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On weakly harmonic maps from Finsler to Riemannian manifolds

Heiko von der Mosel [a,*], Sven Winklmann [b]

[a] *Institut für Mathematik, RWTH Aachen, Germany*
[b] *Centro di Ricerca Matematica Ennio De Giorgi, Scuola Normale Superiore, Pisa, Italy*

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Abstract

We prove global $C^{0,\alpha}$-estimates for harmonic maps from Finsler manifolds into regular balls of Riemannian target manifolds generalizing results of Giaquinta, Hildebrandt, and Hildebrandt, Jost and Widman from Riemannian to Finsler domains. As consequences we obtain a Liouville theorem for entire harmonic maps on simple Finsler manifolds, and an existence theorem for harmonic maps from Finsler manifolds into regular balls of a Riemannian target.

Résumé

Nous démontrons des $C^{0,\alpha}$-estimations globales pour des applications harmoniques des variétés de Finsler dans des boules régulières des variétés riemanniennes en généralisant des résultats de Giaquinta, Hildebrandt et de Hildebrandt, Jost et Widman pour des domaines de Riemann, à des domaines de Finsler. En conséquence nous obtenons un théorème de Liouville pour des applications harmoniques entières sur des variétés de Finsler simples, et un théorème d'existence pour des applications harmoniques des variétés de Finsler dans des boules régulières d'une variété riemannienne.

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1. Introduction

Let $\mathscr{M}^m$ be an m-dimensional oriented smooth manifold and $\chi : \Omega \to \mathbb{R}^m$ a local chart on an open subset $\Omega \subset \mathscr{M}$ which introduces local coordinates $(x^1, \ldots, x^m) = (x^\alpha)$, $\alpha = 1, \ldots, m$. We denote by $T\mathscr{M} = \bigcup_{x\in\mathscr{M}} T_x\mathscr{M}$ the tangent bundle consisting of points (x, y), $x \in \mathscr{M}$, $y \in T_x\mathscr{M}$. These points can be identified on $\pi^{-1}(\Omega) \subset T\mathscr{M}$ by bundle coordinates (x^α, y^α), $\alpha = 1, \ldots, m$, where $\pi : T\mathscr{M} \to \mathscr{M}$, $\pi(x, y) := x$, is the natural projection of $T\mathscr{M}$ onto the base manifold $\mathscr{M}$, and where $y = y^\alpha \frac{\partial}{\partial x^\alpha}|_x \in T_x\mathscr{M}$. Whenever possible, we will not distinguish between the point (x, y) and its coordinate representation (x^α, y^α). Moreover, we employ Einstein's summation convention: Repeated

* Corresponding author.
E-mail addresses: heiko@instmath.rwth-aachen.de (H. von der Mosel), s.winklmann@sns.it (S. Winklmann).

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Greek indices are automatically summed from 1 to m. We will also frequently use the abbreviations $f_{y^\alpha} = \frac{\partial f}{\partial y^\alpha}$, $f_{y^\alpha y^\beta} = \frac{\partial^2 f}{\partial y^\alpha \partial y^\beta}$, etc.

A *Finsler structure* F on $\mathscr{M}$ is a function $F : T\mathscr{M} \to [0,\infty)$ with the following properties: $F \in C^\infty(T\mathscr{M} \setminus 0)$, where $T\mathscr{M} \setminus 0 := \{(x,y) \in T\mathscr{M}, y \neq 0\}$ denotes the *slit tangent bundle*;

$$F(x,ty) = tF(x,y) \quad \text{for all } (x,y) \in T\mathscr{M},\ t > 0; \tag{H}$$

and the *fundamental tensor* $g_{\alpha\beta}(x,y) := (\frac{1}{2}F^2)_{y^\alpha y^\beta}(x,y)$ is positive definite for all $(x,y) \in T\mathscr{M} \setminus 0$. The pair $(\mathscr{M}, F)$ is called a *Finsler manifold*.

An explicit fundamental example is given by the *Minkowski space* $(\mathbb{R}^m, F)$, where $F = F(y)$ does not depend on $x \in \mathbb{R}^m$. A manifold $(\mathscr{M}, F)$ is called *locally Minkowskian*, if for every $x \in \mathscr{M}$ there is a local neighborhood Ω of x such that $F = F(y)$ on $T\Omega$. Moreover, any Riemannian manifold $(\mathscr{M}, g)$ with Riemannian metric g is a Finsler manifold with $F(x,y) := \sqrt{g_{\alpha\beta}(x)y^\alpha y^\beta}$. A Finsler manifold $(\mathscr{M}, F)$ with

$$F(x,y) := \sqrt{g_{\alpha\beta}(x)y^\alpha y^\beta} + b_\sigma(x)y^\sigma, \quad \|b\| := \sqrt{g^{\alpha\beta}b_\alpha b_\beta} < 1, \tag{1.1}$$

is called a *Randers space*.

In the present paper we study harmonic mappings $U : (\mathscr{M}, F) \to (\mathscr{N}, h)$ from a Finsler manifold $(\mathscr{M}, F)$ into an n-dimensional Riemannian target manifold $\mathscr{N}^n$ with metric h and with $\partial\mathscr{N} = \emptyset$. What does it mean for U to be harmonic? While it is common knowledge how to measure the differential dU of U in the Riemannian target by means of the metric h, it is not at all obvious how to integrate the most evident choice of energy density $e(U)(x,y) := \frac{1}{2}g^{\alpha\beta}(x,y)\frac{\partial u^i}{\partial x^\alpha}\frac{\partial u^j}{\partial x^\beta}h_{ij}(u)$ over the Finsler manifold. Here, u is the local representation of U with respect to coordinates (x^α), $\alpha = 1,\dots,m$, on $\mathscr{M}$, and (u^i), $i = 1,\dots,n$, on $\mathscr{N}$; h_{ij} are the coefficients of the Riemannian metric h, and $(g^{\alpha\beta})$ denotes the inverse matrix of $(g_{\alpha\beta})$. In fact, the fundamental tensor $g_{\alpha\beta}$ does *not* establish a well-defined Riemannian metric on $\mathscr{M}$ since it depends not only on $x \in \mathscr{M}$ but also on $y \in T_x\mathscr{M}$. In other words, on each tangent space $T_x\mathscr{M}$, $x \in \mathscr{M}$, one has a whole m-dimensional continuum of possible choices of inner products formally written as $g_{\alpha\beta}(x,y)\,dx^\alpha \otimes dx^\beta$ for $y \in T_x\mathscr{M} \setminus \{0\}$.

We are going to describe in Section 2 how to overcome this conceptual problem by incorporating the “reference directions” $(x,[y]) := \{(x,ty)\colon t > 0\}$ as base points for larger vector bundles sitting over the sphere bundle $S\mathscr{M} = \{(x,[y])\colon (x,y) \in T\mathscr{M} \setminus \{0\}\}$. The resulting general integration formula (Proposition 2.2) yields in particular the integral energy $E(U)$ whose critical points are harmonic mappings. It turns out that for scalar mappings $E(U)$ is proportional to the Rayleigh quotients studied by Bao, Lackey [1] in connection with eigenvalue problems on Finsler manifolds. For mappings into Riemannian manifolds $E(U)$ coincides with Mo’s variant [23] of energy. Mo established a formula for the first variation of the energy, and proved among other things that the identity map from a locally Minkowskian manifold to the same manifold with a flat Riemannian metric is harmonic. Shen and Zhang [29] generalized Mo’s work to Finsler target manifolds, derived the first and second variation formulae, proved non-existence of non-constant stable harmonic maps between Finsler manifolds, and provided with the identity map an example of a harmonic map defined on a flat Riemannian manifold with a Finsler target thus reversing Mo’s setting.

In contrast to these investigations focused on geometric properties of harmonic maps whose existence and smoothness is generally assumed, Tachikawa [31] has studied the variational problem for harmonic maps into Finsler spaces, starting from Centore’s [4] formula for the energy density, which can be regarded as a special case of Jost’s [18–20] general setting of harmonic maps between metric spaces. In particular, Tachikawa [31] has shown a partial regularity result for energy minimizing and therefore harmonic maps from $\mathbb{R}^m$ into a Finsler target manifold for $m = 3, 4$. More recently, Souza, Spruck, and Tenenblat [30] proved Bernstein theorems and the removability of singularities for minimal graphs in particular Randers spaces (cf. (1.1) above) if $\|b\| < 1/\sqrt{3}$, since then the underlying partial differential equation can be shown to be of mean curvature type studied intensively by L. Simon and many others. For $b > 1/\sqrt{3}$ the equation ceases to be elliptic, and there are minimal cones singular at their vertex.

Here we address the basic question: Do harmonic maps with a Finslerian domain exist, and under what circumstances? To answer this question in the affirmative we draw from earlier results by Giaquinta, Hildebrandt, Jost, Kaul and Widman, in particular [9,14,12], on harmonic maps between Riemannian manifolds with image contained in so-called regular balls.[1] A geodesic ball $\mathscr{B}_L(Q) := \{P \in \mathscr{N}\colon \operatorname{dist}(P,Q) \leqslant L\}$ on $\mathscr{N}$ with center $Q \in \mathscr{N}$ and radius $L > 0$ is called *regular*, if it does not intersect the cut-locus of Q and if $L < \frac{\pi}{2\sqrt{\kappa}}$, where

$$\kappa := \max\Bigl\{0, \sup_{\mathscr{B}_L(Q)} K_{\mathscr{N}}\Bigr\} \tag{1.2}$$

is an upper bound on the sectional curvature $K_{\mathscr{N}}$ of $\mathscr{N}$ within $\mathscr{B}_L(Q)$. It is well-known that on simply connected manifolds $\mathscr{N}$ with $K_{\mathscr{N}} \leqslant 0$ all geodesic balls are regular, and that for $\mathscr{N} := S^n$ all geodesic balls contained in an open hemisphere are regular. If $\mathscr{N}$ is compact, connected, and oriented with an even dimension n and $0 < K_{\mathscr{N}} \leqslant \kappa$, then all geodesic balls of radius $L < \frac{\pi}{2\sqrt{\kappa}}$ are regular, whereas for simply connected manifolds of arbitrary dimension with sectional curvature pinched between $\kappa/4$ and κ any geodesic ball with radius less than $\frac{\pi}{2\sqrt{\kappa}}$ is regular; see e.g. [11, pp. 229, 230, 254].

Introducing also the lower curvature bound

$$\omega := \min\Bigl\{0, \inf_{\mathscr{B}_L(Q)} K_{\mathscr{N}}\Bigr\}, \tag{1.3}$$

we can state our results on *weakly harmonic maps*, i.e. on bounded $W^{1,2}$-solutions of the Euler–Lagrange equation of the energy $E(U)$ (for a detailed definition see Section 3).

Theorem 1.1 *(Interior $C^{0,\alpha}$-estimate). Let $(\mathscr{M}^m, F)$ be a Finsler manifold, and let $(\mathscr{N}^n, h)$ be a complete Riemannian manifold with $\partial\mathscr{N} = \emptyset$. Suppose that $\chi : \Omega \to B_{4d}$ is a local coordinate chart of $\mathscr{M}$ which maps Ω onto the open ball $B_{4d} \equiv B_{4d}(0) := \{x \in \mathbb{R}^m\colon |x| < 4d\}$, and suppose that the components of the Finsler metric $g_{\alpha\beta}(x,y)$ satisfy*

$$\lambda|\xi|^2 \leqslant g_{\alpha\beta}(x,y)\xi^\alpha\xi^\beta \leqslant \mu|\xi|^2 \tag{1.4}$$

for all $\xi \in \mathbb{R}^m$ and all $(x,y) \in T\Omega \setminus 0 \cong B_{4d} \times \mathbb{R}^m \setminus \{0\}$ with constants $0 < \lambda \leqslant \mu < +\infty$. Moreover, let $\mathscr{B}_L(Q) \subset \mathscr{N}$ be a regular ball. Finally, assume that $U : \mathscr{M} \to \mathscr{N}$ is a weakly harmonic map with $U(\Omega) \subset \mathscr{B}_L(Q)$. Let u denote the local representation of U with respect to χ and a normal coordinate chart around Q. Then U is Hölder continuous, and we have the estimate

$$\text{Höl}_{\alpha,B_d} u := \sup_{x,y \in B_d} \frac{|u(x) - u(y)|}{|x-y|^\alpha} \leqslant C d^{-\alpha} \tag{1.5}$$

with constants $0 < \alpha < 1$ and $C > 0$ depending only on m, λ, μ, L, ω and κ, but not on $d > 0$. Here, ω and κ are the bounds on the sectional curvature of $\mathscr{N}$ on $\mathscr{B}_L(Q)$ from (1.2) *and* (1.3)*, respectively.*

Letting $d \to \infty$ in (1.5) we immediately obtain the following Liouville theorem for harmonic maps from simple Finsler manifolds generalizing [14, Thm. 1]. Here, a Finsler manifold $(\mathscr{M}, F)$ is called *simple* if there exists a global coordinate chart $\chi : \mathscr{M} \to \mathbb{R}^m$ for which the Finsler metric satisfies condition (1.4) for all $\xi \in \mathbb{R}^m$, $(x,y) \in T\mathscr{M} \setminus \{0\} \cong \mathbb{R}^m \times \mathbb{R}^m \setminus \{0\}$, with constants $0 < \lambda \leqslant \mu < +\infty$.

