Existence and blow up of solutions to certain classes of two-dimensional nonlinear Neumann problems

L’existence et l’explosion de solutions de certaines problèmes Neumann bi-dimensionnels et non linéaires

Kai Medville, Michael S. Vogelius *

Department of Mathematics, Rutgers University, New Brunswick, NJ 08903, USA

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Abstract

In this paper we study, analytically and numerically, the existence and blow up of solutions to two-dimensional boundary value problems of the form $\Delta u_{\lambda} = 0$ in $\Omega$, $\frac{\partial u_{\lambda}}{\partial n} = Du_{\lambda} + \lambda f(u_{\lambda})$ on $\partial \Omega$. We place particular emphasis on $f(u) = \sinh(u) = \left( e^u - e^{-u} \right)/2$, in which case the nonlinear flux boundary condition is frequently associated with the names of Butler and Volmer.

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1. Introduction

Let $\Omega$ be a smooth ($C^\infty$), bounded domain in $\mathbb{R}^2$, and consider the elliptic boundary value problem

\[
\Delta u_{\lambda} = 0 \quad \text{in } \Omega, \\
\frac{\partial u_{\lambda}}{\partial n} = Du_{\lambda} + \lambda \sinh(u_{\lambda}) \quad \text{on } \partial \Omega.
\]

* Corresponding author. 
E-mail address: vogelius@math.rutgers.edu (M.S. Vogelius).

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This is a simplified model problem, the likes of which frequently show up in connection with corrosion/oxidation modeling. Such problems also appear when modeling the electronic properties of certain semiconductor interface systems, for instance MIS (metal-insulator) or MOS (metal-oxide) semiconductor systems [1]. For the latter modeling it is often assumed that the surface electric field is controlled by the use of certain gaseous ambients. The exponential character of the flux boundary condition in general reflects the fact that the charged particles (the electrons) are thought to be regulated by a Boltzmann statistics. For a discussion of some practical aspects of these problems, and some references to the applied literature we refer the reader to [1, 7], and [17]. For a discussion of related problems with "interior" exponential terms, see [2].

For \( \lambda < \min\{-D, 0\} \) the solution of (1) is trivial: zero is the only solution. Our focus is thus on certain nontrivial (nonzero) solutions corresponding to \( \lambda > \min\{-D, 0\} \), and in particular on the asymptotic behavior of these solutions as \( \lambda \) approaches zero. As it turns out there is a significant difference between the solution structure for \( -D < \lambda < 0 \) (if such an interval exists) and the solution structure for \( 0 < \lambda \). For any value of \( \lambda \) in the interval \( -D < \lambda < 0 \) we establish the existence of finitely many (and at least one) nontrivial solutions, whereas for any \( \lambda > 0 \) we establish the existence of infinitely many nontrivial solutions. Our existence proof is based on a variational technique, the likes of which have been used in several related contexts (see [9,14] and [16]). Maybe more surprising than the dichotomy of the solution structure is the difference in the asymptotic behavior of these solutions as \( \lambda \) approaches 0 from below and above, respectively.

For \( \lambda > 0 \) we show that any of the (infinitely many) families of solutions we construct generically contains a subsequence along which the nonlinear flux components \( \lambda \sinh(u_\lambda) \) converge to a finite, nontrivial sum of delta functions \( \sum_{i=1}^{K} C_i \delta_{x_i} \). Along the same subsequence the functions \( u_\lambda \) will, modulo a possible eigen-component (that can only appear for a countable set of \( D \)'s) have a finite limit at all but a finite set of boundary points (generically \( \{x_i\}_{i=1}^{K} \)). Finally, under very minimal assumptions, we derive a set of \( K \) necessary conditions for the point-mass locations \( x_i \).

For \( \lambda < 0 \) we show that any family of solutions which does not converge to 0, as \( \lambda \to 0_- \), contains a subsequence which blows up pointwise almost everywhere as \( \lambda \to 0_- \). We also show that along such a subsequence the nonlinear flux components \( \lambda \sinh(u_\lambda) \) blow up in \( H^{-1/2}(\partial \Omega) \), and even after a rescaling they converge weakly to a distribution, which is not supported at a finite set of points.

It is interesting to note that for \( -D < \lambda < 0 \) we establish an upper bound for \( \| u_\lambda \|^2_{H^1(\Omega)} \) of the order \( (\log \frac{1}{1+\lambda})^2 \), as \( \lambda \to 0_- \), whereas for the solutions we construct corresponding to \( 0 < \lambda \) we establish an upper bound for the "essential" part of \( \| u_\lambda \|^2_{H^1(\Omega)} \) of the order \( \log \frac{1}{1+\lambda} \). We do not claim that the solutions we variationally construct necessarily represent all solutions to the problem (1) – in certain situations (e.g. for an annulus, or for a disk and \( D \) negative) it is not hard to find an extra family of solutions for \( \lambda > 0 \) such that the "essential" part of \( \| u_\lambda \|^2_{H^1(\Omega)} \) grows like \( (\log \frac{1}{1+\lambda})^2 \) as \( \lambda \to 0_+ \), and the functions \( u_\lambda \) as well as the boundary flux components \( \lambda \sinh(u_\lambda) \) blow up almost everywhere. In the situations mentioned above these additional solutions possess higher order bifurcations, whereas the other solutions do not appear to possess any. We provide some numerical data that cast extra light on this phenomenon.

We have already in earlier papers [5,10] and [12] analyzed the special case \( \frac{\partial u}{\partial n} = \lambda \sinh(u) \) (i.e., \( D = 0 \)). That analysis included a study of the existence structure as well as a study of the asymptotic behavior as \( \lambda \to 0_+ \). In that case there are no nontrivial solutions for \( \lambda < 0 \), and so one does not encounter the quite remarkable difference in existence structure and blow-up behavior between \( \lambda \to 0_- \) and \( \lambda \to 0_+ \). For \( D = 0 \) we have presented strong numerical evidence that the fluxes \( \lambda \sinh(u_\lambda) \) might occasionally (depending on the domain \( \Omega \)) converge to a sum of delta functions plus a regular part, as \( \lambda \to 0_+ \). [12] This should be compared to the results in this paper which (with only minor additional assumptions) show that, for \( D \neq 0 \), the functions \( \lambda \sinh(u_\lambda) \) can only converge to a pure sum of delta functions – the limit of the fluxes \( Du_\lambda + \lambda \sinh(u_\lambda) \), on the other hand, will always contain a regular part as well.

Of particular interest for \( D \) are Steklov eigenvalues, i.e., the case when \( D = D_k^s \) for any of the countable set of nonnegative values \( 0 = D_1 \leq D_2 \leq \cdots \) for which the linear boundary value problem

\[
\Delta \phi = 0 \quad \text{in } \Omega, \quad \frac{\partial \phi}{\partial n} = D \phi \quad \text{on } \partial \Omega,
\]

possesses nontrivial solutions. We may think of the corresponding bound states \( \phi_k \) as associated with impurities and defects. As stated in [1] such bound states are known to "strongly affect the electrical properties of the bulk semiconductor" – the results in the present paper (in particular the estimates of the finite dimensional projection \( PD u_\lambda \)) provide some qualitative and quantitative clarification of this statement.
In the case of $D = 0$, and a domain in the shape of a disk, it is possible to give explicit (and surprisingly simple) formulas for what we believe to be all the solutions to (1), see [5]. We have not been able to derive similar explicit formulas for $D \neq 0$.

Finally we present some numerical calculations and some heuristic arguments related to the solution structure of the boundary value problem

$$\Delta u_\lambda = 0 \quad \text{in } \Omega,$$
$$\frac{\partial u_\lambda}{\partial n} = Dv_\lambda + \lambda f(u_\lambda) \quad \text{on } \partial \Omega,$$

for several other functions $f$ (odd and with $f'(0) = 1$, $f(t) > 0$ for $t > 0$).

2. The Butler–Volmer case

In this and the following two subsections we provide a very careful analysis of the existence and the asymptotic behavior of solutions to the boundary value problem (1). As already mentioned, nonlinear boundary flux conditions of this (exponential) type are frequently found in the corrosion literature; they are often associated with the names of Butler and Volmer.

When we talk about a solution to (1) we shall always mean a real function $u_\lambda \in H^1(\Omega)$ which satisfies the boundary value problem in the weak sense that

$$\int_\Omega \nabla u_\lambda \nabla v \, dx = D \int_{\partial \Omega} u_\lambda v \, d\sigma_x + \lambda \int_{\partial \Omega} \sinh(u_\lambda) v \, d\sigma_x,$$

for any $v \in H^1(\Omega)$. The fact that we restrict attention to domains $\Omega$ that are two dimensional ensures that $e^v$ is in $L^p(\partial \Omega)$, $1 < p < \infty$, for any $v \in H^1(\Omega)$. It furthermore ensures that the mapping $v \mapsto e^v|_{\partial \Omega}$ is compact from $H^1(\Omega)$ to $L^p(\partial \Omega)$, $1 < p < \infty$. These facts are both essential for our present analysis, in particular as far as existence of solutions to (1) is concerned. Very classical results from elliptic regularity theory ensure that any weak, finite energy solution is a classical solution to (1); indeed it is $C^\infty(\overline{\Omega})$.

Before proceeding let us present some computational results that provide intuition concerning the kind of results we might expect to be able to prove. For our computational experiments we take $\Omega$ to be the unit disk, and we first take $D = 2$ (a Steklov eigenvalue). Based on a boundary integral formulation, a collocation method, and a “continuation scheme” we now calculate what we believe to be all the nontrivial solutions to (1) (modulo rotations). For details about the numerical implementations, see [11] and [12]. In the left frame of Fig. 1 we display the $H^1(\Omega)$-norm

$$\|u_\lambda\|_{H^1(\Omega)} = \langle u_\lambda, u_\lambda \rangle_{H^1}^{1/2} = \left( \int_\Omega |\nabla u_\lambda|^2 \, dx + \int_{\partial \Omega} u_\lambda^2 \, d\sigma_x \right)^{1/2},$$

as a function of $\lambda$ for all these solutions. In the right frame of Fig. 1 we display the energy

$$E_\lambda(u_\lambda) = \frac{1}{2} \int_\Omega |\nabla u_\lambda|^2 \, dx - \frac{D}{2} \int_{\partial \Omega} u_\lambda^2 \, d\sigma_x - \lambda \int_{\partial \Omega} (\cosh(u_\lambda) - 1) \, d\sigma_x,$$

Fig. 1. Left frame: $H^1(\Omega)$-norm as a function of $\lambda$ for different solutions to (1). Right frame: energies $E_\lambda$ for the same solutions.
Fig. 2. Left frame: normal boundary flux component $\lambda \sinh(u_\lambda)$ for $\lambda$ negative. Right frame: normal boundary flux component $\lambda \sinh(u_\lambda)$ for $\lambda$ positive. In both frames $D = 2$.

Fig. 3. Left frame: normal boundary flux component $\lambda \sinh(u_\lambda)$ for $\lambda$ negative (smallest flux: $\lambda \sim -0.64 \times 10^{-1}$, largest flux $\lambda \sim -0.56 \times 10^{-4}$). Right frame: normal boundary flux component $\lambda \sinh(u_\lambda)$ for $\lambda$ positive (smallest flux: $\lambda \sim 0.85 \times 10^{-1}$, largest flux $\lambda \sim 0.75 \times 10^{-4}$). In both frames $D = 1.5$.

as a function of $\lambda$ for all these solutions. It is quite easy to show that there are no nontrivial solutions for $\lambda < \min\{-D, 0\}$. We note that if $u_\lambda$ is a solution to (1) so is $-u_\lambda$. We furthermore note that if $\Omega$ is a disk and $u_\lambda$ is a solution to (1), then so is any rotation of $u_\lambda$. We do not consider these to be essentially different solutions. In Fig. 1 there appears to be finitely many essentially different solutions for any fixed $\lambda$ in the interval $-D = -2 < \lambda < 0$, whereas there appears to be infinitely many essentially different solutions for any $\lambda > 0$ (we have only “traced” solutions that bifurcate from the trivial (0) solution before $\lambda = 6$, but the same pattern would persist for “later” solutions). The $H^1(\Omega)$ norm (and also the energy) of all nontrivial solutions “blows up” as $\lambda$ approaches 0.

In Fig. 2 we display the nonlinear part of the normal boundary currents $\lambda \sinh(u_\lambda)$ for 8 values of $\lambda$. The left frame corresponds to $\lambda$ negative (between $-0.79 \times 10^{-1}$ and $-0.70 \times 10^{-4}$) and the flux components are taken along the branch emanating from the trivial solution at $\lambda = -1$. The right frame corresponds to $\lambda$ positive (between $0.85 \times 10^{-1}$ and $0.75 \times 10^{-4}$) and the flux components are taken along the branch emanating from the trivial solution at $\lambda = 1$.

The blow-up behavior of the solutions is clearly considerably different depending on whether $\lambda \to 0_-$, or whether $\lambda \to 0_+$. The blow-up behavior is not significantly affected by whether $D$ is a Steklov eigenvalue or not, as evidenced by Fig. 3, each frame of which depicts the nonlinear part of the normal boundary currents $\lambda \sinh(u_\lambda)$ for 8 values of $\lambda$. Only this time $D = 1.5$, and we follow solution branches emanating from the trivial solution at $\lambda = -0.5$ and $\lambda = 1.5$, respectively. Nontrivial solutions corresponding to $\lambda < 0$ (only possibly when $D > 0$) blow up pointwise almost everywhere, and the flux components $\lambda \sinh(u_\lambda)$ seem to blow up on entire intervals as $\lambda \to 0_-$, whereas it is tempting to conjecture that the corresponding flux components for $\lambda \to 0_+$ blow up at only a finite number of boundary points (provided $\Omega$ is simply connected, and $D$ is positive).

The purpose of the next two sections is to carefully analyze the existence structure and the blow-up patterns, as partially illustrated in Figs. 1–3. For obvious reasons we separate this analysis into two different parts, one concerning $\lambda < 0$, and one concerning $\lambda > 0$.

A common tool for the two analyzes is the energy functional $E_\lambda(\cdot)$ and the associated “restricted” functional $J_\lambda(\cdot)$, defined by

$$J_\lambda(v) = \begin{cases} \inf_{t > 0} E_\lambda(tv) & \text{for } \lambda < 0, \\ \sup_{t > 0} E_\lambda(tv) & \text{for } \lambda > 0, \end{cases}$$
\(v \in H^1(\Omega) \setminus \{0\}\). In the more general setting of (3) the expression \(\cosh v - 1\) (in the energy \(E_\lambda(\cdot)\)) would be replaced by \(F(v)\), where the nonnegative even function \(F\) is given by

\[
F(t) = \int_0^t f(s) \, ds.
\]

We also define

\[
\Sigma = \{w \in H^1(\Omega) : \|w\|_{H^1(\Omega)} = 1\}.
\]

The functional \(J_\lambda(\cdot)\) is even and continuous on \(H^1(\Omega) \setminus \{0\}\) (see [10] Lemma 2.4). Since \(J_\lambda(\cdot)\) is homogeneous (of degree 0) it is occasionally convenient to regard it as just a functional on \(\Sigma\). It is not difficult to see that

\[
v \mapsto J_\lambda(v)\; \text{ maps } H^1(\Omega) \setminus \{0\} \; \text{onto the interval } [J_\lambda(1), 0] \subset (-\infty, 0) \text{ for } \lambda < 0.
\]

In particular, for \(\lambda < 0\) we have \(J_\lambda(v) = 0\) for any \(v\) that vanishes identically on \(\partial \Omega\). A simple calculation yields that

\[
J_\lambda(1) = \inf_{\partial \Omega} \left( -\frac{D}{2} t^2 - \lambda (\cosh(t) - 1) \right) = 0 \quad \text{for } \lambda < \min(-D, 0), \quad \text{and}
\]

\[
J_\lambda(1) = \inf_{\partial \Omega} \left( -\frac{D}{2} t^2 - \lambda (\cosh(t) - 1) \right) < 0 \quad \text{for } \min(-D, 0) < \lambda < 0.
\]

As a consequence \(J_\lambda(\cdot) = 0\) for \(\lambda < \min(-D, 0)\). It is equally easy to show that

\[
v \mapsto J_\lambda(v)\; \text{ maps } H^1(\Omega) \setminus \{0\} \; \text{into the interval } [0, \infty) \text{ for } \lambda > 0.
\]

Concerning \(\lambda > 0\) we see that \(J_\lambda(v) = \infty\) if and only if \(v\) vanishes identically on \(\partial \Omega\). For \(\lambda > \max\{0, -D\}\) we also calculate \(J_\lambda(1) = 0\). We thus conclude that the range of \(J_\lambda(H^1(\Omega) \setminus \{0\})\) is unbounded for any \(\lambda > 0\) and that the range equals the interval \([0, \infty)\) for any \(\lambda > \max\{0, -D\}\).

Following the same arguments as in [10] we may show that \(J_\lambda(\cdot)\) is smooth on the set \(\{v: J_\lambda(1) \leq J_\lambda(v) < 0\}\) for \(\min(-D, 0) < \lambda < 0\), and that \(J_\lambda(\cdot)\) is smooth on the set \(\{v: 0 < J_\lambda(v) < \infty\}\) for \(0 < \lambda\). In each case we do this by showing that there exists a unique value \(t(v) > 0\) such that \(J_\lambda(v) = E_\lambda(t(v)v)\). By the chain rule,

\[
J_\lambda'(v)[w] = E_\lambda'(t(v)v)[v]t'(v)[w] + E_\lambda'(t(v)v)[w]t(v) = E_\lambda'(t(v)v)[w]t(v),
\]

where we have used the fact that \(E_\lambda'(t(v)v)[v] = \frac{d}{dt}_{t=v} E_\lambda(t(v)v) = 0\), due to the definition of \(t(v)\). Since \(J_\lambda'(v)[v] = \frac{d}{dt}_{t=1} J_\lambda(t(v)v) = 0\) we now arrive at the following equivalences for any \(v\), with \(J_\lambda(v) \in \mathbb{R} \setminus \{0\}\),

\[
\exists \alpha \text{ such that } J_\lambda'(v)[\cdot] = \alpha(\cdot, v)_{H^1} \iff J_\lambda(v)[\cdot] = 0 \iff E_\lambda'(t(v)v)[\cdot] = 0.
\]

\(J_\lambda(v) \in \mathbb{R} \setminus \{0\}\) ensures that the positive number \(t(v)\) is well defined. In other words: if \(v\) is a critical point for \(J_\lambda(\cdot)\) on \(\Sigma\), corresponding to a nonzero critical value, then \(u = t(v)v \neq 0\) is a critical point for \(E_\lambda(\cdot)\) in \(H^1(\Omega)\). Conversely, if \(u \neq 0\) is a critical point for \(E_\lambda(\cdot)\) in \(H^1(\Omega)\), with \(J_\lambda(u) \neq 0\), then \(v = u/\|u\|_1\) is a critical point for \(J_\lambda(\cdot)\) on \(\Sigma\). Such critical points are weak-\(e\), and by elliptic regularity, also strong solutions to the boundary value problem (1).

For the existence-analysis (in order to establish the existence of critical point for \(J_\lambda(\cdot)\) on \(\Sigma\)) it is crucial that \(J_\lambda(\cdot)\) has a certain compactness property. For the exponential nonlinearity we consider here this condition is a fairly direct consequence of the compactness of the mapping

\[
H^1(\Omega) \ni v \mapsto \cosh(v)|_{\partial \Omega} \in L^p(\partial \Omega).
\]

**Lemma 1** (Palais–Smale condition). Given any two sequences \(v_n \in \Sigma\), \(\alpha_n \in \mathbb{R}\) with \(J_\lambda(v_n) \to c \neq 0\) and \(J_\lambda'(v_n)[\cdot] - \alpha_n(\cdot, v_n)_{H^1} \to 0\) in \([H^1(\Omega)]^*\), as \(n \to \infty\), we may conclude that \(\alpha_n \to 0\), and that there exists a subsequence (for simplicity also indexed by \(n\)) and a function \(v_\infty \in \Sigma\), so that \(v_n \to v_\infty\) in \(H^1(\Omega)\), as \(n \to \infty\).

