Available online at www.sciencedirect.com
ScienceDirect
Ann. I. H. Poincaré – AN 25 (2008) 1187–1208
ANNALES DE L'INSTITUT HENRI POINCARÉ
ANALYSE NON LINÉAIRE
www.elsevier.com/locate/anihpc

Well-posedness results for a model of damage in thermoviscoelastic materials

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Received 14 November 2006; accepted 27 May 2007

Available online 9 February 2008

Abstract

This paper deals with a phase transitions model describing the evolution of damage in thermoviscoelastic materials. The resulting system is highly non-linear, mainly due to the presence of quadratic dissipative terms and non-smooth constraints on the variables. Existence and uniqueness of a solution are proved, as well as regularity results, on a suitable finite time interval.

Keywords: Damage; Thermoviscoelasticity; Well-posedness results; Non-linear evolution inclusions

1. Introduction

This paper deals with the phenomenon of damage in thermoviscoelastic materials. It is known that a material loses its stiffness during the damage process. Consequently, deformations become uncontrolled and the material breaks. In the last years, Frémond has proposed a macroscopic model describing the damaging process in continuous media using the phase transitions approach and accounting for microscopic movements [11]. In particular, a phase parameter χ characterizes the state of damage of the material. More precisely, the phase parameter χ satisfies the constraint

$$\chi \in [0, 1], \tag{1.1}$$

where $\chi = 1$ and $\chi = 0$ correspond to the undamaged and completely damaged material, respectively. In an intermediate situation it is $\chi \in (0, 1)$. The resulting isothermal model consists into two partial differential equations describing the evolution of the phase parameter and of the deformations. Some analytical results have been obtained both in the one-dimensional setting and in the three-dimensional case [13,4,5]. However, all these results are local in time, as the existence of a solution is proved still the damaging process is not complete. This is mainly due to the degeneracy of the stiffness of the material during the process leading uncontrolled deformations. To overcome this difficulty, our idea is to include some constitutive relation in the model characterizing the behaviour of the material once it is completely damaged in some region. In a recent contribution [12] the authors introduce a model in which it is prescribed as a constraint an uniform bound for the deformations velocity. In the present paper, we propose a model in which it is

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0294-1449/$ – see front matter
doi:10.1016/j.anihpc.2007.05.009

required that when the material is completely damaged some viscosity effects remain (cf. also [5]). In particular, we are able to control deformations when the damaging process is completed in some region of the body, even if the model itself does not ensure any a priori bound on the deformations velocity. Hence, dealing with viscoelastic materials (cf., e.g., [9,10]), it turns out to be interesting to extend the damage Frémond model to non-isothermal situations accounting for thermal effects. Thus, a novelty of the present contribution, with respect to the others in the literature concerning the Frémond model for damage, is the fact that we take thermal effects into account and, consequently, we introduce an energy balance equation in the resulting system.

Now, let us briefly describe the derivation of the model (see also [3]). We consider a thermoviscoelastic material located in a bounded smooth domain $\Omega \subset \mathbf{R}^3$, with boundary $\Gamma := \partial\Omega$, and investigate the damage evolution during a finite time interval $(0, T)$. We use the notation $Q := \Omega \times (0, T)$, and $Q_t = \Omega \times (0, t)$, with $t \in (0, T)$. Let us fix as state variables of our model the absolute temperature θ, the symmetric strain tensor $\varepsilon(\mathbf{u})$ ($\mathbf{u}$ stands for the vector of small displacements), the phase parameter χ related to the quantity of damaged material, and the gradient of damage $\nabla\chi$, accounting for local interactions. Then, we specify the free energy Ψ as follows

$$\Psi\big(\theta, \chi, \nabla\chi, \varepsilon(\mathbf{u})\big) = -c_s\theta\log\theta + \frac{1}{2}\chi\varepsilon(\mathbf{u})K\varepsilon(\mathbf{u}) + w(1-\chi) + \frac{\nu}{2}|\nabla\chi|^2 + \alpha(\theta)\chi\,\mathrm{tr}\,\varepsilon(\mathbf{u}) + I_{[0,1]}(\chi). \tag{1.2}$$

The indicator function $I_{[0,1]}$ accounts for the constraint (1.1) on the phase parameter, as it is $I_{[0,1]}(\chi) = 0$ if $\chi \in [0, 1]$ and $I_{[0,1]}(\chi) = +\infty$ otherwise. The term $\alpha(\theta)\chi$, acting when the material is not completely damaged, represents a thermal expansion coefficient, while K stands for the stiffness matrix. As it is natural, the energy terms associated to deformations disappear once the material is completely damaged, i.e. when $\chi = 0$. Moreover, $c_s > 0$ denotes the heat capacity of the system, $w > 0$ is related to the cohesion energy of the material (which is considered independent of χ), and ν is a positive constant. On a second step, we introduce a pseudo-potential of dissipation Φ depending on suitable dissipative variables, describing the evolution of the thermomechanical system. We consider as dissipative variables the macroscopic velocities $\varepsilon(\mathbf{u}_t)$, the gradient of the temperature $\nabla\theta$ related to the heat flux, and the time derivatives χ_t and $\nabla\chi_t$ related to the microscopic velocities (see [11]).

$$\Phi\big(\nabla\theta, \chi_t, \nabla\chi_t, \varepsilon(\mathbf{u}_t)\big) = \frac{\mu}{2}|\chi_t|^2 + \frac{\eta}{2}|\nabla\chi_t|^2 + \frac{\delta}{2}\varepsilon(\mathbf{u}_t)S\varepsilon(\mathbf{u}_t) + \frac{\lambda}{2\theta}|\nabla\theta|^2 + I_{(-\infty,0]}(\chi_t), \tag{1.3}$$

where μ, η, δ and λ are positive constants, and S is a symmetric and positive definite matrix. The indicator function $I_{(-\infty,0]}(\chi_t)$ represents a constraint on the sign of χ_t, which is forced to be non-positive. Indeed, $I_{(-\infty,0]}(\chi_t) = 0$ if $\chi_t \leqslant 0$, while $I_{(-\infty,0]}(\chi_t) = +\infty$ otherwise. This corresponds to describe an irreversible damaging process as χ cannot increase, i.e. the material cannot repair itself once it is damaged (cf., e.g., [4] and [6]). Hence, before writing the universal balance laws of continuum thermomechanics, i.e., the energy balance and the momentum balance, we specify the constitutive relations for the involved physical quantities. They are derived by Ψ and Φ, in accordance with the second principle of thermodynamics.

The entropy s is given by

$$s = -\frac{\partial\Psi}{\partial\theta} = c_s(\log\theta + 1) - \alpha'(\theta)\chi\,\mathrm{tr}\,\varepsilon(\mathbf{u}), \tag{1.4}$$

and the internal energy e is

$$e = \Psi + \theta s. \tag{1.5}$$

The heat flux $\mathbf{q}$ is assumed to be governed by the Fourier law. We derive it by the pseudo-potential of dissipation introducing the dissipative vector

$$\mathbf{Q}^d = -\frac{\partial\Phi}{\partial\nabla\theta}$$

related to $\mathbf{q}$ by $\mathbf{q} = \theta\mathbf{Q}^d$. Thus, we recover

$$\mathbf{q} = -\lambda\nabla\theta. \tag{1.6}$$

Then, we introduce the stress tensor σ which is supposed to be the sum of non-dissipative and dissipative contributions

$$\sigma = \sigma^{nd} + \sigma^{d} = \frac{\partial\Psi}{\partial\varepsilon(\mathbf{u})} + \frac{\partial\Phi}{\partial\varepsilon(\mathbf{u}_t)} = \chi K\varepsilon(\mathbf{u}) + \alpha(\theta)\chi\mathbf{1} + \delta S\varepsilon(\mathbf{u}_t) \tag{1.7}$$

(**1** denotes the identity matrix) and two internal microscopic forces given by the sum of non-dissipative and dissipative components as well

$$B = B^{nd} + B^{d} = \frac{\partial \Psi}{\partial \chi} + \frac{\partial \Phi}{\partial \chi_t}, \tag{1.8}$$

$$\mathbf{H} = \mathbf{H}^{nd} + \mathbf{H}^{d} = \frac{\partial \Psi}{\partial \nabla\chi} + \frac{\partial \Phi}{\partial \nabla\chi_t}. \tag{1.9}$$

In particular, we have

$$B = \frac{1}{2}\varepsilon(\mathbf{u})K\varepsilon(\mathbf{u}) - w + \alpha(\theta)\operatorname{div}\mathbf{u} + \partial I_{[0,1]}(\chi) + \mu\chi_t + \partial I_{(-\infty,0]}(\chi_t), \tag{1.10}$$

and

$$\mathbf{H} = \nu\nabla\chi + \eta\nabla\chi_t. \tag{1.11}$$

We recall that $\partial I_{[0,1]}$ is the subdifferential of the indicator function $I_{[0,1]}$ and it is defined for $\chi \in [0,1]$ by: $\partial I_{[0,1]}(\chi) = 0$ if $\chi \in (0,1)$, $\partial I_{[0,1]}(0) = (-\infty, 0]$, and $\partial I_{[0,1]}(1) = [0, +\infty)$. Analogously, we have $\partial I_{(-\infty,0]}(\chi_t) = 0$ if $\chi_t < 0$, while $\partial I_{(-\infty,0]}(0) = [0, +\infty)$.

Now, we write the balance laws of continuum thermomechanics. The energy balance equation reads

$$e_t + \operatorname{div}\mathbf{q} = r + \sigma\varepsilon(\mathbf{u}_t) + B\chi_t + \mathbf{H}\cdot\nabla\chi_t. \tag{1.12}$$

Note on the right-hand side of (1.12) the heat source r and the mechanically induced heat sources, which are related to macroscopic and microscopic stresses. In the sequel, for the sake of simplicity, we let $r = 0$. In the approach by Frémond [11], (1.12) is derived through a generalization of the principle of virtual power including microscopic movements responsible for the phase transition, i.e. in this case the damaging process. Then, the classical momentum balance is written accounting also for macroscopic accelerations and assuming that no external volume forces act on the body

$$\mathbf{u}_{tt} - \operatorname{div}\sigma = 0. \tag{1.13}$$

Analogously, it is recovered a microscopic balance equation accounting for microscopic accelerations (see [7])

$$\chi_{tt} + B - \operatorname{div}\mathbf{H} = 0. \tag{1.14}$$

The above equations (1.12), (1.13), and (1.14) are completed by suitable boundary conditions. We let (here $\mathbf{n}$ is the outward normal unit vector to the boundary)

$$\mathbf{q}\cdot\mathbf{n} = 0 \quad \text{in } \Gamma \times (0,T), \tag{1.15}$$

$$\mathbf{H}\cdot\mathbf{n} = 0 \quad \text{in } \Gamma \times (0,T), \tag{1.16}$$

$$\mathbf{u} = 0 \quad \text{in } \Gamma \times (0,T). \tag{1.17}$$

Now, we substitute in (1.12)–(1.14), (1.15)–(1.17) the constitutive relations written in terms of Ψ and Φ. Applying the chain rule, we get in $\Omega \times (0,T)$

$$\big(c_s - \theta\alpha''(\theta)\chi\operatorname{div}\mathbf{u}\big)\theta_t - \lambda\Delta\theta - \alpha'(\theta)\theta(\chi_t\operatorname{div}\mathbf{u} + \chi\operatorname{div}\mathbf{u}_t) = \mu|\chi_t|^2 + \eta|\nabla\chi_t|^2 + \delta S\varepsilon(\mathbf{u}_t)\varepsilon(\mathbf{u}_t), \tag{1.18}$$

$$\mathbf{u}_{tt} - \operatorname{div}\big(\chi K\varepsilon(\mathbf{u}) + \alpha(\theta)\chi\mathbf{1} + \delta S\varepsilon(\mathbf{u}_t)\big) = 0, \tag{1.19}$$

$$\chi_{tt} + \mu\chi_t - \eta\Delta\chi_t - \nu\Delta\chi + \partial I_{[0,1]}(\chi) + \partial I_{(-\infty,0]}(\chi_t) \ni w - \frac{1}{2}\varepsilon(\mathbf{u})K\varepsilon(\mathbf{u}) - \alpha(\theta)\operatorname{div}\mathbf{u} \tag{1.20}$$

and in $\Gamma \times (0,T)$

$$\partial_n\theta = 0, \quad \partial_n\chi = \partial_n\chi_t = 0, \quad \mathbf{u} = 0. \tag{1.21}$$

Then, we fix initial assumptions (holding in Ω)

$$\theta(0) = \theta_0, \tag{1.22}$$

$$\chi(0) = \chi_0 \in (0,1], \qquad \chi_t(0) = \chi_1, \tag{1.23}$$

$$\mathbf{u}(0) = \mathbf{u}_0, \qquad \mathbf{u}_t(0) = \mathbf{u}_1. \tag{1.24}$$

