

C^1 -regularity of the Aronsson equation in \mathbf{R}^2

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Abstract

For a nonnegative, uniformly convex $H \in C^2(\mathbf{R}^2)$ with $H(0) = 0$, if $u \in C(\Omega)$, $\Omega \subset \mathbf{R}^2$, is a viscosity solution of the Aronsson equation (1.7), then $u \in C^1(\Omega)$. This generalizes the C^1 -regularity theorem on infinity harmonic functions in \mathbf{R}^2 by Savin [O. Savin, C^1 -regularity for infinity harmonic functions in dimensions two, Arch. Ration. Mech. Anal. 176 (3) (2005) 351–361] to the Aronsson equation.

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Résumé

Si $H \in C^2(\mathbf{R}^2)$ est une fonction uniformément convexe telle que $H(0) = 0$, et si $u \in C(\Omega)$, $\Omega \subset \mathbf{R}^2$, est une solution de viscosité de l'équation d'Aronsson (1.7), alors $u \in C^1(\Omega)$. Ceci généralise à l'équation d'Aronsson le théorème de C^1 -régularité de Savin [O. Savin, C^1 -régularité pour les fonctions ∞ -harmoniques dans \mathbf{R}^2 , Arch. Ration. Mech. Anal. 176 (3) (2005) 351–361] pour les fonctions ∞ -harmoniques dans \mathbf{R}^2 .

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1. Introduction

Calculus of variations in L^∞ was initiated by Aronsson [1–4] in 1960s. Thanks to both the development of theory of viscosity solutions of elliptic equations by Crandall and Lions (cf. [17]) and several applications to applied fields (cf. Barron [8], Barron and Jensen [9], Aronsson, Crandall and Juutinen [7], and Crandall [13]), there have been great interests in the last few years to study the minimization problem of the *supremal* functional:

$$F(u, \Omega) = \operatorname{ess\,sup}_{x \in \Omega} H(x, u(x), \nabla u(x)), \quad \Omega \subset \mathbf{R}^n, \quad u \in W^{1,\infty}(\Omega, \mathbf{R}^l). \quad (1.1)$$

Barron, Jensen and Wang [10] have established both necessary and sufficient conditions for the sequentially lower semicontinuity property of the supremal functional F in $W^{1,\infty}$, which are suitable L^∞ -versions of Morrey's qua-

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siconvexity (cf. [20]) for integral functionals. In the scalar case ($l = 1$), Barron, Jensen and Wang [11] (see also Crandall [12]) have established, under appropriate conditions, the existence of *absolute minimizers* and proved that any absolute minimizer is a viscosity solution of the Aronsson equation

$$-\nabla_x(H(x, u(x), \nabla u(x))) \cdot H_p(x, u(x), \nabla u(x)) = 0, \quad x \in \Omega. \quad (1.2)$$

Among other results in [22], the second author has showed that the convexity of $H(\cdot, p)$ are sufficient for viscosity solutions of the Aronsson equation (1.2) to be absolute minimizers of F .

Partially motivated by [22] and Crandall, Evans and Garipey [16], Garipey, Wang and Yu [18] have established the equivalence between absolute minimizers of F and viscosity solutions of Aronsson equation (1.2) for quasiconvex Hamiltonians $H = H(p) \in C^2(\mathbf{R}^n)$ by introducing the comparison principle of *generalized cones* (see Theorem 2.1 in Section 2 below).

In this paper, we are mainly interested in the regularity (e.g. differentiability or C^1) of viscosity solutions of the Aronsson equation.

Before stating the main results, we would like to review some of previous results for $H(p) = |p|^2$, $p \in \mathbf{R}^n$. It is well known (cf. [5,6,19]) that Eq. (1.2) is the infinity-Laplace equation, of which a viscosity solution is called an infinity harmonic function,

$$-\Delta_\infty u := -\sum_{i,j=1}^n u_i u_j u_{ij} = 0, \quad \text{in } \Omega, \quad (1.3)$$

and the absolute minimality is called absolute minimal Lipschitz extension (or AMLE) property:

for any open subset $U \Subset \Omega$ and $v \in W^{1,\infty}(U) \cap C(\bar{U})$ with $v = u$ on ∂U , we have

$$\|\nabla u\|_{L^\infty(U)} \leq \|\nabla v\|_{L^\infty(U)}. \quad (1.4)$$

Aronsson [5] proved that any C^2 -infinity harmonic function satisfies the AMLE property. Jensen [19] has proved the equivalence between an infinity harmonic function and the AML property, and the unique solvability of the Dirichlet problem of Eq. (1.3).

Crandall, Evans and Garipey [16] have recently showed that $u \in C^0(\Omega)$ is an infinity harmonic function iff u enjoys comparison with cones in Ω :

for any open subset $U \Subset \Omega$, $a, b \in \mathbf{R}$, and $x_0 \in \Omega$,

$$u(x) \leq (\geq) a + b|x - x_0| \quad \text{on } \partial(U \setminus \{x_0\}) \Rightarrow u(x) \leq (\geq) a + b|x - x_0| \quad \text{in } U. \quad (1.5)$$

In a very recent important paper [21], Savin has utilized [16] and Crandall and Evans [15] to prove that *any C^0 -infinity harmonic function in $\Omega \subset \mathbf{R}^2$ is in $C^1(\Omega)$* .

In this paper, we extend the main theorems of [21] to the Aronsson equation for a class of Hamiltonian functions $H \in C^2(\mathbf{R}^2)$.

First we extend [15] and obtain the following theorem on blow-up limits of viscosity solutions of the Aronsson equation on \mathbf{R}^n for $n \geq 2$, which may have its own interests.

Theorem A. Assume that $H \in C^2(\mathbf{R}^n)$ is nonnegative and uniformly convex, i.e. there is $\alpha_H > 0$ such that

$$p^T \cdot H_{pp}(p) \cdot p \geq \alpha_H |p|^2, \quad \forall p \in \mathbf{R}^n, \quad (1.6)$$

and $H(0) = 0$. Suppose that $u \in C^0(\Omega)$, $\Omega \subset \mathbf{R}^n$, is a viscosity solution of the Aronsson equation:

$$-H_p(\nabla u(x)) \otimes H_p(\nabla u(x)) : \nabla^2 u(x) = 0, \quad \text{in } \Omega, \quad (1.7)$$

then for any $x \in \Omega$, there exists a $e_{x,r} \in \mathbf{R}^n$, with $H(e_{x,r}) = S^+(H, u, x)$ (see Section 2 for the definition of $S^+(H, u, x)$), such that

$$\lim_{r \rightarrow 0} \max_{B_r(x)} \frac{|u(y) - u(x) - e_{x,r} \cdot (y - x)|}{r} = 0. \quad (1.8)$$

Based on Theorem A and the main theorem of [18], we are able to make necessary modifications of the idea of [21] to prove

Theorem B. Under the same assumptions of Theorem A. If $u \in C^0(\Omega)$, $\Omega \subset \mathbf{R}^2$, is a viscosity solution of the Aronsson equation (1.7), then $u \in C^1(\Omega)$.

A compactness argument from the proof of Theorem B yields

Theorem C. Under the same assumptions of Theorem A. There exists a monotonically nondecreasing function $\rho : [0, 1] \rightarrow \mathbf{R}_+$, with $\lim_{r \rightarrow 0} \rho(r) = 0$, such that if u is a viscosity solution of the Aronsson equation (1.7) in $B_1 \subset \mathbf{R}^2$, with $\|H(\nabla u)\|_{L^\infty(B_1)} \leq C$, then

$$\text{osc}_{B_r} |\nabla u| \leq \rho(r), \quad \forall r \leq \frac{1}{2}. \tag{1.9}$$

A direct consequence of Theorem C is the following Liouville property.

Theorem D. Under the same assumptions of Theorem A. If $u \in C^0(\mathbf{R}^2)$ is a viscosity solution of the Aronsson equation (1.7) in \mathbf{R}^2 and

$$|u(x)| \leq C(1 + |x|),$$

then u is linear.

Due to the local Lipschitz continuity for viscosity solutions of (1.7), the uniform convexity condition (1.6) can be slightly relaxed. In fact, we have the following remark.

Remark E. Theorems B, C, and D remain to be true, provided that $H \in C^2(\mathbf{R}^2)$ satisfies:

- (i) $H(0) = 0$ and $H(p) \geq 0$ for any $p \in \mathbf{R}^2$.
- (ii) H is coercive, i.e. $\lim_{|p| \rightarrow +\infty} H(p) = +\infty$.
- (iii) $(H_{p_i p_j})(p)$ is positive definite for any $p \in \mathbf{R}^2$.

The paper is written as follows. In Section 2, we review the main theorems of [18]. In Section 3, we outline some basic properties of the Aronsson equation (1.7) and prove of Theorem A. In Section 4, we establish three key propositions which are needed to prove Theorem B. In Section 5, we prove both Theorems B, C, D, and Remark E.

2. Review of the main theorems by [18]

In this section, we review the main theorems of [18] which are needed for the paper. We assume that the Hamiltonian function $H(x, u, p) = H(p) \in C^2(\mathbf{R}^n)$ depends only on the p -variable. First, we recall a few definitions.

Definition 2.1. A function $u \in W_{\text{loc}}^{1,\infty}(\Omega)$ is an absolute minimizer of the supremal functional F , if for any open subset $U \Subset \Omega$ and any $v \in W^{1,\infty}(U) \cap C(\bar{U})$ with $v = u$ on ∂U , then

$$F(u, U) = \|H(\nabla u(x))\|_{L^\infty(U)} \leq F(v, U) = \|H(\nabla v(x))\|_{L^\infty(U)}. \tag{2.1}$$

Definition 2.2. For any $k \geq 0$, the generalized cone centered at the origin with slope k , $C_k^H(x)$, is defined by

$$C_k^H(x) = \max_{\{p \in \mathbf{R}^n : H(p) = k\}} p \cdot x, \quad x \in \mathbf{R}^n, \tag{2.2}$$

where \cdot is the inner product in \mathbf{R}^n .

Definition 2.3. A upper semicontinuous function $u \in \text{USC}(\Omega)$ enjoys comparison with generalized cones from above (abbreviated $u \in \text{CGCA}(\Omega)$) if we have, for any $V \Subset \Omega$, $x_0 \in \Omega$, and $k \geq 0$,

$$u(x) \leq u(x_0) + C_k^H(x - x_0) \quad \forall x \in \partial V \Rightarrow u(x) \leq u(x_0) + C_k^H(x - x_0), \quad \forall x \in V. \tag{2.3}$$

A lower semicontinuous function $v \in \text{LSC}(\Omega)$ enjoys comparison with generalized cones from below (abbreviated $v \in \text{CGCB}(\Omega)$) if $-v$ enjoys comparison with generalized cones from above, with H replaced by \hat{H} ($\hat{H}(p) \equiv H(-p)$ for all $p \in \mathbf{R}^n$). A continuous function $w \in C(\Omega)$ enjoys comparison with generalized cones (or abbreviated $w \in \text{CGC}(\Omega)$) if $w \in \text{CGCA}(\Omega) \cap \text{CGCB}(\Omega)$.