**Proof.** Since \(J_\lambda'(v_n)[\cdot] - \alpha_n(\cdot, v_n)_{H^1} \to 0\) in \([H^1(\Omega)]^*\), and since \(J_\lambda'(v_n)[v_n] = 0\), it follows immediately that

\[
-\alpha_n = J_\lambda'(v_n)[v_n] - \alpha_n(v_n, v_n)_{H^1} \to 0,
\]
as desired. It then also follows that
\[ J'_k(v_n)[·] \to 0 \quad \text{in} \quad [H^1(\Omega)]'. \tag{6} \]

Defining \( u_n = t(v_n) v_n \), we now proceed to prove that the sequence \( \{u_n\} \) is bounded in \( H^1(\Omega) \). We may without loss of generality suppose that the sequence \( \{t(v_n)\} \) is strictly positive, since \( 0 < J'_k(v_n) < \infty \) for \( n \) sufficiently large. Our definition of \( t(v_n) \) (stationarity of \( E_{\lambda}(t(v_n)) \) with respect to \( t \)) implies
\[
\frac{1}{2} \left( \int_\Omega |t(v_n) \nabla v_n|^2 \, dx - D \int_{\partial \Omega} (t(v_n) v_n)^2 \, d\sigma_x \right) = \frac{\lambda}{2} \int_\Omega t(v_n) v_n \sinh(t(v_n) v_n) \, d\sigma_x.
\]
This in turn gives the inequality
\[
|c| + 1 > |J(v_n)| = |E_{\lambda}(t(v_n) v_n)| = |\lambda| \int_{\partial \Omega} \left( \frac{u_n}{2} \sinh(u_n) - \cosh(u_n) + 1 \right) \, d\sigma_x,
\]
with \( C_0 = \max_{x \in \mathbb{R}} (2 \cosh(x) + x^2 - \frac{2}{e} \sinh(x) - 1) < \infty \). From this it is easy to see that both \( \int_{\partial \Omega} u_n^2 \, d\sigma_x \) and \( \int_{\partial \Omega} \cosh(u_n) \, d\sigma_x \) are bounded by constants depending on \( c, \lambda, \) and \( \Omega \). Invoking the energy convergence (the fact that \( E_{\lambda}(u_n) \to c \)) we see that \( \int_{\Omega} |\nabla u_n|^2 \, dx \) is bounded by a constant with the same set of dependencies. In summary
\[ 0 < t(v_n) = \|u_n\|_{H^1(\Omega)} \leq C(c, \lambda, \Omega). \]

We may now extract a subsequence (also indexed by \( n \)) such that \( u_n \rightharpoonup u_\infty \) weakly in \( H^1(\Omega) \) and \( t(v_n) \to b \), for some \( u_\infty \in H^1(\Omega) \) and some \( b \geq 0 \). Note that \( c \neq 0 \) implies that \( b > 0 \), and the weak \( H^1(\Omega) \) convergence implies that \( \int_{\partial \Omega} (u_n - u_\infty)^2 \, d\sigma_x \to 0 \). Therefore,
\[
\int_{\Omega} |\nabla u_n - \nabla u_\infty|^2 \, dx = (E'_{\lambda}(u_n) - E'_{\lambda}(u_\infty))[u_n - u_\infty] + D \int_{\partial \Omega} (u_n - u_\infty)^2 \, d\sigma_x
\]
\[ + \lambda \int_{\partial \Omega} (u_n - u_\infty)(\sinh(u_n) - \sinh(u_\infty)) \, d\sigma_x
\]
\[ = E'_{\lambda}(u_n)[u_n - u_\infty] - E'_{\lambda}(u_\infty)[u_n - u_\infty] + o(1).
\]
The last equality follows from the \( L^2(\partial \Omega) \) convergence and Trudinger’s inequality, that is,
\[
\int_{\partial \Omega} (\sinh(u))^2 \, d\sigma_x \leq C_1 c_2 \|u\|_{H^1(\Omega)}^2,
\]
see Lemma 2.1 of [10]. By (5), \( E'_{\lambda}(u_n)[·] = \frac{1}{t(v_n)} J'_{\lambda}(v_n)[·] \) and so from (6), the fact that \( \lim t(v_n) = b > 0 \), and the weak convergence \( u_n \rightharpoonup u_\infty \to 0 \), we now conclude that \( u_n \rightharpoonup u_\infty \) in \( H^1(\Omega) \). It follows that \( v_n = \frac{1}{t(v_n)} u_n \rightharpoonup \frac{1}{b} u_\infty = v_\infty \) in \( H^1(\Omega) \), and that \( v_\infty \in \Sigma \). \( \square \)

Let \( \{(D_k, \phi_k)\}_{k=1}^{\infty} \) denote the Steklov eigenvalues and eigenvectors for (2). The eigenvalues \( D_k \) form a non-decreasing sequence \( 0 = D_1 \leq D_2 \leq \cdots \), with \( D_k \to \infty \) as \( k \to \infty \). There may be repeated values in this sequence, since each eigenvalue appears as many times as its (finite) multiplicity indicates. The \( \phi_k \) may be selected so that \( \int_{\partial \Omega} \phi_j \phi_k \, d\sigma_x = \delta_{jk} \). The functions \( \phi_k|_{\partial \Omega} \) now form an orthonormal basis for \( L^2(\partial \Omega) \). The \( \phi_k \) are then also orthogonal in \( H^1(\Omega) \). Corresponding to any \( D \in \mathbb{R} \) we define the projection operator
\[
P_D u = \sum_{D_k=D} \langle u, \phi_k \rangle_{L^2(\partial \Omega)} \phi_k = \sum_{D_k=b} \langle u, \phi_k \rangle_{H^1} \phi_k.
\]
P_D is a projection onto the “eigenspace” \( V_D \), associated with \( D \). Note that we interpret this to mean that \( P_D = 0 \) (and \( V_D = \{0\} \)) if \( D \) is not a Steklov eigenvalue. In the case of the unit disk we have \( D_{2k+1} = k, k \geq 0 \) and \( D_{2k} = k \),
\( k \geq 1 \), as seen in Fig.1 (where \( D = 2 \)). In general, it is well known that the eigenvalues \( D_k \) grow according to a classical Weyl asymptotics. Indeed, under the assumption that \( \Omega \) is simply connected, it is extremely easy to see that \( ck \leq D_k \leq Ck \), as \( k \to \infty \) (chose a conformal transformation to map \( \Omega \) onto the unit disk, build a min-max characterization of the original eigenvalues and compare with the known eigenvalues for the problem (2) on the unit disk). Much more detailed results concerning the asymptotic behavior of \( D_k \) and \( \phi_k \) have been established in [15].

It will frequently be convenient to decompose solutions to (1) as

\[ u_\lambda = P_D u_\lambda + (I - P_D) u_\lambda = P_D u_\lambda + w_\lambda, \]

where \( D \) is the same as the “shift” which appears in (1). The following lemma will then play a crucial role.

**Lemma 2.** Let \( D \in \mathbb{R} \) be fixed, and suppose \( u_\lambda, \lambda \neq 0 \), is a solution to (1), i.e., a solution to

\[ \Delta u_\lambda = 0 \quad \text{in} \; \Omega, \quad \frac{\partial u_\lambda}{\partial n} = D u_\lambda + \lambda \sinh(u_\lambda) \quad \text{on} \; \partial \Omega. \]

Let \( w_\lambda \) denote the function \( w_\lambda = (I - P_D) u_\lambda \). There exists a constant \( C \), depending on \( \Omega \) and \( D \), but independent of \( \lambda \), and \( u_\lambda \) such that

\[ \| P_D u_\lambda \|_{H^1(\Omega)} \leq C \left( \| w_\lambda \|_{H^1(\Omega)}^2 + 1 \right). \]

**Proof.** From the definition of \( u_\lambda \), and integration by parts, we immediately get

\[ \lambda \int_{\partial \Omega} \sinh(u_\lambda) P_D u_\lambda \, d\sigma_x = -D \int_{\partial \Omega} u_\lambda P_D u_\lambda \, d\sigma_x + \int_{\partial \Omega} \frac{\partial u_\lambda}{\partial n} P_D u_\lambda \, d\sigma_x \]

\[ = -\int_{\partial \Omega} u_\lambda \frac{\partial P_D u_\lambda}{\partial n} \, d\sigma_x + \int_{\partial \Omega} \frac{\partial u_\lambda}{\partial n} P_D u_\lambda \, d\sigma_x = 0. \]

For \( \lambda \neq 0 \) we thus conclude

\[ \int_{\partial \Omega} \sinh(u_\lambda) P_D u_\lambda \, d\sigma_x = 0. \quad (7) \]

By insertion of the identity

\[ \sinh(u_\lambda) = \sinh(P_D u_\lambda + w_\lambda) = \sinh(P_D u_\lambda) \cosh(w_\lambda) + \cosh(P_D u_\lambda) \sinh(w_\lambda) \]

into (7), rearrangement, and use of the estimate \( | \cosh(P_D u_\lambda) P_D u_\lambda | \leq | \sinh(P_D u_\lambda) P_D u_\lambda + e^{-1} \), we now obtain

\[ \int_{\partial \Omega} \sinh(P_D u_\lambda) P_D u_\lambda \cosh(w_\lambda) \, d\sigma_x = -\int_{\partial \Omega} \cosh(P_D u_\lambda) P_D u_\lambda \sinh(w_\lambda) \, d\sigma_x \]

\[ \leq \int_{\partial \Omega} | \cosh(P_D u_\lambda) P_D u_\lambda | | \sinh(w_\lambda) | \, d\sigma_x \]

\[ \leq \int_{\partial \Omega} \left( | \sinh(P_D u_\lambda) P_D u_\lambda + e^{-1} | | \sinh(w_\lambda) | \right) \, d\sigma_x. \]

Since \( \cosh(w_\lambda) - | \sinh(w_\lambda) | = e^{-|w_\lambda|} \) this last inequality immediately leads to

\[ \int_{\partial \Omega} \sinh(P_D u_\lambda) P_D u_\lambda e^{-|w_\lambda|} \, d\sigma_x \leq e^{-1} \int_{\partial \Omega} | \sinh(w_\lambda) | \, d\sigma_x. \]

An application of Cauchy–Schwarz’s inequality yields
\[
\left( \int_{\partial \Omega} \left( \sinh(P_D u_\lambda) P_D u_\lambda \right)^{1/2} d\sigma_x \right)^2 \leq \int_{\partial \Omega} \sinh(P_D u_\lambda) P_D u_\lambda \, e^{-|w_\lambda|} \, d\sigma_x \cdot \int_{\partial \Omega} e^{|w_\lambda|} \, d\sigma_x \\
\leq e^{-1} \int_{\partial \Omega} \left| \sinh(w_\lambda) \right| \, d\sigma_x \cdot \int_{\partial \Omega} e^{|w_\lambda|} \, d\sigma_x \leq C e^{C \|w_\lambda\|^2_{H^1(\Omega)}}. \quad (8)
\]

For the last estimate we used a version of Trudinger’s inequality, asserting that for a (smooth) bounded, two-dimensional domain \( \Omega \) there exists a constant \( C \) such that
\[
\int_{\partial \Omega} e^{\|w\|} \, d\sigma_x \leq C e^{C \|w\|^2_{H^1(\Omega)}}, \quad \forall w \in H^1(\Omega);
\]
see for example [10] or [17]. Since \( e^{\|p/2\|} \leq (\sinh(p))^2 + C \) it follows from (8) that
\[
\int_{\partial \Omega} e^{\|P_D u_\lambda\|^2/2} \, d\sigma_x \leq C \left( e^{C \|w_\lambda\|^2_{H^1(\Omega)} + 1} \right) \leq 2C e^{C \|w_\lambda\|^2_{H^1(\Omega)}} \leq e^{C \|w_\lambda\|^2_{H^1(\Omega)} + \log 2C},
\]
and so an application of Jensen’s inequality (applied to the convex function \( p \to e^{p/2} \)) leads to
\[
e^{\int_{\partial \Omega} \left( |P_D u_\lambda|^2/2 \right) (d\sigma_x/|\partial \Omega|)} \leq e^{\int_{\partial \Omega} e^{\|P_D u_\lambda\|^2/2} \frac{d\sigma_x}{|\partial \Omega|}} \leq e^{C \|w_\lambda\|^2_{H^1(\Omega)} + 1}.
\]

By taking a logarithm on both sides we get
\[
\int_{\partial \Omega} |P_D u_\lambda| \, d\sigma_x \leq C \left( \|w_\lambda\|^2_{H^1(\Omega)} + 1 \right).
\]

Since \( P_D u_\lambda \) is either 0 or lies in the finite dimensional Steklov eigenspace associated with \( D \) (where all norms are equivalent) we now conclude that
\[
\|P_D u_\lambda\|_{H^1(\Omega)} \leq C \left( \|w_\lambda\|^2_{H^1(\Omega)} + 1 \right),
\]
with \( C \) independent of \( \lambda \) and \( u_\lambda \). Here we use that \( \| \cdot \|_{L^1(\partial \Omega)} \) is a norm on the eigenspace \( V_D \) (since \( \phi \in V_D \) vanishes identically if \( \phi|_{\partial \Omega} = 0 \)). This completes the proof of the lemma. \( \Box \)

2.1. Negative \( \lambda \)

To consider the very simplest case first, suppose \( \lambda \leq \min\{0, -D\} \), and suppose \( u_\lambda \) is a solution to (1). Green’s formula then immediately gives
\[
\int_{\Omega} |\nabla u_\lambda|^2 \, dx = D \int_{\partial \Omega} u_\lambda^2 \, d\sigma_x + \lambda \int_{\partial \Omega} \sinh(u_\lambda) u_\lambda \, d\sigma_x = (D + \lambda) \int_{\partial \Omega} u_\lambda^2 \, d\sigma_x + \lambda \int_{\partial \Omega} (\sinh(u_\lambda) - u_\lambda) u_\lambda \, d\sigma_x \leq 0.
\]
For the last inequality we used that \( D + \lambda \leq 0 \), that \( \lambda \leq 0 \), and that \( (\sinh(x) - x)x \geq 0 \). We may thus conclude that \( u_\lambda \) is a constant. However, if we additionally suppose that \( D \neq 0 \) or \( \lambda \neq 0 \), then the only constant, \( z \), for which \( Dz + \lambda \sinh(z) = 0 \) is \( z = 0 \). In summary:

**Proposition 1.** For \( \lambda \leq \min\{0, -D\} \) the only solution to (1) is, with one exception, \( u_\lambda = 0 \). The exception is \( D = \lambda = 0 \), in which case any constant is a solution to (1).

The case \( \min\{0, -D\} < \lambda \leq 0 \) is more interesting (and complicated). Except for possibly \( \lambda = 0 \), there are now always nontrivial solutions to (1). Corresponding to the segment \( \min\{0, -D\} < \lambda < 0 \) we may actually prove the following result about existence and about a general upper bound for the \( H^1(\Omega) \)-norm of solutions.
Proposition 2. Suppose the shift $D$, appearing in (1), is positive. There exist constants $C_1 > 0$ and $C_2 > 0$, depending only on $D$ and $\Omega$, such that

$$
\|u_\lambda\|_{H^1(\Omega)}^2 \leq C_1 \left( \log \frac{1}{|\lambda|} \right)^2 + C_2,
$$

for any $-D < \lambda < 0$, and any solution $u_\lambda$ to (1). Furthermore, there exists at least one family of solutions $U_\lambda \neq 0$, $-D < \lambda < 0$, and two constants $c_1 > 0$ and $c_2$ such that

$$
c_1 \left( \log \frac{1}{|\lambda|} \right)^2 + c_2 \leq \|U_\lambda\|_{H^1(\Omega)}^2 \leq C_1 \left( \log \frac{1}{|\lambda|} \right)^2 + C_2.
$$

Proof. The fact that

$$
\int_\Omega |\nabla u_\lambda|^2 \, dx - D \int_{\partial \Omega} u_\lambda^2 \, d\sigma_x - \lambda \int_{\partial \Omega} \sinh(u_\lambda) u_\lambda \, d\sigma_x = 0
$$

for any solution $u_\lambda$ to (1), and any $\lambda < 0$, may be rewritten

$$
\int_\Omega |\nabla u_\lambda|^2 \, dx + |\lambda| \int_{\partial \Omega} \sinh(u_\lambda) u_\lambda \, d\sigma_x = D \int_{\partial \Omega} u_\lambda^2 \, d\sigma_x. \quad (9)
$$

It is well known that, given any $\epsilon > 0$, there exists a constant $C_\epsilon$, such that

$$
\|u\|_{L^2(\partial \Omega)} \leq \epsilon \|u\|_{H^{1/2}(\partial \Omega)}^2 + C_\epsilon \|u\|_{\overline{H}^{1/2}(\partial \Omega)}^2 \leq \epsilon \|u\|_{H^{1/2}(\partial \Omega)}^2 + C_\epsilon \|u\|_{L^1(\partial \Omega)}^2. \quad (10)
$$

For the last inequality we used that $L^1(\partial \Omega)$ embeds continuously into $H^{-1}(\partial \Omega)$, since $\partial \Omega$ is one-dimensional. It is also well known that

$$
\|u\|_{H^{1/2}(\partial \Omega)}^2 \leq C \|u\|_{H^1(\Omega)}^2 \leq C \left( \|\nabla u\|_{L^2(\Omega)}^2 + \|u\|_{L^1(\partial \Omega)}^2 \right). \quad (11)
$$

By selecting $\epsilon$ sufficiently small we obtain from a combination of (9), (10) and (11), that

$$
\frac{1}{2} \int_\Omega |\nabla u_\lambda|^2 \, dx + |\lambda| \int_{\partial \Omega} \sinh(u_\lambda) u_\lambda \, d\sigma_x \leq C \|u_\lambda\|_{L^1(\partial \Omega)}^2. \quad (12)
$$

The function $x \to g(x) = \sinh(x)x$ is convex, and so Jensen’s inequality immediately asserts that

$$
g \left( \int_{\partial \Omega} |u_\lambda| \frac{d\sigma}{|\partial \Omega|} \right) \leq \int_{\partial \Omega} g \left( |u_\lambda| \frac{d\sigma}{|\partial \Omega|} \right) = \int_{\partial \Omega} \sinh(u_\lambda) u_\lambda \, d\sigma_x \frac{d\sigma}{|\partial \Omega|}.
$$

By a combination of this inequality and (12) we obtain

$$
|\lambda| e^g |u_\lambda| \frac{d\sigma}{|\partial \Omega|} \leq |\lambda| g \left( \int_{\partial \Omega} |u_\lambda| \frac{d\sigma}{|\partial \Omega|} \right) + |\lambda| \cosh(1) \leq C \left( \int_{\partial \Omega} |u_\lambda| \, d\sigma_x \right)^2 + |\lambda| \cosh(1),
$$

which immediately leads to

$$
\int_{\partial \Omega} |u_\lambda| \, d\sigma_x \leq C_1 \log \frac{1}{|\lambda|} + C_2,
$$

for $-D < \lambda < 0$. From (12) it now follows that

$$
\int_\Omega |\nabla u_\lambda|^2 \, dx \leq C_1 \left( \log \frac{1}{|\lambda|} \right)^2 + C_2,
$$

and so, in summary

$$
\|u_\lambda\|_{H^1(\Omega)}^2 \leq C_1 \left( \log \frac{1}{|\lambda|} \right)^2 + C_2.
$$
This completes the proof of the first part of this proposition. The second part is much simpler. We just note that for any $-D < \lambda < 0$ there exists exactly one positive solution to

$$\sinh z_\lambda = -\frac{D}{\lambda} z_\lambda.$$ 

Here we have used the fact that $-D < \lambda < 0 \Rightarrow -D/\lambda > 1$. This solution satisfies

$$c_1 \log \frac{1}{|\lambda|} + c_2 < z_\lambda,$$

for some constants $c_1 > 0$ and $c_2$, and so the constant function $U_\lambda(x) = z_\lambda$ is easily seen to be a nonzero solution to (1) with the desired lower bound.

Based on Fig. 1 one might expect that any sequence of solutions $u_{\lambda_n}, \lambda_n \to 0_-$, which does not degenerate to the 0 solution for $\lambda_n$ sufficiently near 0, must contain a subsequence such that $\|u_{\lambda_n}\|^2_{H^1(\Omega)}$ is bounded from below by

$$c_1 (\log \frac{1}{|\lambda_n|})^2$$

as $\lambda_n \to 0_-$. We are not quite able to prove that, but we can establish the following weaker result. This result also shows that the only possible blow up behavior as $\lambda \to 0_-$ is blow-up almost everywhere. The family $U_\lambda$ constructed in Lemma 2 does blow up everywhere.

**Proposition 3.** Suppose the shift $D$, appearing in (1), is positive. There exists a constant $c_1 > 0$ such that whenever $u_{\lambda_n}, -D < \lambda_n < 0, \lambda_n \to 0_-$, is a family of solutions to (1) with the property that $\|u_{\lambda_n}\|_{H^1(\Omega)}$ does not converge to 0 as $\lambda_n \to 0_-$, then we may extract a subsequence, for simplicity also denoted $u_{\lambda_n}$, with

$$c_1 \log \frac{1}{|\lambda_n|} \leq \|u_{\lambda_n}\|^2_{H^1(\Omega)} \quad \text{as } \lambda_n \to 0_-.$$ (13)

We may extract this subsequence so that $u_{\lambda_n}$ converges pointwise to $\pm \infty$ almost everywhere in $\Omega$, and so that $u_{\lambda_n}|_{\partial \Omega}$ converges pointwise to $\pm \infty$ on a set of positive one dimensional surface measure. By appropriate extraction of the subsequence we may also arrange that $w_{\lambda_n} = (I - PD)u_{\lambda_n}$ converges pointwise to $\pm \infty$ almost everywhere in $\Omega$, that $w_{\lambda_n}|_{\partial \Omega}$ converges pointwise to $\pm \infty$ on a set of positive one dimensional surface measure, and that

$$\exists \gamma_n \to \infty \text{ such that } \frac{\lambda_n \sinh(u_{\lambda_n})}{\gamma_n} \rightharpoonup \mu \neq 0, \quad \text{weakly in } H^{-1/2}(\partial \Omega), \quad \text{as } \lambda_n \to 0_-.$$ (14)

**Remarks.** A distribution $\mu \in H^{-1/2}(\partial \Omega)$ cannot consist of Dirac delta functions. By comparison with Theorem 2 later in this paper the structure of the (rescaled flux-component) limit $\mu$, for $\lambda \to 0_-$, is thus completely different from the finite sum of Dirac delta masses that generically emerges as the limit of the flux-component $\lambda \sinh(u_{\lambda})$ (for the variationally constructed solutions) as $\lambda \to 0_+$. Broadly speaking the last statement of Proposition 3 means that, as $\lambda$ approaches $0_-$, the flux-component $\lambda \sinh(u_{\lambda})$ “blows up on a thicker set” than is the case (for the variationally constructed solutions) when $\lambda$ approaches $0_+$. This is exactly what we evidenced in Fig. 2. Similarly we also see that $w_{\lambda}$ blows up almost everywhere in $\Omega$, and on a set of positive measure on $\partial \Omega$, as $\lambda \to 0_-$, whereas Theorem 2 implies that (for the variationally constructed solutions) $w_{\lambda}$ generically only blows up at a finite number of points on $\partial \Omega$, as $\lambda \to 0_+$.