Remark 1.1. Let us discuss the thermodynamical consistence of the model. Explicitly writing (1.12), accounting for the prescribed constitutive relations (1.6)–(1.9), by the chain rule we have

$$\theta\left(s_t + \operatorname{div}\mathbf{Q}^q - \frac{r}{\theta}\right) = B^d\chi_t + \mathbf{H}^d\cdot\nabla\chi_t + \sigma^d\varepsilon(\mathbf{u}_t) - \mathbf{Q}^d\cdot\nabla\theta$$
$$= \partial\Phi\big(\nabla\theta, \chi_t, \nabla\chi_t, \varepsilon(\mathbf{u}_t)\big)\cdot\big(\nabla\theta, \chi_t, \nabla\chi_t, \varepsilon(\mathbf{u}_t)\big) \geqslant 0, \tag{1.25}$$

where $\partial\Phi$ denotes the subdifferential of Φ with respect to the dissipative variables $(\nabla\theta, \chi_t, \nabla\chi_t, \varepsilon(\mathbf{u}_t))$. Now, Φ is a convex, non-negative function, attaining its minimum 0 for $(\nabla\theta, \chi_t, \nabla\chi_t, \varepsilon(\mathbf{u}_t)) = (0,0,0,0)$. Thus, its subdifferential is a maximal monotone graph with $(0,0,0,0) \in \partial\Phi(0,0,0,0)$, from which the inequality in (1.25) easily follows. Hence, as the absolute temperature is $\theta > 0$, (1.25) yields the Clausius–Duhem inequality

$$s_t + \operatorname{div}\mathbf{Q}^d - \frac{r}{\theta} \geqslant 0.$$

Now, concerning the doubly non-linear character of (1.20), we actually observe that if $\chi_t \leqslant 0$ and, e.g., $\chi_0 = 1$, we have for any solution $\chi \leqslant 1$ a.e. in Q. Thus, if the solution χ is sufficiently regular, we can deduce that there exists $\hat{t} \in (0, T]$ such that $\chi \in [0,1]$ a.e. in $Q_{\hat{t}}$ just proving that $\chi \geqslant 0$. Indeed (see [4]), provided the solution χ is smooth enough, we have

$$\chi(t) - 1 = \chi_0 - 1 + \int_0^t \chi_t(s)\,ds$$

from which it follows

$$\|\chi - 1\|_{L^\infty(Q_t)} \leqslant \int_0^t \|\chi_t\|_{L^\infty(\Omega)} \leqslant c_\Omega t^{1/2}\|\chi_t\|_{L^2(0,T;H^2(\Omega))}, \tag{1.26}$$

with c_Ω denoting the embedding constant of $H^2(\Omega)$ into $L^\infty(\Omega)$ (in the three-dimensional case). Thus, to prove that

$$\|\chi - 1\|_{L^\infty(Q_t)} \leqslant 1, \tag{1.27}$$

it is sufficient to bound, e.g., χ_t in $L^2(0,T;H^2(\Omega))$ and choose t sufficiently small in (1.26). In particular, restricting our analysis to a suitable time interval $(0,\hat{t})$, we are allowed to omit the constraint on χ in (1.20) and deal directly with the differential inclusion

$$\chi_{tt} + \mu\chi_t - \eta\Delta\chi_t - \nu\Delta\chi + \partial I_{(-\infty,0]}(\chi_t) \ni w - \frac{1}{2}\varepsilon(\mathbf{u})K\varepsilon(\mathbf{u}) - \alpha(\theta)\operatorname{div}\mathbf{u}. \tag{1.28}$$

By using a fixed point argument combined with an a priori estimates and passage to the limit technique we are able to prove that there exists a solution to our initial and boundary value problem in a suitable time interval (Theorem 2.1). Then, uniqueness follows by contracting estimates. Finally, further regularity results are established under suitable assumptions on the data of the problem (Theorem 2.2). Let us remark that the local character of our results is essentially related to the presence of highly non-linear terms in the resulting system (see also [3] and [7]). In Section 2 we derive the variational formulation of the problem and state the main results. Section 3 is devoted to the proof of the existence result. In particular, it is proved the positivity of the temperature which is a crucial point in showing the thermodynamic consistency of the model (cf. Remark 1.1). The uniqueness result is detailed in Section 4. Finally, in Section 5, we get additional regularity on the solution.

2. Analytical formulation and main results

In this section, we present the analytical problem we are going to solve, which is recovered by (1.18)–(1.19), (1.28) and (1.21), (1.22)–(1.24). We make some simplification. In particular, we consider $\mathbf{u}$ as a scalar quantity u (so that ∇u stands for deformation) and let $\alpha(\theta) = \alpha\theta$, with $\alpha \in \mathbf{R}$ and $\mathbf{a} = (\alpha,\alpha,\alpha)$. The physical constants are taken $c_s = \nu = \lambda = \mu = \alpha = \delta = \eta = 1$. The stiffness matrix K and the viscosity matrix S are assumed equal to the identity matrix. Hence, we introduce the Hilbert triplet $V \hookrightarrow H \hookrightarrow V'$, with $H := L^2(\Omega)$ identified as usual with its dual space, and $V := H^1(\Omega)$. Moreover, we denote by $(\cdot,\cdot)$ the scalar product in H and by ${}_{X'}\langle\cdot,\cdot\rangle_X$ the duality pairing between the space X and its topological dual X'. Then, the associated Riesz isomorphism $J: V \to V'$ is related to the scalar product in V $((\cdot,\cdot))$ and in V' $((\cdot,\cdot))_*$ as follows

$$ {}_{V'}\langle Jv_1, v_2\rangle_V := ((v_1, v_2)), \qquad ((u_1, u_2))_* := {}_{V'}\langle u_1, J^{-1}u_2\rangle_V, \tag{2.1}$$

for $v_i \in V$, $u_i \in V'$, $i = 1, 2$. In addition, we set $V_0 = H_0^1(\Omega)$ and $W = \{v \in H^2(\Omega)\colon \partial_n v = 0 \text{ on } \Gamma\}$. We denote by $\|\cdot\|_X$ both the norm in a Banach space X and in some power of it X^p.

We aim to investigate the following PDE's system

$$\theta_t - \Delta\theta = \theta\chi\mathbf{a}\cdot\nabla u_t + \theta\chi_t\mathbf{a}\cdot\nabla u + |\chi_t|^2 + |\nabla\chi_t|^2 + |\nabla u_t|^2, \tag{2.2}$$

$$\chi_{tt} + \chi_t - \Delta\chi_t - \Delta\chi + \partial I_{(-\infty,0]}(\chi_t) \ni w - \frac{1}{2}|\nabla u|^2 - \theta\mathbf{a}\cdot\nabla u, \tag{2.3}$$

$$u_{tt} - \operatorname{div}(\nabla u_t + \chi\nabla u + \theta\chi\mathbf{a}) = 0, \tag{2.4}$$

combined with the initial and boundary conditions expressed by (1.21) and (1.22)–(1.24). Hence, to simplify notation, we introduce the operator β

$$\beta(\chi_t) := (\mathrm{Id} + \partial I_{(-\infty,0]})(\chi_t). \tag{2.5}$$

However, let us point out that our results can be applied to a fairly general maximal monotone operator not necessarily coercive (see (2.9)–(2.10) below).

Actually, we address the above system in the duality between V' and V for (2.2)–(2.3) and between V_0' and V_0 for (2.4). In particular, in this abstract framework, we have to specify the meaning of the operators $-\Delta$ and $-\operatorname{div}$ in Eqs. (2.2)–(2.3) and (2.4). More precisely, in (2.2)–(2.3) the operator $-\Delta$ stands for the realization of the Laplace operator with homogeneous Neumann boundary conditions

$$-\Delta : V \to V', \quad {}_{V'}\langle -\Delta u, v\rangle_V = \int_\Omega \nabla u\cdot\nabla v \quad \forall u, v \in V,$$

while $-\operatorname{div}$ in (2.4) is defined by

$$-\operatorname{div} : H^3 \to V_0', \quad {}_{V_0'}\langle -\operatorname{div}\mathbf{v}, u\rangle_{V_0} = \int_\Omega \mathbf{v}\cdot\nabla u \quad \forall\mathbf{v}\in H^3,\ \forall u \in V_0.$$

Let us observe that if u and χ belong to $H^2(\Omega)$ (this assumption could be relaxed), then there holds (here $-\Delta$ is the Laplace operator)

$$-\operatorname{div}(\chi\nabla u) \in H \quad \text{and} \quad -\operatorname{div}(\chi\nabla u) = -\chi\Delta u - \nabla\chi\cdot\nabla u.$$

This fact can be proved by means of an approximation-density procedure. Thus, in such a regularity framework, the term $-\operatorname{div}(\chi\nabla v)$ makes sense in H, hence almost everywhere in Ω. Analogously, also the term $-\Delta v$ can be understood as an L^2-function once we have $v \in H^2(\Omega)$.

Now, concerning the Cauchy conditions (1.22)–(1.24), we assume the following hypotheses

$$\theta_0 \in H, \quad \theta_0 > 0 \text{ a.e. in } \Omega, \quad \theta_0^{-1} \in L^1(\Omega), \tag{2.6}$$

$$\chi_0 \in W, \qquad \chi_1 \in V, \tag{2.7}$$

$$u_0 \in H^2(\Omega)\cap V_0, \qquad u_1 \in V_0. \tag{2.8}$$

Moreover, we suppose that

$$\beta : \mathbf{R} \to 2^{\mathbf{R}} \text{ is a maximal monotone operator}, \quad \text{with } 0 \in \beta(0) \text{ and } \operatorname{dom}\beta \subseteq (-\infty, 0]. \tag{2.9}$$

Standard convex analysis results (see, e.g., [2]) ensure that there exists a functional

$$\hat\beta : \mathbf{R} \to [0, +\infty] \quad \text{proper, convex, lower semicontinuous}, \quad \text{with } \beta = \partial\hat\beta \text{ and } \hat\beta(0) = 0 = \min\hat\beta. \tag{2.10}$$

Then, we can state the main result of the paper.

Theorem 2.1. *Let the assumptions* (2.6)–(2.10) *hold. Then, there exist* $\tau \in (0, T]$ *and a unique quadruple of functions* (θ, χ, u, ξ) *with regularity*

$$\theta \in H^1(0, \tau; V') \cap C^0\big([0, \tau]; H\big) \cap L^2(0, \tau; V), \tag{2.11}$$

$$\theta^{-1} \in L^\infty\big(0, \tau; L^1(\Omega)\big), \tag{2.12}$$

$$\chi \in H^2(0, \tau; H) \cap W^{1,\infty}(0, \tau; V) \cap H^1(0, \tau; W), \tag{2.13}$$

$$u \in H^2(0, \tau, H) \cap W^{1,\infty}(0, \tau; V_0) \cap H^1\big(0, \tau; H^2(\Omega)\big), \tag{2.14}$$

$$\xi \in L^2(0, \tau; H), \tag{2.15}$$

fulfilling (1.22)–(1.24) *and*

$$\langle \theta_t, v\rangle + (\nabla\theta, \nabla v) = \big(\theta\chi\mathbf{a}\cdot\nabla u_t + \theta\chi_t\mathbf{a}\cdot\nabla u + |\chi_t|^2 + |\nabla\chi_t|^2 + |\nabla u_t|^2, v\big) \quad \forall v \in V \textit{ a.e. in } (0, \tau), \tag{2.16}$$

$$\chi_{tt} - \Delta\chi_t - \Delta\chi + \xi = w - \frac{1}{2}|\nabla u|^2 - \theta\mathbf{a}\cdot\nabla u \quad \textit{a.e. in } Q_\tau, \tag{2.17}$$

$$\xi \in \beta(\chi_t) \quad \textit{a.e. in } Q_\tau, \tag{2.18}$$

$$u_{tt} - \Delta u_t - \operatorname{div}(\chi\nabla u + \theta\chi\mathbf{a}) = 0 \quad \textit{a.e. in } Q_\tau, \tag{2.19}$$

and such that

$$\chi \in [0, 1] \quad \textit{a.e. in } Q_\tau, \tag{2.20}$$

$$\theta > 0 \quad \textit{a.e. in } Q_\tau. \tag{2.21}$$

Now, by strengthening some hypotheses on the data, we address the improvement of the regularity of the solution provided by Theorem 2.1. Hence, suppose moreover

$$\theta_0 \in V, \tag{2.22}$$

$$\chi_1 \in W, \quad \chi_1 \in \operatorname{dom}\beta \text{ a.e. in } \Omega, \tag{2.23}$$

$$\text{there exists } \xi \in H \text{ such that } \xi \in \beta(\chi_1) \text{ a.e. in } \Omega, \tag{2.24}$$

$$u_1 \in H^2(\Omega) \cap V_0. \tag{2.25}$$

Then, the following regularity result holds.