Definition 2.4. A function $u \in \text{USC}(\Omega)$ is a viscosity subsolution of the Aronsson equation (1.7), if for any $(x_0, \phi) \in \Omega \times C^2(\Omega)$

$$u(x_0) - \phi(x_0) \geq u(x) - \phi(x), \quad \forall x \in \Omega$$

then

$$-H_p(\nabla\phi) \otimes H_p(\nabla\phi) : \nabla^2\phi|_{x=x_0} \leq 0. \quad (2.4)$$

A function $u \in \text{LSC}(\Omega)$ is a viscosity supersolution of the Aronsson equation (1.7), if $-u$ is a viscosity subsolution of (1.7) with H replaced by \hat{H} , where $\hat{H}(p) = H(-p)$ for $p \in \mathbf{R}^n$. A function $u \in C(\Omega)$ is a viscosity solution of the Aronsson equation (1.7), if u is both a viscosity subsolution and supersolution of (1.7).

Now we state the main theorem proved by [18] on the equivalence among absolute minimizers, viscosity solutions of the Aronsson equation (1.7), and comparison with generalized cones.

Theorem 2.1. (See [18].) Assume that H satisfies:

H1. $H \in C^2(\mathbf{R}^n)$, $H(0) = 0$, and $H(p) \geq 0$ for any $p \in \mathbf{R}^n$.

H2. H is quasi-convex, i.e.

$$H(tp + (1-t)q) \leq \max\{H(p), H(q)\}, \quad \forall p, q \in \mathbf{R}^n, 0 \leq t \leq 1. \quad (2.5)$$

H3. H is coercive:

$$\lim_{|p| \rightarrow +\infty} H(p) = +\infty. \quad (2.6)$$

Then the following three statements are equivalent:

(a) $u \in W_{\text{loc}}^{1,\infty}(\Omega)$ is an absolute minimizer of $F(v, \cdot) = \|H(\nabla v)\|_{L^\infty(\cdot)}$.

(b) $u \in \text{CGC}(\Omega)$.

(c) $u \in C(\Omega)$ is a viscosity solution of the Aronsson equation (1.7).

In particular, Theorem 2.1 implies that any viscosity solution of (1.7) enjoys comparison with $C_k^H(\cdot)$. This plays a crucial role in the paper.

In [18] Proposition 3.1, the following properties of $\text{CGC}(\Omega)$ have been established.

Proposition 2.2. (See [18].) Under the same assumptions as in Theorem 2.1. If $u \in \text{CGC}(\Omega)$, then $u \in W_{\text{loc}}^{1,\infty}(\Omega)$, and, for any $x_0 \in \Omega$ and $0 < r < d(x_0, \partial\Omega)$, we have (i)

$$S_r^+(H, u, x_0) := \inf\{k \geq 0 \mid u(x) \leq u(x_0) + C_k^H(x - x_0), \forall x \in \partial B_r(x_0)\} \quad (2.7)$$

exists and is monotonically nondecreasing with respect to $r > 0$. In particular,

$$S^+(H, u, x_0) = \lim_{r \downarrow 0^+} S_r^+(H, u, x_0) \quad (2.8)$$

exists for any $x_0 \in \Omega$ and is upper semicontinuous; (ii) $S_r^-(H, u, x_0) := S_r^+(\hat{H}, -u, x_0)$ exists and is monotonically nondecreasing with respect to $r > 0$, where $\hat{H}(p) = H(-p)$ for $p \in \mathbf{R}^n$. In particular, $S^-(H, u, x_0) = \lim_{r \downarrow 0^+} S_r^-(H, u, x_0)$ exists for any $x_0 \in \Omega$ and is upper semicontinuous; and (iii) for any $x_0 \in \Omega$, we have $S^+(H, u, x_0) = S^-(H, u, x_0)$. Moreover, if $\nabla u(x_0)$ exists for $x_0 \in \Omega$, then we have

$$H(\nabla u(x_0)) = S^+(H, u, x_0) = S^-(H, u, x_0). \quad (2.9)$$

Proposition 2.3. (See [18].) Assume $u \in \text{CGC}(\Omega)$. For any $x_0 \in \Omega$, $0 < r < d(x_0, \partial\Omega)$, if $x_r \in \partial B_r(x_0)$ satisfies

$$u(x_r) - u(x_0) = C_{S_r^+(H,u,x_0)}^H(x_r - x_0) \tag{2.10}$$

then for any $0 < R < d(x_r, \partial\Omega)$ we have

$$S_R^+(H, u, x_r) \geq S_r^+(H, u, x_0). \tag{2.11}$$

For this paper, we also need the comparison property of viscosity solutions for (1.7) with linear functions.

Proposition 2.4. Under the same assumptions as in Theorem 2.1, let $\psi(x) = a \cdot x + b$ ($a \in \mathbf{R}^n$, $b \in \mathbf{R}$) be a linear function such that if $a \neq 0$ then $H_p(a) \neq 0$. For any open subset $U \Subset \Omega$, if $u \in C^0(\Omega)$ is a viscosity solution of the Aronsson equation (1.7) and $u \leq \psi$ on ∂U , then $u \leq \psi$ in U .

Proof. If $a = 0$, then $\psi = b$ so that $\psi(x) = b + C_0^H(x)$ is a generalized cone with slope $k = 0$. Hence Proposition 2.4 follows from Theorem 2.1.

If $a \neq 0$, then we have $|H_p(a)| = \delta > 0$ for some $\delta > 0$. Suppose that the conclusion were false, then there exist an open subset $\hat{V} \Subset \Omega$ and $\hat{x} \in V$ such that $u = \psi$ on $\partial \hat{V}$, and

$$\max_{\hat{V}}(u - \psi) = (u - \psi)(\hat{x}) > 0.$$

Let $R > 0$ be such that $V \subset B_R$, and $\epsilon > 0$ be sufficiently small. Then we have

$$u(x) \leq \psi(x) + \epsilon(R^2 - |x|^2) \quad \text{on } \partial \hat{V},$$

but

$$u(\hat{x}) > \psi(\hat{x}) + \epsilon(R^2 - |\hat{x}|^2).$$

For simplicity, we may assume

$$\{u - (\psi + \epsilon(R^2 - |x|^2))\}(\hat{x}) = \max_{\hat{V}}(u - (\psi + \epsilon(R^2 - |x|^2))).$$

Since $\psi + \epsilon(R^2 - |x|^2) \in C^2(\hat{V})$ and u is a viscosity solution of (1.7), we have

$$0 \geq - \sum_{i,j} H_{p_i}(a - 2\epsilon\hat{x}) H_{p_j}(a - 2\epsilon\hat{x}) (-2\epsilon\delta_{ij}) = 2\epsilon |H_p(a - 2\epsilon\hat{x})|^2. \tag{2.12}$$

This is impossible, since

$$|H_p(a - 2\epsilon\hat{x})| \geq \frac{1}{2} |H_p(a)| = \frac{\delta}{2} > 0$$

if $\epsilon > 0$ is sufficiently small. This completes the proof of Proposition 2.4. \square

3. Some preliminary results of (1.7) and proof of Theorem A

This section is devoted to the proof of Theorem A. To do it, we need several lemmas. The first lemma asserts that the assumption of Theorem A implies that of Theorem 2.1.

Lemma 3.1. Suppose that $H \in C^2(\mathbf{R}^n)$ is a nonnegative, uniformly convex function, with $H(0) = 0$. Then H satisfies the conditions (H1)–(H3) in Theorem 2.1. Moreover, $H(p) > 0$ and $H_p(p) \neq 0$ for all $0 \neq p \in \mathbf{R}^n$.

Proof. Since a uniformly convex function is quasi-convex, H satisfies H2. Since H is uniformly convex, there exists a $\alpha_H > 0$ such that

$$H_{pp}(p) \geq \alpha_H I_n, \quad \forall p \in \mathbf{R}^n. \tag{3.1}$$

This, combined with the Taylor expansion up to second order, implies

$$H(p) \geq H(q) + H_p(q) \cdot (p - q) + \frac{\alpha_H}{2} |p - q|^2, \quad \forall p, q \in \mathbf{R}^n. \tag{3.2}$$

It follows from $0 = H(0) = \min_{p \in \mathbf{R}^n} H(p)$ that $H_p(0) = 0$. Therefore, by (3.2), we have

$$H(p) \geq \frac{\alpha_H}{2} |p|^2, \quad \forall p \in \mathbf{R}^n,$$

this yields H3.

Suppose that there exists $0 \neq p_0 \in \mathbf{R}^n$ such that $H(p_0) = 0$. Then the convexity and nonnegativity of H imply $H(tp_0) = 0$ for all $0 \leq t \leq 1$. This implies

$$0 = \frac{d^2}{dt^2} \Big|_{t=0^+} H(tp_0) = p_0^T \cdot H_{pp}(0) \cdot p_0 \geq \alpha_H |p_0|^2,$$

this is impossible.