Theorem 1.2 *(Liouville Theorem). Suppose that $(\mathscr{M}, F)$ is a simple Finsler manifold and that $(\mathscr{N}, h)$ is a complete Riemannian manifold with $\partial\mathscr{N} = \emptyset$. Furthermore, suppose that $\mathscr{B}_L(Q)$ is a regular ball in $\mathscr{N}$. Then any harmonic map $U : \mathscr{M} \to \mathscr{N}$ with $U(\mathscr{M}) \subset \mathscr{B}_L(Q)$ is constant.*

Extending the Hölder estimates to the boundary, and combining them with well-known gradient estimates and linear theory we obtain

Theorem 1.3 *(Global $C^{2,\alpha}$-estimates). Let $(\mathscr{M}^m, F)$ be a compact Finsler manifold, $\Phi : \mathscr{M} \to \mathscr{B}_L(Q) \subset \mathscr{N}$ of class $C^{2,\alpha}$, $\mathscr{B}_L(Q)$ a regular ball in the Riemannian target manifold $(\mathscr{N}^n, h)$, $\partial\mathscr{N} = \emptyset$. Then there is a constant C*

[1] Using Jost's method [21] to prove Hölder regularity of generalized harmonic mappings, Eells and Fuglede later considered weakly harmonic maps from *Riemannian polyhedra* into Riemannian manifolds [5–8]. A Finsler manifold, however, does not fall into the category of Riemannian polyhedra.

depending only on $\kappa, \omega, m, n, \lambda, \mu, \alpha$, *and* Φ *such that* $\|U\|_{C^{2,\alpha}(\mathscr{M},\mathscr{N})} \leqslant C$ *for all harmonic maps* $U:\mathscr{M}\to\mathscr{N}$ *with* $U(\mathscr{M})\subset\mathscr{B}_L(Q)$ *and* $U|_{\partial\mathscr{M}}=\Phi|_{\partial\mathscr{M}}$.

Theorem 1.3 together with a uniqueness theorem modeled after the corresponding result of Jäger and Kaul [16] can be employed to prove the existence of harmonic maps with boundary data contained in a regular ball by virtue of the Leray–Schauder-degree theory:

Corollary 1.4. *If for a given mapping* $\Phi\in C^{1,\alpha}(\partial\mathscr{M},\mathscr{N})$ *there is a point* $Q\in\mathscr{N}$ *such that* $\Phi(\partial\mathscr{M})$ *is contained in a regular ball about* Q *in* $\mathscr{N}$, *then there exists a harmonic mapping* $U:\mathscr{M}\to\mathscr{N}$ *with image* $U(\mathscr{M})$ *contained in that regular ball, and with* $U|_{\partial\mathscr{M}}=\Phi$.

This result is optimal in the sense that the less restrictive inequality $L\leqslant\frac{\pi}{2\sqrt{\kappa}}$ in the definition of a regular ball admits an example of a boundary map $\Phi:\partial(\mathscr{M}^m)\to\mathscr{N}^n:=S^n$ with $\Phi(\partial\mathscr{M})\subset\mathscr{B}_L(Q)$, $L=\frac{\pi}{2\sqrt{\kappa}}$, $n=m\geqslant 7$, and $\mathscr{M}$ a Riemannian manifold, such that Φ *cannot* be extended to a harmonic map of int$(\mathscr{M})$ into $\mathscr{N}$; see [15, Sec. 2].

The proof of Theorem 1.1, which will be carried out in detail in Section 3, consists of a local energy estimate and a subtle iteration procedure based on the observation that $|u|^2$ is a subsolution of an appropriate linear elliptic equation. We learnt about this approach from M. Pingen's work [26,27], who utilized ideas of Caffarelli [3] and M. Meier [22] to study not only harmonic maps between Riemannian manifolds, but also parabolic systems and singular elliptic systems. With this elegant method we can completely avoid the use of mollified Green's functions in contrast to [9], or [5,6].

In Section 4 we sketch the ideas how to extend the Hölder estimates to the boundary. For the gradient estimate we refer to the Campanato method described in [9, Sec. 7], again avoiding any arguments based on Green's functions. Once having established these estimates, the higher order estimates in Theorem 1.3 follow from standard linear theory, see e.g. [10]. Finally, Corollary 1.4 can be proved in the same way as the corresponding existence theorem in [12]. Therefore, details will be left to the reader. In addition, more detailed but straightforward computations regarding the transformation behavior of several geometric quantities introduced in Section 2 were suppressed here to shorten the presentation; they can be found in the extended preprint version [32] of this article.

2. Basic concepts from Finsler geometry and preliminary results

Fundamental tensor and Cartan tensor

Properties (i)–(iii) of the Finsler structure F presented in the introduction imply that $F(x,\cdot):T_x\mathscr{M}\to[0,\infty)$ defines a *Minkowski norm* on each tangent space $T_x\mathscr{M}$, $x\in\mathscr{M}$. Moreover, from the homogeneity relation (H) together with Euler's Theorem on homogeneous functions we infer $g_{\alpha\beta}(x,y)y^\alpha y^\beta=(FF_{y^\alpha y^\beta}+F_{y^\alpha}F_{y^\beta})y^\alpha y^\beta=F^2(x,y)$ for all $(x,y)\in T\mathscr{M}\setminus 0$, which in particular implies $F(x,y)>0$ for all $(x,y)\in T\mathscr{M}\setminus 0$. The coefficients of the *Cartan tensor* are given by[2] $A_{\alpha\beta\gamma}(x,y):=\frac{F}{2}\frac{\partial g_{\alpha\beta}}{\partial y^\gamma}(x,y)=\frac{F}{4}(F^2)_{y^\alpha y^\beta y^\gamma}$. The Cartan tensor measures the deviation of a Finsler structure from a Riemannian one in the following sense: The Finsler structure is Riemannian, i.e., $F(x,y)^2=g_{\alpha\beta}(x)y^\alpha y^\beta$, if and only if the coefficients of the Cartan tensor vanish. The following transformation laws for the fundamental tensor and the Cartan tensor under coordinate changes $\tilde{x}^p=\tilde{x}^p(x^1,\dots,x^m)$, $p=1,\dots,m$, on $\mathscr{M}$ can easily be deduced (see [32, Lemma 2.1]):

$$\tilde{g}_{pq}=\frac{\partial x^\alpha}{\partial\tilde{x}^p}\frac{\partial x^\beta}{\partial\tilde{x}^q}g_{\alpha\beta}\quad\text{and}\quad\tilde{A}_{pqr}=\frac{\partial x^\alpha}{\partial\tilde{x}^p}\frac{\partial x^\beta}{\partial\tilde{x}^q}\frac{\partial x^\gamma}{\partial\tilde{x}^r}A_{\alpha\beta\gamma},\tag{2.1}$$

respectively.

The Sasaki metric

Let $\pi^*T\mathscr{M}$ be the pull-back of $T\mathscr{M}$ and likewise, $\pi^*T^*\mathscr{M}$ be the pull-back of the co-tangent bundle $T^*\mathscr{M}$ under π. That is, e.g., one works with the bundle $\pi^*T\mathscr{M}:=\bigcup_{(x,y)\in T\mathscr{M}\setminus 0}T_x\mathscr{M}$ with fibers given by $(\pi^*T\mathscr{M})_{(x,y)}=$

[2] We follow here the convention used in [2].

$T_{\pi(x,y)}\mathscr{M} = T_x\mathscr{M}$ for all $(x,y) \in T\mathscr{M} \setminus 0$. The vector bundles $\pi^*T\mathscr{M}$ and $\pi^*T^*\mathscr{M}$ have two globally defined sections, namely the *distinguished section*

$$\ell(x,y) := \ell^\alpha(x,y)\frac{\partial}{\partial x^\alpha} := \frac{y^\alpha}{F(x,y)}\frac{\partial}{\partial x^\alpha} \tag{2.2}$$

and the *Hilbert form*

$$\omega := \omega_\alpha(x,y)\,dx^\alpha := \frac{\partial F}{\partial y^\alpha}(x,y)\,dx^\alpha. \tag{2.3}$$

(Here, with a slight abuse of notation, $\frac{\partial}{\partial x^\alpha}$ and dx^α are regarded as sections of $\pi^*T\mathscr{M}$ and $\pi^*T^*\mathscr{M}$, respectively.) The homogeneity condition (H) implies that ℓ and ω are naturally dual to each other, i.e., $\omega(\ell) = 1$, and one obtains (see [32, p. 9])

$$g_{\alpha\beta}(x,y)\ell^\alpha\ell^\beta = 1, \qquad g^{\alpha\beta}(x,y)\omega_\alpha\omega_\beta = 1, \tag{2.4}$$

where $(g^{\alpha\beta})$ denotes the inverse matrix of $(g_{\alpha\beta})$. The introduction of the *formal Christoffel symbols*

$$\gamma^\alpha_{\beta\rho} = \frac{1}{2}g^{\alpha\sigma}\left(\frac{\partial g_{\rho\sigma}}{\partial x^\beta} + \frac{\partial g_{\sigma\beta}}{\partial x^\rho} - \frac{\partial g_{\beta\rho}}{\partial x^\sigma}\right) \tag{2.5}$$

and the coefficients N^α_β of a non-linear connection on $T\mathscr{M} \setminus 0$, the so-called *Ehresmann connection*, defined by

$$\frac{1}{F}N^\alpha_\beta := \gamma^\alpha_{\beta\kappa}\ell^\kappa - A^\alpha_{\beta\kappa}\gamma^\kappa_{\rho\sigma}\ell^\rho\ell^\sigma \tag{2.6}$$

gives rise to the following local sections of $T(T\mathscr{M} \setminus 0)$ and $T^*(T\mathscr{M} \setminus 0)$:

$$\frac{\delta}{\delta x^\beta} := \frac{\partial}{\partial x^\beta} - N^\alpha_\beta\frac{\partial}{\partial y^\alpha} \quad \text{and} \quad \delta y^\alpha := dy^\alpha + N^\alpha_\beta\,dx^\beta.$$

It is easily checked that $\{\frac{\delta}{\delta x^\alpha}, F\frac{\partial}{\partial y^\alpha}\}$ and $\{dx^\alpha, \frac{\delta y^\alpha}{F}\}$ form local bases for the tangent bundle and co-tangent bundle of $T\mathscr{M} \setminus 0$, respectively, which are naturally dual to each other. The reason to introduce these new bases is their nice behavior under coordinate transformations as stated in the following lemma; for the straightforward but tedious proof we refer to [32, Appendix].

