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Existence by minimisation of solitary water waves on an ocean of infinite depth

Existence par minimisation d'ondes solitaires à la surface d'un océan infiniment profond

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Abstract

The abstract minimisation method introduced in a recent work by E. Séré, J.F. Toland and the author [Minimisation methods for quasi-linear problems, with an application to periodic water waves, preprint] gives a new proof of the existence of capillary-gravity solitary water waves on the surface of a two-dimensional ocean of infinite depth. This problem was first studied by Iooss and Kirrmann [Arch. Rational Mech. Anal. 136 (1998) 1–19] in the setting of normal form theory for reversible infinite-dimensional "spatial" dynamical systems.
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Résumé

La méthode de minimisation abstraite introduite dans un travail récent de E. Séré, J.F. Toland et l'auteur, donne une nouvelle preuve de l'existence d'ondes solitaires à la surface d'un océan infiniment profond, sous l'action de la gravité et de tension superficielle. Ce problème a été étudié par Iooss et Kirrmann [Arch. Rational Mech. Anal. 136 (1998) 1–19] à l'aide de la théorie des formes normales pour les systèmes dynamiques "spatiaux", réversibles et de dimension infinie.
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1. Introduction

A two-dimensional layer of water of infinite depth is considered, on the surface of which a wave is propagating without changing its form. We work in a frame of reference (x, y) that follows the wave, where x is the propagation coordinate and y is the height coordinate. The surface is supposed to be the graph of a function $y = \eta(x)$. Each molecule of water below the surface, i.e. located at some (x, y) with $y \leqslant \eta(x)$, moves with speed $\vec{v}(x, y) \in \mathbb{R}^2$ in a constant gravity field $(0, -\lambda)$ pointing downward ($\lambda > 0$). Since we assume that the wave moves to the right with velocity $\nu > 0$, the asymptotic speed in the referential frame that follows the wave is given by $\lim_{y\to-\infty} \vec{v}(x, y) = (-\nu, 0)$. The density of the water is constant, say equal to 1, and the flow is irrotational, so that $\operatorname{div}\vec{v} \equiv 0$ and $\operatorname{rot}\vec{v} \equiv 0$. Moreover $\vec{v}(x, \eta(x)) \parallel (1, \eta'(x))$ for all $x \in \mathbb{R}$. The free boundary being subjected to surface tension, the following Bernoulli condition holds:

$$\frac{1}{2}\left|\vec{v}\big(x, \eta(x)\big)\right|^2 + \lambda\eta(x) - \beta\frac{\eta''(x)}{(1 + \eta'(x)^2)^{3/2}} = \frac{\nu^2}{2} \quad \forall x \in \mathbb{R},$$

where $\beta > 0$ is the surface-tension intensity and the constant $\nu^2/2$ on the right has been chosen so that the quiescent state

$$\eta \equiv 0, \quad \vec{v} \equiv (-\nu, 0)$$

is a solution (the "trivial" one). We shall first study periodic waves of large period P and then let $P \to \infty$ to get solitary waves (that is, $\lim_{|x|\to\infty} \eta(x) = 0$). The energy carried by a periodic wave over a period is given by

$$\frac{1}{2}\int_0^P \int_{-\infty}^{\eta(x)} \left\{\left|\vec{v}(x, y)\right|^2 - \nu^2\right\} dx\, dy + \frac{\lambda}{2}\int_0^P \eta^2(x)\, dx + \beta\int_0^P \sqrt{1 + \eta'(x)^2}\, dx,$$

where we have subtracted ν^2 so that the first integral is finite (this is related to the choice of the constant on the right-hand side of the Bernoulli condition).

The papers dealing with water waves are innumerable. The main approaches rely on experiments, model equations, bifurcation theory (local and global), numerical analysis, the Calculus of Variations, centre manifold theory, normal form theory, etc. Also other similar physical settings have been studied: three-dimensional problems, fixed upper surface, finite depth, no surface tension, many liquids and interfaces, etc. Besides periodic and solitary water waves, generalised solitary water waves (i.e. asymptotic to ripples) and fronts have been considered.