Suppose that there exists $0 \neq p_1 \in \mathbf{R}^n$ such that $H(p_1) > 0$, but $H_p(p_1) = 0$. Then (3.2) implies

$$0 = H(0) \geq H(p_1) - H_p(p_1) \cdot p_1 = H(p_1).$$

This is also impossible. Hence the proof of Lemma 3.1 is complete. \square

Denote by $\mathbf{S}^{n-1} \subset \mathbf{R}^n$ the unit sphere of \mathbf{R}^n , and let $H^{-1}(c) = \{p \in \mathbf{R}^n : H(p) = c\}$ ($c > 0$) be the level set of H . Now we have

Lemma 3.2. *Under the same assumptions as in Lemma 3.1. Then for any $c > 0$, the Gauss map $v(p) = \frac{H_p(p)}{|H_p(p)|} : H^{-1}(c) \rightarrow \mathbf{S}^{n-1}$ is a C^1 diffeomorphism.*

Proof. It follows from Lemma 3.1 and the implicit function theorem that for any $c > 0$, $H^{-1}(c) \subset \mathbf{R}^n$ is a strictly convex C^2 -hypersurface. Hence, by (3.1), $v \in C^1(H^{-1}(c), \mathbf{S}^{n-1})$ is one to one, and onto. In particular, v is a C^1 -diffeomorphism. \square

Lemma 3.3. *Under the same assumptions as in Lemma 3.1. For any $k \geq 0$, $C_k^H(\cdot)$ is convex and positively homogeneous of degree one. Moreover, $C_k^H(x) \in C^1(\mathbf{R}^n \setminus \{0\})$.*

Proof. It follows from the definition that $C_k^H(\cdot)$ is the supremum of a family of linear functions so that it convex and satisfies

$$C_k^H(tx) = tC_k^H(x), \quad \forall t > 0, \forall x \in \mathbf{R}^n.$$

To see $C_k^H \in C^1(\mathbf{R}^n \setminus \{0\})$, it suffices to prove that for $k > 0$ $C_k^H \in C^1(\mathbf{S}^{n-1})$. By the Lagrange Multiple Theorem, for any $x \in \mathbf{S}^{n-1}$ there exists a $\hat{p} \in H^{-1}(k)$ such that

$$C_k^H(x) = \hat{p} \cdot x, \quad \text{and} \quad \frac{H_p(\hat{p})}{|H_p(\hat{p})|} = x \tag{3.3}$$

so that $v(\hat{p}) = x$ or $\hat{p} = v^{-1}(x) \in C^1(\mathbf{S}^{n-1})$. Therefore $C_k^H(x) = v^{-1}(x) \cdot x \in C^1(\mathbf{S}^{n-1})$. \square

Lemma 3.4. *Under the same assumptions as in Lemma 3.1. If $u \in C^0(\mathbf{R}^n)$ is a viscosity solution of the Aronsson equation (1.7) satisfying $S^+(H, u, 0) = \|H(\nabla u)\|_{L^\infty(\mathbf{R}^n)} < \infty$, then there exists $p_0 \in \mathbf{R}^n$, with $H(p_0) = S^+(H, u, 0)$, such that*

$$u(x) = u(0) + p_0 \cdot x, \quad \forall x \in \mathbf{R}^n. \tag{3.4}$$

Proof. First we claim that

$$S_r^+(H, u, 0) = S^+(H, u, 0), \quad \forall r > 0, \tag{3.5}$$

$$S_R^+(H, u, y) \leq S^+(H, u, 0), \quad 0 \neq y \in \mathbf{R}^n, \forall R > 0. \tag{3.6}$$

To see (3.6), observe that for any $y \in \mathbf{R}^n$ and $x \in \mathbf{R}^n$, with $|x - y| = R$, we have

$$\begin{aligned} u(x) &= u(y) + \int_0^1 \nabla u(tx + (1-t)y) \cdot (x-y) dt \\ &\leq u(y) + \int_0^1 \max_{H(p) \leq \|H(\nabla u)\|_{L^\infty}} p \cdot (x-y) \\ &= u(y) + C_{\|H(\nabla u)\|_{L^\infty}}^H(x-y). \end{aligned}$$

This implies

$$S_R^+(H, u, y) \leq \|H(\nabla u)\|_{L^\infty(\mathbf{R}^n)} = S^+(H, u, 0).$$

It is easy to see that (3.5) follows from (3.6) and the fact $S_r^+(H, u, 0) \geq S^+(H, u, 0)$.

If $S^+(H, u, 0) = 0$, then $H(\nabla u(x)) = 0$ for a.e. $x \in \mathbf{R}^n$. Hence $\nabla u(x) = 0$ for a.e. $x \in \mathbf{R}^n$ so that u is constant and (3.4) holds with $p_0 = 0$.

We may assume that $S^+(H, u, 0) > 0$. For $R > 1$, there exists $x_R \in \partial B_R$ such that

$$u(x_R) = u(0) + C_{S^+(H,u,0)}^H(x_R). \tag{3.7}$$

On the other hand, for any $x \in \mathbf{R}^n$, we have, by (3.6),

$$u(x_R) - u(x) \leq C_{S^+(H,u,0)}^H(x_R - x) \leq C_{S^+(H,u,0)}^H(x_R - x). \tag{3.8}$$

(3.7) and (3.8) imply

$$u(x) \geq u(0) + |x_R| \left\{ C_{S^+(H,u,0)}^H\left(\frac{x_R}{|x_R|}\right) - C_{S^+(H,u,0)}^H\left(\frac{x_R - x}{|x_R|}\right) \right\}. \tag{3.9}$$

For any $x \in \mathbf{R}^n$, let $R > 1$ be sufficiently large so that $x_R - x \neq 0$. By Lemma 3.3, we have that $C_{S^+(H,u,0)}^H \in C^1(\mathbf{R}^n \setminus \{0\})$ is convex. Hence we have

$$C_{S^+(H,u,0)}^H\left(\frac{x_R}{|x_R|}\right) - C_{S^+(H,u,0)}^H\left(\frac{x_R - x}{|x_R|}\right) \geq \nabla C_{S^+(H,u,0)}^H\left(\frac{x_R - x}{|x_R|}\right) \cdot \frac{x}{|x_R|}. \tag{3.10}$$

Combining (3.9) with (3.10), we obtain

$$u(x) \geq u(0) + \nabla C_{S^+(H,u,0)}^H\left(\frac{x_R - x}{|x_R|}\right) \cdot x, \quad \forall x \in \mathbf{R}^n. \tag{3.11}$$

We may assume that there exists a $q \in \mathbf{S}^{n-1}$ such that $\lim_{R \rightarrow \infty} \frac{x_R}{|x_R|} = q$. It is clear that $\lim_{R \rightarrow \infty} \frac{x_R - x}{|x_R|} = q$. Hence (3.11) yields

$$u(x) \geq u(0) + \nabla C_{S^+(H,u,0)}^H(q) \cdot x, \quad \forall x \in \mathbf{R}^n. \tag{3.12}$$

Applying the same argument with u and H replaced by $-u$ and $\hat{H}(p) = H(-p)$ for $p \in \mathbf{R}^n$, we have that there is a $\hat{q} \in \mathbf{S}^{n-1}$ such that

$$u(x) \leq u(0) + \nabla C_{S^+(H,u,0)}^H(\hat{q}) \cdot x, \quad \forall x \in \mathbf{R}^n. \tag{3.13}$$

Comparing (3.12) with (3.13), we conclude that

$$\nabla C_{S^+(H,u,0)}^H(q) = \nabla C_{S^+(H,u,0)}^H(\hat{q}) = p_0$$

and

$$u(x) = u(0) + p_0 \cdot x, \quad \forall x \in \mathbf{R}^n.$$

It is easy to see that $H(p_0) = S^+(H, u, 0)$. Hence the proof of Lemma 3.4 is complete. \square

Now we are ready to give the proof of Theorem A.

Proof of Theorem A. For simplicity, we assume that $x_0 = 0$ and $u(0) = 0$. It follows from Theorem 2.1 and Proposition 2.2 that $u \in W_{loc}^{1,\infty}(\Omega)$. For any $r_k \downarrow 0$, let $v_k(x) = \frac{u(r_k x)}{r_k} : B_{r_k^{-1}} \rightarrow \mathbf{R}$. Then for any $R > 0$, we have

$$\|v_k\|_{L^\infty(B_R)} \leq \|\nabla u\|_{L^\infty(B_{r_k R})} R \leq CR, \quad \|\nabla v_k\|_{L^\infty(B_R)} \leq \|\nabla u\|_{L^\infty(B_{r_k R})} \leq C. \tag{3.14}$$

Therefore we may assume that there exists $v_\infty \in L_{loc}^\infty(\mathbf{R}^n)$, with $\nabla v_\infty \in L^\infty(\mathbf{R}^n)$, such that for any $R > 0$

$$\|v_k - v_\infty\|_{C^0(B_R)} \rightarrow 0, \quad \nabla v_k \rightarrow \nabla v_\infty \text{ weak}^* \text{ in } L^\infty(B_R). \tag{3.15}$$

Since v_k is a viscosity solution of (1.7) on $B_{r_k^{-1}}$, we see that v_∞ is a viscosity solution of (1.7) on \mathbf{R}^n . Moreover, by the lower semicontinuity (cf. [10]), we have

$$\begin{aligned} \|H(\nabla v_\infty)\|_{L^\infty(B_R)} &\leq \liminf_{k \rightarrow \infty} \|H(\nabla v_k)\|_{L^\infty(B_R)} = \liminf_{k \rightarrow \infty} \|H(\nabla u)\|_{L^\infty(B_{r_k R})} \\ &= \liminf_{k \rightarrow \infty} \|S^+(H, u, x)\|_{L^\infty(B_{r_k R})} = S^+(H, u, 0). \end{aligned} \tag{3.16}$$

In particular, we have

$$S^+(H, v_\infty, 0) \leq \|H(\nabla v_\infty)\|_{L^\infty(\mathbf{R}^n)} \leq S^+(H, u, 0). \tag{3.17}$$

On the other hand, for any $r > 0$ there exists a $x_r^k \in \partial B_r$ such that

$$v_k(x_r^k) - v_k(0) = C_{S_r^+(H, v_k, 0)}^H(x_r^k). \tag{3.18}$$

It is readily seen that

$$S_r^+(H, v_k, 0) = S_{r_k r}^+(H, u, 0) \geq S^+(H, u, 0). \tag{3.19}$$

After passing to subsequences, we may assume that there exists $x_r \in \partial B_r$ such that $\lim_{k \rightarrow \infty} x_r^k = x_r$. Then (3.18) and (3.19) imply

$$v_\infty(x_r) - v_\infty(0) \geq C_{S^+(H, u, 0)}^H(x_r). \tag{3.20}$$

This implies

$$S_r^+(H, v_\infty, 0) \geq S^+(H, u, 0), \quad \forall r > 0. \tag{3.21}$$

Taking r into zero, (3.17) and (3.21) imply

$$S^+(H, v_\infty, 0) = \|H(\nabla v_\infty)\|_{L^\infty(\mathbf{R}^n)} = S^+(H, u, 0). \tag{3.22}$$

Hence Lemma 3.4 implies that $v_\infty = a \cdot x$, for some $a \in \mathbf{R}^n$, and we have

$$\lim_{r_k \rightarrow 0} \left\| \frac{u(r_k x) - u(0)}{r_k} - a \cdot x \right\|_{L^\infty(B_1)} = 0. \tag{3.23}$$

This implies (1.8). Suppose not, then there exists $r_k \downarrow 0^+$, $x_k \in \mathbf{R}^n$ with $|x_k| = r_k$, and $\delta > 0$ such that for any $e \in H^{-1}(S^+(H, u, 0))$, we have

$$\frac{|u(x_k) - e \cdot x_k|}{r_k} \geq \delta, \quad \forall k \geq 1. \tag{3.24}$$

But, on the other hand, there exists $a \in H^{-1}(S^+(H, u, 0))$ such that after passing to subsequences, $v_k(x) \equiv \frac{u(x_k x)}{r_k} \rightarrow a \cdot x$ uniformly on B_1 . This contradicts (3.24). Therefore the proof of Theorem A is complete. \square

4. Propositions

This section is to establish the differentiability for any viscosity solution of the Aronsson equation (1.7) by showing that there is a unique blow-up limit at any point for $n = 2$: for any $x_0 \in \Omega$, $\lim_{r \downarrow 0} e_{x_0, r}$ exists and its limit is continuous with respect to x_0 .