Theorem 2.2. *Assume* (2.22)–(2.25) *in addition to* (2.6)–(2.10). *Then, there exist* $\widehat{T} \in (0, T]$ *and a unique quadruple of functions* (θ, χ, u, ξ) *with regularity*

$$\theta \in H^1(0, \widehat{T}; H) \cap C^0\big([0, \widehat{T}]; V\big) \cap L^2(0, \widehat{T}; W), \tag{2.26}$$

$$\chi \in W^{2,\infty}(0, \widehat{T}; H) \cap H^2(0, \widehat{T}; V) \cap W^{1,\infty}(0, \widehat{T}; W), \tag{2.27}$$

$$u \in W^{2,\infty}(0, \widehat{T}; H) \cap H^2(0, \widehat{T}; V_0) \cap W^{1,\infty}\big(0, \widehat{T}; H^2(\Omega)\big), \tag{2.28}$$

$$\xi \in L^\infty(0, \widehat{T}; H) \tag{2.29}$$

fulfilling (1.22)–(1.24), (2.17)–(2.21)*, and*

$$\theta_t - \Delta\theta = \theta\chi\mathbf{a}\cdot\nabla u_t + \theta\chi_t\mathbf{a}\cdot\nabla u + |\chi_t|^2 + |\nabla\chi_t|^2 + |\nabla u_t|^2 \quad \textit{a.e. in } Q_{\widehat{T}}. \tag{2.30}$$

The proof of these results will be carried out throughout the remainder of the paper: the existence of a local solution is derived by means of a fixed point technique; the uniqueness result is established by some contracting estimates and the regularity result is obtained by performing proper a priori estimates.

3. The existence result

To prove the existence result stated by Theorem 2.1, we apply the Schauder fixed point theorem to a suitable operator $\mathcal{T}$ we are going to construct.

First step: definition of $\mathcal{T}$.

For $R > 0$, let

$$\mathcal{X} := \big\{(u,\chi) \in H^1\big(0,\tau; W_0^{1,4}(\Omega)\big) \times H^1\big(0,\tau; W^{1,4}(\Omega)\big),\ \chi \in [0,1] \text{ a.e. in } Q_\tau,\\ \big\|(u,\chi)\big\|_{H^1(0,\tau;W_0^{1,4}(\Omega))\times H^1(0,\tau;W^{1,4}(\Omega))} \leqslant R\big\}, \tag{3.1}$$

where $\tau \in (0,T]$ will be chosen later. First, we fix an arbitrary $(\hat{u},\hat{\chi}) \in \mathcal{X}$ and we substitute (u,χ) in (2.16) by $(\hat{u},\hat{\chi})$. Note, in particular, that $|\hat{\chi}_t|^2 + |\nabla\hat{\chi}_t|^2 + |\nabla\hat{u}_t|^2$ belongs to $L^1(0,\tau;H)$. Standard results in the theory of parabolic equations (see, e.g., [1]) ensure that there exists a unique

$$\theta := \mathcal{T}_1(\hat{u},\hat{\chi}) \in \big[W^{1,1}(0,\tau;H) + H^1(0,\tau;V')\big] \cap C^0\big([0,\tau];H\big) \cap L^2(0,\tau,;V) \tag{3.2}$$

solving the corresponding equation (2.16) with the associated Cauchy condition (1.22). Then, we consider (2.19) and replace θ and χ by $\theta = \mathcal{T}_1(\hat{u},\hat{\chi})$ and $\hat{\chi}$, respectively. We denote by

$$u := \mathcal{T}_2\big(\mathcal{T}_1(\hat{u},\hat{\chi}),\hat{\chi}\big) \in H^2(0,\tau;H) \cap W^{1,\infty}(0,\tau;V_0) \cap H^1\big(0,\tau;H^2(\Omega)\big)$$

the corresponding solution satisfying (1.24) (see, e.g., [5] for existence and uniqueness results related to this kind of equations). Finally, we consider $\theta = \mathcal{T}_1(\hat{u},\hat{\chi})$ and $u = \mathcal{T}_2(\theta,\hat{\chi})$ in (2.17). The theory of evolution equations associated to maximal monotone operators (see, e.g., [2]) ensure that the corresponding system (2.17)–(2.18)–(1.23) admits a unique pair (χ,ξ) of solutions, with

$$\chi := \mathcal{T}_3(\theta,u) \in H^2(0,\tau;H) \cap W^{1,\infty}(0,\tau;V) \cap H^1(0,\tau;W)$$

and $\xi \in L^2(0,\tau;H)$. By the above construction, it results well-defined an operator $\mathcal{T}$ obtained by the composition of $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3$, i.e.

$$\mathcal{T}(\hat{u},\hat{\chi}) = \big(u = \mathcal{T}_2(\theta,\hat{\chi}), \chi = \mathcal{T}_3(\theta,u)\big), \quad \text{where } \theta = \mathcal{T}_1(\hat{u},\hat{\chi}). \tag{3.3}$$

Second step: a priori estimates.

Let us proceed by performing some (formal) a priori estimates on the above defined functions (θ,χ,u,ξ). Actually, we should exploit the following estimates on suitable regularized versions of the equations and then passing to the limit with respect to the approximating parameters. However, for the sake of simplicity, we prefer to formally proceed, as the arguments we apply to prove compactness and continuity of $\mathcal{T}$ are mostly the same we should use to pass to the limit in the regularized versions of the estimates.

First a priori estimate. We first deal with (2.16), in which we now intend that $\hat{u}$ and $\hat{\chi}$ are written in place of u and χ. Test (2.16) by θ and integrate over $(0,t)$, with $t \in (0,\tau)$ (cf. (3.1)). We have

$$\frac{1}{2}\big\|\theta(t)\big\|_H^2 + \|\nabla\theta\|_{L^2(0,t;H)}^2 \leqslant \frac{1}{2}\|\theta_0\|_H^2 + \sum_{j=1}^{3}\big|I_j(t)\big|, \tag{3.4}$$

where the integrals $I_j(t)$ are handled as follows. Using Hölder's and Young's inequalities, the uniform bound of $\hat{\chi}$ (cf. (3.1)), and Sobolev's embedding $V \hookrightarrow L^4(\Omega)$, we get

$$I_1(t) = \int_0^t\!\!\int_\Omega \theta\hat{\chi}\mathbf{a}\cdot\nabla\hat{u}_t\theta \leqslant c\int_0^t \|\theta\|_{L^4(\Omega)}\|\nabla\hat{u}_t\|_{L^4(\Omega)}\|\theta\|_H\\ \leqslant \frac{1}{4}\|\theta\|_{L^2(0,t;V)}^2 + c\int_0^t \|\nabla\hat{u}_t\|_{L^4(\Omega)}^2\|\theta\|_H^2. \tag{3.5}$$

We warn that here and in the sequel, we employ the same symbol c for different positive constants even in the same formula, in regard of simplicity. Now, note that by definition of $\mathcal{X}$ the function $\|\nabla\hat{u}_t\|_{L^4(\Omega)}^2$ belongs to $L^1(0,\tau)$. Then, let us recall Sobolev's embedding $W^{1,4}(\Omega) \hookrightarrow L^\infty(\Omega)$. Thus, analogously proceeding, we infer that

$$I_2(t) = \int_0^t \int_\Omega \theta \hat{\chi}_t \mathbf{a} \cdot \nabla \hat{u} \theta \leqslant c \int_0^t \|\theta\|_{L^4(\Omega)} \|\hat{\chi}_t\|_{L^\infty(\Omega)} \|\nabla \hat{u}\|_{L^4(\Omega)} \|\theta\|_H$$

$$\leqslant \frac{1}{4} \|\theta\|^2_{L^2(0,t;V)} + c \int_0^t \|\hat{\chi}_t\|^2_{W^{1,4}(\Omega)} \|\theta\|^2_H, \tag{3.6}$$

as $\|\nabla \hat{u}\|_{L^\infty(0,\tau;L^4(\Omega))} \leqslant c$, by (3.1). In addition, there holds $\|\hat{\chi}_t\|^2_{W^{1,4}(\Omega)} \in L^1(0,\tau)$. Finally, we specify the last integral as

$$I_3(t) = \int_0^t \int_\Omega \big(|\hat{\chi}_t|^2 + |\nabla \hat{\chi}_t|^2 + |\nabla \hat{u}_t|^2\big)\theta \leqslant \int_0^t \big(\|\hat{\chi}_t\|^2_{L^4(\Omega)} + \|\nabla \hat{\chi}_t\|^2_{L^4(\Omega)} + \|\nabla \hat{u}\|^2_{L^4(\Omega)}\big) \|\theta\|_H, \tag{3.7}$$

and we point out that $\|\hat{\chi}_t\|^2_{L^4(\Omega)} + \|\nabla \hat{\chi}_t\|^2_{L^4(\Omega)} + \|\nabla \hat{u}\|^2_{L^4(\Omega)}$ is bounded in $L^1(0,\tau)$. Thus, combining (3.4) with (3.5)–(3.7), summing up $\|\theta\|^2_{L^2(0,t;H)}$ to both sides of (3.4), we can apply a generalized version of Gronwall's lemma (see, e.g., [1]) to deduce

$$\|\theta\|_{L^\infty(0,\tau;H) \cap L^2(0,\tau;V)} \leqslant c. \tag{3.8}$$

Now, let us deal with (2.19) in which $\hat{\chi}$ and $\theta = \mathcal{T}_1(\hat{u}, \hat{\chi})$ are introduced.

Second a priori estimate. We test (2.19) by $-\Delta u_t$ and integrate over $(0,t)$. We have

$$\frac{1}{2} \big\|\nabla u_t(t)\big\|^2_H + \|\Delta u_t\|^2_{L^2(0,t;H)} \leqslant \frac{1}{2} \|\nabla u_1\|^2_H + \sum_{j=4}^{7} \big|I_j(t)\big|. \tag{3.9}$$

Then, we handle the integrals $I_j(t)$. By use of Hölder's and Young's inequalities, we get

$$I_4(t) = \int_0^t \int_\Omega |\nabla \hat{\chi} \cdot \nabla u \Delta u_t| \leqslant \int_0^t \|\nabla \hat{\chi}\|_{L^4(\Omega)} \|\nabla u\|_{L^4(\Omega)} \|\Delta u_t\|_H$$

$$\leqslant \frac{1}{8} \|\Delta u_t\|^2_{L^2(0,t;H)} + c\left(1 + \int_0^t \|\Delta u_t\|^2_{L^2(0,s;H)}\, ds\right), \tag{3.10}$$

where we have exploited

$$\big\|\nabla u(s)\big\|^2_{L^4(\Omega)} \leqslant c \big\|\Delta u(s)\big\|^2_H \leqslant c\left(1 + \int_0^s \|\Delta u_t\|^2_H\right)$$

and that

$$\|\nabla \hat{\chi}\|_{L^\infty(0,\tau;L^4(\Omega))} \leqslant c.$$

Analogously, on account of the uniform bound of $\hat{\chi}$ (cf. (3.1)), we obtain

$$I_5(t) = \int_0^t \int_\Omega |\hat{\chi} \Delta u \Delta u_t| \leqslant c \int_0^t \|\Delta u\|_H \|\Delta u_t\|_H$$

$$\leqslant \frac{1}{8} \|\Delta u_t\|^2_{L^2(0,t;H)} + c\left(1 + \int_0^t \|\Delta u_t\|^2_{L^2(0,s;H)}\, ds\right). \tag{3.11}$$

The last two integrals in (3.9) are treated as follows (cf. (3.1) and (3.8))

$$I_6(t) = \int_0^t \int_\Omega |\mathbf{a} \cdot \nabla \theta \hat{\chi} \Delta u_t| \leqslant \frac{1}{8} \|\Delta u_t\|^2_{L^2(0,t;H)} + c\|\theta\|^2_{L^2(0,t;V)} \leqslant \frac{1}{8} \|\Delta u_t\|^2_{L^2(0,t;H)} + c \tag{3.12}$$

and

$$I_7(t)=\int_0^t\int_\Omega |\theta\mathbf{a}\cdot\nabla\hat{\chi}\,\Delta u_t| \leqslant \frac{1}{8}\|\Delta u_t\|^2_{L^2(0,t;H)}+c\int_0^t \|\theta\|^2_V\|\nabla\hat{\chi}\|^2_{L^4(\Omega)}$$
$$\leqslant \frac{1}{8}\|\Delta u_t\|^2_{L^2(0,t;H)}+c. \tag{3.13}$$

Thus, an application of Gronwall's lemma to (3.9) combined with (3.10)–(3.13) leads to

$$\|u\|_{W^{1,\infty}(0,\tau;V_0)\cap H^1(0,\tau;H^2(\Omega))}\leqslant c. \tag{3.14}$$

Note that, by a comparison in (2.19), we also infer

$$\|u_{tt}\|_{L^2(0,\tau;H)}\leqslant c. \tag{3.15}$$

Further a priori estimates. Now, we perform some a priori estimates on (2.17) where u and θ are fixed by the previous arguments. We are still proceeding formally as, also in this case, we should deal with the regularized version of (2.17) obtained introducing the Yosida approximation of the operator β and then passing to the limit with respect to the approximating parameter. However, as it is a fairly standard procedure in the theory of (parabolic) equations associated with maximal monotone operators we directly proceed formally.