Lemma 2.1. *Let $\tilde{x}^p = \tilde{x}^p(x^1,\dots,x^m)$, $p = 1,\dots,m$, be a local coordinate change on $\mathscr{M}$ and let $\tilde{y}^p = \frac{\partial \tilde{x}^p}{\partial x^\alpha}y^\alpha$ be the induced coordinate change on $T\mathscr{M}$. Then*

$$\frac{\delta}{\delta \tilde{x}^p} = \frac{\partial x^\alpha}{\partial \tilde{x}^p}\frac{\delta}{\delta x^\alpha}, \quad \frac{\partial}{\partial \tilde{y}^p} = \frac{\partial x^\alpha}{\partial \tilde{x}^p}\frac{\partial}{\partial x^\alpha}, \qquad d\tilde{x}^p = \frac{\partial \tilde{x}^p}{\partial x^\alpha}dx^\alpha, \quad \delta\tilde{y}^p = \frac{\partial \tilde{x}^p}{\partial x^\alpha}\delta y^\alpha. \tag{2.7}$$

As an important consequence we deduce from (2.1) and (2.7) that

$$G = g_{\alpha\beta}(x,y)\,dx^\alpha \otimes dx^\beta + g_{\alpha\beta}(x,y)\frac{\delta y^\alpha}{F(x,y)} \otimes \frac{\delta y^\beta}{F(x,y)}$$

defines a Riemannian metric on $T\mathscr{M} \setminus 0$, the so-called *Sasaki metric*. It induces a splitting of $T(T\mathscr{M} \setminus 0)$ into horizontal subspaces spanned by $\{\frac{\delta}{\delta x^\alpha}\}$ and vertical subspaces spanned by $\{F\frac{\partial}{\partial y^\alpha}\}$, respectively. By a straightforward computation (see Appendix of [32]) one deduces that with respect to this splitting F is horizontally constant, i.e.,

$$\frac{\delta F}{\delta x^\alpha} = 0. \tag{2.8}$$

The sphere bundle $S\mathscr{M}$

We continue with some remarks on scaling invariance. Denote by $S\mathscr{M} = \{(x,[y])\colon (x,y) \in T\mathscr{M} \setminus 0\}$ the *sphere bundle* which consists of the rays $(x,[y]) := \{(x,ty)\colon t > 0\}$. Since the objects $g_{\alpha\beta}$, $\frac{\delta y^\alpha}{F}$, G, etc. are invariant under the scaling $(x,y) \mapsto (x,ty)$, $t > 0$, they naturally make sense on $S\mathscr{M}$. To be more precise, consider the indicatrix bundle

$I := \{(x,y) \in T\mathscr{M} \setminus 0\colon F(x,y) = 1\}$. I is a hypersurface of $T\mathscr{M} \setminus 0$ which can be identified with $S\mathscr{M}$ via the diffeomorphism $\iota\colon S\mathscr{M} \to I$, $\iota(x,[y]) = (x, \frac{y}{F(x,y)})$. Also note that I carries an orientation, since $\nu := y^\alpha \frac{\partial}{\partial y^\alpha}$ is a globally defined unit normal vector field along I. Indeed, by (2.4), ν has unit length: $G(\nu,\nu) = g_{\alpha\beta} y^\tau y^\sigma \frac{\delta y^\alpha}{F}(\frac{\partial}{\partial y^\tau}) \frac{\delta y^\beta}{F}(\frac{\partial}{\partial y^\sigma}) = g_{\alpha\beta} \frac{y^\tau}{F} \frac{y^\sigma}{F} \delta^\alpha_\tau \delta^\beta_\sigma \underset{(2.4)}{=} 1$. Furthermore, since F is horizontally constant by (2.8), the differential of F is given by $dF = \frac{\delta F}{\delta x^\alpha} dx^\alpha + F \frac{\partial F}{\partial y^\alpha} \frac{\delta y^\alpha}{F} = \frac{\partial F}{\partial y^\alpha} \delta y^\alpha$, and therefore, for any tangent vector $X = X^\alpha \frac{\delta}{\delta x^\alpha} + Y^\alpha F \frac{\partial}{\partial y^\alpha}$ on $T\mathscr{M} \setminus 0$ we find $dF(X) = \frac{\partial F}{\partial y^\alpha} F Y^\alpha$, which then leads to $d(\log F)(X) = \frac{dF(X)}{F} = g_{\alpha\beta}(x,y) \frac{y^\beta}{F} Y^\alpha = G(\nu, X)$. In particular, if X is tangent to I at $(x,y) \in I$, i.e., $X = \frac{dc}{dt}(0)$ for some smooth curve $c\colon (-\varepsilon,\varepsilon) \to I$ with $c(0) = (x,y)$, we obtain $G(\nu, X) = d(\log F)(X) = \frac{d}{dt}(\log F)(c(t))|_{t=0} = 0$, where we have used in the last equation that $F = 1$ on I.

Hence, we can think of $S\mathscr{M} \subset T\mathscr{M} \setminus 0$ as being an oriented $(2m-1)$-dimensional submanifold of $T\mathscr{M} \setminus 0$ to which the above objects pull back. In particular, the Sasaki metric induces a Riemannian metric $G_{S\mathscr{M}}$ with a volume form $dV_{S\mathscr{M}}$ on $S\mathscr{M}$. $dV_{S\mathscr{M}}$ will be of particular importance in the definition of harmonic mappings from Finsler manifolds.

Orthonormal frames

For later purposes let us write down some of the preceding formulas in orthonormal frames: Let $\{e_\sigma\}$ be an oriented local g-orthonormal frame for $\pi^* T\mathscr{M}$ (i.e. $g(e_\sigma, e_\tau) = \delta_{\sigma\tau}$), such that $e_m = \ell$ is the distinguished section defined in (2.2). Let $\{\omega^\sigma\}$ be the dual frame for $\pi^* T^*\mathscr{M}$ such that $\omega^m = \omega$ is the Hilbert form (2.3). Then we have local expansions of the form $e_\sigma = u^\alpha_\sigma \frac{\partial}{\partial x^\alpha}$ and $\omega^\sigma = v^\sigma_\alpha\, dx^\alpha$. Since $e_m = \ell$ and $\omega^m = \omega$ we find $u^\alpha_m = \ell^\alpha = \frac{y^\alpha}{F}$ and $v^m_\alpha = F_{y^\alpha}$. Also note the relations $u^\sigma_\beta v^\alpha_\sigma = \delta^\alpha_\beta$, and $u^\alpha_\sigma v^\sigma_\beta = \delta^\alpha_\beta$, $u^\alpha_\sigma u^\beta_\tau g_{\alpha\beta}(x,y) = \delta_{\sigma\tau}$. Hence,

$$\det(v^\sigma_\alpha) = +\sqrt{\det(g_{\alpha\beta})(x,y)}, \tag{2.9}$$

where the positive sign is due to the specific orientation of the frame. We can now introduce local G-orthonormal bases $\{\hat{e}_\sigma, \hat{e}_{m+\sigma}\}$ for $T(T\mathscr{M} \setminus 0)$ and $\{\omega^\sigma, \omega^{m+\sigma}\}$ for $T^*(T\mathscr{M} \setminus 0)$ which are dual to each other:

$$\hat{e}_\sigma = u^\alpha_\sigma \frac{\delta}{\delta x^\alpha}, \quad \hat{e}_{m+\sigma} = u^\alpha_\sigma F \frac{\partial}{\partial y^\alpha}, \quad \sigma = 1,\dots,m, \tag{2.10}$$

and

$$\omega^\sigma = v^\sigma_\alpha\, dx^\alpha, \quad \omega^{m+\sigma} = v^\sigma_\alpha \frac{\delta y^\alpha}{F}, \quad \sigma = 1,\dots,m. \tag{2.11}$$

In these frames, the Sasaki metric takes the form $G = \delta_{\sigma\tau}\omega^\sigma \otimes \omega^\tau + \delta_{\sigma\tau}\omega^{m+\sigma} \otimes \omega^{m+\tau}$ and its volume form on $T\mathscr{M} \setminus 0$ is given by

$$dV_{T\mathscr{M}\setminus 0} = \omega^1 \wedge \cdots \wedge \omega^m \wedge \omega^{m+1} \wedge \cdots \wedge \omega^{2m}. \tag{2.12}$$

Since F is horizontally constant by (2.8), and $v^m_\alpha = F_{y^\alpha}$, one easily verifies the relation $\omega^{2m} = d(\log F)$. Thus, ω^{2m} vanishes on the indicatrix bundle I, which means that $\hat{e}_{2m}$ is a unit normal to I and $\hat{e}_1,\dots,\hat{e}_{2m-1}$ are tangential. Note that $\hat{e}_{2m}$ coincides with the above defined normal vector field ν. In particular, we may specify the orientation of I such that $\{\hat{e}_1,\dots,\hat{e}_{2m-1}\}$ is positively oriented. It follows that $dV_{S\mathscr{M}}$ is given by $dV_{S\mathscr{M}} = \omega^1 \wedge \cdots \wedge \omega^m \wedge \omega^{m+1} \wedge \cdots \wedge \omega^{2m-1}$. In other words, $dV_{S\mathscr{M}}$ can be obtained by plugging ν into the last slot of $dV_{T\mathscr{M}\setminus 0}$, i.e.,

$$dV_{S\mathscr{M}}(X_1,\dots,X_{2m-1}) = dV_{T\mathscr{M}\setminus 0}(X_1,\dots,X_{2m-1},\nu) \tag{2.13}$$

for all vector fields $X_1,\dots,X_{2m-1}$ tangential to $S\mathscr{M} \subset T\mathscr{M} \setminus 0$.

The volume $dV_{S\mathscr{M}}$ in local coordinates

For local computations, in particular for the derivation of the Euler–Lagrange equations for weakly harmonic mappings, we need to derive an expression for the volume element $dV_{S\mathscr{M}}$ in local coordinates.

Let $\chi : \Omega \to \mathbb{R}^m$ be a local coordinate chart of $\mathscr{M}$ with coordinates $(x^1, \dots, x^m)$. We consider the mapping $\Phi : \Omega \times S^{m-1} \to I \subset T\mathscr{M} \setminus 0$ with $\Phi(x,\theta) := (x, \frac{y}{F(x,y)})$, where

$$y = y(x,\theta) := y^\alpha(\theta) \frac{\partial}{\partial x^\alpha}\bigg|_x, \tag{2.14}$$

and y^α are Cartesian coordinates of $\theta \in S^{m-1}$, i.e.,

$$\theta = \big(y^1(\theta), \dots, y^m(\theta)\big). \tag{2.15}$$

Let $(\theta^1, \dots, \theta^{m-1})$ be local coordinates for S^{m-1}. Then we compute

$$d\Phi\left(\frac{\partial}{\partial x^\alpha}\right) = \frac{\partial}{\partial x^\alpha} - \frac{1}{F^2}\frac{\partial F}{\partial x^\alpha} y^\beta \frac{\partial}{\partial y^\beta}, \quad \alpha = 1, \dots, m, \tag{2.16}$$

and

$$d\Phi\left(\frac{\partial}{\partial \theta^A}\right) = \left(\frac{1}{F}\frac{\partial y^\beta}{\partial \theta^A} - \frac{1}{F^2}\frac{\partial F}{\partial y^\gamma}\frac{\partial y^\gamma}{\partial \theta^A} y^\beta\right)\frac{\partial}{\partial y^\beta}, \quad A = 1, \dots, m-1. \tag{2.17}$$

Here note carefully that, on the left-hand side, $\frac{\partial}{\partial x^\alpha}$ and $\frac{\partial}{\partial \theta^A}$ are considered as tangent vectors to Ω and S^{m-1} with respect to (x^α) and (θ^A), respectively, whereas on the right-hand side $\frac{\partial}{\partial x^\alpha}$ and $\frac{\partial}{\partial y^\alpha}$ are tangent vectors of $T\mathscr{M}$ associated with the bundle coordinates (x^α, y^α).