To do this, we follow the clever idea of [21] and the strategy is as follows. Consider the generic case that $x_0 \in \Omega$ such that $S^+(H, u, x_0) > 0$ and u is nonlinear near x_0 . Then there exists $r_0 > 0$ and a linear segment $[y_0, z_0] \subset B_{r_0/6}(x_0)$ and a linear function $l(x) = a_0 \cdot x + b_0$ such that for some $x_0 \in (y_0, z_0)$, one may

have either (i) $u(x) \geq l(x), \forall x \in [y_0, z_0]; u(x_0) = l(x_0); u(y_0) > l(y_0), u(z_0) > l(z_0)$, or (ii) $u(x) \leq l(x), \forall x \in [y_0, z_0]; u(x_0) = l(x_0); u(y_0) < l(y_0), u(z_0) < l(z_0)$. Assume the first case happens, then the first observation is that there is an affine plane P containing the graph of l such that y_0, z_0 belong to two different connected components of $\{u > P\} \cap B_{r_0}(x_0)$ (see Lemma 4.2 below). Since Theorem A implies that there exists $e_{x_0,r} \in \mathbf{R}^2$ such that $\max_{x \in B_r(x_0)} \frac{|u(x) - u(x_0) - e_{x_0,r} \cdot (x - x_0)|}{r} = o(1)$ for small $r > 0$, we want to prove $|e_{x_0,r} - a_0| = o(1)$ for small $r > 0$ (see Lemmas 4.3 and 4.4 below). The proof of Lemma 4.5 is different from that of [21] Lemma 3, here we use the discrete gradient flow which has originated from [15] when they studied the regularity of the infinity Laplace equation. A continuous version of this type of gradient flow by [15] was communicated to us by Crandall [14].

The following proposition is the crucial step to prove Theorem B.

Proposition 4.1. *Under the same assumptions as in Theorem A, let $u \in C^{0,1}(B_6)$, $B_6 \subset \mathbf{R}^2$, be a viscosity solution of the Aronsson equation (1.7). For any $\epsilon > 0$, there is $\delta = \delta(\epsilon, H, \|\nabla u\|_{L^\infty(B_6)}) > 0$ such that if*

$$\|u - e_6 \cdot x\|_{L^\infty(B_6)} \leq \delta, \quad 1 \leq H(e_6) \leq 2, \tag{4.1}$$

and if, for $0 < r \leq \frac{1}{2}$,

$$\|u - e_{0,6r} \cdot x\|_{L^\infty(B_{6r})} \leq \delta r, \tag{4.2}$$

for some $e_{0,6r} \in \mathbf{R}^2$ with $H(e_{0,6r}) = S^+(H, u, 0)$. Then

$$|e_6 - e_{0,6r}|^2 \leq C\epsilon \tag{4.3}$$

for some $1 \leq C = C(H) < \infty$.

Proof. We divide the proof into two cases.

Case A. *u is nonlinear in B_r :* There are a line segment $[z_1, z_2] \subset B_r$, a linear function $l(x) = a_0 \cdot x + b_0$, $x \in [z_1, z_2]$ with $a_0 = \frac{u(z_2) - u(z_1)}{|z_2 - z_1|}$, and a $z_3 \in (z_1, z_2)$ such that either (1)

$$u \geq l \quad \text{on } [z_1, z_2], \quad u(z_1) > l(z_1), \quad u(z_3) = l(z_3), \quad u(z_2) > l(z_2),$$

or (2)

$$u \leq l \quad \text{on } [z_1, z_2], \quad u(z_1) < l(z_1), \quad u(z_3) = l(z_3), \quad u(z_2) < l(z_2).$$

For simplicity, assume that (1) holds. Then we have

Lemma 4.2. *Under the same notations. Then there exists $e \in \mathbf{R}^2$, with $H(e) = S^+(H, u, z_3)$, such that z_1 and z_2 belong to two distinct connected components of the set $\{y \in \mathbf{R}^2: u(y) > u(z_3) + e \cdot (y - z_3)\} \cap B_6$.*

Proof of Lemma 4.2. It follows from Theorem A that for $s_k \downarrow 0$ there exists $e \in \mathbf{R}^2$, with $H(e) = S^+(H, u, z_3)$, such that

$$\lim_{s_k \downarrow 0} \frac{\|u(y) - u(z_3) - e \cdot (y - z_3)\|_{L^\infty(B_{s_k}(z_3))}}{s_k} = 0. \tag{4.4}$$

We first claim that $z_1, z_2 \in \{y \in \mathbf{R}^2: u(y) > u(z_3) + e \cdot (y - z_3)\}$. In fact, by the assumption (1) and (4.4), we have, for any $z_k \in [z_1, z_2] \cap \partial B_{s_k}(z_3)$,

$$\begin{aligned} (a_0 - e) \cdot \left(\frac{z_k - z_3}{s_k} \right) &= \frac{l(z_k) - l(z_3) - e \cdot (z_k - z_3)}{s_k} \\ &\leq \frac{u(z_k) - u(z_3) - e \cdot (z_k - z_3)}{s_k} \rightarrow 0, \end{aligned}$$

as $k \rightarrow \infty$. Hence, we have $(a_0 - e) \cdot (z - z_3) = 0$ for any $z \in [z_1, z_2]$. This implies

$$u(z_i) - u(z_3) > l(z_i) - l(z_3) = a_0 \cdot (z_i - z_3) = e \cdot (z_i - z_3), \quad i = 1, 2.$$

Now, suppose that z_1, z_2 belong to the same connected component of $\{y \in \mathbf{R}^2: u(y) > u(z_3) + e \cdot (y - z_3)\} \cap B_6$. Then there exists a polygonal line $\Gamma \subset \{y \in \mathbf{R}^2: u(y) > u(z_3) + e \cdot (y - z_3)\} \cap B_6$ joining z_1 to z_2 . Let $S = \Gamma \cup [z_1, z_2]$ be the closed curve and $U \subset B_6$ be the open set such that $S = \partial U$. Without loss of generality, we assume that there exists a $\beta > 0$ such that

$$B_\beta^+(z_3) = B_\beta(z_3) \cap \{y \in \mathbf{R}^2: 0 < \angle(y - z_3, z_2 - z_1) < \pi\} \subset U.$$

Note that there exists an $\delta_0 > 0$ such that $u(y) - u(z_3) - e \cdot (y - z_3) \geq \delta_0$ for any $y \in \Gamma$. Therefore there are a small $\epsilon > 0$ and a unit vector $v \in \mathbf{R}^2$, with $\angle(v, z_2 - z_1) = \frac{\pi}{2}$ and $e + \epsilon v \neq 0$, such that

$$u(y) \geq u(z_3) + (e + \epsilon v) \cdot (y - z_3), \quad \forall y \in S.$$

Note that $\psi(y) = u(z_3) + (e + \epsilon v) \cdot (y - z_3)$ is linear and $\nabla \psi = e + \epsilon v \neq 0$ so that Lemma 3.1 implies that $H(\nabla \psi) > 0$ and $H_p(\nabla \psi) \neq 0$. Therefore, Proposition 2.3 implies

$$u(y) \geq u(z_3) + (e + \epsilon v) \cdot (y - z_3), \quad \forall y \in U.$$

In particular, we have

$$\lim_{s_k \downarrow 0} \max_{y \in B_{s_k}(z_3) \cap B_\beta^+(z_3)} \frac{|u(y) - u(z_3) - e \cdot (y - z_3)|}{s_k} \geq \epsilon > 0,$$

this contradicts (4.4). Therefore Lemma 4.2 is proven. \square

Now we want to prove

$$|e_6 - e|^2 \leq C\epsilon. \tag{4.5}$$

To prove (4.5), we may assume $|e_6 - e| \geq \frac{\epsilon}{5}$. Denote $f = e_6 - e$ and form the strip $\mathcal{S} := \{y \in \mathbf{R}^2 \mid |f \cdot (y - z_3)| \leq 2\delta\}$. It is easy to see the width of \mathcal{S} is at most $\frac{4\delta}{\epsilon}$. Moreover, (4.1) implies

$$|u(y) - u(z_3) - e_6 \cdot (y - z_3)| \leq 2\delta, \quad \forall y \in B_6. \tag{4.6}$$

One can check

$$\begin{aligned} \{y \in \mathbf{R}^2 \mid f \cdot (y - z_3) < -2\delta\} \cap B_6 &\subset \{y \in \mathbf{R}^2: u(y) < u(z_3) + e \cdot (y - z_3)\}, \\ \{y \in \mathbf{R}^2 \mid f \cdot (y - z_3) > 2\delta\} \cap B_6 &\subset \{y \in \mathbf{R}^2: u(y) > u(z_3) + e \cdot (y - z_3)\}. \end{aligned}$$

Therefore, by Lemma 4.2, there is a connected component of $\{y \in \mathbf{R}^2: u(y) > u(z_3) + e \cdot (y - z_3)\} \cap B_6$, called U , that intersects B_1 and is contained in the strip \mathcal{S} .

We claim that $U \not\subset B_6$. For, otherwise, we have $u(y) = u(z_3) + e \cdot (y - z_3)$ for all $y \in \partial U$, Proposition 2.3 implies $u \equiv u(z_3) + e \cdot (y - z_3)$ in U . This contradicts with the definition of U . Therefore there exists a polygonal line Γ inside U starting in B_1 and exiting B_6 .

Now we can find a point $z_4 \in B_6$, with $|z_4 - z_3| = 3$ and $z_4 - z_3 \perp f$, such that

$$(A1) \quad \|u(y) - u(z_3) - e_6 \cdot (y - z_3)\|_{L^\infty(B_2(z_4))} \leq 2\delta, \quad 1 \leq H(e_6) \leq 2.$$

$$(A2) \quad \{y \in \mathbf{R}^2: u(y) > u(z_3) + e \cdot (y - z_3)\} \cap B_6 \text{ has a connected component } U \subset \mathcal{S} \text{ that contains a polygonal line } \Gamma \text{ connecting the two arcs } \mathcal{S} \cap \partial B_2(z_4).$$

The estimate (4.5) follows from the following two lemmas.

