), p^{q+}, \eta^{q+}\big) = 0 = H_u^q\big(u^-, y(\tau), p^{q-}, \eta^{q-}\big).$$

The alternative Hamiltonian H^q being affine in the variables p^q and η^q, we have

$$0 = \int_0^1 \big\{\sigma H_{uu}^q\big(u^\sigma, y(\tau), p^{q+}, \eta^{q+}\big) + (1-\sigma) H_{uu}^q\big(u^\sigma, y(\tau), p^{q-}, \eta^{q-}\big)\big\}[u]\,\mathrm{d}\sigma$$
$$+ \int_0^1 \big\{\big[p^q\big] f_u\big(u^\sigma, y(\tau)\big) + \big[\eta^q\big] G_{J,u}^{(q)}\big(u^\sigma, y(\tau)\big)\big\}\,\mathrm{d}\sigma. \tag{7.38}$$

Using the jump of p^q given by (7.9), and the fact that by hypothesis, first-order components of the state constraint do not have touch points, we easily get that

$$\big[p^q\big] f_u\big(u^\sigma, y(\tau)\big) + \big[\eta^q\big] G_{J,u}^{(q)}\big(u^\sigma, y(\tau)\big) = \big(\big[\eta^q\big] - \tilde{\nu}^q\big) G_{J,u}^{(q)}\big(u^\sigma, y(\tau)\big). \tag{7.39}$$

In addition, (2.43) and (3.8) imply that $H_{uu}^q(u^\sigma, y, p^{q\pm}, \eta^{q\pm})$ is uniformly positive definite, for all $\sigma \in [0,1]$, therefore, multiplying on the right (7.38) by $[u]$, and using (7.39), we obtain that

$$\alpha\big|[u]\big|^2 \leqslant \big(\tilde{\nu}^q - \big[\eta^q\big]\big) \int_0^1 G_{J,u}^{(q)}\big(u^\sigma, y(\tau)\big)[u]\,\mathrm{d}\sigma. \tag{7.40}$$

Note that point (i) is equivalent to the condition $[\eta_i^{q_i}] - \tilde{\nu}_i^{q_i} = 0$ for all $i = 1, \dots, r+s$. Therefore, the implication (i) $\Rightarrow$ (iii) follows from (7.40). Conversely, if (iii) holds, i.e. $[u] = 0$, then (7.38)–(7.39) yields

$$\big(\big[\eta^q\big] - \tilde{\nu}^q\big) G_{J,u}^{(q)}\big(u(\tau), y(\tau)\big) = 0,$$

implying (i) by (2.30). This shows the equivalence (iii) $\Leftrightarrow$ (i). Let us show now (iii) $\Leftrightarrow$ (ii). The implication (iii) $\Rightarrow$ (ii) is trivial. If (ii) holds, then

$$0 = G_J^{(q)}\big(u^+, y(\tau)\big) - G_J^{(q)}\big(u^-, y(\tau)\big) = \int_0^1 G_{J,u}^{(q)}\big(u^\sigma, y(\tau)\big)[u]\,\mathrm{d}\sigma. \tag{7.41}$$

By (7.40), it follows that $[u] = 0$, i.e. (iii) holds, which completes the proof. $\square$

7.3. Well-posedness of the shooting algorithm

We say that the shooting algorithm is (locally) well-posed in the neighborhood of a local solution, if the Jacobian of the shooting mapping is invertible. This allows us to apply locally a Newton method in order to find a zero of the shooting mapping with a very high precision, and low cost. If the additional conditions (7.21)–(7.31) are satisfied, we obtain a stationary point of $(\mathcal{P})$, and if the second-order sufficient condition (6.25) holds, we obtain a local solution of $(\mathcal{P})$.

The first step to study the well-posedness of the shooting algorithm is to compute the Jacobian of the shooting mapping. We denote by π^0 the variation of p^0, σ_τ^i the variation of τ for each $\tau \in \mathcal{T}^i$, $i = 1, \dots, r+s$, $\gamma_\tau^{i,j}$ the variations of alternative jump parameters at entry times $\nu_\tau^{i,j}$ for $\tau \in \mathcal{T}_{\text{en}}^i$, $i = 1, \dots, r$, $j = 1, \dots, q_i$, and γ_τ^i the variations of jump parameters at touch times ν_τ^i for $\tau \in \mathcal{T}_i^{\text{to}}$, $i = 1, \dots, r$ and $q_i \geqslant 2$. All of them will be called *variations of shooting parameters*.