**Proof of Proposition 3.** In order to prove the first statement of this proposition (concerning the lower bound on $\|u_{\lambda_n}\|^2_{H^1(\Omega)}$) it suffices to prove that there exists a (small) constant $c_1 > 0$ such that if $u_{\lambda_n}$ is a sequence of solutions to (1) with

$$\|u_{\lambda_n}\|^2_{H^1(\Omega)} \log \frac{1}{|\lambda_n|} < c_1 \quad \text{as } \lambda_n \to 0_-,$$ (14)

then

$$\|u_{\lambda_n}\|_{H^1(\Omega)} \to 0 \quad \text{as } \lambda_n \to 0_-.$$ (15)
Now suppose (14) is satisfied for some sufficiently small positive $c_1$; in order to verify (15) we start by estimating
\[ \|w_{\lambda_n}\|_{L^2(\partial \Omega)} = \|(I - P_D)u_{\lambda_n}\|_{L^2(\partial \Omega)} \]
\[ \|w_{\lambda_n}\|^2_{L^2(\partial \Omega)} = \sum_{k: D_k \neq D} \alpha_{k, \lambda_n}^2 \quad \text{with} \quad \alpha_{k, \lambda_n} = \int_{\partial \Omega} u_{\lambda_n} \phi_k \, d\sigma_x. \tag{16} \]

Straightforward integration by parts gives that
\[ D_k \int_{\partial \Omega} u_{\lambda_n} \phi_k \, d\sigma_x = \int_{\Omega} \nabla u_{\lambda_n} \nabla \phi_k \, dx = D \int_{\partial \Omega} u_{\lambda_n} \phi_k \, d\sigma_x + \lambda_n \int_{\partial \Omega} \sinh u_{\lambda_n} \phi_k \, d\sigma_x, \]
and so
\[ \alpha_{k, \lambda_n} = \int_{\partial \Omega} u_{\lambda_n} \phi_k \, d\sigma_x = \frac{\lambda_n}{D_k - D} \int_{\partial \Omega} \sinh u_{\lambda_n} \phi_k \, d\sigma_x \quad \text{for} \quad D_k \neq D, \tag{17} \]
and
\[ \int_{\partial \Omega} \sinh u_{\lambda_n} \phi_k \, d\sigma_x = 0 \quad \text{for} \quad D_k = D. \tag{18} \]
From (16) and (17) it follows that
\[ \|w_{\lambda_n}\|^2_{L^2(\partial \Omega)} = \sum_{k: D_k \neq D} \alpha_{k, \lambda_n}^2 \leq C \lambda_n^2 \sum_{k: D_k \neq D} \left| \int_{\partial \Omega} \sinh u_{\lambda_n} \phi_k \, d\sigma_x \right|^2 \quad \text{with} \quad C = \left( \min_{D_k \neq D} |D_k - D| \right)^{-2}. \tag{19} \]

We also have the estimate
\[ \sum_{k: D_k \neq D} \left| \int_{\partial \Omega} \sinh u_{\lambda_n} \phi_k \, d\sigma_x \right|^2 = \int_{\partial \Omega} \sinh^2 u_{\lambda_n} \, d\sigma_x \leq C_1 e^{C_2 \|u_{\lambda_n}\|^2_{H^1(\Omega)}}, \]
(see for example Lemma 2.1 of [10]). Due to the assumption (14) it follows that
\[ \|u_{\lambda_n}\|^2_{H^1(\Omega)} < c_1 \log \frac{1}{|\lambda_n|} \leq \frac{1}{C_2} \log \frac{1}{|\lambda_n|} \quad \text{as} \quad \lambda_n \to 0-, \]
provided $c_1 < \frac{1}{C_2}$ (incidentally, this is the only “smallness” restriction on $c_1$). By a combination of these last two estimates we get
\[ \sum_{k: D_k \neq D} \left| \int_{\partial \Omega} \sinh u_{\lambda_n} \phi_k \, d\sigma_x \right|^2 = \| \sinh u_{\lambda_n} \|^2_{L^2(\partial \Omega)} \leq \frac{C_1}{|\lambda_n|}, \]
which after insertion into (19) leads to
\[ \|w_{\lambda_n}\|^2_{L^2(\partial \Omega)} \leq C |\lambda_n|, \tag{20} \]
as $\lambda_n \to 0-$. We easily see that $w_{\lambda_n} = (I - P_D)u_{\lambda_n}$ satisfies the boundary value problem
\[ \Delta w_{\lambda_n} = 0 \quad \text{in} \quad \Omega, \quad \frac{\partial w_{\lambda_n}}{\partial n} = D w_{\lambda_n} + \lambda_n \sinh u_{\lambda_n} \quad \text{on} \quad \partial \Omega. \tag{21} \]
Due to the estimate
\[ |\lambda_n| \| \sinh u_{\lambda_n} \|^2_{L^2(\partial \Omega)} \leq C_1 |\lambda_n|^{1/2}, \]
(which was proven previously) and the estimate (20) we now conclude that
\[ \frac{\partial w_{\lambda_n}}{\partial n} \to 0 \quad \text{in} \quad L^2(\partial \Omega), \]
as $\lambda_n \to 0-$. It follows, by elliptic estimates, that
\[ w_{\lambda_n} \to 0 \quad \text{in} \quad H^{3/2}(\Omega). \tag{22} \]
If $P_D = 0$ this leads to the desired conclusion (15). If $P_D \neq 0$ it still remains to show that $P_D u_{\lambda_n} \to 0$ in $H^1(\Omega)$. Since $\Omega$ is two dimensional, we may use the Trace Theorem and Sobolev’s Imbedding Theorem, together with (22), to conclude that

$$(I - P_D)u_{\lambda_n} = w_{\lambda_n} \to 0 \quad \text{in } L^\infty(\partial \Omega),$$

as $\lambda_n \to 0_-$. From (18) we have that

$$\int_{\partial \Omega} \sinh(w_{\lambda_n} + P_D u_{\lambda_n}) P_D u_{\lambda_n} \, d\sigma_x = \int_{\partial \Omega} \sinh(u_{\lambda_n}) P_D u_{\lambda_n} \, d\sigma_x = 0,$$

which, due to the formula $\sinh(w + v) = \sinh w \cosh v + \sinh v \cosh w$, translates into

$$\int_{\partial \Omega} \sinh(P_D u_{\lambda_n}) \cosh w_{\lambda_n} P_D u_{\lambda_n} \, d\sigma_x = -\int_{\partial \Omega} \sinh w_{\lambda_n} \cosh(P_D u_{\lambda_n}) P_D u_{\lambda_n} \, d\sigma_x.$$

(24)

We also have the estimate

$$\left| \int_{\partial \Omega} \sinh w_{\lambda_n} \cosh(P_D u_{\lambda_n}) P_D u_{\lambda_n} \, d\sigma_x \right| \leq \| \sinh w_{\lambda_n} \|_{L^\infty(\partial \Omega)} \int_{\partial \Omega} \cosh(P_D u_{\lambda_n}) |P_D u_{\lambda_n}| \, d\sigma_x$$

$$\leq \| \sinh w_{\lambda_n} \|_{L^\infty(\partial \Omega)} \left( \int_{\partial \Omega} \sinh(P_D u_{\lambda_n}) P_D u_{\lambda_n} \, d\sigma_x + |\partial \Omega| e^{-1} \right).$$

By insertion of this into (24), and use of the facts that $\cosh w_{\lambda_n} \geq 1$, and $\sinh w_{\lambda_n} \to 0$ in $L^\infty(\partial \Omega)$ as $\lambda_n \to 0_-$ (cf. (23)), we now obtain

$$\frac{1}{2} \int_{\partial \Omega} \sinh(P_D u_{\lambda_n}) P_D u_{\lambda_n} \, d\sigma_x \leq \| \sinh w_{\lambda_n} \|_{L^\infty(\partial \Omega)} |\partial \Omega| e^{-1},$$

for $\lambda_n < 0$ sufficiently close to 0. Since

$$\| P_D u_{\lambda_n} \|_{L^2(\partial \Omega)}^2 \leq \int_{\partial \Omega} \sinh(P_D u_{\lambda_n}) P_D u_{\lambda_n} \, d\sigma_x,$$

we have therefore verified that

$$P_D u_{\lambda_n} \to 0 \quad \text{in } L^2(\partial \Omega) \quad \text{as } \lambda_n \to 0_-.$$

Since the range of the projection $P_D$ is finite dimensional, all (well defined) norms are equivalent on this space, and so

$$P_D u_{\lambda_n} \to 0 \quad \text{in } H^1(\Omega) \quad \text{as } \lambda_n \to 0_-.$$  

(25)

A combination of (22) and (25) now yields

$$\| u_{\lambda_n} \|_{H^1(\Omega)} \to 0 \quad \text{as } \lambda_n \to 0_-,$$

which is exactly the desired conclusion (15).

We proceed to the proof of the second statement of this proposition. Since $\lambda_n$ is negative, and since $x \sinh x \geq 0$, we calculate

$$\int_{\Omega} |\nabla u_{\lambda_n}|^2 \, dx = D \int_{\partial \Omega} u_{\lambda_n}^2 \, d\sigma_x + \lambda_n \int_{\partial \Omega} u_{\lambda_n} \sinh u_{\lambda_n} \, d\sigma_x \leq D \int_{\partial \Omega} u_{\lambda_n}^2 \, d\sigma_x.$$

(26)

As a consequence

$$\| u_{\lambda_n} \|_{H^1(\Omega)}^2 = \int_{\Omega} |\nabla u_{\lambda_n}|^2 \, dx + \int_{\partial \Omega} u_{\lambda_n}^2 \, d\sigma_x \leq (D + 1) \| u_{\lambda_n} \|_{L^2(\partial \Omega)}^2.$$

(27)
Let \( u_{\lambda_n}, \lambda_n \to 0_- \), be a sequence which satisfies the lower bound (13) and define

\[
\tilde{u}_{\lambda_n} = u_{\lambda_n} / \| u_{\lambda_n} \|_{L^2(\partial \Omega)}.
\]

Due to this definition, and (27),

\[
\| \tilde{u}_{\lambda_n} \|_{L^2(\partial \Omega)} = 1 \quad \text{and} \quad \| \tilde{u}_{\lambda_n} \|_{H^1(\Omega)} \leq C.
\]

By extraction of a subsequence, for simplicity also referred to as \( \tilde{u}_{\lambda_n} \), we may obtain

\[
\tilde{u}_{\lambda_n} \rightharpoonup u_0 \quad \text{weakly in} \quad H^1(\Omega) \quad \text{and} \quad \tilde{u}_{\lambda_n} \to u_0 \quad \text{in} \quad L^2(\partial \Omega),
\]

as \( \lambda_n \to 0_- \). For the second property we relied on the compactness of the trace map from \( H^1(\Omega) \) to \( L^2(\partial \Omega) \). Since \( \tilde{u}_{\lambda_n} \), and therefore also \( u_0 \), are all harmonic in \( \Omega \) we may conclude that

\[
\tilde{u}_{\lambda_n} \to u_0 \quad \text{in} \quad C^0(\Omega_c),
\]

for any compact subdomain \( \Omega_c \) of \( \Omega \). Being harmonic in \( \Omega \) (and nonzero, since \( \| u_0 \|_{L^2(\partial \Omega)} = \lim \| \tilde{u}_{\lambda_n} \|_{L^2(\partial \Omega)} = 1 \) the function \( u_0 \) is different from zero almost everywhere inside \( \Omega \) (and on a set of positive surface measure on \( \partial \Omega \)). From the definition of \( \tilde{u}_{\lambda_n} \), the fact that \( \| u_{\lambda_n} \|_{L^2(\partial \Omega)} \to \infty \) (since \( c_1 \log \frac{1}{\lambda_n} \leq \| u_{\lambda_n} \|_{H^1(\Omega)}^2 \leq C \| u_{\lambda_n} \|_{L^2(\partial \Omega)}^2 \) ) and (28) it now follows that

\[
u_{\lambda_n} \to \pm \infty \quad \text{almost everywhere in} \quad \Omega,
\]

and (by extraction of a subsequence) that

\[
u_{\lambda_n} \to \pm \infty \quad \text{on a set of positive one dimensional surface measure on} \quad \partial \Omega,
\]
as \( \lambda_n \to 0_- \). The “limit” is \(+ \infty\) where \( u_0 \) is positive, \(- \infty\) where \( u_0 \) is negative.

It only remains to prove the validity of the very last statements of the Proposition. As before let \( w_{\lambda_n} = (I - PD)u_{\lambda_n} \).

From integration by parts, (21), and (18) it follows immediately that

\[
\int_{\partial \Omega} |\nabla w_{\lambda_n}|^2 \, d\sigma_x = D \int_{\partial \Omega} w_{\lambda_n}^2 \, d\sigma_x + \lambda_n \int_{\partial \Omega} \sinh(u_{\lambda_n})w_{\lambda_n} \, d\sigma_x = D \int_{\partial \Omega} w_{\lambda_n}^2 \, d\sigma_x + \lambda_n \int_{\partial \Omega} \sinh(u_{\lambda_n})w_{\lambda_n} \, d\sigma_x
\]

\[
\leq D \int_{\partial \Omega} w_{\lambda_n}^2 \, d\sigma_x.
\]

(29)

Lemma 2 and (29) now yield

\[
\| u_\lambda \|_{H^1(\Omega)} = (\| PDu_\lambda \|_{H^{1/2}(\partial \Omega)}^2 + \| w_\lambda \|_{H^{1/2}(\partial \Omega)}^2)^{1/2} \leq C(\| w_\lambda \|_{H^{1}(\Omega)} + \| w_\lambda \|_{H^{1/2}(\partial \Omega)}^2) + 1)
\]

\[
\leq C(\| w_\lambda \|_{L^2(\partial \Omega)}^2 + \| w_\lambda \|_{H^{1/2}(\partial \Omega)}^2) + 1).
\]

If the sequence of numbers \( \| w_{\lambda_n} \|_{L^2(\partial \Omega)} \) contains a bounded subsequence then it follows immediately from this estimate that the sequence \( \| u_{\lambda_n} \|_{H^1(\Omega)} \) contains a bounded subsequence. However this would contradict the estimate (13), which we have already proven, and therefore we may conclude that

\[
\| w_{\lambda_n} \|_{L^2(\partial \Omega)} \to \infty \quad \text{as} \quad \lambda_n \to 0_-.
\]

(30)

Since \( w_{\lambda_n} \) is a solution to the boundary value problem (21), we arrive at the estimates

\[
\| \lambda_n \sinh(u_{\lambda_n}) \|_{H^{-1/2}(\partial \Omega)} \leq \| Dw_{\lambda_n} + \lambda_n \sinh(u_{\lambda_n}) \|_{H^{-1/2}(\partial \Omega)} + \| Dw_{\lambda_n} \|_{H^{-1/2}(\partial \Omega)}
\]

\[
= \| \frac{\partial w_{\lambda_n}}{\partial n} \|_{H^{-1/2}(\partial \Omega)} + \| Dw_{\lambda_n} \|_{H^{-1/2}(\partial \Omega)} \leq C \| w_{\lambda_n} \|_{H^1(\Omega)},
\]

and therefore, due to (29),

\[
\frac{\| \lambda_n \sinh(u_{\lambda_n}) \|_{H^{-1/2}(\partial \Omega)}}{\| w_{\lambda_n} \|_{L^2(\partial \Omega)}} \leq C \frac{\| \lambda_n \sinh(u_{\lambda_n}) \|_{H^{-1/2}(\partial \Omega)}}{\| w_{\lambda_n} \|_{H^1(\Omega)}} \leq C.
\]

(31)
We now define
\[ \tilde{w}_{\lambda_n} = \frac{w_{\lambda_n}}{\| w_{\lambda_n} \|_{L^2(\partial \Omega)}}. \]
Due to this definition, and (29),
\[ \| \tilde{w}_{\lambda_n} \|_{L^2(\partial \Omega)} = 1 \quad \text{and} \quad \| \tilde{w}_{\lambda_n} \|_{H^1(\Omega)} \leq C. \quad (32) \]
Because of the estimates (31), (32), and the compactness of the trace map from \( H^1(\Omega) \) to \( L^2(\partial \Omega) \), we may extract a subsequence, for simplicity also denoted \( \lambda_n \), so that
\[ \tilde{w}_{\lambda_n} \rightharpoonup w_0 \quad \text{weakly in} \quad H^1(\Omega), \]
\[ \tilde{w}_{\lambda_n}|_{\partial \Omega} \to w_0|_{\partial \Omega} \quad \text{in} \quad L^2(\partial \Omega), \]
and
\[ \lambda_n \sinh(u_{\lambda_n}) \| w_{\lambda_n} \|_{L^2(\partial \Omega)} \rightharpoonup \mu, \quad \text{weakly in} \quad H^{-1/2}(\partial \Omega), \quad (33) \]
as \( \lambda_n \to 0 \). The functions \( \tilde{w}_{\lambda_n} \) satisfy
\[ \Delta \tilde{w}_{\lambda_n} = 0 \quad \text{in} \quad \Omega, \]
\[ \frac{\partial \tilde{w}_{\lambda_n}}{\partial n} = D \tilde{w}_{\lambda_n} + \frac{\lambda_n \sinh(u_{\lambda_n})}{\| w_{\lambda_n} \|_{L^2(\partial \Omega)}} \quad \text{on} \quad \partial \Omega. \]
Furthermore
\[ \int_{\partial \Omega} \tilde{w}_{\lambda_n} \phi \, d\sigma_x = 0, \]
for any \( \phi \) that solves
\[ \Delta \phi = 0 \quad \text{in} \quad \Omega, \]
\[ \frac{\partial \phi}{\partial n} = D \phi \quad \text{on} \quad \partial \Omega. \]
The function \( w_0 \) is nonzero (since \( \| w_0 \|_{L^2(\partial \Omega)} = \lim \| \tilde{w}_{\lambda_n} \|_{L^2(\partial \Omega)} = 1 \)) and it satisfies
\[ \Delta w_0 = 0 \quad \text{in} \quad \Omega, \]
\[ \frac{\partial w_0}{\partial n} = Dw_0 + \mu \quad \text{on} \quad \partial \Omega, \]
in a weak, variational sense. From the same argument we used in connection with \( u_{\lambda_n} \) it now follows that
\[ w_{\lambda_n} \to \pm \infty \quad \text{almost everywhere in} \quad \Omega, \]
and (by extraction of a subsequence) that
\[ w_{\lambda_n} \to \pm \infty \quad \text{on a set of positive one dimensional surface measure on} \quad \partial \Omega, \]
as \( \lambda_n \to 0 \). Here we use that \( \| w_{\lambda_n} \|_{L^2(\partial \Omega)} \to \infty \), according to (30). We may also conclude that \( \mu \neq 0 \), because if \( \mu \) vanished identically, then \( w_0 \) would be a Steklov eigenvector, and the orthogonality relationship for the \( \tilde{w}_{\lambda_n} \) would give that
\[ \int_{\partial \Omega} (w_0)^2 \, d\sigma_x = \lim \int_{\partial \Omega} \tilde{w}_{\lambda_n} w_0 \, d\sigma_x = 0. \]
However, this contradicts the fact that \( \| w_0 \|_{L^2(\partial \Omega)} = 1 \).
A combination of (30) and (33) now completes the proof of the proposition. \( \square \)

Proposition 2 already asserts the existence of one nontrivial solution to (1) for \( \lambda \) in the range \( -D < \lambda < 0 \). There is a very useful variational characterization of a (potentially) larger class of solutions, which we shall now introduce, and which we shall also use extensively in the next section, for \( \lambda > 0 \). For this purpose we use the energy \( E_{\lambda}(\cdot) \), and in particular the “restricted” functional
\[ J_{\lambda}(v) = \inf_{t > 0} E_{\lambda}(tv). \quad (34) \]
The functionals \( J_{\lambda}, \lambda < 0 \), are bounded by \( -\infty < J_{\lambda}(1) \leq J_{\lambda}(\cdot) \leq 0 \). We already showed in the previous section that if \( w^* \) is a critical point for \( J_{\lambda}(\cdot) \) on
\[ \Sigma = \{ w \in H^1(\Omega): \| w \|_{H^1(\Omega)} = 1 \}, \]
with \( J_{\lambda}(w^*) < 0 \), then \( J_{\lambda}(t^* w^*) = E_{\lambda}(t^* w^*) \) for some \( t^* > 0 \), and \( u^* = t^* w^* \) is a critical point for \( E_{\lambda} \) in \( H^1(\Omega) \) (see also Lemma 2.5 of [10]). Such critical points are weak- and, by elliptic regularity, also strong solutions to the boundary value problem (1).
In order to establish existence of solutions it thus suffices to find nonzero critical values (and corresponding critical points) for \( J_{\lambda} \) on \( \Sigma \). To do this we employ a (by now) fairly standard result in critical point theory, cf. [9], [14] or [16]. Briefly stated this result asserts that all nonzero values of the form
\[ c_{\lambda}(\cdot) = \inf_{A \in \mathcal{A}} \sup_{w \in A} J_{\lambda}(w) \]
are critical values of $J_\lambda$. The infimum in $A$ is taken over the collection $\mathcal{A}_k$ of subsets $A \subset \Sigma$ that are compact, even and of genus $(A) \geq k \geq 1$.

The most essential prerequisite in order to be able to apply this result is to verify an appropriate compactness property of the functional $J_\lambda(\cdot)$. In the present context the required property is the Palais–Smale Condition verified in Lemma 1. For more details we refer the reader to [10].