We test (2.17) by $-\Delta\chi_t$ and integrate over $(0,t)$. We get

$$\frac{1}{2}\big\|\nabla\chi_t(t)\big\|^2_H+\|\Delta\chi_t\|^2_{L^2(0,t;H)}+\frac{1}{2}\big\|\Delta\chi(t)\big\|^2_H+\int_0^t\int_\Omega \xi(-\Delta\chi_t)$$
$$\leqslant \frac{1}{2}\|\nabla\chi_1\|^2_H+\frac{1}{2}\|\Delta\chi_0\|^2_H+\sum_{j=8}^{10}\big|I_j(t)\big| \tag{3.16}$$

where, in particular, the monotonicity of β yields for a.a. t (the notation is formal)

$$\int_\Omega \xi(-\Delta\chi_t)\geqslant 0 \tag{3.17}$$

(see [15, Lemma 4.1], for a rigorous justification). Then, we estimate the right-hand side of (3.16) as follows

$$I_8(t)=\int_0^t\int_\Omega w\Delta\chi_t\leqslant \frac{1}{4}\|\Delta\chi_t\|^2_{L^2(0,t;H)}+c, \tag{3.18}$$

$$I_9(t)=\int_0^t\int_\Omega \frac{1}{2}|\nabla u|^2\Delta\chi_t\leqslant \frac{1}{4}\|\Delta\chi_t\|^2_{L^2(0,t;H)}+c\int_0^t\|\nabla u\|^4_{L^4(\Omega)}\leqslant \frac{1}{4}\|\Delta\chi_t\|^2_{L^2(0,t;H)}+c, \tag{3.19}$$

$$I_{10}(t)=\int_0^t\int_\Omega \theta\mathbf{a}\cdot\nabla u\Delta\chi_t\leqslant \frac{1}{4}\|\Delta\chi_t\|^2_{L^2(0,t;H)}+c\int_0^t\|\theta\|^2_V\|\nabla u\|^2_{L^4(\Omega)}$$
$$\leqslant \frac{1}{4}\|\Delta\chi_t\|^2_{L^2(0,t;H)}+c \tag{3.20}$$

thanks to (3.14) and (3.8). Now, we combine (3.17)–(3.20) in (3.16) and we obtain

$$\|\Delta\chi\|_{H^1(0,\tau;H)}\leqslant c, \tag{3.21}$$
$$\|\nabla\chi\|_{W^{1,\infty}(0,\tau;H)}\leqslant c. \tag{3.22}$$

Then, we test (2.17) by χ_{tt} and integrate over $(0,t)$. We find

$$\|\chi_{tt}\|^2_{L^2(0,t;H)} + \frac{1}{2}\big\|\nabla\chi_t(t)\big\|^2_H + \int_\Omega \hat\beta\big(\chi_t(t)\big) \leqslant \frac{1}{2}\|\nabla\chi_1\|^2_H + \int_\Omega \hat\beta(\chi_1) + \sum_{j=11}^{12}\big|I_j(t)\big|, \tag{3.23}$$

where we have used the chain rule for $\hat\beta$, see [8, Lemma 3.3]. Moreover, (3.21) leads to

$$I_{11}(t) = \int_0^t\!\!\int_\Omega \Delta\chi\,\chi_{tt} \leqslant \frac{1}{4}\|\chi_{tt}\|^2_{L^2(0,t;H)} + c\|\Delta\chi\|^2_{L^2(0,t;H)} \leqslant \frac{1}{4}\|\chi_{tt}\|^2_{L^2(0,t;H)} + c. \tag{3.24}$$

Concerning $I_{12}(t)$, we can argue as in the derivation of (3.18)–(3.20). We get

$$I_{12}(t) = \int_0^t\!\!\int_\Omega \left(w - \frac{1}{2}|\nabla u|^2 - \theta\mathbf{a}\cdot\nabla u\right)\chi_{tt} \leqslant \frac{1}{4}\|\chi_{tt}\|^2_{L^2(0,t;H)} + c. \tag{3.25}$$

Then, we combine (3.24)–(3.25) in (3.23) and we deduce that

$$\|\chi_{tt}\|_{L^2(0,\tau;H)} \leqslant c. \tag{3.26}$$

Note that, thanks to elliptic regularity results, (3.26), (3.21), and (3.22) yield

$$\|\chi\|_{H^2(0,\tau;H)\cap W^{1,\infty}(0,\tau;V)\cap H^1(0,\tau;W)} \leqslant c. \tag{3.27}$$

Finally, a comparison in (2.17) leads to

$$\|\xi\|_{L^2(0,\tau;H)} \leqslant c. \tag{3.28}$$

Third step: the existence of a fixed point of $\mathcal{T}$.

Now, we are in the position of showing that $\mathcal{T}$ fulfills the assumptions of the Schauder Theorem (cf. (3.1) and (3.3)). At first, we prove that it maps $\mathcal{X}$ into itself, at least for a suitable choice of τ. Thanks to (3.27), by using standard interpolation tools (see, e.g., [16]), we get

$$\|\chi\|_{W^{1,8/3}(0,\tau;W^{1,4}(\Omega))} \leqslant c_1, \tag{3.29}$$

where by c_1 (and then c_2) we denote a positive constant depending on R. Thus, by Hölder's inequality, we obtain

$$\|\chi\|_{H^1(0,\tau;W^{1,4}(\Omega))} \leqslant \tilde c_1\tau^{1/8}\|\chi\|_{W^{1,8/3}(0,\tau;W^{1,4}(\Omega))} \leqslant R, \tag{3.30}$$

where the constant $\tilde c_1$ is positive provided, e.g., $\tau \leqslant R^8(\tilde c_1 c_1)^{-8}$. Analogously proceeding, on account of (3.14), we get

$$\|u\|_{W^{1,8/3}(0,\tau;W_0^{1,4}(\Omega))} \leqslant c_2, \tag{3.31}$$

and hence

$$\|u\|_{H^1(0,\tau;W_0^{1,4}(\Omega))} \leqslant \tilde c_2\tau^{1/8}\|u\|_{W^{1,8/3}(0,\tau;W_0^{1,4}(\Omega))} \leqslant R, \tag{3.32}$$

provided, e.g., $\tau \leqslant R^8(\tilde c_2 c_2)^{-8}$ ($\tilde c_2 > 0$).

Thus, to verify that $\mathcal{T}$ maps $\mathcal{X}$ into itself, it remains to show that $\chi \in [0,1]$ a.e. in Q_τ, at least for a suitable choice of τ.

Recalling that $\operatorname{dom}\beta \subseteq (-\infty, 0]$ and that $\chi_0 \in (0,1]$, we only have to prove that $\chi \geqslant 0$ a.e. in Q_τ, as χ cannot increase. To this aim, we may suppose that there exists $\delta \in (0,1)$ such that $\chi_0(x) \geqslant \delta$ $\forall x \in \Omega$ and show that $\|\chi - \chi_0\|_{L^\infty(Q_\tau)} \leqslant \delta$ (cf. also [4] and [5]). Owing to the regularity of χ, we proceed as follows (cf. (1.26)–(1.27)). Let τ to be chosen such that

$$\|\chi - \chi_0\|_{L^\infty(Q_\tau)} \leqslant \int_0^\tau \|\chi_t\|_{L^\infty(\Omega)} \leqslant c_\Omega\tau^{1/2}\|\chi_t\|_{L^2(0,\tau;W^{1,4}(\Omega))} \leqslant c_\Omega\tau^{1/2}R \leqslant \delta, \tag{3.33}$$

with c_Ω denoting the embedding constant of $W^{1,4}(\Omega)$ into $L^\infty(\Omega)$. Eventually, we may choose

$$\tau = \min\big\{R^8(\tilde c_1 c_1)^{-8}, R^8(\tilde c_2 c_2)^{-8}, \delta^2(c_\Omega R)^{-2}\big\}. \tag{3.34}$$

Let us point out that τ depends only on the data of the problem (and on R).

Concerning the compactness of the operator $\mathcal{T}$ with respect to the topology induced on $\mathcal{X}$ by $H^1(0,\tau;W_0^{1,4}(\Omega))\times H^1(0,\tau;W^{1,4}(\Omega))$, this easily follows by (3.14), (3.15), and (3.27).

Now, it remains to prove that $\mathcal{T}$ is continuous with respect to the topology induced on $\mathcal{X}$ by $H^1(0,\tau;W_0^{1,4}(\Omega))\times H^1(0,\tau;W^{1,4}(\Omega))$. We proceed as follows: we consider a sequence

$$(\hat{u}_n,\hat{\chi}_n)\to(\hat{u},\hat{\chi})\quad\text{in }\mathcal{X},\tag{3.35}$$

and show that

$$\mathcal{T}(\hat{u}_n,\hat{\chi}_n)\to\mathcal{T}(\hat{u},\hat{\chi})\quad\text{in }\mathcal{X}.$$

Let us specify some notation. Let θ_n be the solution of the problem (2.16)–(1.22), once $\hat{u}_n$ and $\hat{\chi}_n$ are fixed, i.e. $\theta_n:=\mathcal{T}_1(\hat{u}_n,\hat{\chi}_n)$. Analogously, let $u_n:=\mathcal{T}_2(\theta_n,\hat{\chi}_n)$ be the solution of (2.19)–(1.24), with θ_n and $\hat{\chi}_n$ fixed; let $(\chi_n:=\mathcal{T}_3(\theta_n,u_n),\xi_n)$ be the solution of (2.17)–(2.18)–(1.23), once θ_n and u_n are fixed. By the above a priori estimates (cf. (3.8), (3.14), (3.15), (3.27), and (3.28)), we can find a constant c independent of n such that

$$\|\theta_n\|_{L^\infty(0,\tau;H)\cap L^2(0,\tau;V)}\leqslant c,\tag{3.36}$$

$$\|u_n\|_{H^2(0,\tau;H)\cap W^{1,\infty}(0,\tau;V_0)\cap H^1(0,\tau;H^2(\Omega))}\leqslant c,\tag{3.37}$$

$$\|\chi_n\|_{H^2(0,\tau;H)\cap W^{1,\infty}(0,\tau;V)\cap H^1(0,\tau;W)}\leqslant c,\tag{3.38}$$

$$\|\xi_n\|_{L^2(0,\tau;H)}\leqslant c.\tag{3.39}$$

Thus, well-known weak and weak-star convergence results yield, at least for suitable subsequences,

$$\theta_n\overset{*}{\rightharpoonup}\theta\quad\text{in }L^\infty(0,\tau;H)\cap L^2(0,\tau;V),\tag{3.40}$$

$$u_n\overset{*}{\rightharpoonup}u\quad\text{in }H^2(0,\tau;H)\cap W^{1,\infty}(0,\tau;V_0)\cap H^1\big(0,\tau;H^2(\Omega)\big),\tag{3.41}$$

$$\chi_n\overset{*}{\rightharpoonup}\chi\quad\text{in }H^2(0,\tau;H)\cap W^{1,\infty}(0,\tau;V)\cap H^1(0,\tau;W),\tag{3.42}$$

$$\xi_n\rightharpoonup\xi\quad\text{in }L^2(0,\tau;H).\tag{3.43}$$

In particular, by strong compactness (cf. [14,19]), we can also infer

$$u_n\to u\quad\text{in }H^1\big((0,\tau);W_0^{1,4}(\Omega)\big),\tag{3.44}$$

$$\chi_n\to\chi\quad\text{in }H^1\big((0,\tau);W^{1,4}(\Omega)\big).\tag{3.45}$$

Now, we show that $\theta=\mathcal{T}_1(\hat{u},\hat{\chi})$. We use (3.40) and (3.35) to pass to the limit in (2.16), written for $\hat{u}_n$, $\hat{\chi}_n$, and θ_n. Hence, a comparison in (2.16) gives $\theta_{nt}\overset{*}{\rightharpoonup}\theta_t$ in $L^1(0,\tau;H)+L^2(0,\tau;V')$. Thus, we get that θ solves the limit equation (where $\hat{u}$ and $\hat{\chi}$ are fixed), and, by uniqueness of the solution, it is identified with $\mathcal{T}_1(\hat{u},\hat{\chi})$.