Given a vector $\zeta \in \mathbb{R}^{(r+s)*}$ and $J := \{i_1 < \cdots < i_s\} \subset \{1, \dots, r+s\}$, the vector ζ_J denotes the row vector of component $(\zeta_{i_1}, \dots, \zeta_{i_s})$. We denote by $\bar{I}(t)$ the complement of $I(t)$ in $\{1, \dots, r+s\}$. With a set of variation of shooting parameters is associated a (unique by (A2)–(A3)) linearized trajectory and multipliers (z, v, π^q, ζ^q) solution of (arguments (u, y, p^q, η^q) and time are omitted):

$$\dot z = f_y z + f_u v \quad \text{on } [0,T] \text{ a.e.}; \quad z(0) = 0, \tag{7.42}$$

$$\dot\pi^q = -\big(H^q_{yy} z + H^q_{yu} v + \pi^q f_y + \zeta^q G^{(q)}_y\big) \quad \text{on } [0,T] \setminus \mathcal{T} \text{ a.e.}, \tag{7.43}$$

$$\pi^q(0) = \pi^0, \tag{7.44}$$

$$0 = H^q_{uy} z + H^q_{uu} v + \pi^q f_u + \zeta^q G^{(q)}_u \quad \text{on } [0,T] \setminus \mathcal{T} \text{ a.e.}, \tag{7.45}$$

$$0 = G^{(q)}_{I(t),u} v + G^{(q)}_{I(t),y} z \quad \text{on } [0,T] \setminus \mathcal{T} \text{ a.e.}, \tag{7.46}$$

$$0 = \zeta^q_{\bar I(t)} \quad \text{on } [0,T] \setminus \mathcal{T} \text{ a.e.} \tag{7.47}$$

and, for all $\tau \in \bigcup_{i=1}^r \mathcal{T}^i$, setting $\nu^{i,0}_\tau := 0$ for $\tau \in \mathcal{T}^i_{\text{en}}$:

$$\begin{aligned}\big[\pi^q(\tau)\big] = &- \sum_{i \leqslant r:\ \tau \in \mathcal{T}^i_{\text{en}}} \sum_{j=1}^{q_i} \big\{\nu^{i,j}_\tau g^{(j-1)}_{i,yy}\big(y(\tau)\big) z(\tau) + \big(\gamma^{i,j}_\tau + \sigma^i_\tau \nu^{i,j-1}_\tau\big) g^{(j-1)}_{i,y}\big(y(\tau)\big)\big\} \\ &- \sum_{i \leqslant r:\ \tau \in \mathcal{T}^i_{\text{to}}} \big\{\nu^i_\tau g_{i,yy}\big(y(\tau)\big) z(\tau) + \gamma^i_\tau g_{i,y}\big(y(\tau)\big) + \sigma^i_\tau \nu^i_\tau g^{(1)}_{i,y}\big(y(\tau)\big)\big\}.\end{aligned} \tag{7.48}$$

Lemma 7.4. *Let (u, y, p^q, η^q) be the trajectory associated with a zero of the shooting mapping, and assume that* (A2)–(A4) *hold and that $\mathcal{T}^i_{\text{to}} = \emptyset$ for all i such that $q_i = 1$. Let π^0, (σ^i_τ), $(\gamma^{i,j}_\tau)$, and (γ^i_τ) be a set of variations of shooting parameters and denote by (z, v, π^q, ζ^q) the linearized trajectory and multipliers solution of* (7.42)–(7.48). *Then this set of variations of shooting parameters belongs to the kernel of the Jacobian of the shooting mapping, iff*:

$$\pi^q(T) = \phi_{yy}\big(y(T)\big) z(T), \tag{7.49}$$

and, for all junction time $\tau \in \mathcal{T}$ and all $i = 1, \dots, r+s$:

$$0 = g^{(j)}_{i,y}\big(y(\tau)\big) z(\tau) \quad \textit{if } \tau \in \mathcal{T}^i_{\text{en}} \textit{ and } q_i \geqslant 1,\ j = 0, \dots, q_i - 1, \tag{7.50}$$

$$0 = g_{i,y}\big(y(\tau)\big) z(\tau) \quad \textit{if } \tau \in \mathcal{T}^i_{\text{to}} \textit{ and } q_i \geqslant 2, \tag{7.51}$$

$$0 = g^{(q_i)}_{i,(u,y)}\big(u(\tau), y(\tau)\big)\big(v(\tau^-), z(\tau)\big) + \sigma^i_\tau \frac{\mathrm{d}}{\mathrm{d}t} g^{(q_i)}_i(u, y)\bigg|_{t=\tau^-} \quad \textit{if } \tau \in \mathcal{T}^i_{\text{en}}, \tag{7.52}$$

$$0 = g^{(q_i)}_{i,(u,y)}\big(u(\tau), y(\tau)\big)\big(v(\tau^+), z(\tau)\big) + \sigma^i_\tau \frac{\mathrm{d}}{\mathrm{d}t} g^{(q_i)}_i(u, y)\bigg|_{t=\tau^+} \quad \textit{if } \tau \in \mathcal{T}^i_{\text{ex}}, \tag{7.53}$$

$$0 = g^{(1)}_{i,y}\big(y(\tau)\big) z(\tau) + \sigma^i_\tau g^{(2)}_i\big(u(\tau), y(\tau)\big) \quad \textit{if } \tau \in \mathcal{T}^i_{\text{to}} \textit{ and } q_i \geqslant 2. \tag{7.54}$$

The proof of this result follows from the linearization of the shooting equations (for the jump of π^q at entry times, see [4, Lemma 3.7]).