We now proceed to show that this construction, for $-D < \lambda < 0$, gives rise to at most finitely many critical values, in complete agreement with Fig. 1. We also show that these critical values tend to $-\infty$ as $\lambda$ tends to $0^-$, so that the $H^1(\Omega)$ norms of the corresponding critical points for $E_\lambda(\cdot)$ tend to $\infty$, and these therefore represent solutions that “blow up” as described in Proposition 3.

**Proposition 4.** Suppose $D > 0$, and define $K^* = \max\{k: -D + D_k < 0\}$. Let $c_k(\lambda), -D < \lambda < 0, k \in \mathbb{N}$, be given by

$$c_k(\lambda) = \inf_{A \in \mathcal{A}_k} \sup_{w \in A} J_\lambda(w),$$

with

$$J_\lambda(w) = \inf_{t > 0} E_\lambda(tw) \leq 0,$$

and $E_\lambda$ as above. Then

$$c_k(\lambda) = 0 \quad \text{for any } k \geq K^* + 1, \quad -D < \lambda < 0.$$ (35)

Furthermore there exist positive constants $a_i, b_i, i = 1, 2$ such that

$$-a_1 \left( \log \frac{1}{|\lambda|} \right)^2 - b_1 \leq c_k(\lambda) \leq -a_2 \log \frac{1}{|\lambda|} + b_2,$$ (36)

for $-D < \lambda < 0$, and any $1 \leq k \leq K^*$.

**Proof.** Suppose $A$ is an even, compact subset of $\Sigma$, with genus $(A) \geq k \geq K^* + 1$. Then

$$A \cap \text{span}\{\phi_1, \phi_2, \ldots, \phi_{K^* - 1}, \phi_{K^*}\} \neq \emptyset;$$ (37)

otherwise the mapping

$$A \ni v \to (\langle \phi_1, v \rangle_{H^1}, \langle \phi_2, v \rangle_{H^1}, \ldots, \langle \phi_{K^* - 1}, v \rangle_{H^1}, \langle \phi_{K^*}, v \rangle_{H^1})$$

would be an odd, continuous map from $A$ to $\mathbb{R}^{K^*} \setminus 0$, contradicting the fact that genus $(A) \geq K^* + 1$. From (37) (and the orthogonality of the Steklov eigenvectors $\phi_k$) it follows immediately that there exists $v \in A$ ($v \neq 0$) such that

$$D_{K^* + 1} \int_{\partial \Omega} v^2 \, d\sigma_x \leq \int_{\Omega} |\nabla v|^2 \, dx.$$ (38)

Since $D \leq D_{K^* + 1}$ (and $\lambda < 0$) we get

$$E_\lambda(tv) = \frac{t^2}{2} \left[ \int_{\Omega} |\nabla v|^2 \, dx - D \int_{\partial \Omega} v^2 \, d\sigma_x \right] - \lambda \int_{\partial \Omega} (\cosh(tv) - 1) \, d\sigma_x \geq -\lambda \int_{\partial \Omega} (\cosh(tv) - 1) \, d\sigma_x \geq 0,$$

for any $t > 0$. In other words, there exists $v \in A$ with $J_\lambda(v) = 0$, and thus it follows that

$$\sup_{w \in A} J_\lambda(w) = 0.$$ 

Since this identity holds for any compact, even subset of $\Sigma$ with genus $(A) \geq K^* + 1$, it follows that

$$c_k(\lambda) = \inf_{A \in \mathcal{A}_k} \sup_{w \in A} J_\lambda(w) = 0.$$ 

---

1. The value of genus $(A)$ is by definition the smallest integer $m$, such that there exists a continuous, odd map from $A$ into $\mathbb{R}^m \setminus \{0\}$. 


for any \( k \geq K^* + 1 \). This proves the statement (35), and we now proceed with the verification of the estimates in (36). Since
\[
c_1(\lambda) \leq c_2(\lambda) \leq \cdots \leq c_{K^*-1}(\lambda) \leq c_{K^*}(\lambda),
\]
it suffices to verify that
\[
a_1 \left( \log \frac{1}{|\lambda|} \right)^2 - b_1 \leq c_1(\lambda) \quad \text{and} \quad c_{K^*}(\lambda) \leq -a_2 \log \frac{1}{|\lambda|} + b_2.
\]
We start with the lower bound for \( c_1(\lambda) \). The inequality
\[
\sup_{w \in A} J_\lambda(w) \geq \inf_{w \in \Sigma} J_\lambda(w) = \inf_{w \in H^1(\Omega)} E_\lambda(w),
\]
which holds for any subset \( A \subseteq \Sigma \), immediately implies that
\[
c_1(\lambda) = \inf_{A \in A_1} \sup_{w \in A} J_\lambda(w) \geq \inf_{w \in H^1(\Omega)} E_\lambda(w). \tag{39}
\]
For the energy \( E_\lambda(w) \) we have that
\[
E_\lambda(w) = \frac{1}{2} \int_\Omega |\nabla w|^2 \, dx - \frac{D}{2} \int w^2 \, d\sigma - \lambda \int (\cosh(w) - 1) \, d\sigma \geq \int_{\partial \Omega} \left[ -\frac{D}{2} w^2 - \lambda (\cosh(w) - 1) \right] d\sigma,
\]
\[
\geq |\partial \Omega| \inf_{t \in \mathbb{R}} \left[ -\frac{D}{2} t^2 - \lambda (\cosh(t) - 1) \right]. \tag{40}
\]
A straightforward calculation (remember \(-D < \lambda < 0\)) gives that
\[
\inf_{t \in \mathbb{R}} \left[ -\frac{D}{2} t^2 - \lambda (\cosh(t) - 1) \right] = -\frac{D}{2} t_e^2 - \lambda (\cosh(t_e) - 1) \geq -\frac{D}{2} t_e^2,
\]
where \( t_e > 0 \) is the unique positive solution to
\[
sinh(t_e) = -\frac{D}{\lambda} t_e.
\]
This \( t_e \) satisfies, for any \( \epsilon > 0 \), the estimate
\[
0 < t_e \leq (1 + \epsilon) \log \frac{1}{|\lambda|} + C_{\epsilon,D},
\]
where the constant \( C_{\epsilon,D} \) depends on \( \epsilon \) and \( D \), but is independent of \( \lambda \). After insertion into (41) we now get
\[
\inf_{t \in \mathbb{R}} \left[ -\frac{D}{2} t^2 - \lambda (\cosh(t) - 1) \right] \geq -(1 + \epsilon) \frac{D}{2} \left( \log \frac{1}{|\lambda|} \right)^2 - C_{\epsilon,D},
\]
where the constant \( C_{\epsilon,D} \) depends on \( \epsilon \) and \( D \), but is independent of \( \lambda \). In combination with (39) and (40) this gives
\[
c_1(\lambda) \geq -(1 + \epsilon)|\partial \Omega| \frac{D}{2} \left( \log \frac{1}{|\lambda|} \right)^2 - C_{\epsilon,D},
\]
which is a lower bound of the desired form.

We now turn our attention to the upper bound for \( c_{K^*}(\lambda) \). In order to verify this bound it suffices to find a compact, even subset \( A^* \subseteq \Sigma \) with
\[
(i) \quad \text{genus}(A^*) \geq K^*,
\]
\[
(ii) \quad \sup_{w \in A^*} J_\lambda(w) \leq -a_2 \log \frac{1}{|\lambda|} + b_2.
\]
Let \( \{\phi_k\}_{k=1}^{K^*} \) be the Steklov eigenvectors corresponding to eigenvalues \( D_k \), with \( 0 \leq D_k < D \), and define for \( R > 0 \)
\[
A_R^* = \left\{ \sum_{k=1}^{K^*} s_k \phi_k : \left\| \sum_{k=1}^{K^*} s_k \phi_k \right\|_{H^1(\Omega)}^2 = \sum_{k=1}^{K^*} (D_k + 1) s_k^2 = R^2 \right\}.
\]
It follows immediately from the Borsuk–Ulam Theorem that genus\(A_R^w) = K^w\). For any \(w = \sum_{k=1}^{K^w} s_k \phi_k \in A_R^w\) we have that
\[
\frac{\int_\Omega |\nabla w|^2 \, dx}{\int_{\partial \Omega} w^2 \, d\sigma} \leq D_{K^w},
\]
so that
\[
\frac{1}{2} \int \frac{|\nabla w|^2 \, dx}{\partial \Omega} \leq \frac{D - D_{K^w}}{2} \int \frac{w^2 \, dx}{\partial \Omega} = \frac{D - D_{K^w}}{2} \sum_{k=1}^{K^w} s_k^2.
\]

\[
= -\frac{D - D_{K^w}}{2} \sum_{k=1}^{K^w} s_k^2 \left( D_k + 1 \right) R^2 \leq -\frac{D - D_{K^w}}{2(D_{K^w} + 1)} R^2 = -a R^2,
\]

with \(a = (D - D_{K^w})/(2(D_{K^w} + 1)) > 0\). As a consequence of Trudinger's inequality (cf. Lemma 2.1 of [10]) we also have
\[
\int \left( \cosh(w) - 1 \right) \, d\sigma_x \leq \int \left| e^w \right| \, d\sigma_x - |\partial \Omega| \leq C_1 e^{C_2 \|w\|^2_{H^1(\Omega)}} = C_1 e^{C_2 R^2},
\]
for any \(w \in A^w\). A combination of (42) and (43) now gives
\[
E_{\lambda}(w) \leq -a R^2 + C_1 |\lambda| e^{C_2 R^2}
\]
for any \(w \in A^w\), with positive constants \(a, C_1\) and \(C_2\) independent of \(w, R\) and \(\lambda\). By selecting \(R = R(\lambda) = \sqrt{(1/C_2) \log(1/|\lambda| + 1)}\) it follows immediately from (44) that there exist positive constants \(a_2\) and \(b_2\), independent of \(\lambda \in (-D, 0)\), such that
\[
E_{\lambda}(w) \leq -a_2 \log \frac{1}{|\lambda|} + b_2 \quad \forall w \in A^w_{R(\lambda)}.
\]

From the definition of \(J_{\lambda}(\cdot)\) it now follows that
\[
J_{\lambda}(w) \leq E_{\lambda}(R(\lambda)w) \leq -a_2 \log \frac{1}{|\lambda|} + b_2 \quad \forall w \in A^w_{1},
\]
or
\[
\sup_{w \in A^w_{1}} J_{\lambda}(w) \leq -a_2 \log \frac{1}{|\lambda|} + b_2.
\]

The compact, even set \(A^w = A^w_{1} \subset \Sigma\) now satisfies the conditions (i) and (ii). This completes the proof of the upper bound for \(c_{K^w}(\lambda)\), and thus the proof of this proposition. \(\square\)

**Remarks.** From Proposition 4 it follows that \(c_{k}(\lambda) < 0\) for \(1 \leq k \leq K^w\), and \(\lambda < 0\) sufficiently close to 0. As discussed earlier \(c_{k}(\lambda)\) is thus a critical value for \(J_{\lambda}(\cdot)\) on \(\Sigma\) (with corresponding critical point \(w_{k,\lambda}\)). Furthermore there exists \(t_{k,\lambda} > 0\) such that \(w_{k,\lambda} = f_{k,\lambda} w_{k,\lambda}\) is a critical point for \(E_{\lambda}(\cdot)\) in \(H^1(\Omega)\), and thus a solution to the boundary value problem (1). As a consequence of Proposition 4 these solutions satisfy
\[
-a_1 \left( \log \frac{1}{|\lambda|} \right) \leq E_{\lambda}(w_{k,\lambda}) \leq -a_2 \log \frac{1}{|\lambda|} + b_2.
\]

They also satisfy
\[
c_1 \log \frac{1}{|\lambda|} - c_2 \leq \|w_{k,\lambda}\|_{H^1(\Omega)}^2 \leq C_1 \left( \log \frac{1}{|\lambda|} \right)^2 + C_2,
\]
in accordance with Proposition 2 and Proposition 3. From the very definition of the values \(c_{k}(\lambda)\) it is clear that
\[
c_1(\lambda) \leq c_2(\lambda) \leq \cdots \leq c_{K^w}(\lambda) \leq 0, \quad -D < \lambda < 0.
\]
One might expect that, generically, it would be true that
\[ c_1(\lambda) < c_2(\lambda) < \cdots < c_{K^*}(\lambda) < 0, \quad -D < \lambda < 0. \]

If this were the case, i.e., if all the \( c_k(\lambda), 1 \leq k \leq K^* \), were negative, and different, then the critical points corresponding to these critical values represent \( K^* \) essentially different nontrivial solutions to the boundary value problem (1) for \(-D < \lambda < 0\). □

2.2. Positive \( \lambda \)

We continue our consideration of the problem (1) with the nonlinear boundary flux \( Du + \lambda \sinh(u) \), but now for \( \lambda > 0 \). The analysis required has many similarities to that presented in [10,12] (where we considered \( D = 0 \)) and so in certain places we shall, for reasons of brevity, not provide all the details – but instead refer the reader to these papers. According to Fig. 1 we expect to find, for any fixed positive \( \lambda \), an infinite set of essentially different solutions. Indeed we may verify this conjecture using a Lyusternik–Schnirelmann approach, similar to that described in the previous section. This variational approach also provides very precise bounds for the \( H^1 \)-norms of the constructed solutions. We shall establish

**Theorem 1.** Suppose \( D \in \mathbb{R} \). For any fixed \( \lambda > 0 \), there exists an integer \( K_\lambda \) and an infinite set of solutions \( \{u_{k,\lambda}\}_{k=K_\lambda}^{\infty} \) to the problem (1). There exist \( \lambda^* > 0 \), and \( K_0 \), such that these solutions obey the energy estimates

\[ c \leq E_\lambda(u_{k,\lambda}) \leq a_k \log \left( \frac{1}{\lambda} \right) + b_k \]

for \( 0 < \lambda < \lambda^* \), and \( k \geq K_0 \). The constants \( c, a_k, b_k \) are positive, and independent of \( \lambda \).

The existence part of this theorem will be established using the auxiliary functional \( J_\lambda: H^1(\Omega) \to [0, \infty] \), defined by

\[ J_\lambda(v) = \sup_{t > 0} E_\lambda(tv). \]

To be specific we prove the existence of infinitely many critical points, \( v_{k,\lambda} \), for \( J_\lambda \) on the manifold \( \Sigma = \{w \in H^1(\Omega): \int_\Omega |\nabla w|^2 \, dx + \int_{\partial \Omega} w^2 \, d\sigma = 1\} \), with corresponding (different) positive critical values. As seen earlier in Section 2 such critical points immediately lead to solutions \( u_{k,\lambda} = t_{k,\lambda} v_{k,\lambda} \) to (1), with \( E_\lambda(u_{k,\lambda}) = J_\lambda(v_{k,\lambda}) \).

In order to arrive at these critical points we define, for any integer \( k \geq 1 \),

\[ c_k(\lambda) = \inf_{A \in \mathcal{A}_k} \sup_{w \in A} J_\lambda(w), \]

where \( \mathcal{A}_k \) is the collection of compact, even subsets of \( \Sigma \) of genus greater than or equal to \( k \). We observe that \( 0 \leq c_1(\lambda) \leq c_2(\lambda) \leq \cdots \leq c_k(\lambda) \leq c_{k+1}(\lambda) \leq \cdots \).

Since the even functional \( J_\lambda(\cdot), \lambda > 0 \), has the appropriate “smoothness properties” (on the set \( \{w: 0 < J_\lambda(w) < \infty\} \)) and satisfies the Palais–Smale Condition of Lemma 1, the same approach used in the previous section (for \( \lambda \) negative) here implies that any positive \( c_k(\lambda) \) is a critical value for \( J_\lambda(\cdot) \).

To establish the existence of nontrivial solutions to (1) it thus suffices to show that \( c_K > 0 \) for some \( K = K_\lambda \). The bounds in Theorem 1 require estimates for the \( c_k(\lambda) \) as \( \lambda \to 0_+ \). These estimates (as well as the existence of \( K_\lambda \)) are established by the following lemma.

**Lemma 3.** Given \( D \in \mathbb{R} \), let \( K \geq 2 \) be a fixed integer such that \( D_K > D \), where \( D_K \) denotes the \( K \)'th Steklov eigenvalue for the problem (2). Let \( c_k(\lambda) \) be as above. There exist positive constants, \( a_k \) and \( b_k \), depending on \( k, D \) and \( K \), but independent of \( \lambda \), such that for all \( k \geq K \) and all \( 0 < \lambda < D_K - D \),

\[ 0 < c_k(\lambda) \leq a_k \log \frac{1}{\lambda} + b_k. \]

**Remark.** In the course of the proof of this lemma we establish a lower bound that is a bit more precise than \( 0 < c_k(\lambda) \). We actually show that

\[ c_k(\lambda) \geq d(D_K - D - \lambda)^2 > 0. \]
In particular, this implies that

\[ c_k(\lambda) \geq c > 0 \quad \text{for } 0 < \lambda < \frac{D_K - D}{2}. \]

**Proof of Lemma 3.** We start with the lower bound. Let \( A \) be any compact, even subset of \( \Sigma \) with genus \((A) \geq k \geq K\).

As in (37), we know there exists \( v_\ast \in A \) such that \( \langle \phi_j, v_\ast \rangle_{H^1} = 0 \) for \( j = 1, \ldots, K - 1 \). Here \( \phi_j \) is the Steklov eigenvector corresponding to the eigenvalue \( D_j, 1 \leq j \leq K - 1 \). This \( v_\ast \) satisfies the inequality

\[ D_K \int_{\partial \Omega} v_\ast^2 \, d\sigma_x \leq \int_{\Omega} |\nabla v_\ast|^2 \, dx. \]

For \( 0 < \lambda < D_K - D \) we thus calculate

\[
E_\lambda(tv_\ast) = \frac{t^2}{2} \left[ \int_{\Omega} |\nabla v_\ast|^2 \, dx - (D + \lambda) \int_{\partial \Omega} v_\ast^2 \, d\sigma_x \right] - \lambda \int_{\partial \Omega} \left( \cosh(tv_\ast) - 1 - \frac{(tv_\ast)^2}{2} \right) \, d\sigma_x \\
\geq \frac{t^2}{2} B(K, \lambda, D) \int_{\Omega} |\nabla v_\ast|^2 \, dx - \lambda \int_{\partial \Omega} \left( \cosh(tv_\ast) - 1 - \frac{(tv_\ast)^2}{2} \right) \, d\sigma_x, \tag{46}
\]

with \( B(K, \lambda, D) = \min \{1, (1 - (\lambda + D)/D_K)\} \). We also have that

\[
\int_{\partial \Omega} \left( \cosh(v) - 1 - \frac{(v)^2}{2} \right) \, d\sigma_x \leq \int_{\partial \Omega} v^4 \cosh(v) \, d\sigma_x \leq \left( \int_{\partial \Omega} v^8 \, d\sigma_x \right) \frac{1}{2} \left( \int_{\partial \Omega} \cosh(2v) \, d\sigma_x \right) \frac{1}{2} \\
\leq C \left( \int_{\Omega} |\nabla v|^2 \, dx \right)^2 e^{C_2 \|\nabla v\|_{L^2(\Omega)}^2},
\]

for any \( v \) which is orthogonal to \( \phi_1 \) (a constant), and as a consequence

\[
\int_{\partial \Omega} \left( \cosh(tv_\ast) - 1 - \frac{(tv_\ast)^2}{2} \right) \, d\sigma_x \leq C t^4 \left( \int_{\Omega} |\nabla v_\ast|^2 \, dx \right)^2 e^{C_2 \|\nabla tv_\ast\|_{L^2(\Omega)}^2} \leq C_1 t^4 e^{C_2 t^2} \int_{\Omega} |\nabla v_\ast|^2 \, dx. \tag{47}
\]

For the last inequality we have used that \( v_\ast \) lies in \( \Sigma \), so that \( \|\nabla v_\ast\|_{L^2(\Omega)}^2 \leq 1 \). Let \( t_0 > 0 \) be given by

\[ t_0 = c \sqrt{B(K, \lambda, D)}, \]

with \( c \) chosen so small, that

\[ \lambda C_1 e^{C_2 e^{B(K, \lambda, D)}} = \lambda C_1 e^{C_2 e^{\min \{1, (1 - (\lambda + D)/D_K)\}}} < \frac{1}{4}, \]

for all \( 0 < \lambda < D_K - D \) (\( C_1 \) being the constant from (47)). A combination of the estimates (46) and (47), with \( t = t_0 \) now yields

\[ E_\lambda(t_0 v_\ast) \geq \frac{1}{4} t_0^2 B(K, \lambda, D) \int_{\Omega} |\nabla v_\ast|^2 \, dx \geq d \left( 1 - \frac{\lambda + D}{D_K} \right)^2, \tag{48}
\]

where \( d > 0 \) depends on \( D_K \) and \( D \), but is independent of \( \lambda \) (in the interval \( 0 < \lambda < D_K - D \)). For the last inequality we have also used that

\[ \int_{\Omega} |\nabla v_\ast|^2 \, dx \geq d \|v_\ast\|_{H^1(\Omega)}^2 = d > 0, \]

due to the facts that \( v_\ast \) is orthogonal to \( \phi_1 \) (a constant) and that \( v_\ast \) lies in \( \Sigma \). From (48) we immediately conclude that

\[ \sup_{v \in A} J_\lambda(v) \geq J_\lambda(v_\ast) \geq E_\lambda(t_0 v_\ast) \geq d(D_K - D - \lambda)^2 > 0. \]
Since $A$ is an arbitrary compact, even subset of $\Sigma$, with genus($A$) $\geq k$, it follows that
\[
c_k(\lambda) = \inf_{A \in \mathcal{A}_k} \sup_{v \in A} J_k(v) \geq d(D_k - D - \lambda)^2 > 0,
\]
as desired.