Remark 3.1. Actually, we can conclude more on the convergence of θ_n. Indeed, let us take (2.16) written for $\hat{u}_n$ and $\hat{\chi}_n$ and then for $\hat{u}$ and $\hat{\chi}$. We take the difference between the corresponding equations and we test it by $\theta_n-\mathcal{T}_1(\hat{u},\hat{\chi})$. After integrating over $(0,t)$, we get

$$\frac{1}{2}\big\|\big(\theta_n-\mathcal{T}_1(\hat{u},\hat{\chi})\big)(t)\big\|_H^2+\big\|\nabla\big(\theta_n-\mathcal{T}_1(\hat{u},\hat{\chi})\big)\big\|_{L^2(0,t;H)}^2\leqslant\sum_{j=13}^{19}\big|I_j(t)\big|,\tag{3.46}$$

where the integrals $I_j(t)$ are treated as follows. Applying Hölder's and Young's inequalities, and Sobolev's embeddings, we have

$$I_{13}(t)=\int_0^t\int_\Omega\big(\theta_n-\mathcal{T}_1(\hat{u},\hat{\chi})\big)^2\hat{\chi}_n\mathbf{a}\cdot\nabla\hat{u}_{nt}$$

$$\leqslant c\int_0^t\big\|\theta_n-\mathcal{T}_1(\hat{u},\hat{\chi})\big\|_H\big\|\theta_n-\mathcal{T}_1(\hat{u},\hat{\chi})\big\|_{L^4(\Omega)}\|\nabla\hat{u}_{nt}\|_{L^4(\Omega)}$$

$$\leqslant \delta \big\|\theta_n - \mathcal{T}_1(\hat{u},\hat{\chi})\big\|^2_{L^2(0,t;V)} + c\int_0^t \|\nabla \hat{u}_{nt}\|^2_{L^4(\Omega)} \big\|\theta_n - \mathcal{T}_1(\hat{u},\hat{\chi})\big\|^2_H, \tag{3.47}$$

for a suitable positive constant δ to be chosen later. Analogously proceeding, we infer that

$$\begin{aligned} I_{14}(t) &= \int_0^t\!\!\int_\Omega \mathcal{T}_1(\hat{u},\hat{\chi})\,(\hat{\chi}_n - \hat{\chi})\,\mathbf{a}\cdot\nabla\hat{u}_{nt}\big(\theta_n - \mathcal{T}_1(\hat{u},\hat{\chi})\big) \\ &\leqslant c\int_0^t \big\|\mathcal{T}_1(\hat{u},\hat{\chi})\big\|_{L^4(\Omega)}\|\hat{\chi}_n - \hat{\chi}\|_{L^\infty(\Omega)}\|\nabla\hat{u}_{nt}\|_{L^4(\Omega)}\big\|\theta_n - \mathcal{T}_1(\hat{u},\hat{\chi})\big\|_H \\ &\leqslant c\|\hat{\chi}_n - \hat{\chi}\|^2_{L^\infty(0,t;W^{1,4}(\Omega))}\big\|\mathcal{T}_1(\hat{u},\hat{\chi})\big\|^2_{L^2(0,t;V)} + c\int_0^t \|\nabla\hat{u}_{nt}\|^2_{L^4(\Omega)}\big\|\theta_n - \mathcal{T}_1(\hat{u},\hat{\chi})\big\|^2_H. \end{aligned} \tag{3.48}$$

Next, we have

$$\begin{aligned} I_{15}(t) &= \int_0^t\!\!\int_\Omega \mathcal{T}_1(\hat{u},\hat{\chi})\hat{\chi}\mathbf{a}\cdot(\nabla\hat{u}_{nt} - \nabla\hat{u}_t)\big(\theta_n - \mathcal{T}_1(\hat{u},\hat{\chi})\big) \\ &\leqslant c\int_0^t \big\|\mathcal{T}_1(\hat{u},\hat{\chi})\big\|_{L^4(\Omega)}\|\nabla\hat{u}_{nt} - \nabla\hat{u}_t\|_{L^4(\Omega)}\big\|\theta_n - \mathcal{T}_1(\hat{u},\hat{\chi})\big\|_H \\ &\leqslant c\|\hat{u}_{nt} - \hat{u}_t\|^2_{L^2(0,t;W_0^{1,4}(\Omega))} + c\int_0^t \big\|\mathcal{T}_1(\hat{u},\hat{\chi})\big\|^2_V\big\|\theta_n - \mathcal{T}_1(\hat{u},\hat{\chi})\big\|^2_H. \end{aligned} \tag{3.49}$$

Analogously arguing, we infer that

$$\begin{aligned} I_{16}(t) &= \int_0^t\!\!\int_\Omega \big(\theta_n - \mathcal{T}_1(\hat{u},\hat{\chi})\big)^2\hat{\chi}_{nt}\mathbf{a}\cdot\nabla\hat{u}_n \\ &\leqslant c\int_0^t \big\|\theta_n - \mathcal{T}_1(\hat{u},\hat{\chi})\big\|_H\big\|\theta_n - \mathcal{T}_1(\hat{u},\hat{\chi})\big\|_{L^4(\Omega)}\|\hat{\chi}_{nt}\|_{L^\infty(\Omega)}\|\nabla\hat{u}_n\|_{L^4(\Omega)} \leqslant \delta\big\|\theta_n - \mathcal{T}_1(\hat{u},\hat{\chi})\big\|^2_{L^2(0,t;V)} \\ &\quad + c\|\hat{u}_n\|^2_{L^\infty(0,t;W_0^{1,4}(\Omega))}\int_0^t \|\hat{\chi}_{nt}\|^2_{W^{1,4}(\Omega)}\big\|\theta_n - \mathcal{T}_1(\hat{u},\hat{\chi})\big\|^2_H, \end{aligned} \tag{3.50}$$

for a suitable positive constant δ to be chosen later. Moreover

$$\begin{aligned} I_{17}(t) &= \int_0^t\!\!\int_\Omega \mathcal{T}_1(\hat{u},\hat{\chi})(\hat{\chi}_{nt} - \hat{\chi}_t)\mathbf{a}\cdot\nabla\hat{u}_n\big(\theta_n - \mathcal{T}_1(\hat{u},\hat{\chi})\big) \\ &\leqslant c\int_0^t \big\|\mathcal{T}_1(\hat{u},\hat{\chi})\big\|_{L^4(\Omega)}\|\hat{\chi}_{nt} - \hat{\chi}_t\|_{L^\infty(\Omega)}\|\nabla\hat{u}_n\|_{L^4(\Omega)}\big\|\theta_n - \mathcal{T}_1(\hat{u},\hat{\chi})\big\|_H \leqslant c\|\hat{\chi}_{nt} - \hat{\chi}_t\|^2_{L^2(0,t;W^{1,4}(\Omega))} \\ &\quad + c\|\hat{u}_n\|^2_{L^\infty(0,t;W_0^{1,4}(\Omega))}\int_0^t \big\|\mathcal{T}_1(\hat{u},\hat{\chi})\big\|^2_V\big\|\theta_n - \mathcal{T}_1(\hat{u},\hat{\chi})\big\|^2_H \end{aligned} \tag{3.51}$$

and

$$
\begin{aligned}
I_{18}(t) &= \int_0^t\int_\Omega \mathcal{T}_1(\hat u,\hat\chi)\hat\chi_t\mathbf{a}\cdot(\nabla\hat u_n-\nabla\hat u)\big(\theta_n-\mathcal{T}_1(\hat u,\hat\chi)\big)\\
&\leqslant c\int_0^t \big\|\mathcal{T}_1(\hat u,\hat\chi)\big\|_H\|\hat\chi_t\|_{L^\infty(\Omega)}\|\nabla\hat u_n-\nabla\hat u\|_{L^4(\Omega)}\big\|\theta_n-\mathcal{T}_1(\hat u,\hat\chi)\big\|_{L^4(\Omega)}\leqslant\delta\big\|\theta_n-\mathcal{T}_1(\hat u,\hat\chi)\big\|^2_{L^2(0,t;V)}\\
&\quad+c\big\|\mathcal{T}_1(\hat u,\hat\chi)\big\|^2_{L^\infty(0,t;H)}\|\hat\chi_t\|^2_{L^2(0,t;W^{1,4}(\Omega))}\|\hat u_n-\hat u\|^2_{L^\infty(0,t;W^{1,4}_0(\Omega))},
\end{aligned}
\tag{3.52}
$$

for a suitable positive constant δ to be chosen later. Finally, we deal with the difference of the quadratic terms. For simplicity, we let

$$\hat f_n=|\hat\chi_{nt}|^2+|\nabla\hat\chi_{nt}|^2+|\nabla\hat u_{nt}|^2$$

and

$$\hat f=|\hat\chi_t|^2+|\nabla\hat\chi_t|^2+|\nabla\hat u_t|^2.$$

We have

$$I_{19}(t)=\int_0^t\int_\Omega(\hat f_n-\hat f)\big(\theta_n-\mathcal{T}_1(\hat u,\hat\chi)\big)\leqslant\int_0^t\|\hat f_n-\hat f\|_H\big\|\theta_n-\mathcal{T}_1(\hat u,\hat\chi)\big\|_H. \tag{3.53}$$

Note that $\|\hat f_n-\hat f\|_{L^1(0,t;H)}\to 0$ as $n\to+\infty$, thanks to (3.35). Thus, we collect (3.47)–(3.53), on account of the uniform bounds of $\hat u_n$, $\hat\chi_n$, $\mathcal{T}_1(\hat u,\hat\chi)$, $\hat u$, $\hat\chi$ (cf. (3.1) and (3.8)) and the convergence specified by (3.35). Choosing δ small enough (e.g. $\delta\leqslant 1/4$), we can apply Gronwall's lemma to (3.46) and deduce

$$\theta_n\to\mathcal{T}_1(\hat u,\hat\chi)\quad\text{in } L^\infty(0,\tau;H)\cap L^2(0,\tau;V).\qquad\square \tag{3.54}$$

Now, we deal with (2.19) written for u_n, with θ_n and $\hat\chi_n$ fixed. It is a standard matter to pass to the limit as $n\to+\infty$ owing to (3.41), (3.35), and (3.54). Moreover, thanks to the uniqueness of the solution of the problem (2.19)–(1.24), once θ and $\hat\chi$ are fixed, we can identify with $u=\mathcal{T}_2(\theta,\hat\chi)$ and (3.44) holds for the whole sequence.

Next, let us consider (2.17) written for (χ_n,ξ_n), once θ_n and u_n are fixed. We pass to the limit as $n\to+\infty$ in (2.17) thanks to (3.42), (3.43), (3.44), and (3.54). Moreover, owing to (3.45) and (3.43), monotonicity arguments (cf. [8]) ensure that $\xi\in\beta(\chi_t)$. Again, by the uniqueness result holding for the problem (2.17)–(2.18)–(1.23), once θ and u are fixed, we can identify with $\chi=\mathcal{T}_3(\theta,u)$ and extend (3.45) to the whole sequence. Finally, (3.54), (3.44), (3.45) (and the above argument) lead to

$$\mathcal{T}(\hat u_n,\hat\chi_n)\to\mathcal{T}(\hat u,\hat\chi)\quad\text{in } H^1\big(0,\tau;W^{1,4}_0(\Omega)\big)\times H^1\big(0,\tau;W^{1,4}(\Omega)\big), \tag{3.55}$$

which concludes the proof of the continuity of the operator $\mathcal{T}$.

Finally, we complete the proof of the regularity specified by (2.11). To this aim, we perform the following estimate on the component θ of the solution provided by the fixed point procedure. After adding θ to both sides of (2.16), we test it by $J^{-1}\theta_t$ and integrate over $(0,t)$. By the definition of J (cf. (2.1)), using Hölder's and Young's inequalities, Sobolev's embeddings and owing to (2.13) and (2.14), we get

$$
\begin{aligned}
\|\theta_t\|^2_{L^2(0,t;V')}+\big\|\theta(t)\big\|^2_H &\leqslant c\|\theta_0\|^2_H+c\int_0^t\|\theta\|_{V'}\big\|J^{-1}\theta_t\big\|_V\\
&\quad+c\int_0^t\big(\|\chi\|_{L^\infty(\Omega)}\|\nabla u_t\|_H+\|\chi_t\|_{L^4(\Omega)}\|\nabla u\|_{L^4(\Omega)}\big)\|\theta\|_{L^4(\Omega)}\big\|J^{-1}\theta_t\big\|_V\\
&\quad+c\int_0^t\big(\|\chi_t\|_{L^4(\Omega)}\|\chi_t\|_H+\|\nabla\chi_t\|_{L^4(\Omega)}\|\nabla\chi_t\|_H+\|\nabla u_t\|_{L^4(\Omega)}\|\nabla u_t\|_H\big)\big\|J^{-1}\theta_t\big\|_V
\end{aligned}
$$

$$\begin{aligned}&\leqslant \frac{1}{2}\|\theta_t\|^2_{L^2(0,t;V')} + c\|\theta\|^2_{L^2(0,t;V)}\\&\quad\times\left(1+\|\chi\|^2_{L^\infty(Q_t)}\|u_t\|^2_{L^\infty(0,t;V_0)} + \|u\|^2_{L^\infty(0,t;W^{1,4}_0(\Omega))}\|\chi_t\|^2_{L^\infty(0,t;V)}\right)\\&\quad+c\|\chi_t\|^2_{L^\infty(0,t;V)}\|\chi_t\|^2_{L^2(0,t;W^{1,4}(\Omega))} + c\|u_t\|^2_{L^\infty(0,t;V_0)}\|u_t\|^2_{L^2(0,t;W^{1,4}_0(\Omega))}\end{aligned}\tag{3.56}$$

from which (2.11) easily follows.

Fourth step: positivity of θ.

