On a Liouville phenomenon for entire weak supersolutions of elliptic partial differential equations

Autour d’un phénomène de Liouville pour les sursolutions entières faibles d’équations aux dérivées partielles elliptiques

Vasilii V. Kurta

American Mathematical Society (Mathematical Reviews), 416 Fourth Street, P.O. Box 8604, Ann Arbor, MI 48107-8604, USA

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Abstract

We study a new Liouville-type phenomenon for entire weak supersolutions of elliptic partial differential equations of the form $A(u) = 0$ on $\mathbb{R}^n$, $n \geq 2$. Typical examples of the operator $A(u)$ are the $p$-Laplacian for $p > 1$, the mean curvature operator, and their well-known modifications.

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1. Introduction

Liouville’s well-known theorem says that any superharmonic function on $\mathbb{R}^2$ bounded below by a constant is itself a constant. On the other hand it is also well known that for $n \geq 3$ there exist non-constant superharmonic functions on $\mathbb{R}^n$ bounded below by a constant. The purpose of this work is to determine for $n \geq 3$ the ‘sharp distance at infinity’ between the non-constant superharmonic functions on $\mathbb{R}^n$ bounded below by a constant and this constant itself in...
the form of a theorem of Liouville type and to characterize basic properties of quasilinear elliptic partial differential
operators which make it possible to obtain such a theorem for supersolutions of quasilinear elliptic partial differential
equations of the form
\[ A(u) = 0 \]  
(1)
on \mathbb{R}^n, n \geq 2. Typical examples of the operator \( A(u) \) are the \( p \)-Laplacian
\[ \Delta_p(u) := \text{div}(|\nabla u|^{p-2} \nabla u), \quad p > 1, \]  
(2)
its well-known modification (see, e.g., [8, p. 155])
\[ \tilde{\Delta}_p(u) := \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left( \left| \frac{du}{dx_i} \right|^{p-2} \frac{du}{dx_i} \right), \quad p > 1, \]  
(3)
the mean curvature operator
\[ \mathcal{E}(u) := \text{div}\left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right), \]  
(4)
and its well-known modifications.

Note that a Liouville theorem for solutions of linear uniformly elliptic second-order partial differential equations
on \( \mathbb{R}^n, n > 2, \) was first obtained, as a direct consequence of a Harnack inequality, in [1] under some continuity
assumptions on the coefficients of the equations and in [12] without continuity assumptions on the coefficients of
the equations. In the case of quasilinear uniformly elliptic second-order partial differential equations on \( \mathbb{R}^n, n \geq 2, \)
a Liouville theorem is a direct consequence of a Harnack inequality first obtained in [14]. Note also that a Liouville
theorem for mappings of \( \mathbb{R}^n, n > 2, \) with bounded distortion was first obtained in [13] by using the same Harnack
inequality from [14]. Finally, in the case of linear uniformly elliptic second-order partial differential equations on \( \mathbb{R}^2, \)
a Liouville theorem is a direct consequence of a Harnack inequality first obtained in [7].

2. Definitions

Let \( A(u) \) be a differential operator defined formally by
\[ A(u) = \sum_{i=1}^{n} \frac{d}{dx_i} A_i(x, u, \nabla u). \]  
(5)
Here and in what follows, \( n \geq 2. \) We assume that the functions \( A_i(x, \eta, \xi), \ i = 1, \ldots, n, \) satisfy the usual
Carathéodory conditions on \( \mathbb{R}^n \times \mathbb{R}^1 \times \mathbb{R}^n; \) namely, they are continuous in \( \eta \) and \( \xi \) for almost all \( x \in \mathbb{R}^n \)
and measurable in \( x \) for any \( \eta \in \mathbb{R}^1 \) and \( \xi \in \mathbb{R}^n. \)

**Definition 1.** Let \( \alpha > 1 \) be a given number. The operator \( A(u) \) given by (5) belongs to the class \( \mathcal{A}(\alpha) \) if for all \( \eta \in \mathbb{R}^1, \)
all \( \xi, \psi \in \mathbb{R}^n, \) and almost all \( x \in \mathbb{R}^n \) the following two inequalities hold:
\[ 0 \leq \sum_{i=1}^{n} \xi_i A_i(x, \eta, \xi), \]  
(6)
with equality only if \( \xi = 0, \) and
\[ \left| \sum_{i=1}^{n} \psi_i A_i(x, \eta, \xi) \right|^{\alpha} \leq K |\psi|^{\alpha} \left( \sum_{i=1}^{n} \xi_i A_i(x, \eta, \xi) \right)^{\alpha-1}, \]  
(7)
with \( K \) a certain positive constant.