Also notice that $\eta_A := (\frac{\partial y^1}{\partial \theta^A}, \dots, \frac{\partial y^m}{\partial \theta^A})$ and $\eta_m := (y^1(\theta), \dots, y^m(\theta))$ are nothing but the realizations of $\frac{\partial}{\partial \theta^A}$ and θ as vectors in $\mathbb{R}^m$. In particular we may without loss of generality assume that $\{\eta_1, \dots, \eta_m\}$ forms a positively oriented basis of $\mathbb{R}^m$. We recall that the normal of the indicatrix bundle at $\Phi(x,\theta) = (x, \frac{y(x,\theta)}{F(x,y(x,\theta))})$ is given by

$$\nu = \hat{e}_{2m} = \frac{y^\alpha}{F(x,y)}\frac{\partial}{\partial y^\alpha}. \tag{2.18}$$

Combining (2.16), (2.17) and (2.18) we obtain:

$$\begin{aligned} &dV_{T\mathscr{M}\setminus 0}\left(\dots, d\Phi\left(\frac{\partial}{\partial x^\alpha}\right), \dots, d\Phi\left(\frac{\partial}{\partial \theta^A}\right), \dots, \nu\right) \\ &\quad = dV_{T\mathscr{M}\setminus 0}\left(\dots, \frac{\partial}{\partial x^\alpha}, \dots, \frac{1}{F}\frac{\partial y^\beta}{\partial \theta^A}\frac{\partial}{\partial y^\beta}, \dots, \frac{y^\gamma}{F}\frac{\partial}{\partial y^\gamma}\right). \end{aligned}$$

From (2.12), (2.9), and (2.10) we infer the relation

$$dV_{T\mathscr{M}\setminus 0}|_{\Phi(x,\theta)} = \det\big(g_{\alpha\beta}(x,y)\big)\, dx^1 \wedge \dots \wedge dx^m \wedge \delta y^1 \wedge \dots \wedge \delta y^m,$$

since $F(\Phi(x,\theta)) = 1$ for all $x \in \Omega$, $\theta \in S^{m-1}$. Hence, we find

$$dV_{T\mathscr{M}\setminus 0}\left(\dots, d\Phi\left(\frac{\partial}{\partial x^\alpha}\right), \dots, d\Phi\left(\frac{\partial}{\partial \theta^A}\right), \dots, \nu\right) = +\frac{\det(g_{\alpha\beta}(x,y))}{F(x,y)^m}\sqrt{\det(\sigma_{AB})}$$

with $\sigma_{AB} := \sum_{\alpha=1}^m \frac{\partial y^\alpha}{\partial \theta^A}\frac{\partial y^\alpha}{\partial \theta^B} = \eta_A \cdot \eta_B$. Note that the sign is due to the specific orientation of $\{\eta_1, \dots, \eta_m\}$. We recall from (2.13) that $dV_{S\mathscr{M}}$ is obtained by plugging ν into the last slot of $dV_{T\mathscr{M}\setminus 0}$. Hence we arrive at

$$\begin{aligned} &\Phi^* dV_{S\mathscr{M}}\left(\dots, \frac{\partial}{\partial x^\alpha}, \dots, \frac{\partial}{\partial \theta^A}, \dots\right) = dV_{S\mathscr{M}}\left(\dots, d\Phi\left(\frac{\partial}{\partial x^\alpha}\right), \dots, d\Phi\left(\frac{\partial}{\partial \theta^A}\right), \dots\right) \\ &\quad = dV_{T\mathscr{M}\setminus 0}\left(\dots, d\Phi\left(\frac{\partial}{\partial x^\alpha}\right), \dots, d\Phi\left(\frac{\partial}{\partial \theta^A}\right), \dots, \nu\right) = \frac{\det(g_{\alpha\beta}(x,y))}{F(x,y)^m}\sqrt{\det(\sigma_{AB})}. \end{aligned}$$

That is $\Phi^* dV_{S\mathscr{M}} = \frac{\det(g_{\alpha\beta}(x,y))}{F(x,y)^m}\sqrt{\det(\sigma_{AB})}\, dx^1 \wedge \dots \wedge dx^m \wedge d\theta^1 \wedge \dots \wedge d\theta^{m-1}$. Finally, observe that $\sqrt{\det(\sigma_{AB})}\, d\theta^1 \wedge \dots \wedge d\theta^{m-1}$ is the standard volume form $d\sigma$ on S^{m-1}. Thus we have shown: $\Phi^* dV_{S\mathscr{M}} = \frac{\det(g_{\alpha\beta}(x,y))}{F(x,y)^m}\, dx^1 \wedge \dots \wedge dx^m \wedge d\sigma$ on $\Omega \times S^{m-1}$.

Let us summarize this as follows:

Proposition 2.2. *Let $\chi : \Omega \to \mathbb{R}^m$ be a local coordinate chart of $\mathscr{M}$, and let $f : S\mathscr{M} \subset T\mathscr{M} \setminus 0 \to \mathbb{R}$ be an integrable function with support in $\pi^{-1}(\Omega)$. Then we have*

$$\int_{S\mathscr{M}} f(x,y)\,dV_{S\mathscr{M}} = \int_{\Omega} \left(\int_{S^{m-1}} f\left(x, \frac{y}{F(x,y)}\right) \frac{\det(g_{\alpha\beta}(x,y))}{F(x,y)^m}\,d\sigma \right) dx.$$

Here, $d\sigma$ is the standard volume form on S^{m-1}, $dx = dx^1 \wedge \cdots \wedge dx^m$, and $y = y(x,\theta)$ for $(x,\theta) \in \Omega \times S^{m-1}$, as defined in (2.14), (2.15).

The Riemannian case – an example

Let $\alpha_1, \ldots, \alpha_m > 0$ be positive real numbers. As a consequence of the identity (for a derivation see e.g. [32, p. 16])

$$\int_{S^{m-1}} \frac{\alpha_1 \cdots \alpha_m}{(\alpha_1^2\theta_1^2 + \cdots + \alpha_m^2\theta_m^2)^{m/2}}\,d\sigma(\theta) = \operatorname{vol}\bigl(S^{m-1}\bigr), \tag{2.19}$$

we find $\int_{S^{m-1}} \frac{\sqrt{\det(g_{\alpha\beta}(x))}}{(g_{\alpha\beta}(x)\theta^\alpha\theta^\beta)^{m/2}}\,d\sigma(\theta) = \operatorname{vol}(S^{m-1})$ for any positive definite symmetric matrix $(g_{\alpha\beta}(x))$. Hence, if the Finsler structure is Riemannian, i.e., $F^2(x,y) = g_{\alpha\beta}(x)y^\alpha y^\beta$, then we have the relation

$$\frac{1}{\operatorname{vol}(S^{m-1})} \int_{S\mathscr{M}} f(x)\,dV_{S\mathscr{M}} = \int_{\Omega} f(x)\sqrt{\det\bigl(g_{\alpha\beta}(x)\bigr)}\,dx = \int_{\mathscr{M}} f(x)\,dV_{\mathscr{M}} \tag{2.20}$$

for all integrable functions $f : \mathscr{M} \to \mathbb{R}$ with support in Ω and trivial extension to $S\mathscr{M}$.

3. Interior regularity of harmonic mappings

The energy functional

Let $U : \mathscr{M}^m \to \mathscr{N}^n$ be a smooth mapping from the m-dimensional Finsler manifold $(\mathscr{M}, F)$ into an n-dimensional Riemannian manifold $(\mathscr{N}, h)$. Following [23,29], we define an energy density $e(U) : S\mathscr{M} \to [0,\infty)$ as follows:

$$e(U)\bigl(x,[y]\bigr) := \frac{1}{2} g^{\alpha\beta}(x,y) \frac{\partial u^i}{\partial x^\alpha} \frac{\partial u^j}{\partial x^\beta} h_{ij}(u). \tag{3.1}$$

Here, u is the local representation of U with respect to coordinates (x^α) and (u^i) on $\mathscr{M}$ and $\mathscr{N}$, respectively, and h_{ij} are the coefficients of the Riemannian target metric h. Moreover, we extend our summation convention: Repeated Latin indices are automatically summed from 1 to n. The energy $E(U)$ is then given by

$$E(U) := \frac{1}{\operatorname{vol}(S^{m-1})} \int_{S\mathscr{M}} e(U)\,dV_{S\mathscr{M}}. \tag{3.2}$$

Here, integration is with respect to the Sasaki metric on $S\mathscr{M}$. We also need the localized energies $E_\Omega(U) := E(U|\Omega)$ for the restriction of U to an open subset $\Omega \subset \mathscr{M}$. In particular, for mappings between Riemannian manifolds the above definition of energy coincides with the usual one by virtue of our observation (2.20), i.e., $E(U) = \frac{1}{2}\int_{\mathscr{M}} g^{\alpha\beta}(x) \frac{\partial u^i}{\partial x^\alpha} \frac{\partial u^j}{\partial x^\beta} h_{ij}(u)\,dV_{\mathscr{M}}$.

As in the Riemannian case, $U \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathscr{N}) \cap L^\infty(\Omega, \mathscr{N})$ is said to be *weakly harmonic* on $\Omega \Subset \mathscr{M}$ if the first variation of E_Ω vanishes at U, i.e., $\frac{d}{d\varepsilon}\big|_{\varepsilon=0} E_\Omega(U_\varepsilon) = 0$ for all variations U_ε of U of the form $U_\varepsilon = \exp_U(\varepsilon V + o(\varepsilon))$, where V is a smooth vector field along U with compact support in Ω. Here, exp denotes the exponential map on $(\mathscr{N}, h)$. We say that U is (weakly) harmonic on $\mathscr{M}$, if it is (weakly) harmonic on Ω for all $\Omega \Subset \mathscr{M}$.

The weak Euler–Lagrange equation

Let $\chi : \Omega \to \mathbb{R}^m$ be a local coordinate chart of $\mathscr{M}$ and put $D := \chi(\Omega)$. In view of the preceding discussion, in particular (3.1), (3.2) and Proposition 2.2, the energy E is locally given by the quadratic functional $E_\Omega(U) = \frac{1}{2}\int_D S^{\alpha\beta}(x)\frac{\partial u^i}{\partial x^\alpha}\frac{\partial u^j}{\partial x^\beta}h_{ij}(u)\,dx$, where

$$S^{\alpha\beta}(x) = \frac{1}{\mathrm{vol}(S^{m-1})}\int\limits_{S^{m-1}} g^{\alpha\beta}(x,y)\frac{\det(g_{\alpha\beta}(x,y))}{F(x,y)^m}\,d\sigma,$$

and the weak Euler–Lagrange equation of E reads as

$$\int\limits_D S^{\alpha\beta}(x)\frac{\partial u^i}{\partial x^\alpha}\frac{\partial \varphi^i}{\partial x^\beta}\,dx = \int\limits_D \Gamma^l_{ij}(u)S^{\alpha\beta}(x)\frac{\partial u^i}{\partial x^\alpha}\frac{\partial u^j}{\partial x^\beta}\varphi^l\,dx \tag{3.3}$$

for all $\varphi \in C^\infty_c(D, \mathbb{R}^n)$. Here, Γ^l_{ij} denote the Christoffel symbols of the Riemannian metric h. Suppose now that the coefficients $g_{\alpha\beta}$ of the Finsler metric satisfy condition (1.4) Then the following structure conditions hold for Eq. (3.3):

$$\lambda_*|\xi|^2 \leqslant S^{\alpha\beta}(x)\xi_\alpha\xi_\beta \leqslant \mu_*|\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^m \text{ and } x \in D \tag{3.4}$$

with $\lambda_* := \lambda^m\mu^{-1-\frac{m}{2}}$, and $\mu_* := \mu^m\lambda^{-1-\frac{m}{2}}$.