Lemma 4.3. *Under the same notations. There is $\delta = \delta(\epsilon, H) > 0$ such that*

$$H(e) \geq H(e_6) - 2\epsilon. \tag{4.7}$$

Proof of Lemma 4.3. We may assume that $H(e) \leq H(e_6) - \epsilon$. The convexity of H implies

$$H_p(e_6) \cdot f \geq H(e_6) - H(e) \geq \epsilon. \tag{4.8}$$

Now we have

Claim 4.3(a). *There are $\delta = \delta(H) > 0$ and $C = C(H) > 0$, with $C\delta \leq 2$, such that for any $x \in U \cap B_1(z_4)$, we have*

$$C_{H(e)+C\delta}^H(y-x) \geq e \cdot (y-x) + 4\delta, \quad \forall y \in \partial B_1(x). \tag{4.9}$$

Proof of Claim 4.3(a). As $H(e) \leq H(e_6) - \epsilon \leq 2$ and $C\delta \leq 2$, we have $H(p) \leq 4$ for any $p \in H^{-1}(H(e) + C\delta)$. For any $y \in \partial B_1(x)$, let $p_0 \in H^{-1}(H(e) + C\delta)$ be such that

$$C_{H(e)+C\delta}^H(y-x) = p_0 \cdot (y-x).$$

By the Lagrange Multiple Theorem, we have

$$\frac{H_p(p_0)}{|H_p(p_0)|} = y-x.$$

By the convexity, we have

$$C\delta = H(p_0) - H(e) \leq H_p(p_0) \cdot (p_0 - e).$$

Hence we have

$$\begin{aligned} C_{H(e)+C\delta}^H(y-x) - e \cdot (y-x) &= (p_0 - e) \cdot (y-x) = (p_0 - e) \cdot \frac{H_p(p_0)}{|H_p(p_0)|} \\ &\geq \frac{C}{|H_p(p_0)|} \delta \geq \frac{C}{\max_{H(p) \leq 4} |H_p(p)|} \delta \geq 4\delta \end{aligned}$$

provided that $C \geq 4 \max_{\{H(p) \leq 4\}} |H_p(p)|$. This implies (4.9). \square

Claim 4.3(b). *There are $\delta = \delta(H) > 0$ and $C = C(H) > 0$, with $C\delta \leq 2$, such that for any $x \in U \cap B_1(z_4)$, we have*

$$S^+(H, u, x) \leq H(e) + C\delta. \tag{4.10}$$

Proof of Claim 4.3(b). For any $x_0 \in B_1(z_4) \cap U$, we have that $B_1(x_0) \subset B_2(z_4)$ and $u(z_3) + e \cdot (x_0 - z_3) < u(x_0)$. Note that

$$u(y) = u(z_3) + e \cdot (y - z_3) < u(x_0) + e \cdot (y - x_0) \leq u(x_0) + C_{H(e)}^H(y - x_0), \quad \forall y \in \partial U \cap B_1(x_0),$$

and

$$u(y) \leq u(z_3) + e \cdot (y - z_3) + 4\delta < u(x_0) + e \cdot (y - x_0) + 4\delta, \quad \forall y \in \partial B_1(x_0) \cap U. \tag{4.11}$$

It follows from (4.9) that we have

$$u(y) \leq u(x_0) + C_{H(e)+C\delta}^H(y - x_0), \quad \text{on } \partial(U \cap B_1(x_0)).$$

Hence Theorem 2.1 implies

$$u(y) \leq u(x_0) + C_{H(e)+C\delta}^H(y - x_0), \quad \text{in } U \cap B_1(x_0).$$

This, combined with Proposition 2.2, implies (4.10). \square

Now we have

Claim 4.3(c). *There exists $y_0 \in L = \{z_4 + t \frac{H_p(e_6)}{|H_p(e_6)|} : -\frac{1}{3} \leq t \leq -\frac{1}{4}\}$ such that*

$$S^+(H, u, y_0) \geq H(e_6) - 50|H_p(e_6)|\delta. \tag{4.12}$$

Proof of Claim 4.3(c). Since $L \subset B_2(z_4)$, A1 implies

$$u\left(z_4 - \frac{1}{3} \frac{H_p(e_6)}{|H_p(e_6)|}\right) - u\left(z_4 - \frac{1}{4} \frac{H_p(e_6)}{|H_p(e_6)|}\right) \geq \frac{1}{12} e_6 \cdot \frac{H_p(e_6)}{|H_p(e_6)|} - 4\delta. \tag{4.13}$$

On the other hand, let $c = \sup_{x \in L} S^+(H, u, x)$, then we have

$$u\left(z_4 - \frac{1}{3} \frac{H_p(e_6)}{|H_p(e_6)|}\right) - u\left(z_4 - \frac{1}{4} \frac{H_p(e_6)}{|H_p(e_6)|}\right) \leq \frac{1}{12} C_c^H \left(\frac{H_p(e_6)}{|H_p(e_6)|}\right). \tag{4.14}$$

In fact, for any large positive integer n , let $v = \frac{1}{12} \frac{H_p(e_6)}{|H_p(e_6)|}$ and $r_n = \frac{|v|}{n}$ and $x_0 = z_4 - \frac{1}{3} \frac{H_p(e_6)}{|H_p(e_6)|}$, $x_1 = x_0 + \frac{v}{n}, \dots, x_{n-1} = x_0 + \frac{n-1}{n} v$, $x_n = z_4 - \frac{1}{4} \frac{H_p(e_6)}{|H_p(e_6)|}$. Then we have

$$u(x_i) - u(x_{i-1}) \leq C_{S_{r_n}^+(H, u, x_{i-1})}^H(x_i - x_{i-1}), \quad \forall 1 \leq i \leq n. \tag{4.15}$$

The monotonicity of $S_r^+(H, u, x)$ and upper semicontinuity of $S^+(H, u, x)$ imply

$$\lim_{n \rightarrow \infty} \sup_{x \in L} S_{r_n}^+(H, u, x) \leq c \equiv \sup_{x \in L} S^+(H, u, x).$$

Therefore for any $\epsilon > 0$ there is a sufficiently large $n_0 > 0$ such that

$$\max_{0 \leq i \leq n_0-1} S_{r_{n_0}}^+(H, u, x_i) \leq c + \epsilon.$$

Therefore (4.15) implies

$$\begin{aligned} u\left(z_4 - \frac{1}{4} \frac{H_p(e_6)}{|H_p(e_6)|}\right) - u\left(z_4 - \frac{1}{3} \frac{H_p(e_6)}{|H_p(e_6)|}\right) &= \sum_{i=1}^{n_0} (u(x_i) - u(x_{i-1})) \\ &\leq \sum_{i=1}^{n_0} C_{c+\epsilon}^H(x_i - x_{i-1}) \\ &= \frac{1}{12} C_{c+\epsilon}^H \left(\frac{H_p(e_6)}{|H_p(e_6)|}\right). \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, this implies (4.14).

Combining (4.13) with (4.14), we obtain

$$e_6 \cdot H_p(e_6) - 48|H_p(e_6)|\delta \leq C_c^H(H_p(e_6)). \tag{4.16}$$

Now we have

$$C_{H(q)}^H(H_p(q)) = q \cdot H_p(q), \quad \forall 0 \neq q \in \mathbf{R}^2. \tag{4.17}$$

To see (4.17), observe that for $q \neq 0$, Lemma 3.1 implies $H(q) > 0$ and $H_p(q) \neq 0$. If $v \in H^{-1}(H(q))$ is such that $v \cdot H_p(q) = C_{H(q)}^H(H_p(q))$, then the Lagrange Multiple Theorem implies $\frac{H_p(v)}{|H_p(v)|} = \frac{H_p(q)}{|H_p(q)|}$. In particular, the Gauss map $v(v) = v(q)$ and Lemma 3.2 implies $v = q$ and (4.17) follows. (4.16), combined with (4.17), implies

$$C_{H(e_6)}^H(H_p(e_6)) - 48|H_p(e_6)|\delta \leq C_c^H(H_p(e_6)). \tag{4.18}$$

Now we need

$$C_{H(e_6) - 48|H_p(e_6)|\delta}^H(H_p(e_6)) \leq C_{H(e_6)}^H(H_p(e_6)) - 48|H_p(e_6)|\delta. \tag{4.19}$$

To see (4.19), note that for any $q \in H^{-1}(H(e_6) - 48|H_p(e_6)|\delta)$, the convexity of H implies that

$$H(e_6) - 48|H_p(e_6)|\delta = H(q) \geq H(e_6) + H_p(e_6) \cdot (q - e_6)$$

so that

$$q \cdot H_p(e_6) \leq e_6 \cdot H_p(e_6) - 48|H_p(e_6)|\delta.$$

Taking maximum of $q \in H^{-1}(H(e_6) - 48|H_p(e_6)|\delta)$, we obtain (4.19).

From (4.18) and (4.19), we have $c \geq H(e_6) - 48|H_p(e_6)|\delta$ and hence (4.12) is proven. \square

Now we proceed with the discrete gradient flow as follows. Let $t = d(\Gamma, \partial U \cap B_2(z_4)) \leq \frac{2\delta}{|\Gamma|}$ be the step size and y_0 be given by Claim 4.3(c), let $y_i \in B_2(z_4)$ ($1 \leq i \leq m = m(\delta)$) be such that

$$|y_i - y_{i-1}| = t, u(y_i) = u(y_{i-1}) + C_{S^+(H,u,y_{i-1})}^H(y_i - y_{i-1}), \quad 1 \leq i \leq m, \tag{4.20}$$

and

$$d(y_m, \partial B_2(z_4)) \leq t. \tag{4.21}$$

Then it follows from Proposition 2.3 that

$$S^+(H, u, y_i) \geq S^+(H, u, y_0) \geq H(e_6) - 50|H_p(e_6)|\delta, \quad 0 \leq i \leq m. \tag{4.22}$$

To see (4.21), observe that

$$\begin{aligned} u(y_m) - u(y_0) &= \sum_{i=1}^m (u(y_i) - u(y_{i-1})) \geq \sum_{i=1}^m C_{S^+(H,u,y_{i-1})}^H(y_i - y_{i-1}) \\ &\geq \sum_{i=1}^m C_{S^+(H,u,y_0)}^H(y_i - y_{i-1}) \\ &\geq \sum_{i=1}^m C_{H(e_6)-50|H_p(e_6)|\delta}^H(y_i - y_{i-1}) \geq Cmt, \end{aligned} \tag{4.23}$$

where we have used the fact that there exists $C > 0$ depending only on H such that

$$C_{H(e_6)-50|H_p(e_6)|\delta}^H(x) \geq C|x|, \quad \forall |x| = 1. \tag{4.24}$$

Since $|u(y_m) - u(y_0)| \leq \|\nabla u\|_{L^\infty(B_6)}|y_m - y_0|$, (4.23) implies (4.21) holds for some $m = m(\delta) \geq 1$.