To prove the upper bound, we introduce some special functions. Let $\{\sigma_j\}_{j=1}^k$ be a set of $k(\geq 2)$ distinct points on $\partial \Omega$, and let $\epsilon$ and $R$ be two positive numbers. Define
\[
d_{j,\epsilon}(x) := -\log(|x - \sigma_j|^2 + \epsilon^2),
\]
and then the set
\[
G_{\epsilon,R} := \left\{ w = \sum_{j=1}^k \alpha_j d_{j,\epsilon}(x): \int_\Omega |\nabla w|^2 \, dx + \int_{\partial \Omega} w^2 \, d\sigma_x = R^2 \right\}.
\]

**Claim 1.** Given $D, k \geq 2$ and $\{\sigma_j\}_{j=1}^k$ there exist $\lambda^* > 0$, and functions $\epsilon(\lambda) > 0$, $R(\lambda) > 0$, such that for $0 < \lambda < \lambda^*$
\[
\int_\Omega |\nabla v|^2 \, dx - D \int_{\partial \Omega} v^2 \, d\sigma_x - \lambda \int_{\partial \Omega} v \sinh(v) \, d\sigma_x \leq 0 \quad \forall v \in G_{\epsilon(\lambda), R(\lambda)}.
\]
Moreover, the functions $\epsilon(\cdot)$ and $R(\cdot)$ may be chosen so that
\[
\epsilon(\lambda) = O(\lambda) \quad \text{and} \quad R(\lambda) = O\left(\sqrt{\log(1/\lambda)}\right),
\]
as $\lambda$ approaches 0.

For $D \geq 0$ this claim follows directly from Lemma 3.4 in [10]; for $D < 0$ a slightly modified version of the proof of Lemma 3.4 in [10] is required. We do not reproduce the proof here, instead we proceed with the verification of Claim 1.

Claim 1 asserts that
\[
f_\lambda(v) = \frac{1}{2} \left( \int_\Omega |\nabla v|^2 \, dx - D \int_{\partial \Omega} v^2 \, d\sigma_x - \lambda \int_{\partial \Omega} \cosh(t(v^*)v) - 1 \right) \, d\sigma_x
\]
for some $v^* \in G_{\epsilon,R}$. We can also estimate
\[
J_\lambda(v^*) = E_\lambda(t(v^*)v^*) = \frac{1}{2} \left( \int_\Omega |\nabla v^*|^2 \, dx - D \int_{\partial \Omega} (t(v^*))^2 \, d\sigma_x - \lambda \int_{\partial \Omega} (\cosh(t(v^*)v^*) - 1) \, d\sigma_x \right)
\]
\[
\leq \frac{1}{2} \left( \int_\Omega |\nabla v^*|^2 \, dx + |D| \int_{\partial \Omega} (v^*)^2 \, d\sigma_x \right) \leq \frac{1}{2} (t(v^*))^2 \max\{1, |D|\} R^2.
\]

We observe that the function $f(t) = E_\lambda(tv^*)$ has a strictly concave derivative
\[
f'(t) = 1 \int_{\Omega} |\nabla v^*|^2 \, dx - D t \int_{\partial \Omega} (v^*)^2 \, d\sigma_x - \lambda \int_{\partial \Omega} \sinh(tv^*)v^* \, d\sigma_x,
\]
on the interval $(0, \infty)$. To see this, we simply calculate that
\[
f''(t) = -\lambda \int_{\partial \Omega} \sinh(tv^*)(v^*)^3 \, d\sigma_x < 0.
\]
Claim 1 asserts that $f'(1) \leq 0$. Since we also have $f'(0) = 0$ and $f'(t(v^*)) = 0$, the concavity of $f'$ now implies that $t(v^*) \leq 1$. The estimate (49) now yields
\[
c_k(\lambda) \leq J_\lambda(v^*) \leq C_k \log\left(\frac{1}{\lambda}\right),
\]
for $\lambda$ sufficiently small, as desired. $\square$

With this lemma we have established the existence of solutions to (1) for any $\lambda > 0$, indeed we have already shown that any positive $c_k(\lambda)$ corresponds to a nontrivial solution, $u_{k,\lambda}$, with $E_{\lambda}(u_{k,\lambda}) = c_k(\lambda)$. The energy estimates in Theorem 1 follow directly from the upper bound in Lemma 3, and from the remark following the statement of Lemma 3. That our process actually leads to infinitely many essentially different (energy-different) solutions follows from the fact that $c_k(\lambda) \to \infty$ as $k \to \infty$ for any fixed $\lambda > 0$ (see [9]). This verifies Theorem 1.

As before, let $\{D_k\}$ and $\{\phi_k\}$ be the Steklov eigenvalues and normalized eigenfunctions. We remind the reader of the definition of the bounded linear projection operator $P_D$:

$$P_D(\cdot) = \sum_{k: D_k = D} \langle \cdot, \phi_k \rangle_{L^2(\partial\Omega)} \phi_k = \sum_{k: D_k = D} \frac{\langle \cdot, \phi_k \rangle_{H^1(\Omega)}}{\langle \phi_k, \phi_k \rangle_{H^1(\Omega)}} \phi_k.$$

We now have the following $H^1$ bounds concerning the solutions, whose existence are assured by Theorem 1.

**Proposition 5.** Let $\lambda^* < 1$ be a fixed positive number, and let $u_\lambda$, $0 < \lambda < \lambda^*$, be a family of solutions to (1) (with shift $D$) whose energies satisfy the estimate

$$E_\lambda(u_\lambda) \leq a \log \frac{1}{\lambda} + b, \quad (50)$$

for some positive constants $a$ and $b$. Set $w_\lambda = (I - P_D)u_\lambda$, so that $u_\lambda = w_\lambda + P_D u_\lambda$. There exist positive constants $C_1$ and $C_2$, depending on $a$, $b$, $D$ and $\lambda^*$, but otherwise independent of $u_\lambda$ and $\lambda$, such that

$$\|w_\lambda\|_{H^1(\Omega)} \leq C_1 \log \frac{1}{\lambda} + C_2, \quad 0 < \lambda < \lambda^*,$$

and

$$\|P_D u_\lambda\|_{H^1(\Omega)} \leq C_1 \log \frac{1}{\lambda} + C_2, \quad 0 < \lambda < \lambda^*.$$

**Proof.** Integration by parts combined with the upper bound (50) gives

$$\lambda \int_{\partial\Omega} \left( \frac{u_\lambda}{2} \sinh(u_\lambda) - \cosh(u_\lambda) + 1 \right) d\sigma_x = E_\lambda(u_\lambda) \leq a \log \frac{1}{\lambda} + b.$$

Since $|u| e^{\|u\|} \leq C_1 (\frac{1}{4} \sinh(u) - \cosh(u) + 1) + C_2$, it follows that

$$\lambda \int_{\partial\Omega} |u_\lambda| e^{|u_\lambda|} d\sigma_x \leq C_1 \log \frac{1}{\lambda} + C_2.$$

A simple convexity argument (see [10] or [12]) now gives

$$\lambda \int_{\partial\Omega} |\sinh(u_\lambda)| d\sigma_x \leq \lambda \int_{\partial\Omega} \cosh(u_\lambda) d\sigma_x \leq C,$$

for some constant $C$. Testing Eqs. (21) for $w_\lambda$ against the eigenfunctions, $\phi_k$, and integrating by parts, we obtain

$$(D_k - D) \int_{\partial\Omega} w_\lambda \phi_k d\sigma_x = \int_{\partial\Omega} \lambda \sinh(u_\lambda) \phi_k d\sigma_x. \quad (52)$$

As in the proof of Proposition 3 we therefore have

$$\|w_\lambda\|^2_{L^2(\partial\Omega)} = \sum_{k: D_k \neq D} \alpha^2_{k,\lambda},$$

with

$$\alpha_{k,\lambda} = \int_{\partial\Omega} w_\lambda \phi_k d\sigma_x = \frac{1}{D_k - D} \int_{\partial\Omega} \lambda \sinh(u_\lambda) \phi_k d\sigma_x.$$
If $D \neq 0$ it follows immediately that there exists a constant $C$ such that
\[
\|w_\lambda\|_{L^2(\partial \Omega)}^2 = \sum_{k: D_k \neq D} |D_k - D|^{-2} \int_{\partial \Omega} \lambda \sinh(u_\lambda) \phi_k \, d\sigma_x \leq C \left( \sum_{k \neq 1} |D_k|^{-2} \int_{\partial \Omega} \lambda \sinh(u_\lambda) \phi_k \, d\sigma_x \right)^2.
\]

Simply take $C = \max\{|D|^{-2}, \max_{D_k \neq D} |1 - \frac{D}{D_k}|^{-2}\}$. If $D = 0$ it follows similarly that
\[
\|w_\lambda\|_{L^2(\partial \Omega)}^2 = \sum_{k: D_k \neq D} |D_k - D|^{-2} \int_{\partial \Omega} \lambda \sinh(u_\lambda) \phi_k \, d\sigma_x = \sum_{k \neq 1} |D_k|^{-2} \int_{\partial \Omega} \lambda \sinh(u_\lambda) \phi_k \, d\sigma_x^2.
\]

Due to (51), $|\int_{\partial \Omega} \lambda \sinh(u_\lambda) \, d\sigma_x| \leq C$, and so in both cases ($D \neq 0$ and $D = 0$) we have
\[
\|w_\lambda\|_{L^2(\partial \Omega)}^2 \leq C \left( \sum_{k \neq 1} |D_k|^{-2} \int_{\partial \Omega} \lambda \sinh(u_\lambda) \phi_k \, d\sigma_x \right)^2 + 1.
\]

with $C$ depending on $a$, $b$, $D$ and $\lambda^*$, but otherwise independent of $w_\lambda$ and $\lambda$. Now let $W_\lambda$ denote the solution to
\[
\Delta W_\lambda = 0 \quad \text{in } \Omega, \quad \frac{\partial W_\lambda}{\partial n} = F_\lambda = \lambda \sinh(u_\lambda) - \frac{1}{|\partial \Omega|} \int_{\partial \Omega} \lambda \sinh(u_\lambda) \, d\sigma_x \quad \text{on } \partial \Omega,
\]

with $\int_{\partial \Omega} W_\lambda \, d\sigma_x = 0$. Duality and elliptic regularity estimates immediately give that
\[
\|W_\lambda\|_{L^2(\partial \Omega)} \leq C \|F_\lambda\|_{H^{-1}(\partial \Omega)} \leq C \|F_\lambda\|_{L^1(\partial \Omega)} \leq C \|\lambda \sinh(u_\lambda)\|_{L^1(\partial \Omega)}.
\]

Here we have also used the fact that $\partial \Omega$ is one-dimensional to obtain that $L^1(\partial \Omega)$ continues embeds into $H^{-1}(\partial \Omega)$. The function $W_\lambda$ is constructed exactly so that
\[
\|W_\lambda\|_{L^2(\partial \Omega)}^2 = \sum_{k \neq 1} |D_k|^{-2} \int_{\partial \Omega} \lambda \sinh(u_\lambda) \phi_k \, d\sigma_x^2,
\]

and a combination of (53) and (54) with the $L^1$-bound (51) thus gives
\[
\|w_\lambda\|_{L^2(\partial \Omega)}^2 \leq C \left( \|W_\lambda\|_{L^2(\partial \Omega)}^2 + 1 \right) \leq C \left( \|\lambda \sinh(u_\lambda)\|_{L^1(\partial \Omega)}^2 + 1 \right) \leq C.
\]

From integration by parts, and the use of (21) and (52) (if $D$ is a Steklov eigenvalue) we get that
\[
\frac{1}{2} \int_{\Omega} |\nabla w_\lambda|^2 \, dx - \frac{D}{2} \int_{\partial \Omega} w_\lambda^2 \, d\sigma_x - \lambda \int_{\partial \Omega} (\cosh(u_\lambda) - 1) \, d\sigma_x
\]
\[
= \frac{1}{2} \int_{\Omega} |\nabla u_\lambda|^2 \, dx - \frac{D}{2} \int_{\partial \Omega} u_\lambda^2 \, d\sigma_x - \lambda \int_{\partial \Omega} (\cosh(u_\lambda) - 1) \, d\sigma_x
\]
\[
= E_\lambda(u_\lambda) \leq a \log \frac{1}{\lambda} + b.
\]

By a combination of this estimate with (51) and (55) it follows that
\[
\int_{\Omega} |\nabla w_\lambda|^2 \, dx \leq C_1 \log \frac{1}{\lambda} + C_2.
\]

The estimates (55) and (56) give the desired $H^1$ bound for $w_\lambda$. 

The function $x \rightarrow g(x) = \cosh(x)$ is convex, and so Jensen’s inequality, in combination with (51), asserts that

\[
g(\int_{\partial\Omega} |u\lambda| \, d\sigma/|\partial\Omega|) \leq \int_{\partial\Omega} g(|u\lambda|) \, d\sigma/|\partial\Omega| \leq \frac{C}{\lambda}.
\]

In other words

\[
\|u\lambda\|_{L^1(\partial\Omega)} \leq |\partial\Omega| \log \frac{1}{\lambda} + C.
\]

By a combination of this estimate with (55) it follows that

\[
\|PDu\lambda\|_{L^1(\partial\Omega)} \leq \|PDu\lambda - u\lambda\|_{L^1(\partial\Omega)} + \|u\lambda\|_{L^1(\partial\Omega)} = \|w\lambda\|_{L^1(\partial\Omega)} + \|u\lambda\|_{L^1(\partial\Omega)} \\
\leq |\partial\Omega|^{1/2} \|w\lambda\|_{L^2(\partial\Omega)} + \|u\lambda\|_{L^1(\partial\Omega)} \leq |\partial\Omega| \log \frac{1}{\lambda} + C. \tag{57}
\]

We note that $PDu\lambda$ lies in the finite dimensional eigenspace $V_D \subset H^1(\Omega)$ spanned by the “harmonically extended” eigenfunctions ($V_D = \{0\}$ if $D$ is not an eigenvalue). We also note that $\|\cdot\|_{L^1(\partial\Omega)}$ is a norm on $V_D$. Since all norms are equivalent on $V_D$ it now follows from (57) that

\[
\|PDu\lambda\|_{H^1(\partial\Omega)} \leq C_1 \log \frac{1}{\lambda} + C_2,
\]

as desired. $\Box$

We may also establish a lower bound that applies to the solutions constructed in Theorem 1.

**Proposition 6.** There exists a constant $c_1 > 0$ such that whenever $u\lambda_n$, $\lambda_n \rightarrow 0_+$, is a sequence of solutions to (1) with the property that $\|u\lambda_n\|_{H^1(\Omega)}$ does not converge to 0 as $\lambda_n \rightarrow 0_+$, then we may extract a subsequence, for simplicity also denoted $u\lambda_n$, with

\[
c_1 \log \frac{1}{\lambda_n} \leq \|u\lambda_n\|_{H^1(\Omega)}^2 \quad \text{as } \lambda_n \rightarrow 0_+.
\]

**Proof.** It suffices to prove that there exists a constant $c_1 > 0$ such that if $u\lambda_n$, $\lambda_n \rightarrow 0_+$ is a family of solutions to (1) with

\[
\|u\lambda_n\|_{H^1(\Omega)}^2 \sqrt{\log \frac{1}{\lambda_n}} < c_1 \quad \text{as } \lambda_n \rightarrow 0_+,
\]

then

\[
\|u\lambda_n\|_{H^1(\Omega)} \rightarrow 0 \quad \text{as } \lambda_n \rightarrow 0_+.
\]

The argument to show this proceeds exactly as in the proof of Proposition 3. $\Box$

Having thus provided asymptotic bounds for the $H^1$ norm of the variationally constructed solutions, we now continue with a more detailed blow-up analysis of these solutions as $\lambda \rightarrow 0_+$. As we shall see, the fact that $\lambda \sinh(u\lambda)$ is bounded in $L^1(\partial\Omega)$, leads to a completely different blow-up pattern than that, which we saw for $\lambda \rightarrow 0_-$ in the previous section. For $D = 0$ a very detailed blow-up analysis has already been carried out in [10] and [12].

Except for $D = 0$ the analysis is somewhat more complicated (and the results are less complete) when $D$ is a Steklov eigenvalue. Some of the analysis that follows is directly unnecessary if $D$ is not a Steklov eigenvalue. Indeed, in that case $PDu\lambda = 0$ and so $u\lambda = (I - PD)u\lambda = w\lambda$. For reasons of completeness we have decided to proceed in a general framework – however, simplifications for the case when $D$ is not an eigenvalue will be noted when appropriate.

If $D$ is a Steklov eigenvalue, the possibility exists that the contribution of the “mode” corresponding to $D$ becomes unbounded. That is, writing $u\lambda = PDu\lambda + w\lambda$, the norm of the component $PDu\lambda$ can be unbounded (growing like $C_1 \log \frac{1}{\lambda} + C_2$) as $\lambda$ approaches $0_+$. We have already seen in the proof of Proposition 5 that

\[
\|PDu\lambda\|_{L^1(\partial\Omega, d\sigma/|\partial\Omega|)} \leq \log \frac{1}{\lambda} + C_2, \tag{58}
\]
i.e., the constant $C_1$ in front of $\log \frac{1}{\lambda}$ may be taken to be one, if we use the measure $\text{d}\sigma/|\partial \Omega|$ in our definition of the $L^1$-norm. A similar estimate holds for the norm on $L^p(\partial \Omega, \text{d}\sigma/|\partial \Omega|)$, however, it is not clear (except when $D = 0$, and $P_Du_\lambda = \text{const}$) that a similar estimate holds for the $L^\infty$ norm. This in turn makes it somewhat unclear whether $\|\lambda e^{P_Du_\lambda}\|_{L^\infty(\partial \Omega)}$ is uniformly bounded, as $\lambda \to 0_+$, in the case when $D$ is an eigenvalue different from 0. We are, nonetheless, able to establish a partial result in this direction. In order to state this result we first introduce some notation. By $V_D$ we denote, as before, the eigenspace corresponding to $D$, i.e.,

$$V_D = \left\{ \phi \in C^\infty(\overline{\Omega}) : \Delta \phi = 0 \text{ in } \Omega, \frac{\partial \phi}{\partial n} = D\phi \text{ on } \partial \Omega \right\},$$

and given any $\phi \in V_D \setminus \{0\}$ we introduce the set $I_\phi$

$$I_\phi = \{ x \in \partial \Omega : |\phi(x)| < \|\phi\|_{C^0(\partial \Omega)} \},$$

and the set $M_\phi$

$$M_\phi = \partial \Omega \setminus I_\phi = \{ x \in \partial \Omega : |\phi(x)| = \|\phi\|_{C^0(\partial \Omega)} \}.$$

When $D$ is a Steklov eigenvalue different from zero, we expect $I_\phi$ to be almost all of $\partial \Omega$. For instance, when $\Omega$ is a disk, and $D$ is an eigenvalue different from zero, then $M_\phi$ consists of a finite (even) number of equispaced points.