Jacobi field estimates

According to Jost [17], any two points P_1, P_2 of a regular ball $\mathscr{B}_L(Q)$ can be connected by a geodesic completely contained in $\mathscr{B}_L(Q)$. This geodesic is shortest among all curves joining P_1 and P_2 within $\mathscr{B}_L(Q)$. Moreover, it contains no pair of conjugate points. In particular, around each point $P \in \mathscr{B}_L(Q)$ one may introduce a normal coordinate chart $\psi : \mathscr{B}_L(Q) \to \mathbb{R}^n$. Denote by $(v^i) = (v^1, \dots, v^n)$ the corresponding coordinates. Then P has coordinates $(0, \dots, 0)$ and, if $P' \in \mathscr{B}_L(Q)$ has coordinates v, then $\mathrm{dist}(P, P') = |v| < \frac{\pi}{\sqrt{\kappa}}$. Moreover, the following estimates hold for the metric and the Christoffel symbols; see e.g. [15, Section 5]:

$$\{\delta_{ij} - a_\omega(|v|)h_{ij}(v)\}\zeta^i\zeta^j \leqslant \Gamma^l_{ij}(v)v^l\zeta^i\zeta^j \leqslant \{\delta_{ij} - a_\kappa(|v|)h_{ij}(v)\}\zeta^i\zeta^j, \tag{3.5}$$

$$b^2_\kappa(|v|)|\zeta|^2 \leqslant h_{ij}(v)\zeta^i\zeta^j \leqslant b^2_\omega(|v|)|\zeta|^2 \tag{3.6}$$

for all $\zeta \in \mathbb{R}^n$. Here, the functions a_σ and b_σ are defined as follows:

$$a_\sigma(t) = \begin{cases} t\sqrt{\sigma}\,\mathrm{ctg}(t\sqrt{\sigma}) & \text{if } \sigma > 0, 0 \leqslant t < \frac{\pi}{\sqrt{\sigma}}, \\ t\sqrt{-\sigma}\,\mathrm{ctgh}(t\sqrt{-\sigma}) & \text{if } \sigma \leqslant 0, 0 \leqslant t < \infty, \end{cases}$$

and

$$b_\sigma(t) = \begin{cases} \frac{\sin(t\sqrt{\sigma})}{t\sqrt{\sigma}} & \text{if } \sigma > 0, 0 \leqslant t < \frac{\pi}{\sqrt{\sigma}}, \\ \frac{\sinh(t\sqrt{-\sigma})}{t\sqrt{-\sigma}} & \text{if } \sigma \leqslant 0, 0 \leqslant t < \infty. \end{cases}$$

As a consequence of (3.5) and (3.6) we obtain for every positive semi-definite matrix $(S^{\alpha\beta}) \in \mathbb{R}^{m\times m}$, and for every matrix $(p^i_\alpha) \in \mathbb{R}^{n\times m}$

$$S^{\alpha\beta}p^i_\alpha p^i_\beta - a_\omega(|v|)S^{\alpha\beta}p^i_\alpha p^j_\beta h_{ij}(v) \leqslant \Gamma^l_{ij}(v)v^l S^{\alpha\beta}p^i_\alpha p^j_\beta \leqslant S^{\alpha\beta}p^i_\alpha p^i_\beta - a_\kappa(|v|)S^{\alpha\beta}p^i_\alpha p^j_\beta h_{ij}(v), \tag{3.7}$$

$$b^2_\kappa(|v|)S^{\alpha\beta}p^i_\alpha p^i_\beta \leqslant S^{\alpha\beta}p^i_\alpha p^j_\beta h_{ij}(v) \leqslant b^2_\omega(|v|)S^{\alpha\beta}p^i_\alpha p^i_\beta. \tag{3.8}$$

Moreover, if we use normal coordinates centered around Q, then by (3.6) in connection with our assumption $L < \frac{\pi}{2\sqrt{\kappa}}$ we can estimate the distance of two points $P_1, P_2 \in \mathscr{B}_L(Q)$ with coordinates p_1, p_2 by[3]

$$b_\kappa(L)|p_1 - p_2| \leqslant \operatorname{dist}(P_1, P_2) \leqslant b_\omega(L)|p_1 - p_2|. \tag{3.9}$$

Subsolutions of elliptic equations and a local energy estimate

Let $\psi : \mathscr{B}_L(Q) \to \mathbb{R}^n$ be a normal coordinate chart around some point $P \in \mathscr{B}_L(Q)$. We denote by $v = (v^1, \dots, v^n)$ the representation of U with respect to ψ and χ, i.e., $v := \psi \circ U \circ \chi^{-1}$ and we abbreviate $\partial_\alpha = \frac{\partial}{\partial x^\alpha}$. The weak Euler–Lagrange equation then takes the form

$$\int_{B_{4d}} \left\{ S^{\alpha\beta}(x)\partial_\alpha v^l \partial_\beta \varphi^l - f^l(v)\varphi^l \right\} dx = 0 \quad \text{for all } \varphi \in C_c^\infty(B_{4d}, \mathbb{R}^n), \tag{3.10}$$

and hence by approximation for all $\varphi \in W_0^{1,2}(B_{4d}, \mathbb{R}^n) \cap L^\infty(B_{4d}, \mathbb{R}^n)$. Here we have set $f^l(v) := \Gamma_{ij}^l(v) S^{\alpha\beta}(x) \times \partial_\alpha v^i \partial_\beta v^j$. Denoting $\mathbf{E}(v) = S^{\alpha\beta}(x)\partial_\alpha v^i \partial_\beta v^j h_{ij}(v)$ and $\mathbf{P}(v) = S^{\alpha\beta}(x)\partial_\alpha v^l \partial_\beta v^l - f^l(v)v^l$ we infer from (3.7)

$$a_\kappa\big(|v|\big)\mathbf{E}(v) \leqslant \mathbf{P}(v). \tag{3.11}$$

Lemma 3.1 *(Subsolution & local energy estimate).* [4] *Let v be the representation of U with respect to normal coordinates around $P \in \mathscr{B}_L(Q)$. Then*

(i) (*Subsolution*) *If $|v| < \frac{\pi}{2\sqrt{\kappa}}$ on a domain $G \subset B_{4d} \subset \mathbb{R}^m$ then $\partial_\alpha(S^{\alpha\beta}(x)\partial_\beta |v|^2) \geqslant 0$ on G.*
(ii) (*Local energy estimate*) *If $|v| \leqslant L$ on $B_{4R}(x_0) \subset B_{4d}$ then*

$$R^{2-m} \int_{B_R(x_0)} \mathbf{E}(v)\,dx \leqslant C\big[M^2(4R) - M^2(R)\big], \tag{3.12}$$

where $M(r) := \sup_{B_r(x_0)} |v|$, $0 \leqslant r \leqslant 4R$. Here, the constant C depends only on m, λ, μ, κ and L.

Proof. (i) Using $\varphi = v\eta$, $\eta \in C_c^\infty(G)$, $\eta \geqslant 0$, as a test-function in (3.10) we obtain:

$$-\frac{1}{2}\int_G S^{\alpha\beta}(x)\partial_\alpha |v|^2 \partial_\beta \eta\,dx = \int_G \mathbf{P}(v)\eta\,dx. \tag{3.13}$$

Since $a_\kappa(|v|) \geqslant a_\kappa(\frac{\pi}{2\sqrt{\kappa}}) = 0$ on G we infer from (3.11) that $\mathbf{P}(v) \geqslant 0$ on G. This gives the desired result.

(ii) By virtue of part (i) the function $z := M^2(4R) - |v|^2 \geqslant 0$ is a supersolution of the linear elliptic operator $\partial_\beta(S^{\alpha\beta}\partial_\alpha)$ in $G := B_{4R}(x_0)$. Hence Moser's weak Harnack inequality [24, Thm. 3], [10, Thm. 8.18] implies the existence of a constant $C_1 = C_1(m, \lambda_*, \mu_*)$ such that

$$\frac{1}{R^m}\int_{B_{2R}(x_0)} z\,dx \leqslant C_1(m, \lambda_*, \mu_*) \inf_{B_R(x_0)} z. \tag{3.14}$$

Let $w \in W_0^{1,2}(B_{4R}(x_0))$ be a solution of

$$\int_{B_{4R}(x_0)} S^{\alpha\beta}\partial_\alpha \varphi \partial_\beta w\,dx = \frac{1}{R^2}\int_{B_{4R}(x_0)} \chi_{B_{2R}(x_0)}\varphi\,dx \quad \forall \varphi \in W_0^{1,2}\big(B_{4R}(x_0)\big). \tag{3.15}$$

[3] For the right inequality compare the length of the geodesic connecting P_1, P_2 with the length of the image of the straight line under exp using (3.6) in the Riemannian length functional together with $b_\omega(|v|) \leqslant b_\omega(L)$. For the left inequality connect P_1 and P_2 by a minimizing geodesic and use $b_\kappa(|v|) \geqslant b_\kappa(L)$.

[4] In the Euclidean context part (i) of this lemma is due to M. Meier [22, p. 5], for part (ii) compare with [9, Proof of Prop. 1].

Then one has $w \not\equiv 0$, and according to [10, Thm. 8.1] the estimate $\inf_{B_{4R}(x_0)} w \geqslant \inf_{\partial B_{4R}(x_0)}(\min\{w, 0\}) = 0$. Therefore, by the weak Harnack inequality, there is a constant $C_2 = C_2(m, \lambda_*, \mu_*)$ such that

$$0 < \frac{1}{R^m} \int_{B_{2R}(x_0)} w \, dx \leqslant C_2(m, \lambda_*, \mu_*) \inf_{B_R(x_0)} w. \tag{3.16}$$

To estimate the left-hand side from below we choose $\varphi := w$ in (3.15) and obtain from (3.4)

$$\lambda_* \int_{B_{4R}(x_0)} |\nabla w|^2 \, dx \leqslant \frac{1}{R^2} \int_{B_{2R}(x_0)} w \, dx. \tag{3.17}$$

On the other hand, we infer from (3.15) and (3.4) by means of Hölder's inequality

$$\frac{1}{R^2} \int_{B_{2R}(x_0)} \varphi \, dx \leqslant \mu_* \|\nabla w\|_{L^2(B_{4R}(x_0))} \|\nabla \varphi\|_{L^2(B_{4R}(x_0))} \quad \text{for all } \varphi \in W_0^{1,2}\big(B_{4R}(x_0)\big),$$

which together with (3.17) yields for any non-negative $\varphi \in W_0^{1,2}(B_{4R}(x_0))$

$$\frac{1}{R^m} \int_{B_{2R}(x_0)} w \, dx \geqslant \frac{1}{R^{m+2}} \frac{\lambda_* \|\varphi\|^2_{L^1(B_{2R}(x_0))}}{\mu_*^2 \|\nabla \varphi\|^2_{L^2(B_{4R}(x_0))}}. \tag{3.18}$$

To estimate the right-hand side we choose φ to be the function[5]

$$\varphi(x) := \frac{1}{2m}\big[(4R)^2 - |x - x_0|^2\big] \quad \in W_0^{1,2}\big(B_{4R}(x_0)\big), \tag{3.19}$$

which leads to an explicit lower bound for the left-hand side of (3.16) depending only on m, λ_*, μ_*, but not on R. Hence, we find a constant $C_3 = C_3(m, \lambda_*, \mu_*)$ such that

$$0 < C_3 \leqslant w \quad \text{in } B_R(x_0). \tag{3.20}$$

On the other hand, a quantitative version of Stampacchia's maximum principle (see [13, Lemma 2.1]) yields a constant $C_4 = C_4(m, \lambda_*, \mu_*)$ such that

$$0 \leqslant w \leqslant C_4 \quad \text{in } B_{4R}(x_0). \tag{3.21}$$

Inserting $\varphi := wz \in W_0^{1,2}(B_{4R}(x_0))$ as a test-function in (3.15) leads to

$$\begin{aligned} \int_{B_{4R}(x_0)} S^{\alpha\beta} \partial_\alpha z \partial_\beta (w^2) \, dx &\underset{(3.4)}{\leqslant} \int_{B_{4R}(x_0)} S^{\alpha\beta} \partial_\alpha z \partial_\beta (w^2) \, dx + \int_{B_{4R}(x_0)} 2 S^{\alpha\beta} z \partial_\alpha w \partial_\beta w \, dx \\ &= \frac{2}{R^2} \int_{B_{2R}(x_0)} wz \, dx, \end{aligned} \tag{3.22}$$

where we used ellipticity (3.4) and the fact that $z \geqslant 0$ to obtain the inequality on the left. On the other hand, using (3.13) together with (3.11) and the fact that $a_\kappa(|v|) \geqslant a_\kappa(L) > a_\kappa(\frac{\pi}{2\sqrt{\kappa}}) = 0$, we obtain $0 \leqslant \int_{B_{4R}(x_0)} \mathbf{E}(v)\eta \, dx \leqslant \frac{1}{2a_\kappa(L)} \int_{B_{4R}(x_0)} S^{\alpha\beta} \partial_\alpha z \partial_\beta \eta \, dx$ for any $\eta \in W_0^{1,2}(B_{4R}(x_0)) \cap L^\infty(B_{4R}(x_0))$. Applying this to $\eta := w^2$ in combination with (3.20), (3.22), (3.21), and (3.14) we arrive at