It is easy to see that condition (7) is fulfilled whenever the inequality
\[ \left( \sum_{i=1}^{n} A_i^2(x, \eta, \xi) \right)^{\alpha/2} \leq K \left( \sum_{i=1}^{n} \xi_i A_i(x, \eta, \xi) \right)^{\alpha-1} \]  
(8)
holds for all \( \eta \in \mathbb{R}^1 \), all \( \xi, \psi \in \mathbb{R}^n \), and almost all \( x \in \mathbb{R}^n \). Hence, the operator \( A(u) \) given by (5) and satisfying conditions (6) and (8) belongs to the class \( \mathcal{A}(\alpha) \).

**Remark 1.** Conditions (7) and (8) on the behavior of the coefficients of partial differential operators were introduced in [10].

It is not difficult to verify that for any given \( p > 1 \) the differential operators (2) and (3) as well as the differential operator \( A(u) \) given by (5) and satisfying the well-known growth conditions

\[
\left( \sum_{i=1}^{n} A_i^2(x, \eta, \xi) \right)^{1/2} \leq K_1 |\xi|^{p-1} \tag{9}
\]

and

\[
|\xi|^p \leq K_2 \sum_{i=1}^{n} \xi_i A_i(x, \eta, \xi), \tag{10}
\]

with \( K_1, K_2 \) positive constants, belong to the class \( \mathcal{A}(\alpha) \) with \( \alpha = p \).

It is also easy to see that linear divergent elliptic partial differential operators of the form

\[
L := \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j} \left( a_{ij}(x) \frac{\partial}{\partial x_i} \right) \tag{11}
\]

with \( a_{ij}(x) \) measurable bounded coefficients and with the (possibly non-uniformly) positive-definite quadratic form

\[
\sum_{i,j=1}^{n} a_{ij}(x) \xi_i \xi_j \tag{12}
\]

belong to the class \( \mathcal{A}(\alpha) \) with \( \alpha = 2 \) but do not satisfy condition (10) for any fixed \( p > 1 \).

In connection with this we give another example of an operator that belongs to the class \( \mathcal{A}(\alpha) \) with a certain \( \alpha > 1 \) but does not satisfy condition (10) for any fixed \( p > 1 \). Let \( a(x, \eta, \xi) \) be a positive bounded function that satisfies the Carathéodory conditions on \( \mathbb{R}^n \times \mathbb{R}^1 \times \mathbb{R}^n \). It is easy to see that for a given \( p > 1 \) the weighted \( p \)-Laplacian

\[
\bar{\Delta}_p(u) := \text{div}(a(x, u, \nabla u)|\nabla u|^{p-2} \nabla u) \tag{13}
\]

belongs to the class \( \mathcal{A}(\alpha) \) with \( \alpha = p \) but does not satisfy condition (10) for any fixed \( p > 1 \) if the function \( a(x, \eta, \xi) \) is only assumed to be positive.

It can happen that an operator \( A(u) \) given by (5) belongs simultaneously to several different classes \( \mathcal{A}(\alpha) \). For example, the mean curvature operator \( \Xi(u) \) given by (4) belongs to the classes \( \mathcal{A}(\alpha) \) for all \( 1 < \alpha \leq 2 \); similarly its modification for \( p \geq 2 \),

\[
\Xi_p(u) := \text{div} \left( \frac{|\nabla u|^{p-2} \nabla u}{\sqrt{1 + |\nabla u|^2}} \right), \tag{14}
\]

belongs to the classes \( \mathcal{A}(\alpha) \) for all \( \alpha \in (p-1, p] \) and \( p \geq 2 \). Obviously, operators given by (4) and (14) do not satisfy conditions (9)–(10) for any fixed \( p \geq 1 \).

**Definition 2.** Let \( \alpha > 1 \) be a given number, and let the operator \( A(u) \) given by (5) belong to the class \( \mathcal{A}(\alpha) \). A measurable function \( u : \mathbb{R}^n \to \mathbb{R}^1 \) is called an entire weak supersolution of Eq. (1) on \( \mathbb{R}^n \) if \( u \in L^1_{\text{loc}}(\mathbb{R}^n) \), \( |\nabla u| \in L^{\alpha}_{\text{loc}}(\mathbb{R}^n) \), and the integral inequality

\[
\int_{\mathbb{R}^n} \sum_{i=1}^{n} \varphi_{\xi_i} A_i(x, u, \nabla u) \, dx \geq 0 \tag{15}
\]

holds for every non-negative function \( \varphi \in W^{1,\alpha}(\mathbb{R}^n) \) with compact support.
3. Results

**Theorem 1.** Let $n \geq 2$ and $\alpha > 1$ be given numbers such that $\alpha \geq n$. Let the operator $A(u)$ given by (5) belong to the class $A(\alpha)$, and let $u(x)$ be an entire weak supersolution of (1) on $\mathbb{R}^n$ bounded below by a constant. Then $u(x)$ is a constant on $\mathbb{R}^n$.