**Lemma 4.** Suppose $D = D_k$ is a Steklov eigenvalue for the boundary value problem (2). Suppose $u_{\lambda_n}, \lambda_n \to 0_+$, is a sequence of solutions to (1) (corresponding to that same shift $D$) whose energies satisfy the estimate

$$E_\lambda(u_{\lambda_n}) \leq a \log \frac{1}{\lambda_n} + b,$$

for some positive constants $a$ and $b$. Then we have the following results

$D = 0$: There exists a constant $C$ depending only on $a$, $b$ and $|\partial \Omega|$ such that

$$\lambda_n e^{\|P_Du_{\lambda_n}\|_{C^0(\partial \Omega)}} = \lambda_n e^{\frac{1}{\lambda_n} \int_{\partial \Omega} u_{\lambda_n} \text{d}\sigma} \leq C \quad \text{as } \lambda_n \to 0_+,$$

$D \neq 0$: There exists a subsequence, for simplicity also denoted $\lambda_n$, and a Steklov eigenvector $\phi \in V_D \setminus \{0\}$ such that, given any $x_0 \in I_\phi \subset \partial \Omega$, we may find an open neighborhood $\omega_{x_0} \subset \partial \Omega$, of $x_0$, with

$$\sup_{x \in \omega_{x_0}} \lambda_n e^{\|P_Du_{\lambda_n}(x)\|_{C^0(\partial \Omega)}} \to 0 \quad \text{as } \lambda_n \to 0_+.$$

**Proof.** The result for $D = 0$ was already used in [10] and [12], and it follows directly from (58). We proceed with the case $D = D_k \neq 0$. From the estimates (51) and (55) in the proof of Proposition 5 we know that any sequence of solutions

$$u_{\lambda_n} = P_Du_{\lambda_n} + w_{\lambda_n}, \quad \lambda_n \to 0_+,$$

to (1), which satisfies the energy bound assumed in the present lemma, also satisfies

$$\lambda_n \int_{\partial \Omega} e^{\|P_Du_{\lambda_n} + w_{\lambda_n}\|_{C^0(\partial \Omega)}} \text{d}\sigma \leq C,$$

and

$$\int_{\partial \Omega} w_{\lambda_n}^2 \text{d}\sigma \leq C,$$

with $C$ only depending on $a$, $b$, $D$ and $|\partial \Omega|$. Suppose $\|P_Du_{\lambda_n}\|_{C^0(\partial \Omega)} \neq 0$. The sequence

$$\phi_{\lambda_n} = \frac{P_Du_{\lambda_n}}{\|P_Du_{\lambda_n}\|_{C^0(\partial \Omega)}}$$
is bounded in $V_D$. Due to the finite dimensionality of $V_D$ we may thus extract a subsequence, also denoted $\phi_{\lambda_n}$, with

$$\phi_{\lambda_n} \to \phi \in V_D \quad \text{in any norm as } \lambda_n \to 0_+.$$  

(61)

The statement of this lemma is trivial if $\|P_D u_{\lambda_n}\|_{C^0(\partial \Omega)}$ is bounded (or has a bounded subsequence). We may thus suppose that

$$\|P_D u_{\lambda_n}\|_{C^0(\partial \Omega)} \to \infty \quad \text{as } \lambda_n \to 0_+.$$  

The function $\phi$ necessarily has $\|\phi\|_{C^0(\partial \Omega)} = 1$. We may without loss of generality suppose there exists $\bar{x} \in \partial \Omega$ with $\phi(\bar{x}) = 1$. Now suppose $x_0$ is in $I_{\phi}$, and therefore $-1 < \phi(x_0) < 1$. By virtue of (61), there exist open $\partial \Omega$-neighborhoods $\omega_{x_0}$ and $\omega_{\bar{x}}$, of $x_0$ and $\bar{x}$, respectively, and a small positive number $\eta$, such that

$$\sup_{x \in \omega_{x_0}} |\phi_{\lambda_n}(x)| \leq 1 - 2\eta \leq 1 - \eta \leq \inf_{x \in \omega_{\bar{x}}} \phi_{\lambda_n}(x),$$  

(62)

for $n$ sufficiently large (as $\lambda_n$ converges to 0). Based on (59) and (62) we conclude that

$$\sup_{x \in \omega_{x_0}} \lambda_n e^{\|P_D u_{\lambda_n}\|_{C^0(\partial \Omega)} (x)} \int_{\omega_{\bar{x}}} e^{u_{\lambda_n}} \, d\sigma = \sup_{x \in \omega_{x_0}} \lambda_n e^{\|P_D u_{\lambda_n}\|_{C^0(\partial \Omega)} (x)} \int_{\omega_{\bar{x}}} e^{u_{\lambda_n}} \, d\sigma$$

$$\leq \lambda_n e^{-\eta \|P_D u_{\lambda_n}\|_{C^0(\partial \Omega)}} \int_{\omega_{\bar{x}}} e^{|\phi_{\lambda_n}(x) + u_{\lambda_n}} \, d\sigma$$

$$= \lambda_n e^{-\eta \|P_D u_{\lambda_n}\|_{C^0(\partial \Omega)}} \int_{\omega_{\bar{x}}} e^{P_D u_{\lambda_n} + u_{\lambda_n}} \, d\sigma$$

$$\leq C e^{-\eta \|P_D u_{\lambda_n}\|_{C^0(\partial \Omega)}},$$  

(63)

for $\lambda_n$ sufficiently close to 0. Now, we also have that

$$0 < c < \int_{\omega_{\bar{x}}} e^{u_{\lambda_n}} \, d\sigma,$$  

(64)

because if this lower bound did not hold then we could find a subsequence $\lambda_{n_m} \to 0_+$ (for simplicity denoted $\lambda_m$, $m \to \infty$) so that

$$\int_{\omega_{\bar{x}}} e^{u_{\lambda_m}} \, d\sigma \leq |\omega_{\bar{x}}|/2m,$$

and as a consequence

$$\left| \{ x \in \omega_{\bar{x}} : e^{u_{\lambda_m}} \leq 1/m \} \right| \geq |\omega_{\bar{x}}|/2.$$

This would imply that

$$\left| \{ x \in \omega_{\bar{x}} : w_{\lambda_m} \leq - \log m \} \right| \geq |\omega_{\bar{x}}|/2,$$

and thus

$$\int_{\omega_{\bar{x}}} |w_{\lambda_m}|^2 \, d\sigma \geq \frac{|\omega_{\bar{x}}|}{2} \left( \log m \right)^2, \quad \text{as } m \to \infty$$

as $m \to \infty$ ($\lambda_m \to 0_+$). We have now arrived at a contradiction to the estimate (60). A combination of (63) and (64), with the fact that $\|P_D u_{\lambda_n}\|_{C^0(\partial \Omega)}$ converges to $\infty$, immediately leads to the assertion of this lemma. \qed

As already noted in the proof of Proposition 5, any sequence of solutions to (1) ($\lambda_n \to 0_+$) that satisfies the energy bound (50) also satisfies the estimate

$$\lambda_n \int_{\partial \Omega} e^{\pm u_{\lambda_n}} \, d\sigma \leq C.$$
We may thus extract a subsequence, \( \{ u_{\lambda_n} \} \), which in addition to the conclusion of Lemma 4 has
\[
\frac{\lambda_n}{2} e^{u_{\lambda_n}} \bigg|_{\partial \Omega} \to \mu_+ \quad \text{and} \quad \frac{\lambda_n}{2} e^{-u_{\lambda_n}} \bigg|_{\partial \Omega} \to \mu_- \tag{65}
\]
for two nonnegative measures \( \mu_+ \) and \( \mu_- \). The convergence is in the sense of measures (i.e., weak* in the dual of \( C^0(\partial \Omega) \)). With the present blow-up analysis we seek to characterize the limiting behavior of \( u_{\lambda_n} \) modulo its potential “eigenfunction part”, i.e., we study the limiting behavior of \( w_{\lambda_n} = (I-P_D) u_{\lambda_n} \) (if \( D \) is not a Steklov eigenvalue then \( w_{\lambda_n} = u_{\lambda_n} \)). As already observed, \( w_{\lambda} \) solves
\[
\Delta w_{\lambda_n} = 0 \quad \text{in} \ \Omega, \\
\frac{\partial w_{\lambda_n}}{\partial n} = D w_{\lambda_n} + \lambda_n \sinh u_{\lambda_n} \quad \text{on} \ \partial \Omega, 
\]
or in its distributional formulation
\[
\int_{\Omega} w_{\lambda_n} \Delta v \, dx + \int_{\partial \Omega} D w_{\lambda_n} v \, d\sigma_x = - \int_{\partial \Omega} \lambda_n \sinh u_{\lambda_n} v \, d\sigma_x, 
\]
for all \( v \in C^2(\overline{\Omega}) \) with \( \frac{\partial v}{\partial n} = 0 \) on \( \partial \Omega \). As we have already seen, \( w_{\lambda_n} \) and \( u_{\lambda_n} \) satisfy the estimates
\[
\| w_{\lambda_n} \|_{L^2(\partial \Omega)} \leq C \quad \text{and} \quad \| \lambda_n \sinh u_{\lambda_n} \|_{L^1(\partial \Omega)} \leq C.
\]
It now follows quite easily from an application of standard elliptic regularity theory to the boundary value problem (66), and duality, that
\[
\| w_{\lambda_n} \|_{H^{1-\epsilon}(\Omega)} \leq C_\epsilon,
\]
for any \( \epsilon > 0 \). The constant \( C_\epsilon \) is independent of \( \lambda_n \to 0_+ \). By extraction of a subsequence (for simplicity also denoted \( \lambda_n \)) we may thus achieve convergence in \( H^{1-\epsilon}(\Omega) \) for any \( \epsilon > 0 \), i.e., we may achieve
\[
w_{\lambda_n} \to w_0 \quad \text{in} \ H^{1-\epsilon}(\Omega), \tag{67}
\]
for any \( \epsilon > 0 \). The limit \( w_0 \) is a solution to the problem \( \Delta w_0 = 0 \) \( \text{in} \ \Omega \),
\[
\frac{\partial w_0}{\partial n} = D w_0 + (\mu_+ - \mu_-) \quad \text{on} \ \partial \Omega, 
\]
in the distributional sense that
\[
\int_{\Omega} w_0 \Delta v \, dx + \int_{\partial \Omega} D w_0 v \, d\sigma_x = - \int_{\partial \Omega} v (\mu_+ - \mu_-) \, d\sigma_x,
\]
for all \( v \in C^2(\overline{\Omega}) \) with \( \frac{\partial v}{\partial n} = 0 \) on \( \partial \Omega \). We also note that, for \( y \in \Omega \),
\[
w_0(y) = - \int_{\partial \Omega} N(x,y) (D w_0(x) \, d\sigma_x + d(\mu_+ - \mu_-) + \frac{1}{|\partial \Omega|} \int_{\partial \Omega} w_0 \, d\sigma_x),
\]
where \( N(\cdot, y) \) is the “standard” Neumann function, satisfying
\[
\Delta N(\cdot, y) = \delta_y \quad \text{in} \ \Omega, \quad \frac{\partial N(\cdot, y)}{\partial n} = \frac{1}{|\partial \Omega|} \quad \text{on} \ \partial \Omega.
\]
Our blow-up analysis, more specifically, concerns the structure of the measures \( \mu_+ \) and \( \mu_- \). We now define what it means to be a regular and a singular point with respect to the measure \( v = \mu_+ + \mu_- \).

**Definition 1.** We call a point \( x_0 \in \partial \Omega \) a regular point if there exists a function \( \psi \in C^0(\partial \Omega) \) such that \( 0 \leq \psi \leq 1 \), \( \psi \equiv 1 \) in a neighborhood of \( x_0 \), and
\[
\int_{\partial \Omega} \psi \, dv < \frac{\pi}{2}.
\]
A point on $\partial \Omega$ is called singular if it is not regular. We denote by $S$ the set of all singular points. Note that it follows from this definition (and the finiteness of the measure $\nu$) that $S$ is a finite set. We also introduce the notion of nondegeneracy for a Steklov eigenspace.

**Definition 2.** A nonzero Steklov eigenvalue $D_k$ for the boundary value problem (2) is said to have a nondegenerate eigenspace, if any eigenfunction $\phi$ in $V_{D_k} \setminus \{0\}$ attains its extremal values at only a finite number of points (on $\partial \Omega$), i.e., if $M_\phi$ consists of a finite number of points, for any $\phi \in V_{D_k} \setminus \{0\}$.

The blow-up analysis follows along the same lines as in [10] and [12], and depends crucially on the following inequality. Suppose $g$ is a non-trivial, smooth function on $\partial \Omega$ (with $\int_{\partial \Omega} g d\sigma = 0$) and suppose $\nu$ is a classical solution to

\[
\begin{cases}
\Delta \nu = 0 & \text{in } \Omega, \\
\frac{\partial \nu}{\partial n} = g & \text{on } \partial \Omega,
\end{cases}
\]

normalized by $\int_{\partial \Omega} \nu d\sigma = 0$. Then, for any $0 < \delta < \pi$, there exists a constant $C_\delta$, independent of $g$, such that

\[
\int_{\delta \Omega} \exp\left(\frac{(\pi - \delta)|\nu(x)|}{\|g\|_{L^1}}\right) d\sigma_x \leq C_\delta.
\]

This is a special version of an inequality, first proved in [4]. For a proof of this special version, see [10].

**Lemma 5.** Suppose $u_{\lambda_n}, \lambda_n \to 0+$ is a sequence of solutions to (1) (corresponding to the shift $D$) whose energies satisfy the estimate

\[
E_{\lambda_n}(u_{\lambda_n}) \leq a \log \frac{1}{\lambda_n} + b,
\]

for some positive constants $a$ and $b$. Suppose the sequence is selected so that (65) holds for the two regular Borel measures $\mu_{\pm}$. If $D$ is a nonzero Steklov eigenvalue we additionally suppose the sequence has been selected according to the conclusion of Lemma 4. Let $x_0 \in \partial \Omega$ be a regular point with respect to the measure $\nu = \mu_+ + \mu_-$ in the sense of Definition 1. Then we have the following results

\[D\text{ is not a Steklov eigenvalue: There exist } r_0 > 0 \text{ and } C < \infty \text{ such that}
\]

\[
\sup_{y \in \partial \Omega \cap B(x_0, r_0)} |u_{\lambda_n}(y)| \leq \sup_{y \in \Omega \cap B(x_0, r_0)} |u_{\lambda_n}(y)| \leq C.
\]

As a consequence $\lambda_n e^{\pm u_{\lambda_n}} \to 0$ uniformly on $\partial \Omega \cap B(x_0, r_0)$ as $\lambda_n \to 0+$.

\[D = 0: \text{ There exist } r_0 > 0 \text{ and } C < \infty \text{ such that}
\]

\[
\sup_{y \in \partial \Omega \cap B(x_0, r_0)} |w_{\lambda_n}(y)| \leq \sup_{y \in \Omega \cap B(x_0, r_0)} |w_{\lambda_n}(y)| \leq C.
\]

\[D\text{ is a Steklov eigenvalue } \neq 0: \text{ Suppose } x_0 \text{ also lies in } I_\phi \text{ where } \phi \text{ is the Steklov eigenvector arising in the conclusion of Lemma 4. There exist } r_0 > 0 \text{ and } C < \infty \text{ such that}
\]

\[
\sup_{y \in \partial \Omega \cap B(x_0, r_0)} |w_{\lambda_n}(y)| \leq \sup_{y \in \Omega \cap B(x_0, r_0)} |w_{\lambda_n}(y)| \leq C.
\]

Furthermore $\lambda_n e^{\pm w_{\lambda_n}} \to 0$ uniformly on $\partial \Omega \cap B(x_0, r_0)$ as $\lambda_n \to 0+$.

**Proof.** The proof of this lemma when $D$ is not a Steklov eigenvalue is simpler than when $D$ is such an eigenvalue (when $D$ is not an eigenvalue $P_D u_j = 0$ and $u_j = w_j$). We shall thus only consider the case where $D$ is a Steklov eigenvalue. A proof for $D = 0$ has already been given in [10]. The proof for $D \neq 0$ has considerable overlap with that proof, however, for the convenience of the reader we present the details here. Let $x_0 \in \partial \Omega$ be a regular point, with $x_0 \in I_\phi$, where $\phi$ is the Steklov eigenvector arising in the conclusion of Lemma 4. Let $\psi$ be a smooth function with
the properties described in the definition of a regular point; by a simple regularization procedure we may arrange that \( \psi \in C^\infty(\partial \Omega) \). We decompose the function \( w_{\lambda_n} = w_1 + w_2 + w_3 + C^* \), into three harmonic functions and a constant (for simplicity of notation we drop the \( \lambda_n \) label on the functions \( w_i, \, 1 \leq i \leq 3 \), and the constant \( C^* \)). The harmonic functions are normalized by \( \int_{\partial \Omega} w_1 \, d\sigma = \int_{\partial \Omega} w_2 \, d\sigma = \int_{\partial \Omega} w_3 \, d\sigma = 0 \). In addition these harmonic functions satisfy the Neumann boundary conditions

\[
\begin{align*}
\frac{\partial w_1}{\partial n} &= \lambda_n \psi \sinh(u_{\lambda_n}) - \frac{\lambda_n}{|\partial \Omega|} \int_{\partial \Omega} \psi \sinh(u_{\lambda_n}) \, d\sigma, \\
\frac{\partial w_2}{\partial n} &= \lambda_n (1 - \psi) \sinh(u_{\lambda_n}) - \frac{\lambda_n}{|\partial \Omega|} \int_{\partial \Omega} (1 - \psi) \sinh(u_{\lambda_n}) \, d\sigma, \\
\frac{\partial w_3}{\partial n} &= Dw_{\lambda_n} - \frac{D}{|\partial \Omega|} \int_{\partial \Omega} w_{\lambda_n} \, d\sigma.
\end{align*}
\]

By (55)

\[
\frac{1}{|\partial \Omega|} \int_{\partial \Omega} |w_{\lambda_n}| \, d\sigma \leq \frac{1}{|\partial \Omega|^{1/2}} \|w_{\lambda_n}\|_{L^2(\partial \Omega)} \leq C,
\]

and

\[
\left\| w_{\lambda_n} - \frac{1}{|\partial \Omega|} \int_{\partial \Omega} w_{\lambda_n} \, d\sigma \right\|_{L^2(\partial \Omega)} \leq \|w_{\lambda_n}\|_{L^2(\partial \Omega)} \leq C.
\]

It follows immediately that the constant \( C^* = \frac{1}{|\partial \Omega|} \int_{\partial \Omega} w_{\lambda_n} \, d\sigma \) is bounded independently of \( \lambda_n \). From elliptic regularity theory it follows that \( \|w_3\|_{H^{3/2}(\Omega)} \) is bounded independently of \( \lambda_n \). Using Sobolev’s imbedding theorem we conclude that

\[
\|w_3\|_{C^0(\overline{\Omega})} \leq C,
\]

independently of \( \lambda_n \). Again, by elliptic regularity

\[
\|w_2\|_{L^p(\Omega)} \leq C_p \left\| \frac{\partial w_2}{\partial n} \right\|_{L^1(\partial \Omega)} \leq C_p,
\]

for any \( 1 < p < \infty \). Suppose that \( r_1 > 0 \) is picked sufficiently small so that \( \psi \equiv 1 \) on \( B(x_0, r_1) \cap \partial \Omega \). Since

\[
\frac{\partial w_2}{\partial n} = -\frac{\lambda_n}{|\partial \Omega|} \int_{\partial \Omega} (1 - \psi) \sinh(u_{\lambda_n}) \, d\sigma, \quad \text{a bounded constant},
\]

on \( B(x_0, r_1) \cap \partial \Omega \), local elliptic estimates give that

\[
\|w_2\|_{C^0(B(x_0, r_1/2) \cap \partial \Omega)} \leq \|w_2\|_{C^0(B(x_0, r_1/2) \cap \overline{\Omega})} \leq C \left( \left| \frac{\lambda_n}{|\partial \Omega|} \int_{\partial \Omega} (1 - \psi) \sinh(u_{\lambda_n}) \, d\sigma \right| + \|w_2\|_{L^2(\Omega)} \right) \leq C.
\]

Lastly, since

\[
\left| \sinh(x) \right| = \cosh(x) - e^{-|x|}
\]

the convergence in measure of \( \lambda_n \cosh u_{\lambda_n} \) towards \( \mu_+ + \mu_- = \nu \) means that

\[
\lambda_n \int_{\partial \Omega} \psi \left| \sinh(u_{\lambda_n}) \right| \, d\sigma \to \int_{\partial \Omega} \psi \, d\nu < \frac{\pi}{2}.
\]
Therefore, for \( \delta \) sufficiently small,

\[
\left\| \frac{\partial w_1}{\partial n} \right\|_{L^1(\partial \Omega)} = \int_{\partial \Omega} \left| \frac{\partial w_1}{\partial n} \right| \, d\sigma < \pi - 2\delta,
\]

for all sufficiently small \( \lambda_n \). We may thus apply (69) with \( v = w_1 \) and \( g = \partial w_1 / \partial n \) to obtain

\[
\int_{\partial \Omega} e^{p^* |w_1|} \, d\sigma \leq C_{p^*},
\]

for some \( 1 < p^* = (\pi - \delta) / (\pi - 2\delta) \leq (\pi - \delta) / \| \partial w_1 / \partial n \|_{L^1(\partial \Omega)} \).

Suppose \( r_1 \) is chosen sufficiently small that

\[
\sup_{x \in \partial \Omega \cap B(x_0, r_1/2)} \lambda_n e^{\left| PDu_{\lambda_n}(x) \right|} \leq C,
\]

which is possible since \( x_0 \in I_\phi \). Using (70), (71), (72), (73) and the boundedness of the constant \( C^* \), we can now estimate

\[
\lambda_n^* \int_{\partial \Omega \cap B(x_0, r_1/2)} \psi e^{p^* |w_{\lambda_n}|} \, d\sigma \leq C \int_{\partial \Omega \cap B(x_0, r_1/2)} \lambda_n^* e^{p^* (|PDu_{\lambda_n}| + |w_1| + |w_2| + |w_3| + |C^*|)} \, d\sigma \leq C \int_{\partial \Omega} e^{p^* |w_1|} \, d\sigma \leq C,
\]

for some fixed \( p^* > 1 \), and all \( \lambda_n \) sufficiently small. Therefore, Sobolev’s imbedding theorem yields

\[
\left\| \frac{\partial w_1}{\partial n} \right\|_{H^{-s^*}(\partial \Omega \cap B(x_0, r_1/2))} \leq C \left\| \frac{\partial w_1}{\partial n} \right\|_{L^{p^*}(\partial \Omega \cap B(x_0, r_1/2))} \leq C \left( \lambda_n \psi \left| \sinh(u_{\lambda_n}) \right| \right)_{L^{p^*}(\partial \Omega \cap B(x_0, r_1/2))} + \int_{\partial \Omega} \lambda_n \left| \sinh(u_{\lambda_n}) \right| \, d\sigma \leq C,
\]

for some fixed \( s^* < 1/2 \). Interior elliptic estimates and duality now give

\[
\|w_1\|_{H^{1/2-s^*}(\Omega \cap B(x_0, r_1/4))} \leq C \left( \left\| \frac{\partial w_1}{\partial n} \right\|_{H^{-s^*}(\partial \Omega \cap B(x_0, r_1/2))} + \|w_1\|_{L^2(\Omega \cap B(x_0, r_1/2))} \right) \leq C.
\]

Sobolev’s imbedding theorem then yields the local \( C^0 \) bound

\[
\|w_1\|_{C^0(\Omega \cap B(x_0, r_1/4))} \leq \|w_1\|_{C^0(\Omega \cap B(x_0, r_1/4))} \leq C.
\]

A combination of (70), (71), and (74) with the fact that the constant \( C^* \) is bounded leads to the desired estimate for \( w_{\lambda_n} \), with \( r_0 = r_1/4 \). Once this uniform estimate is proven, it follows immediately from the estimate

\[
\lambda_n e^{\left| u_{\lambda_n} \right|} \leq \lambda_n e^{\left| PDu_{\lambda_n} \right| + \left| w_{\lambda_n} \right|},
\]

and the fact that

\[
\sup_{x \in \omega_{\lambda_n}} \lambda_n e^{\left| PDu_{\lambda_n}(x) \right|} \to 0 \quad \text{as } \lambda_n \to 0_+,
\]

that \( \lambda_n e^{\left| u_{\lambda_n} \right|} \) converges uniformly to zero on \( \partial \Omega \cap B(x_0, r_0) \), for \( r_0 \) sufficiently small. \( \square \)

Based on this lemma we are now able to establish the following theorem characterizing the possible limits of \( \lambda_n \sinh(u_{\lambda_n}) \). The proof of this theorem entirely parallels that of Theorem 1 in [12], and we refer the interested reader to that paper for the details.
Theorem 2. Suppose $u_{λ_n}, λ_n → 0_+$, is a sequence of solutions to (1) (corresponding to shift $D$) whose norms satisfy the estimate

$$E_λ(u_{λ_n}) \leq a \log \frac{1}{λ_n} + b,$$

for some positive constants $a$ and $b$. Suppose this sequence has been selected so that

$$λ_n e^{±u_{λ_n}} → 2μ_± \text{ weak }^* \text{ in } C^0(∂Ω)^*,$$

i.e., in the sense of measures, and

$$w_{λ_n} → w_0 \text{ in } H^{1−ε}(Ω), \text{ as } λ_n → 0_+,$$

for any $ε > 0$ (see (65) and (67)). If $D$ is a nonzero Steklov eigenvalue we additionally suppose $λ_n$ has been selected according to the conclusion of Lemma 4. Let $S$ denote the set of singular points relative to the measure $ν = μ_+ + μ_-$. Then we have the following results concerning the limits of $λ_n e^{±u_{λ_n}}|_{∂Ω}$

$D$ is not a Steklov eigenvalue: The nonlinear boundary terms $λ_n e^{±u_{λ_n}}|_{∂Ω}$ converge in the sense of measures towards

$$2μ_± = \sum_{x_i ∈ S} 2α_i^± δ_{x_i}, \text{ with } α_i^± > 0.$$

$D = 0$: The nonlinear boundary terms $λ_n e^{±u_{λ_n}}|_{∂Ω}$ converge in the sense of measures towards

$$2μ_± = \sum_{x_i ∈ S} 2α_i^± δ_{x_i} + d_± e^{±w_0}, \text{ with } α_i^± > 0, d_+ > 0, \text{ and } d_+ · d_− = 0.$$

$D$ is a Steklov eigenvalue $≠ 0$: Suppose additionally that $D$ has a nondegenerate eigenspace in the sense of Definition 2. Then there exists a finite set of points $M ⊂ ∂Ω$ such that the nonlinear boundary terms $λ_n e^{±u_{λ_n}}|_{∂Ω}$ converge in the sense of measures towards

$$2μ_± = \sum_{x_i ∈ S ∪ M} 2α_i^± δ_{x_i}, \text{ with } α_i^± > 0.$$

In all cases we have that $α_i^+ + α_i^− > 0$ for points $x_i ∈ S$ (or $x_i ∈ S ∪ M$), and $w_0$ is $C^∞$ in $Ω \setminus S$ (or $Ω \setminus (S ∪ M)$). $S$ (or $S ∪ M$) may possibly be empty; an empty sum of delta functions should be interpreted as 0.