[5] The specific function φ in (3.19) solves the equation $\Delta\varphi = -1$ on $B_{4R}(x_0)$ thus maximizing the quotient $[\int f]^2 / \|\nabla f\|^2_{L^2}$ among functions f defined on $B_{4R}(x_0)$ with zero boundary data. This is related to the classical problem of *torsional rigidity* of isotropic beams; see [28, Ch. 5], [25]. Note, however, that the L^1-norm in the quotient in (3.18) is taken over the smaller ball $B_{2R}(x_0)$.

$$
\begin{aligned}
C_3^2 \int\limits_{B_R(x_0)} \mathbf{E}(v)\,dx &\underset{(3.20)}{\leqslant} \int\limits_{B_{4R}(x_0)} \mathbf{E}(v) w^2\,dx \leqslant \frac{1}{2a_\kappa(L)} \int\limits_{B_{4R}(x_0)} S^{\alpha\beta}\partial_\alpha z \partial_\beta (w^2)\,dx \\
&\underset{(3.22)}{\leqslant} \frac{1}{a_\kappa(L)R^2} \int\limits_{B_{2R}(x_0)} wz\,dx \underset{(3.21)}{\leqslant} \frac{C_4}{a_\kappa(L)R^2} \int\limits_{B_{2R}(x_0)} z\,dx \\
&\underset{(3.14)}{\leqslant} \frac{C_1C_4R^{m-2}}{a_\kappa(L)} \inf_{B_R(x_0)} z = \frac{C_1C_4R^{m-2}}{a_\kappa(L)}\big[M^2(4R) - M^2(R)\big]. \qquad \square
\end{aligned}
$$

As a starting point for our iteration argument we will use (cf. [22, p. 5])

Lemma 3.2. *Let $G \subset \mathbb{R}^m$ be a domain in $\mathbb{R}^m$ and suppose that $w \in W^{1,2}(B_{4R}(x_0) \cap G)$ is a weak solution of $\partial_\alpha(S^{\alpha\beta}(x)\partial_\beta w) \geqslant 0$ in $B_{4R}(x_0) \cap G$, where the coefficients $S^{\alpha\beta} \in L^\infty(B_{4R}(x_0) \cap G)$ satisfy $\lambda_*|\xi|^2 \leqslant S^{\alpha\beta}(x)\xi_\alpha\xi_\beta \leqslant \mu_*|\xi|^2$ for all $\xi \in \mathbb{R}^m$, $x \in B_{4R}(x_0) \cap G$ with constants $0 < \lambda_* \leqslant \mu_* < +\infty$.*

(i) *If $G = B_{4R}(x_0) \subset \mathbb{R}^m$ then $\sup_{B_R(x_0)} w \leqslant (1-\delta_0)\sup_{B_{4R}(x_0)} w + \delta_0 ⨍_{B_R(x_0)} w\,dx$ with a constant $\delta_0 \in (0,1)$ depending only on m, λ_* and μ_*.*
(ii) *If $\mathscr{L}^m(B_R(x_0) \setminus G) \geqslant \gamma \mathscr{L}^m(B_R(x_0))$ for some constant $\gamma > 0$, then $\sup_{B_R(x_0)\cap G} w \leqslant (1-\delta_0)\sup_{B_{4R}(x_0)\cap G} w + \delta_0 \sup_{B_R(x_0)\cap\partial G} w$ with a constant $\delta_0 \in (0,1)$ depending only on m, λ_*, μ_*, and γ.*

Proof. (i) We can assume that $w \not\equiv 0$, and apply Moser's weak Harnack inequality [10, Thm. 8.18] to the non-negative supersolution $v := \sup_{B_{4R}(x_0)} w - w$ of the elliptic operator $\partial_\alpha(S^{\alpha\beta}\partial_\beta)$ in $B_{4R}(x_0)$ to obtain a constant $C = C(m, \lambda_*, \mu_*) > 0$, such that

$$
\frac{1}{R^m} \int\limits_{B_R(x_0)} v\,dx \leqslant \frac{1}{R^m} \int\limits_{B_{2R}(x_0)} v\,dx \leqslant C \inf_{B_R(x_0)} v \leqslant (C + l_m) \inf_{B_R(x_0)} v
$$

for $l_m := \mathscr{L}^m(B_1(0))$, which implies

$$
\frac{\mathscr{L}^m(B_R(x_0))}{R^m}\Bigg[\sup_{B_{4R}(x_0)} w - ⨍\limits_{B_R(x_0)} w\,dx\Bigg] \leqslant (C + l_m)\Big[\sup_{B_{4R}(x_0)} w - \sup_{B_R(x_0)} w\Big],
$$

and therefore $\sup_{B_R(x_0)} w \leqslant \sup_{B_{4R}(x_0)} w - l_m(C+l_m)^{-1}[\sup_{B_{4R}(x_0)} w - ⨍_{B_R(x_0)} w\,dx]$. Set $\delta_0 = \delta_0(m, \lambda_*, \mu_*) := l_m(C+l_m)^{-1} \in (0,1)$.

(ii) Moser's weak Harnack inequality [10, Thm. 8.26] applied to the non-negative supersolution

$$
v := \sup_{B_{4R}(x_0)\cap G} w - w
$$

yields

$$
\begin{aligned}
&\frac{1}{R^m} \int\limits_{B_{2R}(x_0)\setminus G} \Big(\sup_{B_{4R}(x_0)\cap G} w - \sup_{B_{4R}(x_0)\cap\partial G} w\Big)\,dx + \int\limits_{B_{2R}(x_0)\cap G} \inf\Big\{v, \inf_{B_{4R}(x_0)\cap\partial G} v\Big\}\,dx \\
&\quad\leqslant C \inf_{B_R(x_0)\cap G} v = C\Big(\sup_{B_{4R}(x_0)\cap G} w - \sup_{B_R(x_0)\cap G} w\Big) \leqslant (C + \gamma l_m)\Big(\sup_{B_{4R}(x_0)\cap G} w - \sup_{B_R(x_0)\cap G} w\Big).
\end{aligned}
$$

The second term on the left-hand side is non-negative and the first is bounded from below by $\gamma l_m(\sup_{B_{4R}(x_0)\cap G} w - \sup_{B_{4R}(x_0)\cap\partial G} w)$, which gives the desired result for $\delta_0 := \gamma l_m (C+\gamma l_m)^{-1} \in (0,1)$. $\square$

Iteration procedure

As before suppose that $B_{4R}(x_0) \subset B_{4d}$. Choose $J \in \mathbb{N}$ so large that

$$
L(1 + J^{-1}) < \frac{\pi}{2\sqrt{\kappa}}, \tag{3.23}
$$

and set

$$\varepsilon := \frac{1}{2KJ} \in (0,1) \tag{3.24}$$

with a constant $K = K(\omega, L) \geqslant 1$ yet to be specified. Define l to be the smallest integer such that $(1-\delta_0)^l < \varepsilon^2$ for δ_0 as in Lemma 3.2, and put $s := 4^{-l}$.

Claim 1. *If v is the representation of U with respect to normal coordinates around P in $\mathscr{B}_L(Q)$ with $|v| \leqslant L$, then there exists $i_0 = i_0(L, J, \omega, \kappa, m, \lambda, \mu) \in \mathbb{N}$ such that*

$$\fint_{B_{R_0}(x_0)} |v - \bar{v}_{R_0}|^2\,dx \leqslant L^2 \varepsilon^4 s^{mJ} \quad \text{for } R_0 = 4^{-i_0} R, \tag{3.25}$$

where $\bar{v}_{R_0} := \fint_{B_{R_0}(x_0)} v\,dx$.

Proof. We have $0 < C_5 = C_5(m, \lambda, \mu, L, \kappa) := \lambda_* b_\kappa^2(L) \leqslant \lambda_* b_\kappa^2(|v|)$, and therefore by (3.4), (3.8), and part (ii) of Lemma 3.1 applied to $B_{4r_i}(x_0) \subset B_{4R}(x_0)$, $r_i := 4^{-i}R$, $i \in \mathbb{N}$,

$$\begin{aligned} C_5 \int_{B_{r_i}(x_0)} |\nabla v|^2\,dx &\leqslant \int_{B_{r_i}(x_0)} \lambda_* b_\kappa^2\big(|v|\big)|\nabla v|^2\,dx \underset{(3.4)}{\leqslant} \int_{B_{r_i}(x_0)} b_\kappa^2\big(|v|\big) S^{\alpha\beta}\partial_\alpha v^i \partial_\beta v^i\,dx \\ &\underset{(3.8)}{\leqslant} \int_{B_{r_i}(x_0)} S^{\alpha\beta}\partial_\alpha v^i \partial_\beta v^j h_{ij}(v)\,dx \\ &\underset{(3.12)}{\leqslant} C(m, \lambda, \mu, L, \kappa) r_i^{n-2}\big[M^2(4r_i) - M^2(r_i)\big], \end{aligned} \tag{3.26}$$

which implies by the Poincaré inequality

$$\fint_{B_{r_i}(x_0)} |v - \bar{v}_{r_i}|^2\,dx \leqslant C\big[M^2(4r_i) - M^2(r_i)\big] = C\left[M^2\left(\frac{R}{4^{i-1}}\right) - M^2\left(\frac{R}{4^i}\right)\right].$$

Choosing the integer $p := [\frac{C}{\varepsilon^4 s^{mJ}}] + 1$ we find $i_0 \in \{1, \dots, p\}$ such that

$$\begin{aligned} p \cdot \left[M^2\left(\frac{R}{4^{i_0-1}}\right) - M^2\left(\frac{R}{4^{i_0}}\right)\right] &\leqslant \sum_{i=1}^{p}\left[M^2\left(\frac{R}{4^{i-1}}\right) - M^2\left(\frac{R}{4^i}\right)\right] = M^2(R) - M^2(4^{-p}R) \\ &\leqslant M^2(R) = \big(\sup_{B_R(x_0)} |v|\big)^2 \leqslant L^2. \end{aligned}$$

Thus our choice of p implies $\fint_{B_{R_0}(x_0)} |v - \bar{v}_{R_0}|^2\,dx \leqslant \frac{CL^2}{p} \leqslant L^2\varepsilon^4 s^{mJ}$ for $R_0 := r_{i_0}$. □

For $k = 0, 1, \dots, J$ let $R_k = s^k R_0$ and $P_k = \exp_Q(\frac{k}{J}\bar{u}_{R_0})$, i.e., $P_k \in \mathscr{B}_L(Q)$ corresponds to $k\bar{u}_{R_0}/J$ under normal coordinates around Q, and let $v^{(k)}$ be the representation of U with respect to normal coordinates around P_k. Finally, let $L_0 := L$ and $L_k := (\frac{1}{J} + 1 - \frac{k}{J})L \leqslant L$ for $k = 1, \dots, J$.