In addition to the tangentiality conditions (A5)(i), reducibility condition (A5)(ii) and strict complementarity assumption on boundary arcs (A6)(i) made for pure state constraints in Section 6, we will need the following assumptions, also for the mixed control-state constraints:

(A5) (iii) (Nontangentiality conditions for mixed control-state constraints) For all $i = r+1, \dots, r+s$ and all $\tau^i_{\text{en}} \in \mathcal{T}^i_{\text{en}}$ and $\tau^i_{\text{ex}} \in \mathcal{T}^i_{\text{ex}}$,

$$\frac{\mathrm{d}}{\mathrm{d}t} c_i\big(u(t), y(t)\big)\bigg|_{t=\tau^{i-}_{\text{en}}} > 0, \qquad \frac{\mathrm{d}}{\mathrm{d}t} c_i\big(u(t), y(t)\big)\bigg|_{t=\tau^{i+}_{\text{ex}}} < 0. \tag{7.55}$$

(A6) (ii) (Strict complementarity at touch points)

$$\mathcal{T}^{i,\text{nes}}_{\text{to}} = \emptyset, \quad \text{for all } i = 1, \dots, r+s.$$

(iii) (Strict complementarity for mixed constraints)

$$\lambda_i(t) > 0, \quad \text{for a.a. } t \in \operatorname{int} \Delta_i, \text{ for all } i = r+1, \dots, r+s. \tag{7.56}$$

Assumption (A6)(ii) implies that constraints of order $q_i = 0, 1$ have no touch points.

We will finally make the assumption below:

(A7) The junctions times of different components of the constraint do not coincide (i.e. $i, j \in \{1, \dots, r+s\}$ and $i \neq j$ implies that $\mathcal{T}^i \cap \mathcal{T}^j = \emptyset$).

Remark 7.5. When (A7) holds, for all entry and exit points of state constraints $\tau \in \mathcal{T}^i_{\text{en}} \cup \mathcal{T}^i_{\text{ex}}$, $i = 1, \dots, r$, we have that $q_\tau = q_i$, and assumption (A5)(i) simply says that

$$\frac{\mathrm{d}^{q_i}}{\mathrm{d}t^{q_i}} g_i^{(q_i)}(u, y)\bigg|_{t=\tau^\pm} \neq 0 \quad \text{if } q_i \text{ is odd,}$$

$$\frac{\mathrm{d}^{q_i-1}}{\mathrm{d}t^{q_i-1}} g_i^{(q_i)}(u, y)\bigg|_{t=\tau^\pm} \neq 0 \quad \text{if } q_i \text{ is even,} \tag{7.57}$$

where $\tau^\pm$ denotes τ^- (resp. τ^+) if τ is an entry point (resp. exit point).

Under (A4) and the strict complementarity assumption (A6), using Lemma 2.2, the critical cone $\hat{C}_{L_2}(u)$ defined by (6.22) is the set of $v \in \mathcal{V}$ satisfying (recall that $z_v \in \mathcal{Z}$ is the solution of the linearized state equation (2.22))

$$0 = g_{i,u}^{(q_i)}(u, y)v + g_{i,y}^{(q_i)}(u, y)z_v \quad \text{a.e. on } \mathcal{I}^i_b,\ i = 1, \dots, r+s, \tag{7.58}$$

$$0 = g_{i,y}^{(j)}\big(y(\tau)\big)z_v(\tau), \quad \tau \in \mathcal{T}^i_{\text{en}},\ i = 1, \dots, r,\ j = 0, \dots, q_i - 1, \tag{7.59}$$

$$0 = g_{i,y}\big(y(\tau)\big)z_v(\tau), \quad \tau \in \mathcal{T}^i_{\text{to}},\ i = 1, \dots, r. \tag{7.60}$$

Theorem 7.6 *(Well-posedness of the shooting algorithm). Let* (u, y) *be a local solution of* $(\mathcal{P})$ *satisfying* (A1)–(A7). *Then the shooting algorithm is well-posed in the neighborhood of the trajectory* (u, y), *iff the two conditions below are satisfied*:

(i) *components of the state constraint of order* $q_i \geqslant 3$ *have no boundary arc*;
(ii) *the no-gap sufficient condition* (6.25) *holds, i.e.* $Q(v) > 0$ *for all* $v \in \mathcal{V}$ *satisfying* (7.58)–(7.60) *with the associated linearized state* $z_v \in \mathcal{Z}$ *solution of* (2.22) *and* $Q(v)$ *defined by* (6.26).