**Theorem 2.** Let $n \geq 2$ and $\alpha > 1$ be given numbers such that $n > \alpha$. Let the operator $A(u)$ given by (5) belong to the class $A(\alpha)$, and let $u(x)$ be an entire weak supersolution of (1) on $\mathbb{R}^n$ bounded below by a constant $c$ and such that $u \in L^\infty_{\text{loc}}(\mathbb{R}^n)$. Then either $u(x) = c$ on $\mathbb{R}^n$ or the relation
\[
\liminf_{r \to +\infty} \left[ \sup_{r \leq |x| \leq 2r} (u(x) - c) \right] r^{\frac{\alpha - \sigma}{\alpha - 1}} = +\infty
\]  
holds with any fixed $\nu \in (0, \alpha - 1)$.

**Theorem 3.** Let $n \geq 2$ and $\alpha > 1$ be given numbers such that $n > \alpha$. Let the operator $A(u)$ given by (5) belong to the class $A(\alpha)$, and let $u(x)$ be an entire weak supersolution of (1) on $\mathbb{R}^n$ bounded below by a constant $c$. Then either $u(x) = c$ on $\mathbb{R}^n$ or the relation
\[
\liminf_{r \to +\infty} r^{-\alpha} \int_{r \leq |x| \leq 2r} (u(x) - c)^{\alpha - 1 - \nu} \, dx = +\infty
\]  
holds with any fixed $\nu \in (0, \alpha - 1)$.

Due to the arbitrariness of the constant $c$ in Theorems 2 and 3, the statements of these theorems can be reformulated in a slightly different form.

**Theorem 2'.** Let $n \geq 2$ and $\alpha > 1$ be given numbers such that $n > \alpha$. Let the operator $A(u)$ given by (5) belong to the class $A(\alpha)$, and let $u(x)$ be an entire weak supersolution of (1) on $\mathbb{R}^n$ bounded below by a constant and such that $u \in L^\infty_{\text{loc}}(\mathbb{R}^n)$. Then either $u(x)$ is a constant on $\mathbb{R}^n$ or relation (16) holds with any fixed real number $\nu$ such that $u(x) \geq c$ on $\mathbb{R}^n$ and any fixed $\nu \in (0, \alpha - 1)$.

**Theorem 3'.** Let $n \geq 2$ and $\alpha > 1$ be given numbers such that $n > \alpha$. Let the operator $A(u)$ given by (5) belong to the class $A(\alpha)$, and let $u(x)$ be an entire weak supersolution of (1) on $\mathbb{R}^n$ bounded below by a constant. Then either $u(x)$ is a constant on $\mathbb{R}^n$ or relation (17) holds with any fixed real number $\nu$ such that $u(x) \geq c$ on $\mathbb{R}^n$ and any fixed $\nu \in (0, \alpha - 1)$.

**Remark 2.** It is important to note that for any given $n \geq 2$ and $\alpha > 1$ such that $n > \alpha$ the function
\[
u(x) = (1 + |x|^{\frac{\alpha}{\alpha - 1}})^{\frac{n - \alpha}{\alpha}}
\]  
is an entire weak supersolution of the equation
\[
\Delta_p(u) = 0
\]  
with $p = \alpha$ that is bounded below and is such that relations (16) and (17) hold with any fixed $\nu \in (0, \alpha - 1)$ and, at the same time, the relations
\[
\lim_{r \to +\infty} \left[ \sup_{r \leq |x| \leq 2r} (u(x) - 0) \right] r^{\frac{n - \alpha}{\alpha - 1}} = C_1
\]  
and
\[
\lim_{r \to +\infty} r^{-\alpha} \int_{r \leq |x| \leq 2r} (u(x) - 0)^{\alpha - 1} \, dx = C_2,
\]  
with $C_1, C_2$ certain positive constants, also hold.
Remark 3. The results of this work were announced in [5]. To prove these results we further develop an approach that was proposed for solving similar problems in [6].

Remark 4. The results of Theorem 1 are new only for \( \alpha = n \). Similar results to those of Theorem 1 for entire weak continuous supersolutions of (1) on \( \mathbb{R}^n \) for \( \alpha = n \) were first obtained in [11]. For \( \alpha > n \), the results of Theorem 1 for entire weak supersolutions of (1) on \( \mathbb{R}^n \), which in this case are continuous on \( \mathbb{R}^n \) by the well-known Sobolev imbedding theory, were also first obtained in [11]. Here, we give a new proof of these results from [11] by developing an approach from [6] which does not explicitly use the continuity of entire weak supersolutions of (1) on \( \mathbb{R}^n \).