Remarks. Concerning the structure of the limit of $λ_n e^{±u_{λ_n}}$ and $w_{λ_n}$:

1. When $D = 0$ it was proven in [12] that (since $E_λ(u_{λ_n})$ blow up as $λ_n$ approaches $0_+$) the sets $S$, corresponding to our variationally constructed solutions, are nonempty and the measure $μ_+ + μ_- = \lim λ_n |\sinh(λ_n)]$ as well as the measure $μ_+ − μ_- = \lim λ_n \sinh(u_{λ_n})$ has nonzero delta functions at all the points in $S$. A similar result is easy to verify when $D$ is not a Steklov eigenvalue. We have no proof that the sets $S$, corresponding to our variationally constructed solutions, are nonempty when $D$ is a nonzero Steklov eigenvalue, even though we suspect this to be true.

2. As reported in [12] there is numerical evidence to support the presence of the “regular part” $d_± e^{±w_0}$ in the case $D = 0$, i.e., there is numerical evidence of cases in which not both $d_+$ and $d_−$ are zero.

3. When $D$ is a nonzero Steklov eigenvalue, Theorem 2 at present leaves open the possibility that there may be delta functions in $μ_+$ or $μ_−$ at points (in $M$) outside of $S$. In the next theorem we verify that the point-mass locations for the measures $μ_+ + μ_−$ and $μ_+ − μ_−$ are the same, and that the absolute weight of any point-mass in these two measures is greater than or equal to $2π$. As a corollary to the last statement it follows that $S ∪ M = S$.

4. When $D$ is not a positive Steklov eigenvalue, it is not difficult to see that $w_{λ_n}$ stays uniformly bounded near points in $Ω \setminus S$, and that given any point $x ∈ S$ one may find points $x_n ∈ Ω$, $x_n → x$, so that $|w_{λ_n}(x_n)| → ∞$ as $λ_n → 0_+$ (for the details of a proof of this see for example [10] Lemma 4.8). When $D$ is a positive Steklov eigenvalue (with a nondegenerate eigenspace) similar arguments may be used to prove that $w_{λ_n}$ stays uniformly bounded away from a finite set of boundary points. In many interesting cases the contribution $P_D u_{λ_n}$ vanishes or stays uniformly bounded, and so the variationally defined sequence of solutions, $w_{λ_n}$, blows up “pointwise” in the same places as $w_{λ_n}$. It is this behavior which is markedly different than that which we have seen for $λ_n → 0_−$ (cf. Proposition 3) where we may either find a subsequence along which $u_{λ_n}$ and $w_{λ_n}$ blow up almost everywhere in $Ω$, or the entire original sequence $u_{λ_n}$ converges to zero in $H^1(Ω)$ (and thus, by elliptic estimates, also in $C^0(Ω)$).

We can also obtain information about the strength and locations of the boundary singularities in the limit $w_0$, similar to that which we have already obtained for the case $D = 0$ in [12].
There are two related approaches to establish such results that we know of. One is complex analytic, and was originally introduced in [13] to study “interior” blow-up for positive solutions (to a conceptually related problem). This approach was the basis for the analysis for the case $D = 0$, presented in [10]. The other approach is based on a clever Pohozaev-like integral identity, and its asymptotic limit on shrinking neighborhoods of the singularities; it was originally introduced in [3] and was also used in [6]. Both approaches may be adapted to the present case. Since we have already demonstrated how to apply the first approach on a very related problem (the case $D = 0$) we will here for completeness give a fairly detailed outline of how to apply the second approach. According to Theorem 2 all the point-mass locations of the measures $\lim_{\lambda \to 0} \mu^\pm = 2\mu_\pm$ lie inside the finite set of singular points $S$, or inside the finite set $S \cup M$. In both cases we shall use $\{x_i\}^K_{i=1}$ to denote the set of point-mass locations (of $\mu_+ + \mu_-). Let \{\alpha_i^\pm\}^K_{i=1}$ be the coefficients (weights) associated with the point masses of the measures $\mu_\pm$; we shall use the notation $\alpha_i^+ - \alpha_i^-$ for the coefficients associated with the measure $\mu_+ - \mu_-$. For any fixed $x_i$, let $(r, \theta)$, $0 < r < \pi/2 < \theta < 3\pi/2$, be a polar coordinate system around $x_i$, selected so that $\theta = -\pi/2$ lines up with the outward normal to $\Omega$, and define

$$\phi_i(r, \theta) = \frac{1}{\pi} \left( -\log r + Dr \sin \theta \log r + Dr \cos \theta \left( \theta - \frac{\pi}{2} \right) \right).$$

(75)

Using “complex notation” ($z = r \cos \theta + ir \sin \theta$) this definition is equivalent to

$$\phi_i(z) = -\frac{1}{\pi} \Re \left( \log z + Dz \left( i \log z + \frac{\pi}{2} \right) \right),$$

with $\log z = \log r + i\theta$.

**Theorem 3.** Let the situation be as in Theorem 2. If $D = 0$ suppose $d_\pm = 0$. The weights of the point-masses of the measures $\mu_\pm = \sum_{i=1}^K \alpha_i^\pm \delta_{x_i}$ and $\mu_+ - \mu_- = \sum_{i=1}^K \alpha_i \delta_{x_i}$ satisfy

$$\frac{\alpha_i^2}{2\pi} = (\alpha_i^+ - \alpha_i^-), \quad 1 \leq i \leq K.$$

(76)

Moreover, the point-mass locations $\{x_i\}^K_{i=1}$ satisfy the conditions

$$\frac{\partial}{\partial \tau_x} (w_0(x) - \alpha_i \phi_i(x)) |_{x=x_i} = 0, \quad 1 \leq i \leq K,$$

(77)

where $w_0$ is the limit of the sequence $w_{\lambda_n}$, $\tau_x$ is the tangent to $\partial \Omega$ at the point $x$, and the functions $\phi_i$, $1 \leq i \leq K$, are as defined in (75).

**Proof.** Let $\mathbb{H}$ denote the upper halfplane $\mathbb{H} = \{(y_1, y_2) : y_2 > 0\}$ and let $B(R) = B(0, R)$ denote the disk of radius $R$ centered at the origin. Given a point-mass location $x_i \in \partial \Omega$, let $\Phi : \mathbb{H} \cap B(R) \to \Omega$ be a local conformal straightening of the boundary. By appropriate selection of $\Phi$ and an orthonormal coordinate system we may arrange that $\Phi(0) = x_i$ is also the origin, and that $\nabla \Phi(0) = I$. By selecting $R$ sufficiently small we may assume that $\{x_j\}^K_{j=1} \cap \Phi(\partial \mathbb{H} \cap B(R)) = x_i = 0$. Defining the function $v_{\lambda_n} = w_{\lambda_n} \circ \Phi$

$$\begin{cases}
\Delta v_{\lambda_n} = 0 & \text{in } \mathbb{H} \cap B(R), \\
\frac{\partial v_{\lambda_n}}{\partial y} = h(y) Dv_{\lambda_n} + \lambda_n h(y) \sinh(v_{\lambda_n}) & \text{on } \partial \mathbb{H} \cap B(R),
\end{cases}$$

(78)

where $h(y) = |\det(\nabla \Phi(y))|^{1/2}$. We have arranged that $h(0) = 1$. Moreover, we can choose $h'(0) = h_{y_1}(0)$ arbitrarily: in complex notation, $\Phi$ can be written $\Phi(z) = z + \frac{\gamma}{2} z^2 + O(z^3)$ with $z = y_1 + iy_2$. When written like this, $h'(0)$ can be computed to be $\Re(\gamma)$. Now use the change of variable $\Psi(z) = \frac{1}{k} (e^{kz} - 1)$, for $k$ real, to form a new conformal map $\hat{\Phi} = \Phi \circ \Psi$ (which again locally “straightens” the boundary). $\hat{\Phi}$ is conformal from a half-plane (which may be smaller) and can be expanded as $\hat{\Phi}(z) = z + \frac{\gamma'}{2} z^2 + O(z^3)$ where $\gamma' = \gamma + k$.

Choosing $0 < \epsilon < R$, we multiply (78) by $\frac{\partial}{\partial y_1} v_{\lambda_n}$. Let $\Omega_{\epsilon} = \mathbb{H} \cap B(\epsilon)$, and write the outward unit normal vector $n = (n_1, n_2)$. Integration by parts gives

$$\frac{1}{2} \int_{\Omega_{\epsilon}} \left| \nabla v_{\lambda_n} \right|^2 dy = \int_{\Omega_{\epsilon}} \nabla v_{\lambda_n} \cdot \nabla \frac{\partial}{\partial y_1} v_{\lambda_n} dy.$$
\[ \int \left( Dhv_{\lambda n} \frac{\partial}{\partial y_1} v_{\lambda n} + \lambda_n h \sinh(v_{\lambda n}) \frac{\partial}{\partial y_1} v_{\lambda n} \right) dy_1 + \int \frac{\partial}{\partial n} v_{\lambda n} \frac{\partial}{\partial y_1} v_{\lambda n} \, d\sigma_y \]

\[ = \int \left( \frac{D}{2} h \frac{\partial}{\partial y_1} (v_{\lambda n})^2 + \lambda_n h \frac{\partial}{\partial y_1} \cosh(v_{\lambda n}) \right) dy_1 + \int \frac{\partial}{\partial n} v_{\lambda n} \frac{\partial}{\partial y_1} v_{\lambda n} \, d\sigma_y. \]

A second integration by parts with respect to \( y_1 \) gives

\[ \frac{1}{2} \int_{\partial B(\epsilon) \cap \mathbb{H}} (\nabla v_{\lambda n})^2 n_1 \, d\sigma_y = -\frac{D}{2} \int_{\partial \Omega \cap \mathbb{H}} \frac{\partial h}{\partial y_1} v_{\lambda n}^2 \, dy_1 - \lambda_n \int_{\partial \Omega \cap \mathbb{H}} \frac{\partial h}{\partial y_1} \cosh(v_{\lambda n}) \, dy_1 + \int_{\partial B(\epsilon) \cap \mathbb{H}} \frac{\partial}{\partial n} v \frac{\partial}{\partial y_1} v \, d\sigma_y + \left( \frac{D}{2} h(v_{\lambda n})^2 + \lambda_n h \cosh(v_{\lambda n}) \right) \bigg|_{y_1=\epsilon, y_2=0}^{y_1=-\epsilon, y_2=0}. \]

Now we can take this limit as \( \lambda_n \to 0^+ \), to get the identity

\[ \frac{1}{2} \int_{\partial B(\epsilon) \cap \mathbb{H}} (\nabla v_0)^2 n_1 \, d\sigma_y = -\frac{D}{2} \int_{\partial \Omega \cap \mathbb{H}} \frac{\partial h}{\partial y_1} (v_0)^2 \, dy_1 - (\alpha_i^+ + \alpha_i^-) \left( \frac{\partial h}{\partial y_1}(0) / h(0) \right) + \int_{\partial B(\epsilon) \cap \mathbb{H}} \frac{\partial}{\partial n} v \frac{\partial}{\partial y_1} v \, d\sigma_y + \left( \frac{D}{2} h(y)(v_0(y))^2 \right) \bigg|_{y_1=\epsilon, y_2=0}^{y_1=-\epsilon, y_2=0}. \]

for \( v_0 = \lim v_{\lambda n} \). Here we have used that \( \lambda_n h \cosh(v_{\lambda n}) \) converge uniformly to zero away from a finite set of points (which are disjoint from the points \((\pm \epsilon, 0)\), for \(0 < \epsilon \) sufficiently small). We now decompose \( v_0 \) into a regular and a singular part by first performing this decomposition on \( w_0 \). Using “complex notation” \((z = x_1 + ix_2)\) we define

\[ w_s = -\frac{\alpha_i}{\pi} \text{Re} \left( \log z + Dz \left( i \log z + \frac{\pi}{2} \right) \right), \]

where the logarithm \( \log z \) is chosen to have its “cut” along the negative imaginary axis (which coincides with the outward normal to \( \Omega \)). It is not hard to see that \( w_r = w_0 - w_s \) satisfies \( \Delta w_r = 0 \) in \( \Omega \cap B(\epsilon) \), with \( \frac{\partial}{\partial n} w_r - D w_r \in H^{3/2-\epsilon}(\partial \Omega \cap B(\epsilon)) \) for \( \epsilon \) sufficiently small, and any \( t > 0 \). In other words

\[ w_0 = w_s + w_r, \]

where \( w_r \) is in \( H^{3/2-\epsilon}(\Omega \cap B(\epsilon)) \) for any \( t > 0 \). In particular, \( w_r \) is in \( C^{1,\beta} \) in an \( \overline{\Omega} \) neighborhood of 0 for any \( \beta < 1 \). Define \( v_s = w_s \circ \Phi \) and \( v_r = w_r \circ \Phi \). We thus have the decomposition

\[ v_0 = v_s + v_r \]

with \( v_r \) locally in \( C^{1,\beta} \), for any \( \beta < 1 \), and

\[ v_s(z) = w_s(\Phi(z)), \quad \Phi(z) = z + \frac{a + ib}{2} z^2 + S(z), \]

with \(|S(z)| \leq C|z|^3, \left| \frac{d}{dz} S(z) \right| \leq C|z|^2\), and \( a = h'(0) \). We easily calculate that

\[ \log \Phi(z) + D\Phi(z) \left( i \log \Phi(z) + \frac{\pi}{2} \right) = \log z + \frac{a + ib}{2} z + Dz \left( i \log z + \frac{\pi}{2} \right) + O(|z^2 \log z|), \]

and

\[ \frac{d}{dz} \left( \log \Phi(z) + D\Phi(z) \left( i \log \Phi(z) + \frac{\pi}{2} \right) \right) = \frac{1}{z} + \frac{a + ib}{2} + D \left( i + i \log z + \frac{\pi}{2} \right) + O(|z \log z|), \]

and so in polar coordinates (with \( z = y_1 + iy_2 = r \cos \theta + ir \sin \theta \))

\[ v_s(y) = -\frac{\alpha_i}{\pi} \left( \log r + \frac{a}{2} r \cos \theta - \frac{b}{2} r \sin \theta - Dr \sin \theta \log r - Dr \cos \theta \left( \theta - \frac{\pi}{2} \right) \right) + O(r^2 |\log r|), \]

(80)
and
\[
\nabla_y v_s(y) = -\frac{\alpha_i}{\pi} \left[ (r^{-1} \cos \theta, r^{-1} \sin \theta) + \frac{1}{2} (a, -b) - D \left( \theta - \frac{\pi}{2}, 1 + \log r \right) \right] + O(r \log r), \tag{81}
\]
with \( a = h'(0) \). We now substitute the decomposition \( v_0 = v_s + v_r \) into (79), and consider the case of infinitesimally small \( \epsilon \). It is not difficult to see that some of the terms are \( o(1) \); indeed we obtain (remembering that \( h(0) = 1 \))
\[
\frac{1}{2} \int_{\partial B(\epsilon) \cap \mathbb{H}} \left( |\nabla v_s|^2 + 2 \nabla v_s \nabla v_r \right) n_1 \, d\sigma_y = -\left( \alpha^+_i + \alpha^-_i \right) \frac{\partial h}{\partial y_1}(0) + \left( \frac{D}{2} h(y)(v_s(y)^2 + 2v_s(y)v_r(y)) \right)_{|y_1 = \epsilon, y_2 = 0}
\]
\[
+ \int_{\partial B(\epsilon) \cap \mathbb{H}} \left( \frac{\partial}{\partial n} v_s \frac{\partial}{\partial y_1} v_s + \frac{\partial}{\partial n} v_s \frac{\partial}{\partial y_1} v_r + \frac{\partial}{\partial n} v_r \frac{\partial}{\partial y_1} v_s \right) \, d\sigma_y + o(1). \tag{82}
\]
Using the formula (81) we can now compute the limits
\[
\lim_{\epsilon \to 0} \frac{1}{2} \int_{\partial B(\epsilon) \cap \mathbb{H}} |\nabla v_s|^2 n_1 \, d\sigma_y = \frac{h'(0)}{4\pi} \alpha^2_i,
\]
\[
\lim_{\epsilon \to 0} \int_{\partial B(\epsilon) \cap \mathbb{H}} \nabla v_s \nabla v_r n_1 \, d\sigma_y = -\frac{\alpha_i}{2} \frac{\partial}{\partial y_1} v_r(0),
\]
\[
\lim_{\epsilon \to 0} \int_{\partial B(\epsilon) \cap \mathbb{H}} \frac{\partial}{\partial n} v_s \frac{\partial}{\partial y_1} v_s \, d\sigma_y = \frac{3h'(0)}{4\pi} \alpha^2_i,
\]
\[
\lim_{\epsilon \to 0} \int_{\partial B(\epsilon) \cap \mathbb{H}} \frac{\partial}{\partial n} v_s \frac{\partial}{\partial y_1} v_r \, d\sigma_y = -\alpha_i \frac{\partial}{\partial y_1} v_r(0),
\]
\[
\lim_{\epsilon \to 0} \int_{\partial B(\epsilon) \cap \mathbb{H}} \frac{\partial}{\partial n} v_r \frac{\partial}{\partial y_1} v_s \, d\sigma_y = -\alpha_i \frac{\partial}{\partial y_1} v_r(0).
\]
The point-boundary terms in (82) (at \( y_1 = \pm \epsilon \)) converge to zero as \( \epsilon \to 0 \). For example, by invoking (80),
\[
|v_s(\epsilon, 0)^2 - v_s(-\epsilon, 0)^2| = |v_s(\epsilon, 0) + v_s(-\epsilon, 0)| \cdot |v_s(\epsilon, 0) - v_s(-\epsilon, 0)| \lesssim C|\log(\epsilon)| \cdot |v_s(\epsilon, 0) - v_s(-\epsilon, 0)| \lesssim C|\log(\epsilon)| \epsilon,
\]
so that
\[
|h(\epsilon, 0)v_s(\epsilon, 0)^2 - h(-\epsilon, 0)v_s(-\epsilon, 0)^2| \lesssim |h(\epsilon, 0)| \cdot |v_s(\epsilon, 0)^2 - v_s(-\epsilon, 0)^2| + |h(\epsilon, 0) - h(-\epsilon, 0)| \cdot |v_s(-\epsilon, 0)^2| \lesssim C|\log(\epsilon)| \epsilon + C\epsilon|\log(\epsilon)|^2.
\]
The term involving \( v_s \cdot v_r \) may be estimated similarly. In all we have therefore reduced (82) to the following limiting identity
\[
\frac{h'(0)}{4\pi} \alpha^2_i - \frac{\alpha_i}{2} \frac{\partial}{\partial y_1} v_r(0) + (\alpha^+_i + \alpha^-_i) h'(0) = \frac{3h'(0)}{4\pi} \alpha^2_i \frac{\partial}{\partial y_1} v_r(0) - \frac{3}{2} \alpha_i \frac{\partial}{\partial y_1} v_r(0),
\]
or
\[
\alpha_i \frac{\partial}{\partial y_1} v_r(0) = h'(0) \left( \frac{\alpha^2_i}{2\pi} - (\alpha^+_i + \alpha^-_i) \right).
\]
Since \( h'(0) \) can be chosen arbitrarily, and since \( \frac{\partial}{\partial y_1} v_r(0) = \frac{\partial}{\partial \tau} w_r(0) \) is independent of \( h'(0) \) this identity can only be satisfied if both sides are zero. It follows that
\[
\frac{\alpha^2_i}{2\pi} = \alpha^+_i + \alpha^-_i.
and because $\alpha^+_i + \alpha^-_i > 0$ on $S \cup M$ (so $\alpha_i \neq 0$) it also follows that
\[
\frac{\partial}{\partial \tau} w_r(0) = \frac{\partial}{\partial y_1} v_r(0) = 0.
\]
These are the desired identities. □

**Remarks.** For $\alpha^+_i \geq 0$, $\alpha^+_i + \alpha^-_i > 0$ the equations
\[
\frac{(\alpha^+_i - \alpha^-_i)^2}{2\pi} = \frac{\alpha^2_i}{2\pi} = \alpha^+_i + \alpha^-_i
\]
implies that $|\alpha_i| = |\alpha^+_i - \alpha^-_i| \geq 2\pi$. The case $|\alpha_i| = 2\pi$ arrives if and only if either $\alpha^+_i$ or $\alpha^-_i$ is zero. We suspect (but are unable to prove) that this is the case for solutions satisfying the energy bounds established in Theorem 1. Whenever $\alpha_i = \pm 2\pi$, the corresponding equations
\[
\frac{\partial}{\partial x_i} \left( w_0(x) \mp 2\pi \phi_i(x) \right) \bigg|_{x = x_i} = 0,
\]
may be used to determine the potential singularities. For $D = 0, d_+ = d_- = 0$, when $\phi_i(x) = -\frac{1}{\pi} \log |x - x_i|$, this has been done in a few cases in [12]. Just like in [3] there is a strong relation between these equations and the stationarity of an appropriate “renormalized” energy – we refer the interested reader to [5] and [11] for more details.