Once the junction conditions and the no-gap second-order optimality conditions have been established, and with assumption (A7), Theorem 7.6 is an easy extension of [4, Theorem 3.3] obtained in the scalar case. The next lemma relates the second-order conditions established in Section 6 and the alternative multipliers used in the shooting algorithm.

Lemma 7.7. *Let* (u, y) *be a stationary point of* $(\mathcal{P})$, *satisfying* (A2)–(A4) *and* (A5)(ii). *Then an equivalent expression using the alternative Hamiltonian and multipliers for the quadratic form* $Q(v)$ *defined in* (6.26) *over* $\mathcal{V}$ *is*:

$$\begin{aligned} Q(v) = {} & \int_0^T H^q_{(u,y),(u,y)}\big(u, y, p^q, \eta^q\big)\big((v, z_v), (v, z_v)\big)\,\mathrm{d}t + \phi_{yy}\big(y(T)\big)\big(z_v(T), z_v(T)\big) \\ & + \sum_{i=1}^r \sum_{\tau \in \mathcal{T}^i_{\text{en}}} \sum_{j=1}^{q_i} \nu_\tau^{i,j} g_{i,yy}^{(j-1)}\big(y(\tau)\big)\big(z_v(\tau), z_v(\tau)\big) \\ & + \sum_{i=1}^r \sum_{\tau \in \mathcal{T}^{i,\text{ess}}_{\text{to}}} \nu_\tau^i \left(g_{i,yy}\big(y(\tau)\big)\big(z_v(\tau), z_v(\tau)\big) - \frac{(g_{i,y}^{(1)}(y(t))z_v(t))^2}{\frac{\mathrm{d}^2}{\mathrm{d}t^2} g_i(y(t))|_{t=\tau}} \right). \end{aligned} \tag{7.61}$$

Proof. The contribution of mixed control-state constraints in both (6.26) and (7.61) is equal to $\int_0^T \lambda c_{(u,y),(u,y)}(u, y)((v, z_v), (v, z_v))\,\mathrm{d}t$, therefore, summing over the finitely many state constraints g_i, the proof is identical to [4, Lemma 3.6]. $\square$

Proof of Theorem 7.6. We first prove that if (i) does not hold, the Jacobian of the shooting mapping is singular. So assume that a constraint g_i of order $q_i \geqslant 3$ has a boundary arc $(\tau^i_{\text{en}}, \tau^i_{\text{ex}})$. By assumption (A7) and (5.3), we have that $q_{\tau^i_{\text{en}}} = q_{\tau^i_{\text{ex}}} = q_i$, and hence, by Proposition 5.3, u is continuous until order $q_i - 2 \geqslant 1$. Therefore $\dot{u}$ is continuous at τ^i_{en} and τ^i_{ex}, and consequently, $\frac{\mathrm{d}}{\mathrm{d}t} g^{(q_i)}(u(t), y(t))$ is also continuous, and vanishes at τ^{i-}_{en} and τ^{i+}_{ex}. Taking all variations of jump parameters equal to zero, except $\sigma^i_{\tau^i_{\text{ex}}} \neq 0$, we find by Lemma 7.4 a nonzero element in the kernel of the Jacobian of the shooting mapping. Therefore the shooting algorithm is ill-posed.

We assume now that (i) holds. We will prove that the Jacobian of the shooting mapping is invertible iff (ii) holds. The Jacobian of the shooting mapping is invertible, iff it is one-to-one, i.e. iff the only solution of Eqs. (7.49)–(7.54), where (z, v, π^q, ζ^q) is the solution of (7.42)–(7.48), is $\pi^0 = 0$, $(\sigma^i_\tau) = 0$, $(\gamma^{i,j}_\tau) = 0$, $(\gamma^i_\tau) = 0$. We recognize that (7.42)–(7.48) and (7.49)–(7.51) and (7.54) (which enables, by (A5)(ii), to substitute $-g^{(1)}_{i,y}(y(\tau))z(\tau)/g^{(2)}_i(u(\tau), y(\tau))$ for σ^i_τ in (7.48) for all touch point τ), constitutes the first-order optimality condition for the problem

$$(PQ) \quad \min_{v \in \mathcal{V}} \frac{1}{2} Q(v), \quad v \in \hat{C}_{L_2}(u),$$

with $Q(v)$ given by (7.61) and $\hat{C}_{L_2}(u)$ by (7.58)–(7.60). Here (γ^i_τ) are the multipliers associated with the constraints (7.60), and those associated with the constraints (7.59) are equal to $\gamma^{i,j}_\tau$ if $j = 1$ and $\gamma^{i,j}_\tau + \sigma^i_\tau \nu^{i,j-1}_\tau$ if $j > 1$.