Remark 5. In the case when \( \alpha = p \) and \( A(u) = \Delta_p(u) \), Theorem 1 coincides with well-known Liouville-type theorems for entire superharmonic and \( p \)-superharmonic functions locally bounded on \( \mathbb{R}^n \) (see, e.g., [2, p. 68] and [3, p. 179]). Also, in this case, the results of Theorems 2 and 3 correlate well with certain results in the theory of entire superharmonic and \( p \)-superharmonic functions (see, e.g., [2, pp. 131, 139] and [3, pp. 133, 135]).

4. Proofs

Proof of Theorem 2. The statement of Theorem 2 follows immediately from Theorem 3. In fact, let \( n \geq 2 \) and \( \alpha > 1 \) be given numbers such that \( n > \alpha \). Let the operator \( A(u) \) given by (5) belong to the class \( \mathcal{A}(\alpha) \), and let \( u(x) \) be an entire weak supersolution of (1) on \( \mathbb{R}^n \) bounded below by a constant \( c \), i.e., \( u(x) \geq c \) on \( \mathbb{R}^n \), and such that \( u \in L^\infty_{\text{loc}}(\mathbb{R}^n) \). Hence, by Theorem 3, either \( u(x) = c \) on \( \mathbb{R}^n \) or relation (17) holds with any fixed \( v \in (0, \alpha - 1) \). Further, via the trivial inequality

\[
 r^{-\alpha} \int_{r \leq |x| \leq 2r} (u(x) - c)^{\alpha-1-v} \, dx \leq r^{-\alpha} \left( \sup_{r \leq |x| \leq 2r} (u(x) - c)^{\alpha-1-v} \right) \int_{r \leq |x| \leq 2r} \, dx, \tag{22}
\]

which obviously holds for any \( r > 0 \), it follows from (17) that

\[
 \liminf_{r \to +\infty} \left[ \sup_{r \leq |x| \leq 2r} (u(x) - c)^{\alpha-1-v} \right] r^{n-\alpha} = +\infty. \tag{23}
\]

Then, since

\[
 \sup_{r \leq |x| \leq 2r} (u(x) - c)^{\alpha-1-v} \leq \left[ \sup_{r \leq |x| \leq 2r} (u(x) - c)^{\alpha-1-v} \right]^{\alpha-1-v} \tag{24}
\]

and

\[
 \left[ \sup_{r \leq |x| \leq 2r} (u(x) - c)^{\alpha-1-v} \right] r^{n-\alpha} = \left( \left[ \sup_{r \leq |x| \leq 2r} (u(x) - c)^{\alpha-1-v} \right] r^{\frac{\alpha-1-v}{\alpha-1}} \right)^{\alpha-1-v}, \tag{25}
\]

the validity of (16) follows immediately from that of (23) and (25). \( \square \)

In what follows, a ‘smooth’ function is a \( C^\infty \)-function on \( \mathbb{R}^n \), \( B(r) \) is an open ball on \( \mathbb{R}^n \) of radius \( r > 0 \) centered at the origin, and \( \overline{B}(r) \) is the closure of \( B(r) \).

Proof of Theorem 3. Let \( n \geq 2 \) and \( \alpha > 1 \) be given numbers such that \( n > \alpha \). Let the operator \( A(u) \) given by (5) belong to the class \( \mathcal{A}(\alpha) \), and let \( u(x) \) be an entire weak supersolution of (1) on \( \mathbb{R}^n \) bounded below by a constant \( c \), i.e., \( u(x) \geq c \) on \( \mathbb{R}^n \). Let \( r \) and \( \varepsilon \) be positive numbers, and let \( \zeta : \mathbb{R}^n \to [0, 1] \) be a smooth function which equals 1 on \( \overline{B}(r) \) and 0 outside \( B(2r) \). Substituting, without loss of generality, \( \varphi(x) = (u(x) - c + \varepsilon)^{-v} \zeta^\alpha(x) \) as a test function in inequality (15), where \( v \in (0, \alpha - 1) \) is arbitrary, and integrating by parts, we find

\[
 \alpha \int_{B(2r) \setminus B(r)} \sum_{i=1}^n \zeta_{x_i} A_i(x, u, \nabla u)(u - c + \varepsilon)^{-v} \zeta^{\alpha-1} \, dx \\
 \geq v \int_{B(2r)} \sum_{i=1}^n u \zeta_{x_i} A_i(x, u, \nabla u)(u - c + \varepsilon)^{-v-1} \zeta^\alpha \, dx. \tag{26}
\]
Estimating the left-hand side of (26) by using condition (7) on the coefficients of the operator $A(u)$, we have

$$\begin{align*}
\alpha K^{1/\alpha} & \int_{B(2r) \setminus B(r)} \left( \sum_{i=1}^{n} u_{x_i} A_i(x, u, \nabla u) \right)^{(\alpha-1)/\alpha} |\nabla \zeta|(u - c + \epsilon)^{-v} \xi^{\alpha-1} \, dx \\
& \geq \alpha \int_{B(2r) \setminus B(r)} \left( \sum_{i=1}^{n} \zeta_{x_i} A_i(x, u, \nabla u)(u - c + \epsilon)^{-v} \xi^{\alpha-1} \, dx \right).
\end{align*}$$