Claim 2. *We have $|v^{(k)}| \leqslant L_k$ in $B_{R_k}(x_0)$ for $k = 0, 1, \dots, J$.*

Proof. Clearly, the claim holds for $k = 0$ and we suppose now that it has been shown up to $k-1$, $k \geqslant 1$. Then we estimate on $B_{R_{k-1}}(x_0)$

$$\begin{aligned} |v^{(k)}| &= \operatorname{dist}(U \circ \chi^{-1}, P_k) \leqslant \operatorname{dist}(U \circ \chi^{-1}, P_{k-1}) + \operatorname{dist}(P_{k-1}, P_k) \\ &= |v^{(k-1)}| + \operatorname{dist}(P_{k-1}, P_k) \leqslant L_{k-1} + J^{-1}L \leqslant (1 + J^{-1})L. \end{aligned} \tag{3.27}$$

In particular we have $|v^{(k)}| < \frac{\pi}{2\sqrt{\kappa}}$ by (3.23). Thus we can apply part (i) of Lemma 3.1 and obtain

$$\partial_\alpha\big(S^{\alpha\beta}(x)\partial_\beta|v^{(k)}|^2\big) \geqslant 0 \quad \text{in } B_{R_{k-1}}(x_0).$$

Applying Lemma 3.2 l-times to $w := |v^{(k)}|^2$ yields

$$\sup_{B_{sR_{k-1}}(x_0)} |v^{(k)}|^2 \leqslant (1-\delta_0)^l \sup_{B_{R_{k-1}}(x_0)} |v^{(k)}|^2 + \sum_{i=1}^{l} \tau_i \fint_{B_{\frac{R_{k-1}}{4^i}}(x_0)} |v^{(k)}|^2\,dx,$$

where $\tau_i := \delta_0(1-\delta_0)^{l-i} > 0$ satisfies $\sum_{i=1}^{l}\tau_i = 1-(1-\delta_0)^l$.

For $R^* \in \{R_{k-1}/4^i\colon\ i=1,\dots,l\}$ with

$$\fint_{B_{R^*}(x_0)} |v^{(k)}|^2\,dx = \max_{i=1,\dots,l} \fint_{B_{\frac{R_{k-1}}{4^i}}(x_0)} |v^{(k)}|^2\,dx$$

we can deduce by our choice of l the estimate

$$\begin{aligned}\sup_{B_{sR_{k-1}}(x_0)} |v^{(k)}|^2 &\leqslant \varepsilon^2 \sup_{B_{R_{k-1}}(x_0)} |v^{(k)}|^2 + \big[1-(1-\delta_0)^l\big] \fint_{B_{R^*}(x_0)} |v^{(k)}|^2\,dx \\ &\leqslant \varepsilon^2 \sup_{B_{R_{k-1}}(x_0)} |v^{(k)}|^2 + \big[1-\varepsilon^2(1-\delta_0)\big] \fint_{B_{R^*}(x_0)} |v^{(k)}|^2\,dx \\ &\leqslant 2\varepsilon^2 \sup_{B_{R_{k-1}}(x_0)} |v^{(k)}|^2 + (1-\varepsilon^2) \fint_{B_{R^*}(x_0)} |v^{(k)}|^2\,dx. \end{aligned} \tag{3.28}$$

Observe that by (3.9)

$$\begin{aligned}|v^{(k)}| &= \operatorname{dist}(U\circ\chi^{-1}, P_k) \leqslant \operatorname{dist}(U\circ\chi^{-1}, P_J) + \operatorname{dist}(P_J, P_k) \\ &= \operatorname{dist}(U\circ\chi^{-1}, P_J) + \left(1-\frac{k}{J}\right)|\bar u_{R_0}| \underset{(3.9)}{\leqslant} b_\omega(L)|u-\bar u_{R_0}| + \left(1-\frac{k}{J}\right)L,\end{aligned}$$

which by virtue of Young's inequality leads to

$$|v^{(k)}|^2 \leqslant (1+\varepsilon^{-2})b_\omega^2(L)|u-\bar u_{R_0}|^2 + (1+\varepsilon^2)\left(1-\frac{k}{J}\right)^2 L^2. \tag{3.29}$$

If we use (3.27) to estimate the first term in (3.28), and (3.29) for the second term in (3.28), then we obtain in combination with (3.25) applied to $v := u$

$$\begin{aligned}\sup_{B_{R_k}(x_0)} |v^{(k)}|^2 \;&=\; \sup_{B_{sR_{k-1}}(x_0)} |v^{(k)}|^2 \leqslant 2\varepsilon^2L^2(1+J^{-1})^2 + (1-\varepsilon^4)\left[1-\frac{k}{J}\right]^2 L^2 \\ &\quad + \varepsilon^2\left(\frac{1-\varepsilon^4}{\varepsilon^4}\right) b_\omega^2(L) \fint_{B_{R^*}(x_0)} |u-\bar u_{R_0}|^2\,dx \\ &\underset{(3.25)}{\leqslant} 2\varepsilon^2L^2(1+J^{-1})^2 + (1-\varepsilon^4)\left[1-\frac{k}{J}\right]^2 L^2 + (1-\varepsilon^4)\varepsilon^2 b_\omega^2(L)L^2 \\ &\leqslant L^2\left[8\varepsilon^2 + \left[1-\frac{k}{J}\right]^2 + \varepsilon^2 b_\omega^2(L)\right], \end{aligned} \tag{3.30}$$

where we also used that by $s^J R_0 \leqslant s s^{k-1} R_0 = sR_{k-1} \leqslant R^* \leqslant R_0$

$$\fint_{B_{R^*}(x_0)} |u-\bar u_{R_0}|^2\,dx \leqslant \frac{1}{s^{mJ}} \fint_{B_{R_0}(x_0)} |u-\bar u_{R_0}|^2\,dx.$$

Hence, if we specify $K := \sqrt{2 + \frac{b_\omega(L)^2}{4}}$, we arrive at

$$\sup_{B_{R_k}(x_0)} |v^{(k)}|^2 \leqslant L^2 \left\{ \left[2K\varepsilon + 1 - \frac{k}{J} \right]^2 - 4K\varepsilon \left[1 - \frac{k}{J} \right] - 4K^2\varepsilon^2 + 8\varepsilon^2 + \varepsilon^2 b_\omega^2(L) \right\}$$
$$\leqslant L^2 \left(2K\varepsilon + 1 - \frac{k}{J} \right)^2 \underset{(3.24)}{=} L_k^2.$$

This proves Claim 2. □

In particular we obtain the estimate

$$\operatorname{dist}(U, P_J) = |v^{(J)}| \leqslant \frac{L}{J} \quad \text{in } B_{R_J}(x_0) = B_{s^J 4^{-i_0} R}(x_0),$$

where $s = s(L, J, \omega, m, \lambda, \mu)$, and $i_0 = i_0(L, J, \omega, \kappa, m, \lambda, \mu)$. In view of (3.9) this leads to the following estimate for the oscillation of u:

$$\operatorname{osc}_{B_{R_J}(x_0)} u \underset{(3.9)}{\leqslant} \frac{1}{b_\kappa(L)} \operatorname{osc}_{B_{R_J}(x_0)} U \circ \chi^{-1} \leqslant \frac{2}{b_\kappa(L)} \sup_{B_{R_J}(x_0)} \operatorname{dist}(U \circ \chi^{-1}, P_J) \leqslant \frac{2L}{b_\kappa(L) J}.$$

Since $R_J = s^J 4^{-i_0} R = 4^{-Jl - i_0} R \to 0$ as $J \to \infty$ we can conclude that U is continuous.

Proof of Theorem 1.1. In view of the preceding discussion there exists an integer $i_1 = i_1(m, \lambda, \mu, \omega, \kappa, L)$ such that for all balls $B_{4R}(x_0) \subset B_{4d}$ and for $\tilde{R} := 4^{-i_1} R$ we have

$$\operatorname{osc}_{B_{\tilde{R}}(x_0)} u \leqslant \frac{L}{b_\omega(L)}. \tag{3.31}$$

Let u' be the representation of U with respect to normal coordinates around $U \circ \chi^{-1}(x_0)$, and define

$$\omega'(\rho) := \sup_{B_\rho(x_0)} |u'|^2, \quad 0 < \rho \leqslant \tilde{R}.$$

Using (3.9) and (3.31) we find on $B_\rho(x_0)$ for all $0 < \rho \leqslant \tilde{R}$

$$|u'| = \operatorname{dist}\big(U \circ \chi^{-1}, U \circ \chi^{-1}(x_0)\big) \underset{(3.9)}{\leqslant} b_\omega(L) \big| u - u(x_0) \big| \leqslant b_\omega(L) \operatorname*{osc}_{B_\rho(x_0)} u \leqslant b_\omega(L) \operatorname*{osc}_{B_{\tilde{R}}(x_0)} u \underset{(3.31)}{\leqslant} L. \tag{3.32}$$

Thus (3.26) in the proof of Claim 1 for $v := u'$ and with r_i replaced by $\rho/4$ yields

$$\rho^{2-m} \int_{B_{\rho/4}(x_0)} |\nabla u'|^2 \, dx \leqslant C(m, \lambda, \mu, L, \kappa) \left[\omega'(\rho) - \omega'\left(\frac{\rho}{4}\right) \right], \quad 0 < \rho \leqslant \tilde{R}. \tag{3.33}$$

Next, let $P \in \mathscr{B}_L(Q)$ be the point which corresponds to $\bar{u}_{\rho/4}$ under $\exp_Q$, and let v be the representation of U with respect to normal coordinates around P. Then, again by (3.9) and (3.31)

$$|v| = \operatorname{dist}(U \circ \chi^{-1}, P) \underset{(3.9)}{\leqslant} b_\omega(L) |u - \bar{u}_{\rho/4}| \leqslant b_\omega(L) \operatorname*{osc}_{B_\rho(x_0)} u \underset{(3.31)}{\leqslant} L < \frac{\pi}{2\sqrt{\kappa}}, \tag{3.34}$$

which by iterated application of Lemma 3.2 implies for $\varepsilon > 0$ and $s := 4^{-l}$, where $l = l(m, \lambda_*, \mu_*, \varepsilon)$ is the smallest integer with $(1 - \delta_0)^l < \varepsilon^2$ ($\delta_0 = \delta_0(m, \lambda_* \mu_*)$ as in Lemma 3.2) the estimate

$$\sup_{B_{s\rho}(x_0)} |v|^2 \leqslant 2\varepsilon^2 \sup_{B_\rho(x_0)} |v|^2 + (1 - \varepsilon^2) \fint_{B_{\rho^*}(x_0)} |v|^2 \, dx \tag{3.35}$$

for some $\rho^* \in [s\rho, \rho/4]$, $0 < \rho \leqslant \tilde{R}$ (compare with the proof of Claim 2 above). Using (3.34) and the Poincaré inequality one can show

$$\fint_{B_{\rho^*}(x_0)} |v|^2\,dx \underset{(3.34)}{\leqslant} b_\omega^2(L) \fint_{B_{\rho^*}(x_0)} |u-\bar u_{\rho/4}|^2\,dx \leqslant s^{-m} b_\omega^2(L) \fint_{B_{\rho/4}(x_0)} |u-\bar u_{\rho/4}|^2\,dx$$
$$\leqslant C(m,\lambda_*,\mu_*,\varepsilon,\omega,L)\rho^{2-m} \int_{B_{\rho/4}(x_0)} |\nabla u|^2\,dx,$$

since $s = s(\varepsilon,\delta_0)$. Thus by (3.35) for $0<\rho\leqslant \tilde R$,

$$\sup_{B_{s\rho}(x_0)} |v|^2 \leqslant 2\varepsilon^2 \sup_{B_\rho(x_0)} |v|^2 + C(m,\lambda_*,\mu_*,\varepsilon,\omega,L)\rho^{2-m} \int_{B_{\rho/4}(x_0)} |\nabla u|^2\,dx. \tag{3.36}$$

With $|u|\leqslant L$, (3.4) and (3.8) one has

$$\lambda_* b_\kappa^2(L)|\nabla u|^2 \leqslant S^{\alpha\beta}(x)\partial_\alpha u^i \partial_\beta u^j h_{ij}(u) \leqslant \mu_* b_\omega^2(L)|\nabla u|^2$$

for all $x\in B_R(x_0)$. Replacing u by u' (also with $|u'|\leqslant L$ by (3.32)) one obtains the analogous estimate for $|\nabla u'|^2$ and thus by the invariance of the energy density $e(U)$ (see (3.1)) under change of coordinates

$$\frac{\lambda_* b_\kappa^2(L)}{\mu_* b_\omega^2(L)}|\nabla u'|^2 \leqslant |\nabla u|^2 \leqslant \frac{\mu_* b_\omega^2(L)}{\lambda_* b_\kappa^2(L)}|\nabla u'|^2.$$