Now we have

Claim 4.3(d). *There exist $\delta = \delta(\epsilon) > 0$ and $1 \leq i \leq m$ such that $y_i \in B_1(z_4) \cap U$.*

Proof of Claim 4.3(d). First observe that (4.23) also yields

$$\begin{aligned} u(y_m) - u(y_0) &\geq \sum_{i=1}^m C_{H(e_6)-50|H_p(e_6)|\delta}^H(y_i - y_{i-1}) \\ &\geq C_{H(e_6)-50|H_p(e_6)|\delta}^H\left(\sum_{i=1}^m (y_i - y_{i-1})\right) \\ &= C_{H(e_6)-50|H_p(e_6)|\delta}^H(y_m - y_0), \end{aligned} \tag{4.25}$$

where we have used the triangle inequality $C_k^H(x + y) \leq C_k^H(x) + C_k^H(y)$.

On the other hand, A1 implies

$$u(y_m) - u(y_0) \leq e_6 \cdot (y_m - y_0) + 4\delta. \tag{4.26}$$

Therefore we have

$$C_{H(e_6)-50|H_p(e_6)|\delta}^H(y_m - y_0) \leq e_6 \cdot (y_m - y_0) + 4\delta. \tag{4.27}$$

Note that (4.21) implies $|y_m - z_4| \geq 2 - \frac{2\delta}{|\Gamma|} \geq \frac{4}{3}$ and hence $|y_m - y_0| \geq 1$. Denote $e_m = \frac{y_m - y_0}{|y_m - y_0|}$, then (4.27) implies

$$C_{H(e_6)-50|H_p(e_6)|\delta}^H(e_m) \leq e_6 \cdot e_m + 4\delta. \tag{4.28}$$

Now we claim that there exists $\eta(\delta) > 0$, with $\lim_{\delta \rightarrow 0} \eta(\delta) = 0$, such that

$$\cos \angle(e_m, H_p(e_6)) \geq 1 - \eta(\delta). \tag{4.29}$$

In fact, we may assume that there exists $\hat{e} \in \mathbf{R}^2$, with $|\hat{e}| = 1$, such that $\lim_{\delta \rightarrow 0} e_m = \hat{e}$. Taking δ into zero, (4.28) implies

$$C_{H(e_6)}^H(\hat{e}) = e_6 \cdot \hat{e}.$$

Hence (4.17) implies $\hat{e} = H_p(e_6)$ and (4.29) follows.

Since $H_p(e_6) \cdot e_6 \geq H(e_6) - H(0) \geq 1$, we have $|H_p(e_6)| \geq \frac{1}{|e_6|}$. Moreover, since

$$H(e_6) \leq 2, \quad H(e) \leq \max_{|p| \leq \|\nabla u\|_{L^\infty(B_6)}} H(p),$$

we conclude that there is $C = C(H) > 0$ such that $|e_6| + |e| + |H_p(e_6)| \leq C$, and hence

$$\cos \angle(H_p(e_6), f) \geq \frac{\epsilon}{|H_p(e_6)||f|} \geq \frac{\epsilon}{C}. \tag{4.30}$$

(4.29) and (4.30) imply that if $\delta = \delta(\epsilon, H) > 0$ is sufficiently small, then $y_m \in \{x \in \mathbf{R}^2 \mid f \cdot (y - z_4) \geq 2\delta\}$. From the choice of the step size t , we can conclude that there exists $1 \leq i \leq m$ such that $y_i \in U \cap B_2(z_4)$.

Now we claim that $|y_i - z_4| \leq 1$. For otherwise, we have $|y_i - y_0| \geq \frac{2}{3}$ and the above argument again yields $\cos(\angle(y_i - y_0, H_p(e_6))) \geq 1 - \eta(\delta)$. But this is also impossible if $\delta = \delta(\epsilon) > 0$ is chosen to be sufficiently small. Hence the claim 4.3(d) is proven.

Combining (4.10), (4.12), (4.22), and Claim 4.3(d), we obtain

$$H(e_6) - 50|H_p(e_6)|\delta \leq S^+(H, u, y_i) \leq H(e) + C\delta$$

or

$$H(e) \geq H(e_6) - (C + 50|H_p(e_6)|)\delta.$$

This implies (4.7), provided that $\delta = \delta(\epsilon, H) > 0$ is chosen to be sufficiently small. Therefore the proof of Lemma 4.4 is complete. \square

Let $\alpha = \angle(H_p(e), f) \in [0, \pi]$ be the angle between $H_p(e)$ and f . Now we have

Lemma 4.4. *Under the same notations. For any $\epsilon > 0$, there is a $\delta = \delta(\epsilon, H) > 0$ such that*

$$\left| \alpha - \frac{\pi}{2} \right| \leq 2\epsilon. \tag{4.31}$$

Proof. Without loss of generality, we may assume $\alpha \in [0, \frac{\pi}{2}]$. We may assume $\frac{\pi}{2} - \alpha \geq \epsilon$.

Let $x_\delta = z_4 - \frac{2H_p(e)}{H_p(e) \cdot f} \delta$ be the intersection of $L = \{z_4 + tH_p(e) : t \in \mathbf{R}\}$ and $\{y \in \mathbf{R}^2 : (y - z_4) \cdot f = -2\delta\}$. Observe that

$$|x_\delta - z_4| = \frac{2\delta}{|f| \cos \angle(H_p(e), f)} \leq \frac{2\delta}{\epsilon \sin(\epsilon)} \leq 1$$

provided that $\delta = \delta(\epsilon) > 0$ is chosen to be sufficiently small. This implies $B_1(x_\delta) \subset B_2(z_4)$. By A1, we have

$$u(y) - u(z_3) - e \cdot (y - z_3) \leq u(y) - u(z_3) - e_1 \cdot (y - z_3) + f \cdot (y - z_3) \leq 4\delta, \quad \forall y \in U \cap B_1(x_\delta), \tag{4.32}$$

and

$$u(y) = u(z_3) + e \cdot (y - z_3), \quad \forall y \in \partial U \cap B_1(x_\delta). \tag{4.33}$$

We need

Claim 4.4(a).

$$u(x) \leq u(z_3) + e \cdot (x_\delta - z_3) + C_{H(e)}^H(x - x_\delta), \quad \forall x \in \partial(U \cap B_1(x_\delta)). \tag{4.34}$$

Proof of Claim 4.4(a). First observe that (4.33) implies

$$\begin{aligned} u(x) &= u(z_3) + e \cdot (x_\delta - z_3) + e \cdot (x - x_\delta) \leq u(z_3) + e \cdot (x_\delta - z_3) + \max_{H(p)=H(e)} p \cdot (x - x_\delta) \\ &= u(z_3) + e \cdot (x_\delta - z_3) + C_{H(e)}^H(x - x_\delta), \quad \forall x \in \partial U \cap B_1(x_\delta). \end{aligned} \tag{4.35}$$

On $U \cap \partial B_1(x_\delta)$, we have

$$u(x) \leq u(z_3) + e \cdot (x_\delta - z_3) + e \cdot (x - x_\delta) + 4\delta. \quad \square \tag{4.36}$$

Now we need

Claim 4.4(b). For any $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that

$$c_\delta := \min_{S \cap \partial B_1(x_\delta)} (C_{H(e)}^H(x - x_\delta) - e \cdot (x - x_\delta)) > 4\delta. \tag{4.37}$$

Proof of Claim 4.4(b). Suppose (4.37) were false. Then there exist $\epsilon_0 > 0$ and $\delta_i \rightarrow 0$ such that $c_{\delta_i} \leq 4\delta_i$. Since $e_6, e, \alpha, z_3, z_4, x_\delta$ all depend on δ_i , we write $e_6^i = e_6, e^i = e, f^i = e_6^i - e^i, \alpha_i = \angle(H_p(e^i), f^i), z_3^i = z_3, z_4^i = z_4, S_i = S, x_i = x_\delta$. Then, by the above assumption and Lemma 4.3, we have

$$\alpha_i \leq \frac{\pi}{2} - \epsilon_0, \quad \frac{1}{2} \leq H(e_6^i) - 2\epsilon_0 \leq H(e^i) \leq \max_{|p| \leq \|\nabla u\|_{L^\infty(B_6)}} H(p).$$

Let $y_i \in S_i \cap \partial B_1(x_i)$ such that

$$c_{\delta_i} = C_{H(e^i)}^H(y_i - x_i) - e^i \cdot (y_i - x_i) \leq 4\delta_i.$$

By the Lagrange Multiple Theorem, there exists p_i such that

$$H(p_i) = H(e^i), \quad \frac{H_p(p_i)}{|H_p(p_i)|} = y_i - x_i, \quad C_{H(e^i)}^H(y_i - x_i) = p_i \cdot (y_i - x_i). \tag{4.38}$$

After passing to subsequences, we may assume that $e_6^i \rightarrow e_6, e^i \rightarrow e, f^i \rightarrow f, z_3^i \rightarrow 0, z_4^i \rightarrow z_4, S_i \rightarrow \{x \in \mathbf{R}^2: x \cdot f = 0\}, x_i \rightarrow z_4, p_i \rightarrow p, y_i \rightarrow y$. It is easy to see that $|f| \geq \epsilon_0, \angle(H_p(e), f) \leq \frac{\pi}{2} - \epsilon_0, |z_4| = 3, z_4 \perp f, |y - z_4| = 1, y - z_4 \perp f$,

$$\frac{H_p(p)}{|H_p(p)|} = y - z_4, \quad H(p) = H(e), \quad C_{H(e)}^H(y - z_4) = p \cdot (y - z_4) = e \cdot (y - z_4).$$

This implies $p = e$ and $y - z_4 = \frac{H_p(e)}{|H_p(e)|}$. Therefore $H_p(e) \perp f$, this gives the desired contradiction. Hence (4.37) is true. \square

It follows from (4.35)–(4.37) that for sufficiently small $\delta > 0$, we have

$$u(y) \leq u(z_3) + e \cdot (x_\delta - z_3) + C_{H(e)}^H(y - x_\delta), \quad \forall y \in \partial(U \cap B_1(x_\delta)). \tag{4.39}$$

Therefore, Theorem 2.1 implies

$$u(y) \leq u(z_3) + e \cdot (x_\delta - z_3) + C_{H(e)}^H(y - x_\delta), \quad \forall y \in U \cap B_1(x_\delta). \tag{4.40}$$

By A2, we have that $\{x_\delta + tH_p(e) \mid t \geq 0\} \cap (U \cap B_1(x_\delta)) \neq \emptyset$. But on the other hand, (4.17) and (4.40) imply that if $y_0 = x_\delta + t_0H_p(e) \in U \cap B_1(x_\delta)$, then

$$\begin{aligned} u(y_0) &\leq u(z_3) + e \cdot (x_\delta - z_3) + C_{H(e)}^H(y_0 - x_\delta) \\ &\leq u(z_3) + e \cdot (x_\delta - z_3) + t_0 C_{H(e)}^H(H_p(e)) \\ &= u(z_3) + e \cdot (x_\delta - z_3) + t_0 e \cdot H_p(e) \\ &= u(z_3) + e \cdot (y_0 - z_3). \end{aligned}$$

This is impossible, since $y_0 \in U$. Therefore (4.31) holds and the proof of Lemma 4.4 is complete. \square

Now we return to the proof of (4.5) as follows. From (3.2) of Lemma 3.1, Lemmas 4.3 and 4.4, we have

$$\begin{aligned}
 2\epsilon &\geq H(e_6) - H(e) \geq H_p(e) \cdot f + \frac{1}{2}\alpha_H|f|^2 \\
 &\geq -|H_p(e)||f| \cos \angle(H_p(e), f) + \frac{1}{2}\alpha_H|f|^2 \\
 &\geq -C \sin(2\epsilon) + \frac{1}{2}\alpha_H|f|^2.
 \end{aligned}
 \tag{4.41}$$

Hence we have

$$|f|^2 \leq \alpha_H^{-1} C (\epsilon + \sin(2\epsilon))$$

provided that $\delta = \delta(\epsilon, H) > 0$ is chosen to be sufficiently small. This yields (4.5).