If (ii) holds, i.e. if the second-order sufficient condition (6.25) holds, then by Lemma 7.7 the unique solution of (*PQ*) is zero. By (A2), the cost function of (*PQ*) is a *Legendre form* over $\mathcal{V}$, and hence, the strict positivity of $Q(v)$ over the closed linear space $\hat{C}_{L_2}(u)$ implies its uniform positivity (i.e. there exists $\alpha > 0$ such that $Q(v) \geqslant \alpha \|v\|^2_2$ for all $v \in \hat{C}_{L_2}(u)$). In addition, the set $\hat{C}_{L_2}(u)$ is convex and the linear constraints (7.58)–(7.60) defining $\hat{C}_{L_2}(u)$ are *onto* by Lemma 2.3. Therefore the first-order optimality condition of (*PQ*) is necessary and sufficient for optimality, so (ii) implies that zero is the unique solution of the first-order optimality condition of (*PQ*). Therefore we have $(z, v, \pi^q, \zeta^q) = 0$, and all of π^0, (γ^i_τ), $(\gamma^{i,j}_\tau)$ for $j = 1$, also equal zero by Corollary 4.5 since $[\pi^q(\tau)] = 0$, and we have as well

$$\gamma^{i,j}_\tau + \sigma^i_\tau \nu^{i,j-1}_\tau = 0, \quad \text{for all } j = 2, \dots, q_i,\ i = 1, \dots, r,\ \tau \in \mathcal{T}^i_{\text{en}}. \tag{7.62}$$

Now whenever (i) holds, it holds for all entry/exit times that $q_\tau \leqslant q_i \leqslant 2$, and from assumptions (A5)(i) and (A5)(iii), it follows that $\frac{\mathrm{d}}{\mathrm{d}t} g^{(q_i)}_i(u, y)|_{t=\tau^-}$ is nonzero for all entry points $\tau \in \mathcal{T}^i_{\text{en}}$, for all $i = 1, \dots, r + s$. Therefore, Eqs. (7.52) with $(v, z) = 0$ and (7.62) imply that $\sigma^i_\tau = 0$, for all entry points $\tau \in \mathcal{T}^i_{\text{en}}$, $i = 1, \dots, r + s$, and that $\gamma^{i,j}_\tau = 0$ for all $j = 2, \dots, q_i$, $i = 1, \dots, r$, $\tau \in \mathcal{T}^i_{\text{en}}$. Similarly, we obtain that (7.53) and (7.54) imply that $\sigma^i_\tau = 0$ for all exit and touch points. Therefore, whenever (i)–(ii) holds, the Jacobian of the shooting mapping is one-to-one, hence invertible, and thus the shooting algorithm is well-posed locally around the local solution (u, y).

Assume now that (ii) does not hold. By Theorem 6.1(i), the second-order necessary condition (6.23) holds at the local solution (u, y), implying that $Q(v)$ is nonnegative over $\hat{C}_{L^2}(u)$. Therefore, if (6.25) is not satisfied, this implies that there exists a nonzero optimal solution of (*PQ*), and hence there exists a nonzero solution of its first-order optimality condition. It is then easy to see that the variations of shooting parameters associated as above with this nonzero solution of (*PQ*) are not all zero, and belong to the kernel of the Jacobian of the shooting mapping. This proves that the shooting algorithm is ill-posed. □

8. Final remark: Extension to constraints on the initial and final state

Let us comment on the extension of the results when there are additional equality and/or inequality constraints on the initial and final state:

$$\Psi_i\big(y(0), y(T)\big) = 0, \quad i = 1, \dots, \varrho', \qquad \Psi_i\big(y(0), y(T)\big) \leqslant 0, \quad i = \varrho' + 1, \dots, \varrho, \tag{8.1}$$

with $\Psi : \mathbb{R}^{2n} \to \mathbb{R}^\varrho$ a C^2 mapping $(0 \leqslant \varrho' \leqslant \varrho \leqslant n)$. The results of this paper can easily be generalized, under an additional (strong) controllability assumption (A1$'$) below, having the role of Lemma 2.3 in the proofs, and, for the second-order optimality conditions and the well-posedness of the shooting algorithm, also under an additional assumption that *strict complementarity* holds for the inequality constraints in (8.1). Denote by $\hat{\Psi}$ the mapping composed of the equality and active inequality constraints in (8.1), of dimension $\hat{\varrho}$. Given $\kappa \in [1, +\infty]$ and $(v, x) \in \mathcal{V}_\kappa \times \mathbb{R}^n$, let $z_{v,x}$ denote the (unique) solution in $\mathcal{Z}_\kappa$ of:

$$\dot{z}_{v,x} = f_u(u, y)v + f_y(u, y)z_{v,x}, \quad z_{v,x}(0) = x.$$

(A1′) For $\kappa = 2, \infty$, there exists $\delta > 0$ and $n \in \mathbb{N}^*$ such that the linear mapping $\mathcal{V}_\kappa \times \mathbb{R}^n \to \prod_{i=1}^r W^{q_i,\kappa}(\Delta_i^\delta) \times \prod_{i=r+1}^{r+s} L^\kappa(\Delta_i^n) \times \mathbb{R}^{\hat{\varrho}}$,