Our purpose is to further develop the variational theory of the existence of water waves. We do not aim at new results on water waves, but rather at giving another viewpoint. Garabedian [7] seems to be the first to study gravity (that is, $\beta = 0$) periodic water waves as saddle points of the energy and to sketch a variational proof of their existence. However Turner [12] observed that his proof was incomplete and described the main functional and variational features of the problem. Moreover, by a minimisation under constraint, he established the existence of gravity periodic water waves for a stratified fluid with a free upper surface. His method consists in adding artificial coercivity, in proving a priori estimates for the solutions of the new problem and in deducing that the solutions of small energy belong to a region unaffected by the artificial coercivity. Solitary water waves are then obtained as limits of periodic waves of large periods. With E. Séré and J.F. Toland [5] we applied some of his ideas to Babenko's formalism for periodic water waves [1,2], but using a mountain-pass principle. It is only in [6] that we managed to develop an abstract and relatively simple framework for a class of quasi-linear variational problems of the kind of those arising in the theory of periodic water waves with or without surface tension (see [4] for a generalisation of Babenko's formulation to the surface-tension setting). The works [12] and [6] differ in two respects: the regularisation of the higher order terms and the class of examples they are applied to (although gravity water waves are dealt with in both).

The present proof, wich concerns capillary-gravity solitary waves, relies on the abstract minimisation theory in [6], test functions inspired by solutions of model equations, and a limiting procedure to get solitary waves. The main steps are analogous with those in the variational theory of bifurcation from the essential spectrum (see e.g. [10,11]).

In this way a part of the results of Iooss and Kirrmann [8] obtained by the theory of normal forms is recovered. We have not tried to get, like them, two distinct solitary waves, and it is unclear if the solution found variationally is among those obtained by Iooss and Kirrmann. However it would be interesting to know if multiplicity is also within the reach of the Calculus of Variations.

Finally we mention the work [3] where stability of solitary water waves on an ocean of finite depth is proved variationally.

2. Abstract setting

Let us recall the abstract setting introduced in [6]. Consider a real Hilbert space X_0 with inner-product $\langle\cdot,\cdot\rangle_0$ and norm $\|\cdot\|_0$, and suppose that A is a (possibly unbounded) positive-definite self-adjoint operator on X_0 such that its spectrum is included in $[1,\infty)$. For $k \geqslant 1$ let X_k denote the domain of $A^{k/2}$, which is dense in X_0. Then X_k is a Hilbert space with inner-product and norm defined by $\langle u,w\rangle_k = \langle A^{k/2}u, A^{k/2}w\rangle_0$ and $\|w\|_k = \|A^{k/2}w\|_0$. Moreover $\|w\|_k \leqslant \|w\|_{k+1}$ for all $w \in X_{k+1}$.

For $R_2 > 0$, let $U \subset X_2$ be the open ball $\{w \in X_2\colon\ \|w\|_2 < R_2\}$ and suppose that $\mathcal{K},\mathcal{L} \in C^1(U;\mathbb{R})$. We are interested in the equation

$$\gamma\mathcal{K}'(w) + \mathcal{L}'(w) = 0, \quad w \in U\setminus\{0\},\ \gamma > 0,$$

when the following inequalities hold for constants C_1 to $C_3 > 0$ and C_4 to $C_6 \geqslant 0$:

$$\mathcal{K}(0) = 0 \text{ and } \forall w \in U\colon\quad \mathcal{K}(w) \geqslant C_1\|w\|_1^2, \tag{1a}$$

$$\forall w \in X_4 \cap U\colon\quad \mathcal{K}'(w)Aw \geqslant C_2\|w\|_2^2, \tag{1b}$$

$$\forall w \in U\colon\quad \big|\mathcal{L}(w)\big| \leqslant C_3\|w\|_1^2, \tag{1c}$$

$$\forall w \in X_4 \cap U\colon\quad \mathcal{L}'(w)Aw \geqslant -C_4\|w\|_1^2 - C_5\|w\|_2^2 - C_6\|w\|_1\|w\|_2. \tag{1d}$$