Next we want to prove

$$|e_{0,6r} - e|^2 \leq C\epsilon. \tag{4.42}$$

To prove (4.42), define $v_r(x) = \frac{u(rx)}{r} : B_6 \rightarrow \mathbf{R}$. Then we have

$$\|v_r - e_{0,6r} \cdot x\|_{L^\infty(B_6)} \leq \delta.$$

Moreover, we can check that Lemma 4.2 also implies that $\{y \in \mathbf{R}^2 : u(y) > u(z_3) + e \cdot (y - z_3)\} \cap B_{6r}$ has two connected components that intersect B_r . This implies that for $z'_3 = \frac{z_3}{r}$, $\{y \in \mathbf{R}^2 : v_r(y) > v_r(z'_3) + e \cdot (y - z'_3)\} \cap B_6$ has two connected components that intersect B_1 . Therefore the same argument as above yields (4.42), if we can verify

$$\frac{1}{2} \leq H(e_{0,6r}) \leq 4. \tag{4.43}$$

To see (4.43), first observe that Lemma 4.3 implies that for $\delta > 0$ sufficiently small, we have

$$H(e) \geq H(e_6) - C_H \delta \geq 1 - C_H \delta \geq \frac{3}{4}. \tag{4.44}$$

For any $x \in \partial B_{2r}(z_3) \subset B_{6r}$, we have

$$\begin{aligned}
 u(x) &\leq u(z_3) + e_{0,6r} \cdot (x - z_3) + 2\delta r \\
 &\leq u(z_3) + C_{H(e_{0,6r})}^H (x - z_3) + 2\delta r \\
 &\leq u(z_3) + C_{H(e_{0,6r})+C\delta}^H (x - z_3).
 \end{aligned}
 \tag{4.45}$$

Here we have used the following claim, whose proof is similar to that of Claim 4.4(a),

Claim 4.1(a). *There exists a $C > 0$ such that for any $\alpha > 0$, we have*

$$C_{H(e_{0,6r})}^H(x) + \alpha|x| \leq C_{H(e_{0,6r})+C\alpha}^H(x), \quad \forall x \in B_{6r}. \tag{4.46}$$

It follows from (4.45) that

$$S_{2r}^+(H, u, z_3) \leq H(e_{0,6r}) + C\delta.$$

This implies

$$H(e) = S^+(H, u, z_3) \leq S_{2r}^+(H, u, z_3) \leq H(e_{0,6r}) + C\delta. \tag{4.47}$$

(4.47) and (4.44) imply that for $\delta > 0$ sufficiently small, we have

$$H(e_{0,6r}) \geq H(e) - C\delta \geq \frac{1}{2}. \tag{4.48}$$

On the other hand, (4.1) implies

$$u(x) \leq e_6 \cdot x + \delta \leq C_{H(e_6)}^H(x) + \delta \leq C_{H(e_6)+C\delta}^H(x), \quad \forall x \in \partial B_3.$$

This and (4.46) imply that for $\delta > 0$ sufficiently small, we have

$$H(e_{0,6r}) = S^+(H, u, 0) \leq H(e_6) + C\delta = 2 + C\delta \leq 4. \tag{4.49}$$

It is clear that (4.48) and (4.49) imply (4.43). Therefore (4.42) holds. Note that (4.3) follows from (4.5) and (4.42). This proves Proposition 4.1 in case A.

Next we consider the second case.

Case B. $u(x) \equiv e \cdot x$ near 0 for some $e \in \mathbf{R}^2$: Denote U as the interior of $\{x \in \mathbf{R}^2 \mid u(x) = e \cdot x\}$. If $d(0, \partial U) > \frac{1}{2}$, then we have e_6 is arbitrarily close to $e_{0,6r}$ and hence (4.3) holds trivially. If $d(0, \partial U) \leq \frac{1}{2}$, then we let $x_0 \in \partial U$ be such that $|x_0| = d(0, \partial U)$. Note that

$$\|u(y) - u(x_0) - e_6 \cdot (y - x_0)\|_{L^\infty(B_5(x_0))} \leq 2\delta, \quad 1 \leq H(e_6) \leq 2,$$

and

$$\lim_{r \downarrow 0} \frac{\|u(y) - u(x_0) - e_{x_0,6r}(y - x_0)\|_{L^\infty(B_{6r}(x_0))}}{r} = 0.$$

Hence, by applying Case A in $B_5(x_0)$, we get

$$|e_{x_0,6r} - e_6|^2 \leq C\epsilon.$$

On the other hand, it is easy to see that for $r > 0$ sufficiently small, $e = e_{0,6r} = e_{x_0,6r}$. Hence we have

$$|e_{0,6r} - e_6|^2 \leq C\epsilon$$

provided that $\delta = \delta(\epsilon, H) > 0$ is chosen to be sufficiently small. This implies (4.3). Therefore the proof of Proposition 4.1 is complete. \square

It follows from Proposition 4.1 that we have

Corollary 4.5. *Suppose that $u \in C^{0,1}(B_1)$, $B_1 \subset \mathbf{R}^2$, is a viscosity solution of the Aronsson equation (1.7). Then for any $\epsilon > 0$, there exists $\delta = \delta(\epsilon, H) > 0$ such that if*

$$\|u(x) - u(0) - e_1 \cdot x\|_{L^\infty(B_1)} \leq \delta, \quad 1 \leq H(e_1) \leq 2. \tag{4.50}$$

Then u is differentiable at 0 and

$$|\nabla u(0) - e_1|^2 \leq C\epsilon, \tag{4.51}$$

for some $C = C(H) > 0$.

Proof. By Theorem A, we have that there exists $r_0 = r_0(\delta) > 0$ such that for any $0 < r \leq r_0$ there exists $e_{0,r} \in \mathbf{R}^2$ such that

$$\|u(x) - u(0) - e_{0,r} \cdot x\|_{L^\infty(B_r)} \leq \delta r. \tag{4.52}$$

Therefore we can apply Proposition 4.1 to conclude that there exist $\delta = \delta(\epsilon, H) > 0$ and $C = C(H) > 0$ such that

$$|e_1 - e_{0,r}|^2 \leq C\epsilon, \quad \forall 0 < r \leq r_0. \tag{4.53}$$

This implies

$$|e_{0,r} - e_{0,s}|^2 \leq 2C\epsilon, \quad \forall 0 < r, s \leq r_0.$$

Since $\epsilon > 0$ is arbitrary, this implies $\{e_{0,r}\}_{r>0}$ is a Cauchy sequence. Hence $\lim_{r \downarrow 0} e_{0,r}$ exists, u differentiable at 0, $\nabla u(0) = \lim_{r \downarrow 0} e_{0,r}$, and (4.51) holds. The proof of Corollary 4.5 is complete. \square

5. Proof of Theorems B, C, D, and Remark E

In this section, we apply Proposition 4.1 to give proofs of Theorems B, C and D. We will see that the proof for the second part of Theorem B follows from Proposition 4.1 and Theorem C.

Proof of Theorem B. First Theorem A implies that for any $x \in \Omega$, there exists $e_{x,r} \in \mathbf{R}^2$, with $H(e_{x,r}) = S^+(H, u, x)$, such that for any small $r > 0$

$$\max_{y \in B_r(x)} |u(y) - u(x) - e_{x,r} \cdot (y - x)| \leq \sigma(r)r, \quad \lim_{r \rightarrow 0} \sigma(r) = 0.$$

If $e_{x,r} = 0$, then by Theorem 2.1, we have

$$|u(y) - u(x)| \leq \sigma(r)|y - x|, \quad \forall y \in B_r(x).$$

This implies that u is differentiable at x and $\nabla u(x) = 0$. If $e_{x,r} \neq 0$, then we have, by Lemma 3.1, $H(e_{x,r}) > 0$. Let $v_{x,r}(y) = \frac{u(x+ry) - u(x)}{r}$ and $H_{x,r}(p) = \frac{H(p)}{|H(e_{x,r})|}$, then we can check that $H_{x,r}$ satisfies the conditions of Proposition 4.1, $v_{x,r} \in C^{0,1}(B_1)$ satisfies the Aronsson equation, with H replaced by $H_{x,r}$. Moreover, we have

$$\max_{y \in B_1} |v_{x,r} - e_{x,r} \cdot y| \leq \sigma(r), \quad H_{x,r}(e_{x,r}) = 1.$$

Therefore, by Corollary 4.5, there exists $r_0 > 0$ such that v_{x,r_0} is differentiable at 0. This implies that u is differentiable at any $x \in \Omega$.