In order to complete the proof of the existence part in Theorem 2.1, it remains to establish the positivity of the temperature. The strategy of the proof relies on providing the non-negativity of θ and a bound for the inverse of the temperature $1/\theta$ (cf. (2.12)). Preliminarily, we exploit a maximum principle argument. Thus, we test (2.16) by $-\theta^-$, θ^- denoting the negative part of θ, i.e. $\theta^- := \max\{0, -\theta\}$, and integrate over $(0,t)$. Owing to (2.6) and using Hölder's inequality, we can infer that

$$\begin{aligned}\frac{1}{2}\|\theta^-(t)\|^2_H + \|\nabla\theta^-\|^2_{L^2(0,t;H)} &\leqslant c\int_0^t \|\theta^-\|_H\|\theta^-\|_{L^4(\Omega)}\|\nabla u_t\|_{L^4(\Omega)}\\&\quad + c\int_0^t \|\theta^-\|_H\|\theta^-\|_{L^6(\Omega)}\|\chi_t\|_{L^6(\Omega)}\|\nabla u\|_{L^6(\Omega)}.\end{aligned}\tag{3.57}$$

Hence, we handle the right-hand side of (3.57) by using Young's inequality and Sobolev's embeddings. Recalling that $\|\nabla u\|_{L^6(\Omega)}$ and $\|\chi_t\|_{L^6(\Omega)}$ are bounded in $L^\infty(0,\tau)$ due to (2.13), (2.14), we get

$$\frac{1}{2}\|\theta^-(t)\|^2_H + \frac{1}{2}\|\theta^-\|^2_{L^2(0,t;V)} \leqslant c\int_0^t \left(1+\|\nabla u_t\|^2_{L^4(\Omega)}\right)\|\theta^-\|^2_H.\tag{3.58}$$

Then, since $\|\nabla u_t\|^2_{L^4(\Omega)}$ belongs to $L^1(0,\tau)$ (cf. (2.14)), we can apply to (3.58) Gronwall's lemma and deduce

$$\|\theta^-\|_{L^\infty(0,\tau;H)\cap L^2(0,\tau;V)} \leqslant 0,\tag{3.59}$$

which gives

$$\theta \geqslant 0 \quad \text{a.e. in } Q_\tau.\tag{3.60}$$

Next step is to prove (2.12) (so that combining (2.12) with (3.60) we get (2.21)). For any $\varepsilon > 0$, let us define

$$\theta_\varepsilon := (\theta-\varepsilon)^+ + \varepsilon = \max\{\theta, \varepsilon\}.\tag{3.61}$$

We choose $v = -\theta_\varepsilon^{-2}$ as test function in (2.16) and we integrate over $(0,t)$. Applying the chain rule (see [18] for a detailed justification) and observing that $\theta_\varepsilon(0) \geqslant \theta_0$ a.e. in Ω and $\nabla\theta\cdot\nabla\theta_\varepsilon = |\nabla\theta_\varepsilon|^2$ a.e. in Q_τ, we get (cf. also (2.13), (2.14))

$$\int_\Omega \theta_\varepsilon^{-1}(t) + 8\int_0^t\!\!\int_\Omega \left|\nabla\theta_\varepsilon^{-1/2}\right|^2 \leqslant \int_\Omega \theta_0^{-1} - \int_0^t\!\!\int_\Omega \theta\nabla u_t\cdot\mathbf{a}\chi\theta_\varepsilon^{-2} - \int_0^t\!\!\int_\Omega \theta\chi_t\nabla u\cdot\mathbf{a}\theta_\varepsilon^{-2}.\tag{3.62}$$

We handle the right-hand side of (3.62) recalling also that $0 \leqslant \theta \leqslant \theta_\varepsilon$ a.e. in Q_τ. We obtain

$$\begin{aligned}\int_0^t\!\!\int_\Omega \theta\nabla u_t\cdot\mathbf{a}\chi\theta_\varepsilon^{-2} &\leqslant c\int_0^t\!\!\int_\Omega |\nabla u_t|\theta_\varepsilon^{-1} \leqslant c\int_0^t \|\nabla u_t\|_{L^4(\Omega)}\left\|\theta_\varepsilon^{-1/2}\right\|_{L^4(\Omega)}\left\|\theta_\varepsilon^{-1/2}\right\|_H\\&\leqslant \int_0^t \left\|\theta_\varepsilon^{-1/2}\right\|^2_V + c\int_0^t \|\nabla u_t\|^2_{L^4(\Omega)}\left\|\theta_\varepsilon^{-1/2}\right\|^2_H.\end{aligned}\tag{3.63}$$

Analogously, we can infer that

$$\int_0^t\!\!\int_\Omega \theta\chi_t\nabla u\cdot\mathbf{a}\theta_\varepsilon^{-2} \leqslant c\int_0^t\!\!\int_\Omega |\chi_t||\nabla u|\theta_\varepsilon^{-1} \leqslant c\int_0^t \|\chi_t\|_{L^6(\Omega)}\|\nabla u\|_{L^6(\Omega)}\big\|\theta_\varepsilon^{-1/2}\big\|_{L^6(\Omega)}\big\|\theta_\varepsilon^{-1/2}\big\|_H$$

$$\leqslant \int_0^t \|\theta_\varepsilon^{-1/2}\|_V^2 + c\int_0^t \|\theta_\varepsilon^{-1/2}\|_H^2, \tag{3.64}$$

where again we have used the fact that $\|\nabla u\|_{L^6(\Omega)}$ and $\|\chi_t\|_{L^6(\Omega)}$ are bounded in $L^\infty(0,\tau)$ due to (2.13), (2.14). Next, adding $8\int_0^t \|\theta_\varepsilon^{-1/2}\|_H^2$ to both sides of (3.62), on account of (3.63) and (3.64), we have

$$\int_\Omega \theta_\varepsilon^{-1}(t) + \int_0^t \|\theta_\varepsilon^{-1/2}\|_V^2 \leqslant \int_\Omega \theta_0^{-1} + c\int_0^t \big(1+\|\nabla u_t\|_{L^4(\Omega)}^2\big)\|\theta_\varepsilon^{-1/2}\|_H^2. \tag{3.65}$$

On account of (2.6) and (2.14), we use Gronwall's lemma and we deduce

$$\big\|\theta_\varepsilon^{-1}\big\|_{L^\infty(0,\tau;L^1(\Omega))} \leqslant c. \tag{3.66}$$

The constant c in (3.66) is independent of ε, thus we can apply the monotone convergence theorem as $\varepsilon\to 0^+$ obtaining (2.12) and finally (2.21).

4. The uniqueness result

In this section we prove the uniqueness part in Theorem 2.1. Let us consider two families of solutions $(\theta_i,\chi_i,u_i,\xi_i)$, $i=1,2$, to (2.16)–(2.19) with the associated Cauchy conditions (1.22)–(1.24) defined in some interval $(0,\tau)$ and fulfilling the regularity prescribed by (2.11)–(2.15). Hence, let us denote the difference by

$$\tilde\theta=\theta_1-\theta_2,\quad \tilde\chi=\chi_1-\chi_2,\quad \tilde u=u_1-u_2,\quad \tilde\xi=\xi_1-\xi_2.$$

To prove that $\tilde\theta=\tilde\chi=\tilde u=\tilde\xi=0$, we exploit suitable contracting estimates on the solutions. Before proceeding, we introduce some useful notation. By $\tilde f$ we denote the difference of two functions f_1, f_2. Hence, there holds

$$\widetilde{fg}=f_1g_1-f_2g_2=f_1\tilde g+g_2\tilde f=g_1\tilde f+f_2\tilde g,$$

so that, simplifying notation, in the sequel we omit the subscript writing

$$\widetilde{fg}=f\tilde g+g\tilde f.$$

We first consider (2.16) written for two families of solutions, take the difference, add $\tilde\theta$ to both sides of it, and test it by $J^{-1}\tilde\theta$. After integrating over $(0,t)$, we get

$$\frac12\big\|\tilde\theta(t)\big\|_{V'}^2+\|\tilde\theta\|_{L^2(0,t;H)}^2 \leqslant \sum_{j=20}^{29}\big|I_j(t)\big|, \tag{4.1}$$

where the integrals $I_j(t)$ are treated as follows. By the definition of J (cf. (2.1)), we first have

$$I_{20}(t)=\int_0^t\!\!\int_\Omega \tilde\theta J^{-1}\tilde\theta=\int_0^t \|\tilde\theta\|_{V'}^2. \tag{4.2}$$

Moreover, using Hölder's and Young's inequalities, the uniform bound of χ (cf. (2.20)), and Sobolev's embeddings, we have

$$I_{21}(t)=\int_0^t\!\!\int_\Omega \tilde\theta\chi\mathbf{a}\cdot\nabla u_t J^{-1}\tilde\theta \leqslant c\int_0^t \|\tilde\theta\|_H\|\nabla u_t\|_{L^4(\Omega)}\big\|J^{-1}\tilde\theta\big\|_{L^4(\Omega)}$$

$$\leqslant \delta\|\tilde\theta\|_{L^2(0,t;H)}^2 + c\int_0^t \|u_t\|_{H^2(\Omega)}^2\|\tilde\theta\|_{V'}^2, \tag{4.3}$$

for a suitable positive δ to be chosen later. Note that $\|u_t\|^2_{H^2(\Omega)} \in L^1(0,\tau)$ (cf. (2.14)). Hence, we analogously proceed and we infer that

$$\begin{aligned}I_{22}(t) &= \int_0^t\int_\Omega \theta\tilde{\chi}\mathbf{a}\cdot\nabla u_t J^{-1}\tilde{\theta} \leqslant c\int_0^t \|\theta\|_H\|\tilde{\chi}\|_{L^6(\Omega)}\|\nabla u_t\|_{L^6(\Omega)}\left\|J^{-1}\tilde{\theta}\right\|_{L^6(\Omega)}\\ &\leqslant c\|\tilde{\chi}\|^2_{L^2(0,t;V)} + c\|\theta\|^2_{L^\infty(0,t;H)}\int_0^t \|u_t\|^2_{H^2(\Omega)}\|\tilde{\theta}\|^2_{V'}\\ &\leqslant c\int_0^t \|\tilde{\chi}_t\|^2_{L^2(0,s;V)}\,ds + c\int_0^t \|u_t\|^2_{H^2(\Omega)}\|\tilde{\theta}\|^2_{V'},\end{aligned}\tag{4.4}$$

where we have used the fact that $\|\theta\|_{L^\infty(0,\tau;H)}$ is bounded (cf. (2.11)). Moreover, we have

$$\begin{aligned}I_{23}(t) &= \int_0^t\int_\Omega \theta\chi\mathbf{a}\cdot\nabla\tilde{u}_t J^{-1}\tilde{\theta} \leqslant c\int_0^t \|\theta\|_{L^4(\Omega)}\|\nabla\tilde{u}_t\|_H\left\|J^{-1}\tilde{\theta}\right\|_{L^4(\Omega)}\\ &\leqslant \delta'\|\nabla\tilde{u}_t\|^2_{L^2(0,t;H)} + c\int_0^t \|\theta\|^2_V\|\tilde{\theta}\|^2_{V'},\end{aligned}\tag{4.5}$$

for a suitable positive δ' to be chosen later. Arguing similarly, we infer that

$$\begin{aligned}I_{24}(t) &= \int_0^t\int_\Omega \tilde{\theta}\chi_t\mathbf{a}\cdot\nabla u J^{-1}\tilde{\theta} \leqslant c\int_0^t \|\tilde{\theta}\|_H\|\chi_t\|_{L^\infty(\Omega)}\|\nabla u\|_{L^4(\Omega)}\left\|J^{-1}\tilde{\theta}\right\|_{L^4(\Omega)}\\ &\leqslant c\int_0^t \|\tilde{\theta}\|_H\|\chi_t\|_W\|u\|_{H^2(\Omega)}\|\tilde{\theta}\|_{V'} \leqslant \delta\|\tilde{\theta}\|^2_{L^2(0,t;H)} + c\int_0^t \|\chi_t\|^2_W\|\tilde{\theta}\|^2_{V'},\end{aligned}\tag{4.6}$$

where have used the fact that $\|u\|_{L^\infty(0,\tau;H^2(\Omega))} \leqslant c$. Moreover, we note that (2.13) yields $\|\chi_t\|^2_W \in L^1(0,\tau)$. Since (2.11)–(2.13) imply that $\|\theta\|_H\|u\|_{H^2(\Omega)}$ is bounded in $L^\infty(0,\tau)$, we deduce

$$\begin{aligned}I_{25}(t) &= \int_0^t\int_\Omega \theta\tilde{\chi}_t\mathbf{a}\cdot\nabla u J^{-1}\tilde{\theta} \leqslant c\int_0^t \|\theta\|_H\|\tilde{\chi}_t\|_{L^6(\Omega)}\|\nabla u\|_{L^6(\Omega)}\left\|J^{-1}\tilde{\theta}\right\|_{L^6(\Omega)}\\ &\leqslant \delta''\|\tilde{\chi}_t\|^2_{L^2(0,t;V)} + c\int_0^t \|\tilde{\theta}\|^2_{V'},\end{aligned}\tag{4.7}$$

where the positive constant δ'' will be suitably chosen. Now, using that fact that $\|\chi_t\|^2_{L^\infty(0,t;V)} \leqslant c$, we may infer

$$\begin{aligned}I_{26}(t) &= \int_0^t\int_\Omega \theta\chi_t\mathbf{a}\cdot\nabla\tilde{u} J^{-1}\tilde{\theta} \leqslant c\int_0^t \|\theta\|_{L^6(\Omega)}\|\chi_t\|_{L^6(\Omega)}\|\nabla\tilde{u}\|_H\left\|J^{-1}\tilde{\theta}\right\|_{L^6(\Omega)}\\ &\leqslant c\int_0^t \|\nabla\tilde{u}\|^2_H + c\int_0^t \|\theta\|^2_V\|\tilde{\theta}\|^2_{V'}\\ &\leqslant c\int_0^t \|\nabla\tilde{u}_t\|^2_{L^2(0,s;H)}\,ds + c\int_0^t \|\theta\|^2_V\|\tilde{\theta}\|^2_{V'}.\end{aligned}\tag{4.8}$$