$$(v, x) \to \begin{pmatrix} (g_{i,y}(y(\cdot))z_{v,x}(\cdot)|_{\Delta_i^\delta})_{1\leqslant i\leqslant r} \\ ((c_{i,y}(u(\cdot), y(\cdot))z_{v,x}(\cdot) + c_{i,u}(u(\cdot), y(\cdot))v(\cdot))|_{\Delta_i^n})_{r+1\leqslant i\leqslant r+s} \\ D_{y_0}\hat{\Psi}(y(0), y(T))x + D_{y_T}\hat{\Psi}(y(0), y(T))z_{v,x}(T) \end{pmatrix}$$

is onto, and therefore has a bounded right inverse by the open mapping theorem.

Note that in the absence of mixed control-state constraints, this assumption (A1′) is satisfied e.g. in the case of a linear system, i.e. $f(u, y) = Ay + Bu$, if the pair (A, B) is controllable, the initial and final conditions are fixed $y(0) = y_0$ and $y(T) = y_T$ and satisfy $g_i(y_0) < 0$ and $g_i(y_T) < 0$ for all $i = 1, \dots, r$, and (2.25) holds.

Appendix A

A.1. Tangent and Normal cones in L^∞

Let us recall the characterization of the tangent and normal cones (in the sense of convex analysis) to $\mathcal{K} := L^\infty_-(0, T)$ at point $x \in \mathcal{K}$. The characterization of the tangent cone was obtained by Cominetti and Penot [9]:

$$T_\mathcal{K}(x) = \{h \in L^\infty\colon\ \|\mathbf{1}_{\Delta_n(x)}h_+\|_\infty \to 0 \text{ when } n \to +\infty\}, \tag{A.1}$$

with $\mathbf{1}_{\Delta_n(x)}$ the indicator function of the set $\Delta_n(x)$ defined by (2.7), and $h_+ := \max(h; 0)$ a.e.

Since $\mathcal{K}$ is a cone, the normal cone satisfies $N_\mathcal{K}(x) = \{\lambda \in (L^\infty)^*_+,\ \langle\lambda, x\rangle = 0\}$. Define

$$\mathcal{N}_n(x) := \{y \in L^\infty(0, T);\ y(t) = 0 \text{ for a.a. } t \in \Delta_n(x)\}, \quad n \in \mathbb{N}^*.$$

Then we have the following characterization of $N_\mathcal{K}(x)$.

Lemma A.1. *Let $x \in \mathcal{K}$. Then*

$$N_\mathcal{K}(x) = \left\{\lambda \in (L^\infty)^*_+;\ \langle\lambda, y\rangle = 0,\ \forall y \in \bigcup_{n\in\mathbb{N}^*} \mathcal{N}_n(x)\right\}. \tag{A.2}$$

Proof. "⊂" Let $\lambda \in N_\mathcal{K}(x)$, $n \in \mathbb{N}^*$ and $y \in \mathcal{N}_n(x)$. Then the function $x \pm \frac{1}{n\|y\|_\infty}y$ is nonpositive a.e. on $[0, T]$, and hence, since $\lambda \geqslant 0$,

$$\left\langle \lambda, x \pm \frac{1}{n\|y\|_\infty}y \right\rangle \leqslant 0.$$

Using then that $\langle\lambda, x\rangle = 0$, we obtain that $\pm\langle\lambda, y\rangle \leqslant 0$, i.e. $\langle\lambda, y\rangle = 0$.

"⊃" Assume that $\lambda \in (L^\infty)^*_+$ and $\lambda \in \bigcap_{n\in\mathbb{N}^*}(\mathcal{N}_n(x))^\perp$. Then we have, for all $n \in \mathbb{N}^*$,

$$\langle\lambda, x\rangle = \langle\lambda, \mathbf{1}_{\Delta_n(x)}x\rangle$$

and hence, since $0 \geqslant x(t) \geqslant -\frac{1}{n}$ a.e. on $\Delta_n(x)$,

$$|\langle\lambda, x\rangle| \leqslant \|\lambda\|_{\infty*}\|\mathbf{1}_{\Delta_n(x)}x\|_\infty \leqslant \|\lambda\|_{\infty*}\frac{1}{n}.$$

Letting $n \to +\infty$, we thus obtain that $\langle\lambda, x\rangle = 0$, which achieves the proof. □

We end this section by recalling two results used in the proof of the second-order necessary condition.