(27)

Further, estimating the left-hand side of (27) by Hölder’s inequality, we arrive at

$$\begin{align*}
\alpha K^{1/\alpha} & \left( \int_{B(2r) \setminus B(r)} |\nabla \zeta|^\alpha (u - c + \epsilon)^{\alpha-1-v} \, dx \right)^{1/\alpha} \\
& \times \left( \int_{B(2r) \setminus B(r)} \sum_{i=1}^{n} u_{x_i} A_i(x, u, \nabla u)(u - c + \epsilon)^{-v-1} \xi^{\alpha} \, dx \right)^{(\alpha-1)/\alpha} \\
& \geq \alpha \int_{B(2r) \setminus B(r)} \left( \sum_{i=1}^{n} \zeta_{x_i} A_i(x, u, \nabla u)(u - c + \epsilon)^{-v-1} \xi^{\alpha} \, dx \right) .
\end{align*}$$

(28)

In turn, (26) and (28) imply the inequality

$$\begin{align*}
\alpha K^{1/\alpha} & \left( \int_{B(2r) \setminus B(r)} |\nabla \zeta|^\alpha (u - c + \epsilon)^{\alpha-1-v} \, dx \right)^{1/\alpha} \\
& \times \left( \int_{B(2r) \setminus B(r)} \sum_{i=1}^{n} u_{x_i} A_i(x, u, \nabla u)(u - c + \epsilon)^{-v-1} \xi^{\alpha} \, dx \right)^{(\alpha-1)/\alpha} \\
& \geq v \int_{B(2r)} \sum_{i=1}^{n} u_{x_i} A_i(x, u, \nabla u)(u - c + \epsilon)^{-v-1} \xi^{\alpha} \, dx
\end{align*}$$

(29)

and, therefore, the inequality

$$\begin{align*}
\alpha K & \int_{B(2r) \setminus B(r)} |\nabla \zeta|^\alpha (u - c + \epsilon)^{\alpha-1-v} \, dx \geq v^\alpha \int_{B(r)} \sum_{i=1}^{n} u_{x_i} A_i(x, u, \nabla u)(u - c + \epsilon)^{-v-1} \, dx .
\end{align*}$$

(30)

It is easy to see that the right-hand side of (30) increases monotonically if $\epsilon > 0$ decreases strongly monotonically to zero. Therefore, it follows from (30) that the inequality

$$\begin{align*}
\alpha K & \int_{B(2r) \setminus B(r)} |\nabla \zeta|^\alpha (u - c + \epsilon)^{\alpha-1-v} + \alpha \delta \, dx \geq v^\alpha \int_{B(r)} \sum_{i=1}^{n} u_{x_i} A_i(x, u, \nabla u)(u - c + \epsilon)^{-v-1} \, dx + \alpha \delta
\end{align*}$$

(31)

holds with any $\delta > 0$ and any $\epsilon \in (0, \delta]$. Since for any sequence $\epsilon_k > 0$ monotonically decreasing to zero as $k \to +\infty$ the sequence of functions

$$\Phi_k(x) := |\nabla \zeta|^\alpha (u - c + \epsilon_k)^{\alpha-1-v}$$

(32)

measurable on $\mathbb{R}^n$ converges a.e. on $\mathbb{R}^n$ to the function

$$\Phi(x) := |\nabla \zeta|^\alpha (u - c)^{\alpha-1-v}$$

(33)