Finally, we deal with the difference of quadratic nonlinearities. We get

$$I_{27}(t) = 2\int_0^t\int_\Omega \tilde{\chi}_t \chi_t J^{-1}\tilde{\theta} \leqslant c\int_0^t \|\tilde{\chi}_t\|_{L^4(\Omega)}\|\chi_t\|_H \big\|J^{-1}\tilde{\theta}\big\|_{L^4(\Omega)}$$
$$\leqslant \delta''\|\tilde{\chi}_t\|^2_{L^2(0,t;V)} + c\int_0^t \|\tilde{\theta}\|^2_{V'}. \tag{4.9}$$

Secondly,

$$I_{28}(t) = 2\int_0^t\int_\Omega \nabla\tilde{\chi}_t \cdot \nabla\chi_t\, J^{-1}\tilde{\theta} \leqslant c\int_0^t \|\nabla\tilde{\chi}_t\|_H \|\nabla\chi_t\|_{L^4(\Omega)} \big\|J^{-1}\tilde{\theta}\big\|_{L^4(\Omega)}$$
$$\leqslant \delta''\|\tilde{\chi}_t\|^2_{L^2(0,t;V)} + c\int_0^t \|\chi_t\|^2_W \|\tilde{\theta}\|^2_{V'}. \tag{4.10}$$

Finally, we have

$$I_{29}(t) = 2\int_0^t\int_\Omega \nabla\tilde{u}_t \cdot \nabla u_t J^{-1}\tilde{\theta} \leqslant c\int_0^t \|\nabla\tilde{u}_t\|_H \|\nabla u_t\|_{L^4(\Omega)} \big\|J^{-1}\tilde{\theta}\big\|_{L^4(\Omega)}$$
$$\leqslant \delta'\|\nabla\tilde{u}_t\|^2_{L^2(0,t;H)} + c\int_0^t \|u_t\|^2_{H^2(\Omega)} \|\tilde{\theta}\|^2_{V'}. \tag{4.11}$$

Now, combining (4.2)–(4.11) in (4.1) and choosing δ sufficiently small (e.g., $\delta \leqslant 1/4$), we eventually obtain

$$\frac{1}{2}\big\|\tilde{\theta}(t)\big\|^2_{V'} + \frac{1}{2}\|\tilde{\theta}\|^2_{L^2(0,t;H)} \leqslant c\int_0^t \big(1 + \|\theta\|^2_V + \|u_t\|^2_{H^2(\Omega)} + \|\chi_t\|^2_W\big)\|\tilde{\theta}\|^2_{V'} + 2\delta'\|\nabla\tilde{u}_t\|^2_{L^2(0,t;H)}$$
$$+ 3\delta''\|\tilde{\chi}_t\|^2_{L^2(0,t;V)} + c\int_0^t \big(\|\tilde{\chi}_t\|^2_{L^2(0,s;V)} + \|\nabla\tilde{u}_t\|^2_{L^2(0,s;H)}\big)\,ds. \tag{4.12}$$

Now, we take the difference of (2.17) written for two families of solutions and test it by $\tilde{\chi}_t$. After integrating over $(0,t)$, we have

$$\frac{1}{2}\big\|\tilde{\chi}_t(t)\big\|^2_H + \|\nabla\tilde{\chi}_t\|^2_{L^2(0,t;H)} + \frac{1}{2}\big\|\nabla\tilde{\chi}(t)\big\|^2_H + \int_0^t\int_\Omega \tilde{\xi}\,\tilde{\chi}_t \leqslant \sum_{j=30}^{32}\big|I_j(t)\big|, \tag{4.13}$$

where the integrals $I_j(t)$ will be estimate as follows. Note first that $\int_0^t\int_\Omega \tilde{\xi}\,\tilde{\chi}_t$ in the left-hand side of (4.13) is non-negative, due to the monotonicity of β. Arguing as before, we get

$$I_{30}(t) = \int_0^t\int_\Omega \nabla u \nabla\tilde{u}\,\tilde{\chi}_t \leqslant c\int_0^t \|\nabla u\|_V \|\nabla\tilde{u}\|_H \|\tilde{\chi}_t\|_V \leqslant \delta''\|\tilde{\chi}_t\|^2_{L^2(0,t;V)} + c\int_0^t \|u\|^2_{H^2(\Omega)}\|\nabla\tilde{u}\|^2_H$$
$$\leqslant \delta''\|\tilde{\chi}_t\|^2_{L^2(0,t;V)} + c\int_0^t \|\nabla\tilde{u}_t\|^2_{L^2(0,s;H)}\,ds. \tag{4.14}$$

Moreover

$$I_{31}(t)=\int_0^t\int_\Omega \tilde\theta\mathbf{a}\cdot\nabla u\,\tilde\chi_t\leqslant c\int_0^t\|\tilde\theta\|_H\|\nabla u\|_V\|\tilde\chi_t\|_V\leqslant\delta''\|\tilde\chi_t\|^2_{L^2(0,t;V)}+c_1\|\tilde\theta\|^2_{L^2(0,t;H)},\tag{4.15}$$

where c_1 depends also on $\|u\|_{L^\infty(0,\tau;H^2(\Omega))}$. Similarly

$$\begin{aligned}I_{32}(t)&=\int_0^t\int_\Omega \theta\mathbf{a}\cdot\nabla\tilde u\,\tilde\chi_t\leqslant c\int_0^t\|\theta\|_V\|\nabla\tilde u\|_H\|\tilde\chi_t\|_V\\&\leqslant\delta''\|\tilde\chi_t\|^2_{L^2(0,t;V)}+c\int_0^t\|\theta\|^2_V\|\nabla\tilde u_t\|^2_{L^2(0,s;H)}\,ds.\end{aligned}\tag{4.16}$$

Choosing in (4.14)–(4.16) δ'' sufficiently small (e.g., $\delta''\leqslant 1/4$) and adding $\|\tilde\chi_t\|^2_{L^2(0,t;H)}$ to both sides of (4.13), we deduce

$$\begin{aligned}&\frac12\big\|\tilde\chi_t(t)\big\|^2_H+\frac14\|\tilde\chi_t\|^2_{L^2(0,t;V)}+\frac12\big\|\nabla\tilde\chi(t)\big\|^2_H\\&\quad\leqslant c_1\|\tilde\theta\|^2_{L^2(0,t;H)}+c\int_0^t\big(1+\|\theta\|^2_V\big)\|\nabla\tilde u_t\|^2_{L^2(0,s;H)}+c\|\tilde\chi_t\|^2_{L^2(0,t;H)}.\end{aligned}\tag{4.17}$$

Finally, we write the difference of (2.19) written for two families of solutions and then test it by $\tilde u_t$. Integrating in time, we have

$$\frac12\big\|\tilde u_t(t)\big\|^2_H+\|\nabla\tilde u_t\|^2_{L^2(0,t;H)}=-\int_0^t\int_\Omega(\tilde\chi\nabla u+\chi\nabla\tilde u+\mathbf{a}\tilde\theta\chi+\mathbf{a}\theta\tilde\chi)\cdot\nabla\tilde u_t\geqslant\sum_{j=33}^{36}\big|I_j(t)\big|.\tag{4.18}$$

Proceeding as above we handle the right-hand side of (4.18) as follows (cf. (2.14))

$$I_{33}(t)\leqslant c\int_0^t\|\tilde\chi\|_V\|\nabla u\|_V\|\nabla\tilde u_t\|_H\leqslant\delta'\|\nabla\tilde u_t\|^2_{L^2(0,t;H)}+c\int_0^t\|\tilde\chi_t\|^2_{L^2(0,s;V)}\,ds,\tag{4.19}$$

and (cf. (2.13))

$$I_{34}(t)\leqslant\int_0^t\|\chi\|_{L^\infty(\Omega)}\|\nabla\tilde u\|_H\|\nabla\tilde u_t\|_H\leqslant\delta'\|\nabla\tilde u_t\|^2_{L^2(0,t;H)}+c\int_0^t\|\nabla\tilde u_t\|^2_{L^2(0,s;H)}\,ds.\tag{4.20}$$

Now, it remains to treat the last two integrals

$$I_{35}(t)\leqslant c\int_0^t\|\tilde\theta\|_H\|\chi\|_{L^\infty(\Omega)}\|\nabla\tilde u_t\|_H\leqslant\delta'\|\nabla\tilde u_t\|^2_{L^2(0,t;H)}+c_2\|\tilde\theta\|^2_{L^2(0,t;H)},\tag{4.21}$$

with c_2 depending also on $\|\chi\|_{L^\infty(Q_\tau)}$. Moreover

$$I_{36}(t)\leqslant c\int_0^t\|\theta\|_V\|\tilde\chi\|_V\|\nabla\tilde u_t\|_H\leqslant\delta'\|\nabla\tilde u_t\|^2_{L^2(0,t;H)}+c\int_0^t\|\theta\|^2_V\|\tilde\chi_t\|^2_{L^2(0,s;V)}\,ds.\tag{4.22}$$

For a suitable choice of δ' in (4.19)–(4.22) (e.g., $\delta'\leqslant 1/8$), (4.18) leads to

$$\frac{1}{2}\big\|\tilde{u}_t(t)\big\|_H^2+\frac{1}{2}\|\nabla\tilde{u}_t\|_{L^2(0,t;H)}\leqslant c_2\|\tilde{\theta}\|^2_{L^2(0,t;H)}+c\int_0^t\big(1+\|\theta\|_V^2\big)\|\tilde{\chi}_t\|^2_{L^2(0,s;V)}\,ds$$
$$+c\int_0^t\|\nabla\tilde{u}_t\|^2_{L^2(0,s;H)}\,ds. \tag{4.23}$$

Now, we combine (4.12), (4.17), and (4.23). More precisely, we add (4.12) to (4.17) multiplied by $1/(8c_1)$ and to (4.23) multiplied by $1/(8c_2)$. Then, we choose δ' and δ'' in (4.12) small enough (depending on c_1 and c_2) and eventually get

$$\big\|\tilde{\theta}(t)\big\|^2_{V'}+\|\tilde{\theta}\|^2_{L^2(0,t;H)}+\|\tilde{\chi}_t\|^2_{L^2(0,t;V)}+\big\|\tilde{\chi}_t(t)\big\|^2_H+\|\nabla\tilde{u}_t\|^2_{L^2(0,t;H)}+\big\|\tilde{u}_t(t)\big\|^2_H$$
$$\leqslant c\int_0^t\big(1+\|\theta\|_V^2+\|u_t\|^2_{H^2(\Omega)}+\|\chi_t\|^2_W\big)\|\tilde{\theta}\|^2_{V'}+c\|\tilde{\chi}_t\|^2_{L^2(0,t;H)}$$
$$+c\int_0^t\big(1+\|\theta\|_V^2\big)\|\tilde{\chi}_t\|^2_{L^2(0,s;V)}\,ds+c\int_0^t\big(1+\|\theta\|_V^2\big)\|\nabla\tilde{u}_t\|^2_{L^2(0,s;H)}\,ds. \tag{4.24}$$

Thus, Gronwall's lemma applied to (4.24) ensures that

$$\tilde{\theta}=\tilde{\chi}=\tilde{u}=0\quad\text{a.e. in } Q_\tau. \tag{4.25}$$

Finally, by comparison in (2.17), it follows that $\tilde{\xi}=0$ a.e. in Q_τ too which concludes the proof of the uniqueness result in Theorem 2.1.

5. The regularity result

We perform here some (formal) a priori estimates on the solution (θ,u,χ,ξ) provided by Theorem 2.1. Owing to stronger hypotheses on the initial data (see (2.22)–(2.25)), we will derive proper a priori bounds on the quadruple (θ,u,χ,ξ) in some interval $(0,\widehat{T})$, $\widehat{T}\in(0,\tau]$. Actually, we should establish the following estimates on suitable regularized version of the equations and then pass to the limit, with respect to the approximating parameters. However, for the sake of simplicity, we prefer directly proceed formally.