The continuity of ∇u follows from Theorem C. \square

Proof of Theorem C. First note that $\|H(\nabla u)\|_{L^\infty(B_1)} \leq 1$ implies that there is $C = C(H) > 0$ such that

$$\|\nabla u\|_{L^\infty(B_1)} \leq C. \tag{5.1}$$

Suppose that the conclusion of Theorem C were false. Then there exist $\epsilon_0 > 0$, a family $\{u_k\} \subset C^0(B_1)$ of viscosity solutions of the Aronsson equation (1.7) and $x_k \rightarrow 0$ such that

$$\|\nabla u_k\|_{L^\infty(B_1)} \leq C, \quad |\nabla u_k(x_k) - \nabla u_k(0)| \geq \epsilon_0. \tag{5.2}$$

We may assume that there is a Lipschitz continuous u_∞ on B_1 such that $u_k \rightarrow u_\infty$ uniformly in B_1 . Hence u_∞ is a viscosity solution of (1.7), and

$$\|H(\nabla u_\infty)\|_{L^\infty(B_1)} \leq 1, \quad \|\nabla u_\infty\|_{L^\infty(B_1)} \leq C. \tag{5.3}$$

By Theorem A, we have that for any $\delta > 0$, there exists $r_0 > 0$ and $e_0 \in \mathbf{R}^2$ such that we have

$$\|u_\infty(x) - u_\infty(0) - e_0 \cdot x\|_{L^\infty(B_{r_0})} \leq \frac{\delta}{2}r_0. \tag{5.4}$$

Therefore there exists a sufficiently large $k_0 > 0$ such that

$$\|u_k(x) - u_k(0) - e_0 \cdot x\|_{L^\infty(B_{r_0})} \leq \delta r_0, \quad \forall k \geq k_0. \tag{5.5}$$

This implies that for $k \geq k_0$, we have

$$|u_k(x) - u_k(y) - e_0 \cdot (y - x)| \leq 2\delta r_0, \quad \forall x, y \in B_{r_0}. \tag{5.6}$$

In particular, for $k \geq k_0$, we have

$$\|u_k(x + x_k) - u_k(x_k) - e_0 \cdot x\|_{L^\infty(B_{\frac{r_0}{2}})} \leq 2\delta r_0. \tag{5.7}$$

If $e_0 = 0$, then (5.5), (5.7), and (2.9) imply

$$H(\nabla u_k(0)) = S^+(H, u_k, 0) \leq \max_{|p|=\delta} H(p), \quad H(\nabla u_k(x_k)) = S^+(H, u_k, x_k) \leq \max_{|p|=4\delta} H(p). \tag{5.8}$$

Therefore we know that there exists $\eta(\delta) > 0$, with $\lim_{\delta \rightarrow 0} \eta(\delta) = 0$, such that for k sufficiently large

$$|\nabla u_k|(x_k) + |\nabla u_k|(0) \leq \eta(\delta). \tag{5.9}$$

This contradicts with (5.2), if we choose $\delta > 0$ to be sufficiently small.

If $e_0 \neq 0$, then (5.5), (5.7), and Corollary 4.5 imply that for any $0 < \epsilon \leq \frac{\epsilon_0}{4}$, we can find a sufficiently small $\delta_0 = \delta_0(\epsilon, H(e_0)) > 0$ so that for k sufficiently large, we have

$$|\nabla u_k(x_k) - e_0| \leq \epsilon, \quad |\nabla u_k(0) - e_0| \leq \epsilon.$$

In particular, $|\nabla u_k(x_k) - \nabla u(0)| \leq 2\epsilon < \epsilon_0$. This again contradicts with (5.2). Hence the proof of Theorem C is complete. \square

Proof of Theorem D. First we claim that the linear growth condition at infinity implies

$$\|H(\nabla u)\|_{L^\infty(B_R)} = \sup_{x \in B_R} S^+(H, u, x) \leq C_1. \tag{5.10}$$

In fact, for any $x \in B_R$ and $y \in \mathbf{R}^2$ with $|y - x| = R$, we have

$$u(y) - u(x) \leq 2C(1 + R) \leq 4CR \leq C_{C_1}^H(y - x)$$

for some $C_1 > 0$ depends only on H , where we have used the coercivity condition H3. Hence, $S^+(H, u, x) \leq S_R^+(H, u, x) \leq C_1$.

For $R \uparrow \infty$, let $w_R(x) = \frac{u(Rx)}{R} : B_1 \rightarrow R$. Then w_R satisfies the condition of Theorem C. Hence we have, for any fixed $x_0 \in \mathbf{R}^2$

$$|\nabla u(x_0) - \nabla u(0)| = |\nabla w_R(R^{-1}x_0) - \nabla w_R(0)| \leq C\rho(|x_0|R^{-1}) \rightarrow 0, \tag{5.11}$$

as $R \rightarrow \infty$. This implies $\nabla u(x) \equiv \nabla u(0)$ for all $x \in \mathbf{R}^2$ and u is linear. The proof is now complete. \square

Proof of Remark E. We sketch the argument that Theorems B and C are true under the conditions (i), (ii), and (iii). First note that H satisfies all the conditions of Theorem 2.1, hence we have that u is locally Lipschitz continuous. For simplicity, we may assume that u is Lipschitz continuous on Ω so that

$$m = \|\nabla u\|_{L^\infty(\Omega)} < \infty.$$

Let $\phi \in C^2(\mathbf{R}^2)$ be a nonnegative convex function such that

$$\phi(p) = 0 \quad \text{for } |p| \leq m, \quad (\phi_{p_i p_j}) \geq I_2 \quad \text{for } |p| \geq m + 1. \tag{5.12}$$

Let $\eta \in C^2(\mathbf{R}^2)$ be a nonnegative function such that

$$\eta(p) = 1 \quad \text{for } |p| \leq m + 1, \quad \eta(p) = 0 \quad \text{for } |p| \geq m + 2. \tag{5.13}$$

For $k \geq 1$, define $H_k(p) = H(p)\eta(p) + k\phi(p)$ for $p \in \mathbf{R}^2$. Since

$$H_k(p) = H(p) \quad \text{for } |p| \leq m,$$

u is also a viscosity solution of (1.7) with H replaced by H_k . Moreover, it follows from the condition (iii) that we have

$$\delta_m = \inf \left\{ \sum_{i,j=1}^2 H_{p_i p_j}(p) \xi_i \xi_j : |p| \leq m + 1, |\xi| = 1 \right\} > 0,$$

namely H is uniformly convex in $\{p \in \mathbf{R}^2 : |p| \leq m + 1\}$. We can check that if k is sufficiently large, then $H_k \in C^2(\mathbf{R}^2)$ is uniformly convex. In fact, it is easy to see that

$$\begin{aligned} \nabla^2 H_k(p) &= \nabla^2 H(p) + k\nabla^2 \phi(p) \geq \nabla^2 H(p) \geq \delta_m > 0, \quad \forall |p| \leq m + 1 \\ \nabla^2 H_k(p) &= k\nabla^2 \phi(p) = kI_2, \quad \forall |p| \geq m + 2, \end{aligned}$$

and

$$\begin{aligned} \nabla^2 H_k(p) &\geq \left[k - \max_{m+1 \leq |q| \leq m+2} (2|H_p(q)| |\nabla \eta(q)| + H(q) |\nabla^2 \eta(q)|) \right] I_2 \\ &\geq \frac{k}{2} I_2 > 0, \quad m + 1 \leq |p| \leq m + 2, \end{aligned}$$

provided that $k > 0$ is chosen to be sufficiently large. Therefore Theorems B and C hold. \square

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References

- [1] G. Aronsson, Minimization problems for the functional $\sup_x F(x, f(x), f'(x))$, *Ark. Mat.* 6 (1965) 33–53.
- [2] G. Aronsson, Minimization problems for the functional $\sup_x F(x, f(x), f'(x))$. II, *Ark. Mat.* 6 (1966) 409–431.
- [3] G. Aronsson, Minimization problems for the functional $\sup_x F(x, f(x), f'(x))$. III, *Ark. Mat.* 7 (1969) 509–512.
- [4] G. Aronsson, Extension of functions satisfying Lipschitz conditions, *Ark. Mat.* 6 (1967) 551–561.
- [5] G. Aronsson, On the partial differential equation $u_x^2 u_{xx} + 2u_x u_y u_{xy} + u_y^2 u_{yy} = 0$, *Ark. Mat.* 7 (1968) 395–425.
- [6] G. Aronsson, On certain singular solutions of the partial differential equation $u_x^2 u_{xx} + 2u_x u_y u_{xy} + u_y^2 u_{yy} = 0$, *Manuscripta Math.* 47 (1–3) (1984) 133–151.
- [7] G. Aronsson, M. Crandall, P. Juutinen, A tour of the theory of absolutely minimizing functions, *Bull. Amer. Math. Soc. (N.S.)* 41 (4) (2004) 439–505 (electronic).
- [8] N. Barron, Viscosity solutions and analysis in L^∞ , in: *Nonlinear Analysis, Differential Equations and Control*, Montreal, QC, 1998, in: *NATO Sci. Ser. C Math. Phys. Sci.*, vol. 528, Kluwer Acad. Publ., Dordrecht, 1999, pp. 1–60.
- [9] N. Barron, R. Jensen, Minimizing the L^∞ -norm of the gradient with an energy constraint, *Comm. Partial Differential Equations* 30 (10–12) (2005) 1741–1772.
- [10] N. Barron, R. Jensen, C.Y. Wang, Lower semicontinuity of L^∞ functionals, *Ann. Inst. H. Poincaré Anal. Non Linéaire* 18 (4) (2001) 495–517.
- [11] N. Barron, R. Jensen, C.Y. Wang, The Euler equation and absolute minimizers of L^∞ functionals, *Arch. Ration. Mech. Anal.* 157 (4) (2001) 255–283.
- [12] M. Crandall, An efficient derivation of the Aronsson equation, *Arch. Ration. Mech. Anal.* 167 (4) (2003) 271–279.
- [13] M. Crandall, The infinity-Laplace equation and the elements of calculus of variations in L -infinity, in: *CIME Lecture Notes*, in press.
- [14] M. Crandall, The continuous gradient flow for the infinity-Laplace equation, Private communication.
- [15] M. Crandall, L.C. Evans, A remark on infinity harmonic functions, in: *Proceedings of the USA–Chile Workshop on Nonlinear Analysis*, Via del Mar-Valparaiso, 2000 (electronic), *Electron. J. Differ. Equ. Conf.*, 6, pp. 123–129.
- [16] M. Crandall, C. Evans, R. Gariepy, Optimal Lipschitz extensions and the infinity Laplacian, *Calc. Var. Partial Differential Equations* 13 (2) (2001) 123–139.
- [17] M. Crandall, H. Ishii, P.L. Lions, User’s guide to viscosity solutions of second order partial differential equations, *Bull. Amer. Math. Soc. (N.S.)* 27 (1) (1992) 1–67.
- [18] R. Gariepy, C.Y. Wang, Y. Yu, Generalized cone comparison principle for viscosity solutions of the Aronsson equation and absolute minimizers, *Comm. Partial Differential Equations* 31 (7–9) (2006) 1027–1046.
- [19] R. Jensen, Uniqueness of Lipschitz extensions: minimizing the sup norm of the gradient, *Arch. Ration. Mech. Anal.* 123 (1) (1993) 51–74.
- [20] C.B. Morrey, Quasi-convexity and the lower semicontinuity of multiple integrals, *Pacific J. Math.* 2 (1952) 25–53.
- [21] O. Savin, C^1 -regularity for infinity harmonic functions in dimensions two, *Arch. Ration. Mech. Anal.* 176 (3) (2005) 351–361.
- [22] Y. Yu, L^∞ -variational problems and the Aronsson equations, *Arch. Ration. Mech. Anal.* 182 (1) (2006) 153–180.