**Corollary 1.** Let the situation be as in Theorem 2. Let $S \cup M$ be the finite set of point-mass locations introduced in the case when $D$ is a nonzero Steklov eigenvalue. Then $S \cup M = S$.

**Proof.** Suppose $D$ is a nonzero Steklov eigenvalue, and let $x_i$ be a point in $S \cup M$ (a point-mass location for $\mu_+ + \mu_-$ with weight $\alpha^+_i + \alpha^-_i$). From the previous remarks it follows that $\alpha^+_i + \alpha^-_i \geq |\alpha^+_i - \alpha^-_i| \geq 2\pi$, and so $\int_{\partial \Omega} \psi \, dv = \int_{\partial \Omega} \psi \, d(\mu_+ + \mu_-) \geq 2\pi$ for any $\psi \in C^0(\partial \Omega)$, $0 < \psi \leq 1$, with $\psi \equiv 1$ in a neighborhood of $x_i$. It follows that $x_i$ is a singular point relative to $\nu = \mu_+ + \mu_-$, i.e., $x_i \in S$. □

3. **A general discussion**

So far we have focused on solutions to
\[
\Delta u_{\lambda} = 0 \quad \text{in} \ \Omega, \quad \frac{\partial u_{\lambda}}{\partial n} = D u_{\lambda} + \lambda f(u_{\lambda}) \quad \text{on} \ \partial \Omega, \tag{83}
\]
with $f(x)$ having exponential growth. In the first of the following two sections we briefly discuss, and provide some numerical results for, the case of different $f$. As we shall see, the existence structure described in Sections 2.1 and 2.2 (finitely many solutions for any $\lambda$ in the interval $-D < \lambda < 0$ and infinitely many solutions for any positive $\lambda$) is preserved for a much larger class of odd, superlinear $f$. However, as we shall also see, the “finite point blow-up” observed when $\lambda \to 0_+$ is very much related to exponential growth. It was already noted that the solutions for $f = \sinh$, that we constructed variationally, and that we studied asymptotically in the previous section, do not necessarily represent all solutions for $\lambda > 0$. There may be other solutions whose energy (and “essential” $H^1$-norm squared: $\|w_\lambda\|^2_{H^1(\Omega)}$) grow faster than $\log \frac{1}{\lambda}$ as $\lambda \to 0_+$. In Section 3.2 we discuss two different cases, both with $f = \sinh$, when such “additional” solutions are present (one case has $D$ negative, the other has a domain with a nontrivial topology). Associated to these “high energy” solutions are secondary bifurcations, as illustrated by some of our computational examples.

3.1. **Other nonlinear fluxes**

For the examples in this section, we let the domain $\Omega$ be the unit ball, $\Omega = B(0,1) \subset \mathbb{R}^2$. In order to compare with the numerical results for the exponential case, that are displayed in Fig. 1, we first calculate a bifurcation plot for the case when $f(u) = u + u^3$, and $D = 2$. The left frame in Fig. 4 shows the $H^1(\Omega)$-norm, and the right frame shows the energy $E_\lambda$ as a function of $\lambda$ for seven different solutions to the boundary value problem (83). The Steklov eigenvalues...
Fig. 4. Left frame: the $H^1(\Omega)$-norm as a function of $\lambda$ for different solutions to (83) with $f(u) = u + u^3$. Right frame: the energies $E_\lambda$ for the same solutions.

Fig. 5. The $H^1(\Omega)$-norm of $v_\lambda = \lambda^{1/2} u_\lambda$ as a function of $\lambda$ for different solutions to (83), $f(u) = u + u^3$.

consist of all integers $\geq 0$. Fig. 4 clearly indicates an existence structure much like for the exponential case, namely: finitely many solutions for any $\lambda$ in the interval $-D < \lambda < 0$, and infinitely many solutions for any positive $\lambda$. The nature of the blow-up near $\lambda = 0$ is, however, different from what we witnessed in the exponential case. To support this assertion consider the following formal scaling argument. Define $v_\lambda = \lambda^{1/2} u_\lambda$, then

$$\Delta v_\lambda = 0 \quad \text{in} \quad \Omega, \quad \frac{\partial v_\lambda}{\partial n} = (D + \lambda) v_\lambda + v_\lambda^3 \quad \text{on} \quad \partial \Omega.$$ 

It is to be expected that any of the solution “branches” for this boundary value problem as $\lambda \to 0$ approaches one of the infinitely many solutions to the boundary value problem

$$\Delta v_0 = 0 \quad \text{in} \quad \Omega, \quad \frac{\partial v_0}{\partial n} = D v_0 + v_0^3 \quad \text{on} \quad \partial \Omega.$$ 

That this “limiting behavior” is indeed what transpires is seen from the plots of the $H^1(\Omega)$ norms of the solutions $v_\lambda$, displayed in Fig. 5. The solutions $u_\lambda = \lambda^{-1/2} v_\lambda$ therefore blow up almost everywhere, as do the nonlinear flux terms $\lambda (u_\lambda + u_\lambda^3)$ (whether $\lambda \to 0_-$, or $\lambda \to 0_+$). Similar behavior would be found for any odd, superlinear polynomial. We now briefly turn to the case of a bounded $f$, and the case of an asymptotically linear $f$. In both of these cases we take $D = 0$. Consider first an $f(x)$ that is odd and bounded, and let $\lambda > 0$ be fixed. If we require that $f$ be nondecreasing, then it is not hard to see that there exists a constant $C = C(f, \lambda)$ such that $\|u_\lambda\|_{H^1(\Omega)} \leq C$. If we additionally require that $f'$ be bounded, then there exists a $\lambda^* > 0$ such that the only solution for $\lambda \neq 0$ on the half-axis $\lambda < \lambda^*$ is the zero solution. For more details, see [11]. As a specific example we take $f(u) = \arctan(u)$. The left frame in Fig. 6 shows a plot of the $H^1(\Omega)$-norm for “the first” five nontrivial solutions to the corresponding boundary value problem. Secondly consider an $f$ of the form $f(u) = cu + g(u)$, where the constant $c$ is positive, and where the function $g(u)$ is odd and bounded. Here we can again prove a bound on the $H^1(\Omega)$-norm of $u_\lambda$, for any $\lambda > 0$, such that $c\lambda$ is not a positive integer. The bound deteriorates as $c\lambda$ approaches any positive integer. We also illustrate this with a bifurcation plot. As a specific example we take $f(u) = \frac{1}{2} (u + \sin(u))$. Note that $f'(0) = 1$, and so the bifurcation points from the trivial solution (the Steklov eigenvalues for the “linearized” boundary value problem) remain the nonnegative integers. The right frame in Fig. 6 shows a plot of the $H^1(\Omega)$-norm for “the first” five nontrivial solutions to the corresponding boundary value problem. The plot exhibits the generic behavior, but it has an interesting additional feature (which we believe is associated with the oscillatory behavior of $\sin(u)$) namely: there seem to be an infinite number of solutions to the problem when $c\lambda = \frac{k}{2}$ is a positive integer. This feature is somewhat reminiscent of a feature found in connection with the so-called Gelfand–Liouville Problem (cf. [2] and [8]).
Fig. 6. The $H^1(\Omega)$-norm of “the first” five solutions to (83) with $f(u) = \arctan(u)$ (left frame) and $f(u) = \frac{1}{2}(u + \sin(u))$ (right frame). In both cases $D = 0$.

Fig. 7. $H^1(\Omega)$-norms of solutions to (1), with $D = 0$ on the annulus, $\Omega = B(0, 1) \setminus B(0, 1/2)$. Only primary bifurcations are plotted.

3.2. High energy solutions, secondary bifurcations

In [12] we provided numerical examples of solutions to (1) (i.e., (83) with $f = \sinh$) on simply connected domains, different from a disk, in order to display families of solutions with special properties (we were looking for solutions for $D = 0$, whose flux converges to a measure with a nonzero regular part). To obtain a more “complicated” bifurcation diagram than that shown in Fig. 1 (including a family of solutions whose $H^1(\Omega)$-norm grows faster than $(\log \frac{1}{\lambda})^{1/2}$ as $\lambda \to 0_+$) we now consider (1) on a domain $\Omega$ in the shape of the annulus, $\Omega = B(0, 1) \setminus B(0, 1/2)$. Fig. 7 shows the solutions bifurcating off the zero solution at the (first 10) positive Steklov eigenvalues, for $D = 0$. As usual, we plot the $H^1(\Omega)$-norm of the solutions versus $\lambda$. The solution structure at first sight appears much more complicated than that seen in the bifurcation plot for the unit disk (or any simply connected domain, for that matter). In order to make the solution structure more transparent, it is convenient to divide the solutions into three classes. One class consists of a single branch only, namely the nonconstant radial solution; it is given by

$$u_\lambda = a_\lambda \log r + b_\lambda,$$

with the coefficients $a_\lambda$ and $b_\lambda$ satisfying the equations

$$a_\lambda = \lambda \sinh(b_\lambda), \quad -2a_\lambda = \lambda \sinh(-a_\lambda \log 2 + b_\lambda).$$

This branch bifurcates from the zero solution at the Steklov eigenvalue $D_6 = 3/\log 2 \approx 4.328085$. This radial solution has an $H^1(\Omega)$-norm that blows up faster than the other solutions (at a rate of $\log \frac{1}{\lambda}$ as $\lambda \to 0_+$). The remaining branches, bifurcating from the zero solution at Steklov eigenvalues with nonradial eigenfunctions, can conveniently be divided into two separate classes, A and B.

The separated energy plots look a lot cleaner, (there are no more intersections) but more importantly this separation appears to distinguish the solutions according to the form of the possible limiting measures $\mu = \lim_{\lambda_n \to 0} \partial u_{\lambda_n}/\partial n$. For $\{u_{\lambda_n}\}$ on any of the branches in class A, we have

$$\mu = 2\pi \sum_{i=1}^{2N} (-1)^i \delta_{\sigma_i},$$

and for $\{u_{\lambda_n}\}$ on any of the branches in class B, we have

$$\mu = 2\pi \sum_{i=1}^{2N} (-1)^{i-1} \delta_{\sigma_i} - 2\pi \sum_{i=1}^{2N} (-1)^{i-1} \delta_{\frac{1}{2}\sigma_i}.$$
Here $N$ may be any positive integer and $\{\sigma_i\}_{i=1}^{2N}$ may be any set of $2N$ distinct, equispaced points on the unit circle. Another reasonable way to think about this classification of families of solutions is through eigenvalue and eigenfunction data, rather than the limiting measure. With this in mind, we label each family by a pair $(\gamma, \phi)$ where $\phi$ is a Steklov eigenfunction associated with the eigenvalue $\gamma$, at which the family bifurcates from the 0 solution. For the annulus with outer radius 1 and inner radius $1/R$ for $R > 1$, these Steklov eigenfunctions and values can be found by a separation of variables approach. There are two radial eigenfunctions corresponding to the simple eigenvalues 0 and $\gamma^2/R^2$, at which the family bifurcates from the 0 solution. For the annulus with outer radius 1 and inner radius $1/R$ for $R > 1$, these Steklov eigenfunctions and values can be found by a separation of variables approach. There are two radial eigenfunctions corresponding to the simple eigenvalues 0 and $(R + 1)/\log R$, respectively. As far as nonradial eigenfunctions are concerned, they are all, modulo a rotation, of the form

$$u_m = a_m r^m \sin(m\theta) + b_m r^{-m} \sin(m\theta),$$

for some $m \geq 1$. The Neumann boundary conditions give rise to the two equations

$$ma_m - mb_m = \gamma (a_m + b_m), \quad \text{and}$$

$$mb_m R^{m+1} - ma_m R^{-m+1} = \gamma (b_m R^m + a_m R^{-m}).$$

For each fixed $m \geq 1$ this $2 \times 2$ linear system possesses nontrivial solutions $(a, b)$ for two different values of $\gamma$. For each fixed $m \geq 1$ we thus have two eigenvalue-eigenfunction pairs $(\gamma_m^1, \phi_m^1)$ and $(\gamma_m^2, \phi_m^2)$. Suppose the eigenvalues are ordered such that $\gamma_m^1 < \gamma_m^2$. Then the branch of solutions labelled by the pair $(\gamma_m^1, \phi_m^1)$ falls into the classification A, and the branch of solutions labelled by $(\gamma_m^2, \phi_m^2)$ falls into classification B. The eigenfunction $\phi_m^1$ achieves its extremal values on the outer circle, whereas $\phi_m^2$ achieves its extremal values on the inner circle. For each family of solutions the integer $m$ coincides with the integer $N$, that appears in the limiting measure, i.e., it determines the number of $\delta$-functions that emerge.

The branch of nonconstant radial solutions (whose $H^1(\Omega)$-norm blows up as $\lambda \to 0_+$) is also interesting from the point of view of higher order bifurcations. The simplicity of this family of solutions means we can easily apply a separation of variables argument to identify an infinite (countable) set of $\lambda$-values at which we should expect secondary bifurcations. For details on this calculation, see [11]. The presence of these secondary (as well as higher order) bifurcations can be verified by computational experiments, such as that provided in Fig. 9. Based on our computational experience it appears that the limiting measures, proceeding directly along the secondary bifurcations, for the annulus $\Omega = B(0, 1) \setminus \overline{B(0, 1/2)}$ take the form

$$\mu = \lim_{\lambda \to 0_+} \frac{\partial u_\lambda}{\partial n} = 2\pi \sum_{i=1}^{N} \delta_{\sigma_i} - 2\pi \sum_{i=1}^{N} \delta_{\frac{1}{2}\sigma_i},$$

where $N$ is an arbitrary positive integer, and $\{\sigma_i\}_{i=1}^{N}$ is a set of $N$ distinct, equispaced points on the unit circle. It appears that the secondary bifurcations have $H^1(\Omega)$-norms that blow up like $(\log \frac{1}{\lambda})^{1/2}$ as $\lambda \to 0_+$. Returning to the domain $\Omega = B(0, 1)$, there is a situation where we may apply a similar separation of variables argument, to predict an infinite number of secondary bifurcations. If we let $D = -1$, then we have one family of solutions to (1) that are constant, for $0 < \lambda < 1$. The $H^1(\Omega)$-norm of this family of solutions also blows up at the rate of $(\log \frac{1}{\lambda})^{1/2}$ as $\lambda \to 0_+$. Fig. 10 shows some of the bifurcation diagram in this situation.

With the appearance of these instances of secondary bifurcations a natural question arises: do all (or most) families have secondary bifurcations, or do such bifurcations only emanate from families of solutions whose $H^1(\Omega)$-norm
Blows up faster than $(\log \frac{1}{\lambda})^{1/2}$? The answer to the second question appears to be no (in Fig. 9, the tertiary bifurcation developing seems to disprove this). However, at the same time it appears that a large number of families possess no secondary bifurcations. We examine this phenomenon in more detail in two different situations. We first note that a necessary condition for a bifurcation at a solution $U_\lambda$, is the existence of a nonzero function $h$ that solves

$$\begin{cases}
\Delta h = 0 & \text{in } \Omega, \\
\frac{\partial h}{\partial n} = (D + \lambda \cosh(U_\lambda))h & \text{on } \partial \Omega.
\end{cases}$$

Let $V_\lambda(x) \in C^\infty(\partial \Omega)$ be the function defined by $V_\lambda(x) = D + \lambda \cosh(U_\lambda)$. Given $\lambda > 0$ we define the set of “generalized real eigenvalues” to be the values $\{\gamma(\lambda)\} \subset \mathbb{R}$ for which the equation

$$\begin{cases}
\Delta h = 0 & \text{in } \Omega, \\
\frac{\partial h}{\partial n} = \gamma V_\lambda(x)h & \text{on } \partial \Omega
\end{cases}$$

admits a nonzero solution. Clearly, for a given $\lambda > 0$, the set of generalized real eigenvalues includes the value 1, if and only the necessary condition (84) for bifurcation is satisfied.

We now focus on the possibility of finding secondary bifurcations in two different situations pertaining to the unit disk. In the first situation we take $U_\lambda$ to be the constant solution to (1) with $D = -1$. This is a simple case. Experimentally we know this family of solutions admits bifurcations – as evidenced by the right plot in Fig. 10. But it is instructive to see the correlation with the presence of $\gamma = 1$ as a generalized real eigenvalue for (85). The curves in Fig. 11 depict the first five nonzero, generalized real eigenvalues as a function of $\lambda \in (0, 1)$ (the smallest generalized real eigenvalue is 0). Each of these five curves seem to intersect the $\gamma = 1$ line at a value of $\lambda$, close to which we have previously noticed a secondary bifurcation appear from the constant solution. In this case it is actually not hard to show that each curve gives rise to exactly one intersection with the line $\gamma = 1$, resulting in an infinite (countable) set of potential secondary bifurcations. All the nonzero, generalized real eigenvalues converge to $\infty$ as $\lambda \to 1_-$ and they all converge to zero as $\lambda \to 0_+$.

For the second situation we consider one of the families of exact solutions to (1), constructed in [5]. These solutions pertain to $D = 0$, $\Omega = B(0, 1)$; the particular family we consider bifurcates from the zero solution at $\lambda = 1$, and it has the form

$$U_\lambda(x) = 2 \log|x - \rho(\lambda)\sigma_1| - 2 \log|x - \rho(\lambda)\sigma_2|, \quad (86)$$
with \( \rho(\lambda) = \left(\frac{1+\lambda}{1-\lambda}\right)^{1/2} \) and \( \{\sigma_1, \sigma_2\} = \{(1, 0), (-1, 0)\} \) (or any two diametrically opposed points on the boundary of the unit disk). The fact that we have solutions in closed form is helpful when calculating the generalized eigenvalues. These solutions happen to provide the simplest example of a family, for which the flux converges to a pure sum of \( \delta \)-functions. Fig. 12 shows the five smallest nonzero generalized real eigenvalues for the problem (85) in the case when \( U_\lambda \) is given by (86). We see a marked difference, when compared to Fig. 11. Two of the generalized eigenvalues start at the value 1 (when \( \lambda = 1 \)). One of these remains at 1 for all values \( 0 < \lambda < 1 \), the other clearly falls below. The third generalized real eigenvalue approaches 1 from above as \( \lambda \) nears zero, whereas all the others seem to stay strictly above 1. That the nonzero generalized real eigenvalues converge in pairs to the set of positive integers as \( \lambda \to 1^- \) is consistent with the fact that the positive integers are (double) Steklov eigenvalues for the problem

\[
\Delta \phi = 0 \quad \text{in } \Omega, \quad \frac{\partial \phi}{\partial \mathbf{n}} = \gamma \phi \quad \text{on } \partial \Omega.
\]

The fact that one of the generalized eigenvalues remains 1 for all values \( 0 < \lambda < 1 \) does not mean that we should expect all elements of the family of solutions given by (86) to be a “true” bifurcation point, rather it should be seen as a reflection of the symmetry of the set of solutions. An examination of the eigenfunctions confirms what we should expect: for any given \( \lambda \in (0, 1) \) an eigenfunction corresponding to \( \gamma = 1 \) is \( h = \frac{\partial}{\partial \tau} U_\lambda \). The form of this eigenfunction is consistent with the fact that \( u_\lambda \circ R(\theta) \) is also a solution to the boundary value problem (1) if \( u_\lambda \) is a solution and \( R(\theta) \) is any rotation. The fact that no other generalized eigenvalue ever equals 1 indicates, that we should expect no points of “true” secondary bifurcation from the solution given by (86). We have similar expectations for all the solutions constructed in [5]. Based on our numerical experience we are also inclined to believe that the solution classes denoted A and B (see Fig. 8) which we encountered in connection with the annulus, have no associated secondary bifurcations.

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