Lemma A.2. *The cone $\mathcal{K}$ is polyhedric, i.e. for all $x \in \mathcal{K}$ and all $\lambda \in N_{\mathcal{K}}(x)$,*

$$T_{\mathcal{K}}(x) \cap \lambda^{\perp} = \mathrm{cl}\big(\mathcal{R}_{\mathcal{K}}(x) \cap \lambda^{\perp}\big), \tag{A.3}$$

where $\mathcal{R}_{\mathcal{K}}(x)$ is the radial cone (6.27).

Proof. Let $h \in T_{\mathcal{K}}(x) \cap \lambda^{\perp}$. For $n \in \mathbb{N}^*$, define for a.a. $t \in (0, T)$

$$h_n(t) = \begin{cases} h(t) & \text{a.e. on } [0,T] \setminus \Delta_n(x), \\ h(t)_- & \text{a.e. on } \Delta_n(x) \end{cases}$$

where $h(t)_- = \min(0, h(t))$. For all $0 < \varepsilon < \frac{1}{n\|h\|_\infty}$, it is easily seen that $x + \varepsilon h_n \leqslant 0$ a.e. on $[0, T]$, and hence $h_n \in \mathcal{R}_{\mathcal{K}}(x)$, for all $n \in \mathbb{N}^*$. Moreover, in view of (A.2), we have that $\langle \lambda, h_n \rangle = \langle \lambda, h_- \rangle$. Since $\langle \lambda, h \rangle = \langle \lambda, h_+ \rangle + \langle \lambda, h_- \rangle = 0$, it follows that

$$\big|\langle \lambda, h_- \rangle\big| = \big|\langle \lambda, h_+ \rangle\big| = \big|\langle \lambda, \mathbf{1}_{\Delta_n(x)} h_+ \rangle\big| \leqslant \|\lambda\|_{\infty *} \|\mathbf{1}_{\Delta_n(x)} h_+\|_\infty \to 0$$

when $n \to +\infty$ by (A.1). Hence $\langle \lambda, h_n \rangle = 0$. Finally, $\|h - h_n\|_\infty = \|\mathbf{1}_{\Delta_n(x)} h_+\|_\infty \to 0$ by (A.1) again. So h_n is a sequence in $\mathcal{R}_K(x) \cap \lambda^{\perp}$ that converges to h in L^∞. □

Lemma A.3. *Let $x \in \mathcal{K}$. For any $\lambda \in N_{\mathcal{K}}(x) \cap L^2(0,T)$, the set $T_{\mathcal{K}}(x) \cap \lambda^{\perp}$ is dense in the set $\hat{T}(x) \cap \lambda^{\perp}$, with*

$$\hat{T}(x) := \big\{ w \in L^2(0,T);\ w \leqslant 0 \text{ a.e. on } \Delta(x) \big\}. \tag{A.4}$$

Proof. Let $\hat{w} \in \hat{T}(x) \cap \lambda^{\perp}$. Let w_n be defined a.e. on $[0, T]$ by:

$$w_n(t) = \begin{cases} \max(\min(\hat{w}(t), n), -n) & \text{if } t \in [0,T] \setminus \Delta_n(x), \\ \max(\min(\hat{w}(t), 0), -n) & \text{if } t \in \Delta_n(x). \end{cases}$$

Then $w_n \in L^\infty$, and for all $k \geqslant n$, $\mathbf{1}_{\Delta_k(x)} w_n \leqslant 0$ a.e., and hence by (A.1) $w_n \in T_{\mathcal{K}}(x)$. Since $\lambda \in N_{\mathcal{K}}(x) \cap L^2(0,T)$, $\int_0^T \lambda(t) x(t)\,\mathrm{d}t = 0$ implies that $\lambda(t) = 0$ for a.a. $t \in [0,T] \setminus \Delta(x)$. And then $\int_0^T \lambda(t)\hat{w}(t)\,\mathrm{d}t = 0$ implies, since $\hat{w}(t) \leqslant 0$ on $\Delta(x)$, that $\hat{w}(t) = 0$ for a.a. t such that $\lambda(t) \neq 0$. Consequently, we also have that $w_n(t) = 0$ for a.a. t such that $\lambda(t) \neq 0$, and hence, $\langle \lambda, w_n \rangle = \int_0^T \lambda(t) w_n(t)\,\mathrm{d}t = 0$, i.e. $w_n \in T_{\mathcal{K}}(x) \cap \lambda^{\perp}$. It remains to show that $w_n \to \hat{w}$ for the norm of L^2. If $t \notin \Delta(x)$, for n large enough, $w_n(t) = \max(\min(\hat{w}(t), n), -n) \to \hat{w}(t)$ when $n \to \infty$, and if $t \in \Delta(x)$, since $\hat{w}(t) \leqslant 0$ a.e. on $\Delta(x)$, for all n we have $w_n(t) = \max(\hat{w}(t), -n) \to \hat{w}(t)$. Hence, $w_n(t) \to \hat{w}(t)$ a.e., and $|w_n(t)| \leqslant |\hat{w}(t)|$ for all $t \in [0,T]$, with $\hat{w} \in L^2$. It follows then from the Lebesgue's dominated convergence theorem that $w_n \to \hat{w}$ in L^2, which achieves the proof. □