measurable on $\mathbb{R}^n$, since for sufficiently large $k$

$$|\Phi_k(x)| \leq |\nabla \zeta|^\alpha (u - c + 1)^{\alpha-1-v}$$

(34)
on $\mathbb{R}^n$, and since the function
\[ |\nabla \zeta|^\alpha (u - c + 1)^{\alpha - 1 - \nu} \]
is locally integrable on $\mathbb{R}^n$, then, by Lebesgue’s theorem (see, e.g., [4, p. 303]), for $\varepsilon = \varepsilon_k > 0$ monotonically decreasing to zero we can pass to the limit as $k \to +\infty$ on the left-hand side of (31). As a result, we obtain the inequality
\[ \alpha \kappa \int_{B(2r) \setminus B(r)} |\nabla \zeta|^\alpha (u - c)^{\alpha - 1 - \nu} \, dx \geq \nu^\alpha \int_{B(r)} \sum_{i=1}^n u_{x_i} A_i(x, u, \nabla u) (u - c + \delta)^{-\nu - 1} \, dx, \]
which holds with any $\delta > 0$. Then, for any $r > 0$ and any sequence $\varepsilon_k > 0$ monotonically decreasing to zero as $k \to +\infty$, it follows from (36), by letting $\delta = \varepsilon_k$ and
\[ \Psi_k(x) := \sum_{i=1}^n u_{x_i} A_i(x, u, \nabla u) (u - c + \varepsilon_k)^{-\nu - 1}, \]
that the sequence of integrals
\[ \int_{B(r)} \Psi_k(x) \, dx \]
is bounded above by the positive constant
\[ c_1 = K \left( \frac{\alpha}{\nu} \right)^\alpha \int_{B(2r) \setminus B(r)} |\nabla \zeta|^\alpha (u - c)^{\alpha - 1 - \nu} \, dx, \]
which does not depend on $\varepsilon_k$. Hence, since
\[ \Psi_1(x) \leq \Psi_2(x) \leq \cdots \leq \Psi_k(x) \leq \cdots \]
on $\mathbb{R}^n$, then by Beppo Levi’s theorem (see, e.g., [4, p. 305]), for any $r > 0$ there exists a function $\Theta_r : B(r) \to \mathbb{R}^1$ integrable on $B(r)$ and such that the sequence of functions $\Psi_k(x)$ converges a.e. to $\Theta_r(x)$ on $B(r)$ and
\[ \lim_{k \to +\infty} \int_{B(r)} \Psi_k(x) \, dx = \int_{B(r)} \Theta_r(x) \, dx. \]
Further, it is easy to see that the family of functions $\{\Theta_r\}_{r>0}$ uniquely determines a function $\Psi : \mathbb{R}^n \to \mathbb{R}^1$ which is non-negative, measurable, locally integrable on $\mathbb{R}^n$ and is such that $\Psi(x) = \Theta_r(x)$ on $B(r)$ for all $r > 0$. Therefore, the sequence of functions $\Psi_k(x)$ given by (37) converges a.e. to $\Psi(x)$ on $\mathbb{R}^n$ for any sequence $\varepsilon_k > 0$ monotonically decreasing to zero as $k \to +\infty$. Then, by choosing $\delta = \varepsilon_k$ in (36), where the sequence $\varepsilon_k > 0$ converges monotonically to zero as $k \to +\infty$, and passing to the limit on the right-hand side of (36), we find, due to (41), the inequality
\[ \alpha K \int_{B(2r) \setminus B(r)} |\nabla \zeta|^\alpha (u - c)^{\alpha - 1 - \nu} \, dx \geq \nu^\alpha \int_{B(r)} \Psi(x) \, dx. \]
We divide the rest of the proof into three cases according to the behavior of the right-hand side of (42), which can monotonically approach zero, $+\infty$, or some positive number $I$ as $r$ strongly monotonically approaches $+\infty$.
If the right-hand side of (42) approaches zero as $r \to +\infty$, then, due to the non-negativity of the function $\Psi(x)$, we have that $\Psi(x) = 0$ on $\mathbb{R}^n$. Further, since by (37) and (40) the inequality
\[ \sum_{i=1}^n u_{x_i} A_i(x, u, \nabla u) (u - c + \varepsilon_k)^{-\nu - 1} \leq \Psi(x) \]
holds on $\mathbb{R}^n$ for any sequence $\varepsilon_k > 0$ monotonically decreasing to zero as $k \to +\infty$, then, again, due to the non-negativity of the left-hand side of (43), we obtain that
\[ \sum_{i=1}^n u_{x_i} A_i(x, u, \nabla u) (u - c + \varepsilon_k)^{-\nu - 1} = 0 \]
on \( \mathbb{R}^n \). Hence, by condition (6) on the coefficients of the operator \( A(u) \), the supersolution \( u(x) = \text{const.} \) on \( \mathbb{R}^n \), and, therefore, either \( u(x) = c \) on \( \mathbb{R}^n \) or relation (17) holds with any fixed \( \nu \in (0, \alpha - 1) \).