Now, let us test (2.16) by θ_t and integrate over $(0,t)$, with $0<t<\tau$. We get

$$\|\theta_t\|^2_{L^2(0,t;H)}+\frac{1}{2}\big\|\nabla\theta(t)\big\|^2_H\leqslant\frac{1}{2}\|\nabla\theta_0\|^2_H+\sum_{j=37}^{39}\big|I_j(t)\big|. \tag{5.1}$$

Using Hölder's and Young's inequalities, the uniform bound of χ (cf. (2.20)), and Sobolev's embeddings, we first infer that

$$I_{37}(t)=\int_0^t\!\!\int_\Omega\theta\chi\mathbf{a}\cdot\nabla u_t\theta_t\leqslant c\int_0^t\|\theta\|_{L^4(\Omega)}\|\nabla u_t\|_{L^4(\Omega)}\|\theta_t\|_H$$
$$\leqslant\delta\|\theta_t\|^2_{L^2(0,t;H)}+c\int_0^t\|u_t\|^4_{H^2(\Omega)}+c\int_0^t\|\theta\|^4_V, \tag{5.2}$$

for $\delta>0$ to be chosen later. Analogously, let us handle the second integral as follows

$$I_{38}(t)=\int_0^t\!\!\int_\Omega\theta\chi_t\mathbf{a}\cdot\nabla u\,\theta_t\leqslant c\int_0^t\|\theta\|_{L^4(\Omega)}\|\chi_t\|_{L^\infty(\Omega)}\|\nabla u\|_{L^4(\Omega)}\|\theta_t\|_H$$
$$\leqslant\delta\|\theta_t\|^2_{L^2(0,t;H)}+c\|u\|^4_{L^\infty(0,t;H^2(\Omega))}\int_0^t\|\chi_t\|^4_W+c\int_0^t\|\theta\|^4_V. \tag{5.3}$$

Next, we get

$$I_{39}(t) = \int_0^t\int_\Omega \left(|\chi_t|^2 + |\nabla\chi_t|^2 + |\nabla u_t|^2\right)\theta_t \leqslant \int_0^t \left(\|\chi_t\|^2_{L^4(\Omega)} + \|\nabla\chi_t\|^2_{L^4(\Omega)} + \|\nabla u_t\|^2_{L^4(\Omega)}\right)\|\theta_t\|_H$$
$$\leqslant \delta\|\theta_t\|^2_{L^2(0,t;H)} + c\int_0^t \|\chi_t\|^4_W + c\int_0^t \|u_t\|^4_{H^2(\Omega)}. \tag{5.4}$$

Hence, we combine (5.2)–(5.4) in (5.1), choosing δ sufficiently small (e.g., $\delta \leqslant 1/4$). Owing to (2.11), (2.14), and (2.22), we obtain

$$\|\theta_t\|^2_{L^2(0,t;H)} + \left\|\theta(t)\right\|^2_V \leqslant c + c\int_0^t \|\theta\|^4_V + c\int_0^t \|\chi_t\|^4_W + c\int_0^t \|u_t\|^4_{H^2(\Omega)}. \tag{5.5}$$

Now, we differentiate (2.19) with respect to time. We test by u_{tt} the resulting equation and integrate over $(0,t)$. On account of (2.11), (2.13), and (2.14), we have

$$\frac{1}{2}\left\|u_{tt}(t)\right\|^2_H + \|\nabla u_{tt}\|^2_{L^2(0,t;H)}$$
$$\leqslant \frac{1}{2}\left\|u_{tt}(0)\right\|^2_H - \int_0^t\int_\Omega (\chi\nabla u + \mathbf{a}\theta\chi)_t\cdot\nabla u_{tt}$$
$$= \frac{1}{2}\left\|u_{tt}(0)\right\|^2_H - \int_0^t\int_\Omega (\chi_t\nabla u + \chi\nabla u_t + \mathbf{a}\theta_t\chi + \mathbf{a}\theta\chi_t)\cdot\nabla u_{tt}$$
$$\leqslant \frac{1}{2}\left\|u_{tt}(0)\right\|^2_H + c\int_0^t \|\nabla u_{tt}\|_H\left(\|\chi_t\|_{L^4(\Omega)}\|\nabla u\|_{L^4(\Omega)} + \|\nabla u_t\|_H + \|\theta_t\|_H + \|\theta\|_{L^4(\Omega)}\|\chi_t\|_{L^4(\Omega)}\right)$$
$$\leqslant \frac{1}{2}\left\|u_{tt}(0)\right\|^2_H + \frac{1}{2}\|\nabla u_{tt}\|^2_{L^2(0,t;H)} + c\|u\|^2_{L^\infty(0,t;H^2(\Omega))}\|\chi_t\|^2_{L^2(0,t;V)}$$
$$+ c\|\theta_t\|^2_{L^2(0,t;H)} + c\|\nabla u_t\|^2_{L^2(0,t;H)} + \|\chi_t\|^2_{L^\infty(0,t;V)}\|\theta\|^2_{L^2(0,t;V)}$$
$$\leqslant \frac{1}{2}\left\|u_{tt}(0)\right\|^2_H + \frac{1}{2}\|\nabla u_{tt}\|^2_{L^2(0,t;H)} + c\|\theta_t\|^2_{L^2(0,t;H)} + c. \tag{5.6}$$

Note that $u_{tt}(0)$ is bounded in H by a comparison in (2.19), written for $t=0$, thanks to (2.7)–(2.8), (2.22), (2.23), and (2.25). Hence, using the estimate (5.5) for $\|\theta_t\|^2_{L^2(0,t;H)}$, from (5.6) we deduce

$$\left\|u_{tt}(t)\right\|^2_H + \|\nabla u_{tt}\|^2_{L^2(0,t;H)} \leqslant c + c\int_0^t \|\theta\|^4_V + c\int_0^t \|\chi_t\|^4_W + c\int_0^t \|u_t\|^4_{H^2(\Omega)}. \tag{5.7}$$

Analogously, let us differentiate (2.17) with respect to time and test the resulting equation by χ_{tt}. After integrating in time, we get

$$\frac{1}{2}\left\|\chi_{tt}(t)\right\|^2_H + \|\nabla\chi_{tt}\|^2_{L^2(0,t;H)} + \frac{1}{2}\left\|\nabla\chi_t(t)\right\|^2_H \leqslant \frac{1}{2}\left\|\chi_{tt}(0)\right\|^2_H + \frac{1}{2}\|\nabla\chi_1\|^2_H - \int_0^t\int_\Omega \nabla u\cdot\nabla u_t\,\chi_{tt}$$
$$- \int_0^t\int_\Omega \mathbf{a}\cdot(\theta_t\nabla u + \theta\nabla u_t)\,\chi_{tt}, \tag{5.8}$$

where (using a formal notation) the monotonicity of β yields

$$\beta(\chi_t)'\chi_{tt} = \beta'(\chi_t)|\chi_{tt}|^2 \geqslant 0.$$

Now, by a comparison in (2.17), written for $t = 0$, we have $\|\chi_{tt}(0)\|_H \leqslant c$, thanks to (2.7)–(2.8), (2.22), (2.23), and (2.24). Now, we aim to handle the right-hand side of (5.8). On account of (2.11), (2.13), and (2.14), we can deduce

$$\begin{aligned}\left|\int_0^t\int_\Omega \nabla u\cdot\nabla u_t\chi_{tt}\right| &\leqslant \int_0^t \|\chi_{tt}\|_{L^4(\Omega)}\|\nabla u_t\|_H\|\nabla u\|_{L^4(\Omega)}\\ &\leqslant \frac14\|\chi_{tt}\|^2_{L^2(0,t;V)} + c\int_0^t \|\nabla u_t\|^2_H\|u\|^2_{H^2(\Omega)}\\ &\leqslant \frac14\|\chi_{tt}\|^2_{L^2(0,t;V)} + c.\end{aligned}\tag{5.9}$$

Moreover

$$\begin{aligned}\left|\int_0^t\int_\Omega \mathbf{a}\cdot(\theta_t\nabla u+\theta\nabla u_t)\chi_{tt}\right| &\leqslant c\int_0^t \big(\|\theta_t\|_H\|\nabla u\|_V + \|\theta\|_V\|\nabla u_t\|_H\big)\|\chi_{tt}\|_V\\ &\leqslant \frac14\|\chi_{tt}\|^2_{L^2(0,t;V)} + c\int_0^t\big(\|\theta_t\|^2_H\|u\|^2_{H^2(\Omega)} + \|\nabla u_t\|^2_H\|\theta\|^2_V\big)\\ &\leqslant \frac14\|\chi_{tt}\|^2_{L^2(0,t;V)} + c\|\theta_t\|^2_{L^2(0,t;H)} + c.\end{aligned}\tag{5.10}$$

Combining (5.9)–(5.10) in (5.8), we eventually obtain

$$\begin{aligned}\big\|\chi_{tt}(t)\big\|^2_H + \|\chi_{tt}\|^2_{L^2(0,t;V)} &\leqslant c + c\|\theta_t\|^2_{L^2(0,t;H)}\\ &\leqslant c + c\int_0^t\|\theta\|^4_V + c\int_0^t\|\chi_t\|^4_W + c\int_0^t\|u_t\|^4_{H^2(\Omega)},\end{aligned}\tag{5.11}$$

where the last inequality in (5.11) is derived from the estimate (5.5) for $\|\theta_t\|^2_{L^2(0,t;H)}$.

Now, by comparison in (2.19), taking (2.13)–(2.14) into account, we obtain

$$\begin{aligned}\big\|\Delta u_t(t)\big\|^2_H &\leqslant c\big\|u_{tt}(t)\big\|^2_H + c\big\|\chi(t)\big\|^2_{L^\infty(\Omega)}\big\|\Delta u(t)\big\|^2_H + c\big\|\nabla\chi(t)\big\|^2_{L^4(\Omega)}\big\|\nabla u(t)\big\|^2_{L^4(\Omega)}\\ &\quad + c\big\|\nabla\chi(t)\big\|^2_{L^4(\Omega)}\big\|\theta(t)\big\|^2_{L^4(\Omega)} + c\big\|\chi(t)\big\|^2_{L^\infty(\Omega)}\big\|\nabla\theta(t)\big\|^2_H\\ &\leqslant c + c\big\|u_{tt}(t)\big\|^2_H + c\big\|\theta(t)\big\|^2_V\\ &\leqslant c + c\int_0^t\|\theta\|^4_V + c\int_0^t\|\chi_t\|^4_W + c\int_0^t\|u_t\|^4_{H^2(\Omega)},\end{aligned}\tag{5.12}$$

where the last row in (5.12) is derived from the estimate (5.5) for $\|\theta(t)\|^2_V$ and from the estimate (5.7) for $\|u_{tt}(t)\|^2_H$.

Analogously, we test (2.17) by $-\Delta\chi_t(t)$. Observe firstly that $\int_\Omega \xi(-\Delta\chi_t)$ is non-negative for a.a. t (cf. (3.17)). Then, owing to (2.13)–(2.14), (5.5) and (5.11), we get

$$\begin{aligned}\big\|\Delta\chi_t(t)\big\|^2_H &\leqslant c\big\|\chi_{tt}(t)\big\|^2_H + c\big\|\Delta\chi(t)\big\|^2_H + \big\|\nabla u(t)\big\|^4_{L^4(\Omega)} + c\big\|\nabla u(t)\big\|^2_{L^4(\Omega)}\big\|\theta(t)\big\|^2_{L^4(\Omega)} + c\\ &\leqslant c + c\int_0^t\|\theta\|^4_V + c\int_0^t\|\chi_t\|^4_W + c\int_0^t\|u_t\|^4_{H^2(\Omega)}.\end{aligned}\tag{5.13}$$

Finally, we add (5.5), (5.12), and (5.13); we apply a generalized version of Gronwall's lemma (see, e.g., [17, p. 33]) and we conclude that there exists $\widehat{T}$, with $\widehat{T} \in (0, \tau]$ such that the following upper bounds hold

$$\|\theta\|_{H^1(0,\widehat{T};H)\cap L^\infty(0,\widehat{T};V)} \leqslant c, \tag{5.14}$$

$$\|u\|_{W^{2,\infty}(0,\widehat{T};H)\cap H^2(0,\widehat{T};V_0)\cap W^{1,\infty}(0,\widehat{T};H^2(\Omega))} \leqslant c, \tag{5.15}$$

$$\|\chi\|_{W^{2,\infty}(0,\widehat{T};H)\cap H^2(0,\widehat{T};V)\cap W^{1,\infty}(0,\widehat{T};W)} \leqslant c. \tag{5.16}$$

We complete the proof of the regularity of θ and ξ observing that

$$\|\theta\|_{L^2(0,\widehat{T};W)} \leqslant c$$

by comparison in (2.16) and

$$\|\xi\|_{L^\infty(0,\widehat{T};H)} \leqslant c$$

by comparison in (2.17), owing to (5.14)–(5.16).

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