A.2. First-order optimality condition

If u is a local solution of (2.5) satisfying (2.34), then it is well known that there exist $\eta \in \mathcal{M}([0,T]; \mathbb{R}^{r*})$ and $\lambda \in (L^\infty)^*(0,T; \mathbb{R}^{s*})$ such that

$$DJ(u)v + \big\langle \eta, DG(u)v \big\rangle + \big\langle \lambda, D\mathcal{G}(u)v \big\rangle = 0, \quad \forall v \in \mathcal{U}, \tag{A.5}$$

$$\eta \in N_K\big(G(u)\big), \quad \lambda \in N_{\mathcal{K}}\big(\mathcal{G}(u)\big). \tag{A.6}$$

Lemma A.4. *Assume that u is a local solution of* (2.5) *satisfying* (2.34)*, and that assumption* (2.31) *holds. Then the multiplier λ belongs to $L^\infty(0,T; \mathbb{R}^{s*})$.*

Proof. Let $\tilde{p}$ be the unique solution in $BV(0,T; \mathbb{R}^{n*})$ of:

$$-\mathrm{d}\tilde{p} = H_y(u, y_u, \tilde{p})\,\mathrm{d}t + \mathrm{d}\eta g_y(y_u); \qquad p(T) = \phi_y\big(y_u(T)\big).$$

Then it is not difficult to show that (A.5) writes, with z_v the solution of (2.22):

$$\int_0^T H_u(u, y_u, \tilde{p}) v\,\mathrm{d}t + \big\langle \lambda, c_y(u, y_u) z_v + c_u(u, y_u) v \big\rangle = 0, \quad \forall v \in \mathcal{U}. \tag{A.7}$$

Since u, y and $\tilde{p}$ belong to L^∞, so do the functions $H_u(u(\cdot), y_u(\cdot), \tilde{p}(\cdot))$, $c_u(u(\cdot), y_u(\cdot))$ and $c_y(u(\cdot), y_u(\cdot))$. It follows then from (A.7) that for all $v \in \mathcal{U}$,

$$\left|\langle\lambda, c_u(u, y_u)v\rangle\right| \leqslant \|\lambda\|_{\infty *}\left\|c_y(u, y_u)\right\|_\infty \|z_v\|_\infty + \left\|H_u(u, y_u, \tilde{p})\right\|_\infty \|v\|_1.$$

By Gronwall's lemma, there exists a constant $\kappa > 0$ such that $\|z_v\|_\infty \leqslant \kappa\|v\|_1$, for all $v \in \mathcal{U}$, and hence we obtain that for all $v \in \mathcal{U}$,

$$\left|\langle\lambda, c_u(u, y_u)v\rangle\right| \leqslant \left(\|\lambda\|_{\infty *}\left\|c_y(u, y_u)\right\|_\infty \kappa + \left\|H_u(u, y_u, \tilde{p})\right\|_\infty\right)\|v\|_1 \leqslant \kappa'\|v\|_1. \tag{A.8}$$

By assumption (2.31), for all $w \in L^\infty(0, T; \mathbb{R}^s)$, there exists $v \in \mathcal{U}$ such that $w_i(t) = c_{i,u}(u(t), y_u(t))v(t)$ for a.a. $t \in \Delta_n(c_i(u, y_u))$, for all $i = r+1, \dots, r+s$, and $\|v\|_1 \leqslant M\|w\|_1$ for some constant $M > 0$. Indeed, take e.g. $v(t) = C(t)^\top (C(t)C(t)^\top)^{-1} w(t)$ with $C(t) := c_{I_n^c(t),u}(u(t), y_u(t))$ if $I_n^c(t) \neq \emptyset$, and $v(t) = 0$ otherwise, and $M := \|C^\top (CC^\top)^{-1}\|_\infty$. Since $\lambda \in N_{\mathcal{K}}(\mathcal{G}(u))$, the characterization of the critical cone (A.2) implies that $\langle\lambda, c_u(u, y_u)v\rangle = \langle\lambda, w\rangle$. Then (A.8) yields

$$\left|\langle\lambda, w\rangle\right| \leqslant \kappa''\|w\|_1, \quad \forall w \in L^\infty\left(0, T; \mathbb{R}^s\right). \tag{A.9}$$

Since L^∞ is dense in L^1 and λ is continuous for the norm of L^1, λ can be extended to a continuous linear form over $L^1(0, T; \mathbb{R}^s)$. Therefore λ belong to the dual space $L^\infty(0, T; \mathbb{R}^{s*})$. □

It is not difficult to derive from this result the first-order optimality condition given in Theorem 2.5. See related results in [33,24].

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