If the right-hand side of (42) approaches \(+\infty\) as \( r \to +\infty \), then, due to monotonicity, (42) yields that

\[
\liminf_{r \to +\infty} \int_{B(2r) \setminus B(r)} |\nabla \zeta|^\alpha (u - c)'^{\alpha - 1 - \nu} \, dx = +\infty. \tag{45}
\]

Finally, if the right-hand side of (42) monotonically approaches a certain positive number \( I \) as \( r \) approaches \(+\infty\), i.e.,

\[
\lim_{r \to +\infty} \nu^{\alpha} \int_{B(r)} \Psi(x) \, dx = I > 0, \tag{46}
\]

we again consider inequality (29), just noting here that, due to monotonicity,

\[
\int_{B(2r_k) \setminus B(r_k)} \Psi(x) \, dx \to 0 \tag{47}
\]

for any sequence \( r_k > 0 \) such that \( r_k \to +\infty \). First, we have from (29) the inequality

\[
\alpha K^{1/\alpha} \left( \int_{B(2r) \setminus B(r)} |\nabla \zeta|^\alpha (u - c)'^{\alpha - 1 - \nu} \, dx \right)^{1/\alpha} \times \left( \int_{B(2r) \setminus B(r)} \sum_{i=1}^n u_{x_i} A_i(x, u, \nabla u)(u - c + \varepsilon)^{\nu - 1} \, dx \right)^{(\alpha - 1)/\alpha} \geq \nu \int_{B(r)} \sum_{i=1}^n u_{x_i} A_i(x, u, \nabla u)(u - c + \varepsilon)^{\nu - 1} \, dx. \tag{48}
\]

In (48), let \( \varepsilon = \varepsilon_k > 0 \) converge monotonically to zero as \( k \to +\infty \). Then, by Lebesgue’s theorem (see, e.g., [4, p. 303]), we can pass to the limit on both sides of (48). Namely, we know from the above that for any sequence \( \varepsilon_k > 0 \) monotonically decreasing to zero as \( k \to +\infty \) the sequences of functions \( \Phi_k(x) \) and \( \Psi_k(x) \) measurable and locally integrable on \( \mathbb{R}^n \) and given, respectively, by (32) and (37), converge a.e. on \( \mathbb{R}^n \), respectively, to the functions \( \Phi(x) \) and \( \Psi(x) \) measurable and locally integrable on \( \mathbb{R}^n \). Further, arguing as above and letting \( \varepsilon = \varepsilon_k \) or \( 0 \) monotonically decrease to zero as \( k \to +\infty \), by Lebesgue’s theorem (see, e.g., [4, p. 303]) we can pass to the limit on both sides of (48). As a result, we arrive at the inequality

\[
\alpha K^{1/\alpha} \left( \int_{B(2r) \setminus B(r)} |\nabla \zeta|^\alpha (u - c)'^{\alpha - 1 - \nu} \, dx \right)^{1/\alpha} \left( \int_{B(2r) \setminus B(r)} \Psi(x) \, dx \right)^{(\alpha - 1)/\alpha} \geq \nu \int_{B(r)} \Psi(x) \, dx. \tag{49}
\]

In (49), for \( r = r_k > 0 \) monotonically increasing to \(+\infty\), by passing to the limit as \( r_k \to +\infty \), we obtain from (46), (47), and (49) that

\[
\lim_{r_k \to +\infty} \int_{B(2r_k) \setminus B(r_k)} |\nabla \zeta|^\alpha (u - c)'^{\alpha - 1 - \nu} \, dx = +\infty. \tag{50}
\]

Thus, due to the arbitrariness in the choice of the sequence \( r_k \) in (50), we again arrive at relation (45).

Now, without loss of generality, we choose in (45) the function \( \zeta(x) \) in the form \( \zeta(x) = \psi(|x|/(2r)) \), where \( \psi: [0, +\infty) \to [0, 1] \) is a smooth function that equals 1 on \([0, 1/2]\) and 0 on \([1, +\infty)\) and is such that the inequality

\[
|\nabla \zeta| \leq c_2 r^{-1} \tag{51}
\]

holds on \( \mathbb{R}^n \) with a certain positive constant \( c_2 \) for an arbitrary \( r > 0 \). Relation (17) then follows immediately from (45) and (51). \( \square \)
Proof of Theorem 1. Let \( n \geq 2 \) and \( \alpha > 1 \) be given numbers such that \( \alpha \geq n \). Let the operator \( A(u) \) given by (5) belong to the class \( \mathcal{A}(\alpha) \), and let \( u(x) \) be an entire weak supersolution of (1) on \( \mathbb{R}^n \) bounded below by a constant \( c \), i.e., \( u(x) \geq c \) on \( \mathbb{R}^n \). Let \( r, R, \) and \( \varepsilon \) be positive numbers such that \( R > r \), and let \( \zeta : \mathbb{R}^n \to [0, 1] \) be a smooth function which equals 1 on \( B(R) \) and 0 outside \( B(R) \). Substituting, without loss of generality, \( \varphi(x) = (u(x) - c + \varepsilon)^{-\alpha}\zeta(x) \) as a test function in inequality (15), where \( \nu > \alpha - 1 \) is an arbitrary positive number, and integrating by parts, we have the inequality

\[
\alpha \int_{B(R) \setminus B(r)} \sum_{i=1}^{n} \xi_i A_i(x, u, \nabla u)(u - c + \varepsilon)^{-\alpha} \, dx \geq \nu \int_{B(R)} \sum_{i=1}^{n} u_{x_i} A_i(x, u, \nabla u)(u - c + \varepsilon)^{-\alpha} \, dx. \tag{52}
\]