Together with (3.33) this can be used in (3.36) to infer

$$\sup_{B_{s\rho}(x_0)} |v|^2 \leqslant 2\varepsilon^2 \sup_{B_\rho(x_0)} |v|^2 + C(m,\lambda_*,\mu_*,\varepsilon,\omega,\kappa,L)\big[\omega'(\rho)-\omega'(s\rho)\big]$$

since $s\leqslant 1/4$. We note that (3.9), (3.34), and (3.32) also imply

$$|v| \underset{(3.34)}{\leqslant} b_\omega(L)|u-\bar u_{\rho/4}| \leqslant 2b_\omega(L) \sup_{B_\rho(x_0)} \big|u-u(x_0)\big| \underset{(3.9)}{\leqslant} 2\frac{b_\omega(L)}{b_\kappa(L)} \sup_{B_\rho(x_0)} |u'| \quad \text{in } B_\rho(x_0),$$

because $|u'| = \operatorname{dist}(U\circ\chi^{-1}, U\circ\chi^{-1}(x_0))$, and

$$|u'| \underset{(3.32)}{\leqslant} b_\omega(L)\big|u-u(x_0)\big| \leqslant 2b_\omega(L) \sup_{B_{s\rho}(x_0)} |u-\bar u_{\rho/4}| \leqslant 2\frac{b_\omega(L)}{b_\kappa(L)} \sup_{B_{s\rho}(x_0)} |v| \quad \text{in } B_{s\rho}(x_0),$$

since $|v| = \operatorname{dist}(U\circ\chi^{-1}, \exp_Q \bar u_{\rho/4})$. Therefore from (3.35)

$$\omega'(s\rho) \leqslant C(\kappa,\omega,L)\varepsilon^2\omega'(\rho) + \tilde C(m,\lambda_*,\mu_*,\varepsilon,\omega,\kappa,L)\big[\omega'(\rho)-\omega'(s\rho)\big],$$

which becomes $\omega'(s\rho)\leqslant\theta\omega'(\rho)$ with $\theta = (\tilde C+\frac12)(\tilde C+1)^{-1} < 1$, if we choose $\varepsilon := \sqrt{(2C(\kappa,\omega,L)^{-1}}$. A standard iteration lemma [10, Lemma 8.23] then gives the growth estimate $\omega'(\rho)\leqslant C(\rho/\tilde R)^{2\alpha}\omega'(\tilde R)$ for $0<\rho\leqslant\tilde R$, and according to (3.32) we have

$$\sqrt{\omega'(\rho)} \underset{(3.32)}{\leqslant} C(\omega,L) \operatorname*{osc}_{B_\rho(x_0)} u \leqslant 2C'(\kappa,\omega,L)\sqrt{\omega'(\rho)},$$

hence

$$\operatorname*{osc}_{B_\rho(x_0)} u \leqslant 2C\left(\frac{\rho}{\tilde R}\right)^\alpha \operatorname*{osc}_{B_{\tilde R}(x_0)} u \leqslant C'\left(\frac{\rho}{4R}\right)^\alpha \left(\frac{4R}{\tilde R}\right)^\alpha \operatorname*{osc}_{B_R(x_0)} u \leqslant C''\left(\frac{\rho}{4R}\right)^\alpha \operatorname*{osc}_{B_R(x_0)} u$$

with $\alpha = \alpha(m,\lambda,\mu,L,\omega,\kappa)$ and $C'' = C''(m,\lambda,\mu,L,\omega,\kappa)$. A standard covering argument now leads to the estimate $\text{Höl}_{\alpha,B_d} u \leqslant C$ with C depending on $m,\lambda,\mu,L,\omega,\kappa$ and also on d, and from this the desired estimate (1.5) follows by a simple scaling argument. □

4. Boundary estimates

Let $U:\mathscr{M}\to\mathscr{N}$ be a harmonic mapping which maps a coordinate neighborhood $\Omega\subset\mathscr{M}$ of a point $P\in\partial\mathscr{M}$ into a regular ball $\mathscr{B}_L(Q)\subset\mathscr{N}$, and let $\chi:\overline{\Omega}\to\overline{\Sigma}_{5R}$ be a coordinate chart that maps $\overline{\Omega}$ homeomorphically onto the closure of the set

$$\Sigma_{5R}:=\{x=(x',x^m)\in\mathbb{R}^m:\ |x'|<5R,0<x^m<5R\}$$

with

$$\chi(\partial\mathscr{M}\cap\overline{\Omega})=\Sigma^0_{5R}:=\{x=(x',0)\in\mathbb{R}^m:\ |x'|\leqslant 5R\}.$$

For $x_0\in\Sigma^0_R$ set $S_R(x_0):=B_R(x_0)\cap\{x^m>0\}$ and $S^0_R(x_0):=B_R(x_0)\cap\Sigma^0_{5R}$.

The a priori estimate for the Hölder semi-norm up to the boundary follows by combining the interior estimate (1.5) with the following oscillation estimate, Theorem 4.1, near the boundary to obtain the global oscillation estimate $\operatorname{osc}_{\overline{\Sigma}_R\cap B_\rho(y)}u\leqslant C\rho^\gamma$ for any $y\in\Sigma_R$, where $C=C(\lambda,\mu,L,\omega,\kappa,U|_{\partial\mathscr{M}},m)$ and $\gamma=\gamma(\lambda,\mu,L,\kappa,U|_{\partial\mathscr{M}})\in(0,1)$. Here, as in Theorem 4.1, $u=(u^1,\dots,u^n)$ denotes the normal coordinate representation of U centered at Q. Setting $\sigma(t):=\operatorname{osc}_{S^0_t(x_0)}u$ we formulate

Theorem 4.1. *If $\sigma(R)<L/b_\omega(L)$ and if*

$$2L+b_\omega(L)\sigma(R)<\frac{\pi}{\sqrt{\kappa}},\tag{4.1}$$

then there is $R^*=R^*(\lambda,\mu,L,\omega,\kappa,m)\in(0,R]$ such that for all $\rho\in(0,R^*]$

$$\operatorname*{osc}_{S_\rho(x_0)}u\leqslant C\left[\left(\frac{\rho}{R^*}\right)^\beta\operatorname*{osc}_{S_R(x_0)}u+\sigma(\sqrt{\rho R})\right],\tag{4.2}$$

where $C=C(\lambda,\mu,L,\omega,\kappa,m)$ and $\beta=\beta(\lambda,\mu,m)\in(0,1)$.

Proof. Setting $M_\eta(t):=\sup_{\Sigma^0_t}\operatorname{dist}(U\circ\chi^{-1},\exp_Q\eta)$ for $\eta\in T_Q\mathscr{N}\cong\mathbb{R}^n$, and $M_\eta\equiv M_\eta(R)$, we obtain for $x_0\in\Sigma^0_R$ with $\xi:=u(x_0)$ by (3.9)

$$M_\xi\leqslant b_\omega(L)\sup_{x\in S^0_R(x_0)}\left|u(x)-u(x_0)\right|\leqslant b_\omega(L)\sigma(R)<L<\frac{\pi}{2\sqrt{\kappa}}.\tag{4.3}$$

Thus we can choose $J\in\mathbb{N}$ so large that

$$2L+M_\xi+\frac{3L}{J}<\frac{\pi}{\sqrt{\kappa}},\tag{4.4}$$

which is possible by assumption (4.1). We set $L_0:=L$, and $L_k:=\frac{L}{J}+M_{\xi_k}$ for $1\leqslant k\leqslant J$, where $\xi_k:=(k/J)\xi$. We claim that for normal coordinates $v^{(k)}$ of U centered at $P_k:=\exp_Q\xi_k$ one has

$$|v^{(k)}|\leqslant L_k\quad\text{in } S_{R_k}(x_0)\text{ for } R_k:=\frac{R}{4^{kl}}.\tag{4.5}$$

Here, l is the smallest integer such that

$$(1-\delta_0)^l\leqslant\frac{L^2}{J^2(L+|\xi|)^2},\tag{4.6}$$

where δ_0 is the constant in part (ii) of Lemma 3.2. We prove this claim by induction. (4.5) is valid for $k=0$. Assuming (4.5) for all indices less or equal to $k-1$ we estimate

$$M_{\xi_{k-1}}\leqslant M_\xi+\operatorname{dist}(P_{k-1},P_J)\leqslant M_\xi+\frac{J-(k-1)}{J}L.\tag{4.7}$$

Our induction hypothesis, on the other hand, implies for $x\in S_{R_{k-1}}(x_0)$

$$\left|v^{(k)}(x)\right|\leqslant\operatorname{dist}\bigl(U\circ\chi^{-1}(x),P_{k-1}\bigr)+\operatorname{dist}(P_{k-1},P_k)\leqslant L_{k-1}+\frac{L}{J}.\tag{4.8}$$

In addition, we have by definition of L_k and ξ_k, (4.7), and (4.4)

$$L + |\xi_k| + L_{k-1} + \frac{L}{J} \leqslant L + \frac{k}{J}L + \frac{L}{J} + M_{\xi_{k-1}} + \frac{L}{J}$$
$$\underset{(4.7)}{\leqslant} L + \frac{L}{J}\big[2 + k + J - (k-1)\big] + M_\xi \leqslant 2L + 3\frac{L}{J} + M_\xi \underset{(4.4)}{<} \frac{\pi}{\sqrt{\kappa}},$$

which, together with (4.8) and $|v^{(k)}(x)| \leqslant \operatorname{dist}(U \circ \chi^{-1}(x), Q) + \operatorname{dist}(Q, P_k) \leqslant L + |\xi_k|$ leads to $|v^{(k)}(x)| \leqslant 2^{-1}[L + |\xi_k| + L_{k-1} + L/J] < \pi/(2\sqrt{\kappa})$ for all $x \in S_{R_{k-1}}(x_0)$. Thus, by part (i) of Lemma 3.1, $|v^{(k)}|^2$ is a subsolution of the elliptic operator $\partial_\alpha(S^{\alpha\beta}\partial_\beta)$ on $S_{R_{k-1}}(x_0)$. Applying part (ii) of Lemma 3.2 l-times we obtain by our choice (4.6)

$$\sup_{S_{\frac{R_{k-1}}{4^l}}(x_0)} |v^{(k)}|^2 \leqslant (1-\delta_0)^l \sup_{S_{R_{k-1}}(x_0)} |v^{(k)}|^2 + \sum_{i=0}^{l-1} \delta_0(1-\delta_0)^{l-1-i} \sup_{S^0_{\frac{R_{k-1}}{4^i}}(x_0)} |v^{(k)}|^2$$
$$\leqslant (1-\delta_0)^l\big(L + |\xi|\big)^2 + \big[1 - (1-\delta_0)^l\big]M_{\xi_k}^2 \leqslant \frac{L^2}{J^2} + M_{\xi_k}^2,$$

which implies $|v^{(k)}(x)| \leqslant \frac{L}{J} + M_{\xi_k} = L_k$ for all $x \in S_{R_k}(x_0)$, thus proving our claim (4.5). Specifically, $|v^{(J)}| \leqslant L/J + M_\xi \underset{(4.3)}{<} (1 + 1/J)L \underset{(4.4)}{<} \pi/(2\sqrt{\kappa})$ in $S_{R_J}(x_0)$, and so $|v^{(J)}|^2$ is a subsolution in $S_{R_J}(x_0)$ according to Lemma 3.1. Part (ii) of Lemma 3.2 then implies for $m(t) := \sup_{S_t(x_0)} \operatorname{dist}(U \circ \chi^{-1}, P_J)$ the estimate

$$m^2(\rho) \leqslant (1-\delta_0)m^2(4\rho) + \delta_0 M_\xi^2(4\rho) \quad \text{for all } 0 < \rho \leqslant \frac{R_J}{4}.$$

Iterating as in [10, Lemma 8.23] we obtain $m(\rho) \leqslant K[(\frac{\rho}{R^*})^\beta m(R) + M_\xi(\sqrt{\rho R^*})]$ for $R^* := R^*(\lambda, \mu, L, \omega, \kappa, m) := R_J$ and constants K and $\beta \in (0,1)$ depending only on m, λ, and μ. This together with (4.3) proves (4.2). □

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