Further, we repeat the proof of Theorem 3 word for word from (26) to (30). As a result, we arrive at the inequality

\[
\alpha^\alpha K \int_{B(R) \setminus B(r)} |\nabla \zeta|^\alpha (u - c + \varepsilon)^{-1} \, dx \geq \nu^\alpha \int_{B(r)} \sum_{i=1}^{n} u_{x_i} A_i(x, u, \nabla u)(u - c + \varepsilon)^{-\nu} \, dx. \tag{53}
\]

It follows immediately from (53) that the inequality

\[
\alpha^\alpha \varepsilon^{\alpha - 1 - \nu} K \int_{B(R) \setminus B(r)} |\nabla \zeta|^\alpha \, dx \geq \nu^\alpha \int_{B(r)} \sum_{i=1}^{n} u_{x_i} A_i(x, u, \nabla u)(u - c + \varepsilon)^{-\nu} \, dx \tag{54}
\]

holds with any fixed \( \varepsilon > 0 \) and \( \nu > \alpha - 1 \).

Now, first let \( \alpha > n \). In (54), choosing \( R = 2r \) and the function \( \zeta(x) \) in the form \( \zeta(x) = \psi(|x|/R) \), where \( \psi : [0, +\infty) \to [0, 1] \) is a smooth function that equals 1 on \([0, 1/2]\) and 0 on \([1, +\infty)\) and is such that the inequality (51) holds on \( \mathbb{R}^n \) with a certain positive constant \( c_2 \) for an arbitrary \( R > 0 \), we obtain from (51) and (54) the inequality

\[
c_3 \varepsilon^{n - \alpha} \geq \int_{B(r)} \sum_{i=1}^{n} u_{x_i} A_i(x, u, \nabla u)(u - c + \varepsilon)^{-\nu} \, dx, \tag{55}
\]

which holds with a certain positive constant \( c_3 \) that does not depend on \( r \). Passing to the limit as \( r \to +\infty \) in (55), we find, due to the non-negativity of the integrand, that the equality

\[
\sum_{i=1}^{n} u_{x_i} A_i(x, u, \nabla u)(u - c + \varepsilon)^{-\nu} = 0 \tag{56}
\]

holds on \( \mathbb{R}^n \), and, therefore, by condition (6) on the coefficients of the operator \( A(u) \), that \( u(x) = \text{const.} \) on \( \mathbb{R}^n \).

If \( \alpha = n \), we choose in (54) the function \( \zeta(x) \) in the form \( \zeta(x) = \psi(|x|/R) \) with arbitrary \( R > r > 1 \), where \( \psi : [-\infty, +\infty) \to [0, 1] \) is a smooth function which equals 1 on \([-\infty, 0]\) and 0 on \([1, +\infty)\). It is not difficult to understand (see, e.g., [9, p. 12]) that the inequality

\[
|\nabla \zeta(x)| \leq \frac{c_4}{|x| \ln(R/r)} \tag{57}
\]

holds on \( \mathbb{R}^n \) with a certain positive constant \( c_4 \) for arbitrary \( R > r > 1 \). It then follows from (54) and (57) that the inequality

\[
c_5 \int_{B(R) \setminus B(r)} (|x| \ln(R/r))^{-n} \, dx \geq \int_{B(r)} \sum_{i=1}^{n} u_{x_i} A_i(x, u, \nabla u)(u - c + \varepsilon)^{-\nu} \, dx \tag{58}
\]

holds, and, therefore, so does the inequality

\[
c_6 (\ln(R/r))^{-n + 1} \geq \int_{B(r)} \sum_{i=1}^{n} u_{x_i} A_i(x, u, \nabla u)(u - c + \varepsilon)^{-\nu} \, dx \tag{59}
\]
with arbitrary $R > r > 1$ and certain positive constants $c_5$ and $c_6$ that do not depend on $R$. Passing to the limit as $R \to +\infty$ in (59), we find that the equality

$$\int_{B(r)} \sum_{i=1}^{n} u_{x_i} A_i(x, u, \nabla u)(u - c + \varepsilon)^{-v-1} \, dx = 0$$

(60)

holds with an arbitrary $r > 1$. Passing to the limit as $r \to +\infty$ in (60), we again obtain, due to the non-negativity of the integrand in (60) and by condition (6), that $u(x) = \text{const.}$ on $\mathbb{R}^n$. □

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**References**