In Theorem 1 below, it is assumed that $\gamma_0 > 2C_5/C_2$ and therefore it may be convenient to allow $C_5 = 0$. Observe that $\mathcal{K}'(0) = \mathcal{L}'(0) = 0$ and that $\mathcal{K}(0) = \mathcal{L}(0) = 0$. The next hypothesis is about solutions of a regularised problem: for all $\gamma > 0$, $\varepsilon > 0$ and $w \in U$,

$$\gamma\mathcal{K}'(w) + \mathcal{L}'(w) + \varepsilon A^2 w = 0 \text{ in } X_2^* \text{ implies that } w \in X_4, \tag{1e}$$

where A^2w denotes the functional in X_2^* taking the value $\langle Aw, Au\rangle_0 = \langle u,w\rangle_2$ at any $u \in X_2$. Furthermore we suppose that

$$\begin{gathered}\text{if } \{w_n\} \subset U \text{ converges weakly in } X_2 \text{ to } w \in U,\ \text{then}\\ \liminf_{n\to\infty}\mathcal{K}(w_n) \geqslant \mathcal{K}(w) \text{ and } \liminf_{n\to\infty}\mathcal{L}(w_n) \geqslant \mathcal{L}(w).\end{gathered} \tag{1f}$$

We also need to distinguish the super-quadratic parts of $\mathcal{K}$ and $\mathcal{L}$:

$$\mathcal{M}(w) := \mathcal{K}(w) - \frac{1}{2}\mathcal{K}''(0)(w,w), \qquad \mathcal{N}(w) := \mathcal{L}(w) - \frac{1}{2}\mathcal{L}''(0)(w,w),$$

and we assume that

$$\forall w \in U\colon\quad \big|\mathcal{M}(w)\big| \leqslant C_7\|w\|_1^p \text{ and } \big|\mathcal{N}(w)\big| \leqslant C_8\|w\|_1^p$$

for some constants $p > 2$, $C_7 \geqslant 0$ and $C_8 \geqslant 0$ such that $C_7C_8 > 0$.

Theorem 1. *Let $\gamma_0 > 2C_5/C_2$ be such that there exists $u_* \in U$ with*

$$\gamma_0\mathcal{K}(u_*) + \mathcal{L}(u_*) < 0 = \gamma_0\mathcal{K}(0) + \mathcal{L}(0), \tag{2a}$$

$$\frac{2\gamma_0 C_2 C_4 + C_6^2}{(\gamma_0 C_2)^2 - 2\gamma_0 C_2 C_5} C_1^{-1}\mathcal{K}(u_*) < \frac{1}{2}R_2^2. \tag{2b}$$

Assume also that there exists $C_9 > 0$ *such that*

$$\forall u \in X_2\colon \quad \frac{\gamma_0}{2}\mathcal{K}''(0)(u,u) + \frac{1}{2}\mathcal{L}''(0)(u,u) \geqslant C_9\|u\|_1^2. \tag{3}$$

Then there exists $w \in U\backslash\{0\}$ *and* $\gamma \geqslant \gamma_0$ *such that*

$$\gamma\mathcal{K}'(w) + \mathcal{L}'(w) = 0,$$

$$\gamma_0\mathcal{K}(w) + \mathcal{L}(w) \leqslant \gamma_0\mathcal{K}(u_*) + \mathcal{L}(u_*) < 0$$

and

$$-\gamma_0\mathcal{M}(w) - \mathcal{N}(w) > C_9\left\{\frac{C_9}{\gamma_0 C_7 + C_8}\right\}^{2/(p-2)}. \tag{4}$$

Moreover γ *is bounded above by a constant that depends only on* C_1 *to* C_6, γ_0 *and* R_2.

Proof. Note that for $u \in U \cap X_4$

$$\begin{aligned}
\gamma_0\mathcal{K}'(u)Au + \mathcal{L}'(u)Au &\geqslant \gamma_0 C_2\|u\|_2^2 - C_4\|u\|_1^2 - C_5\|u\|_2^2 - C_6\|u\|_1\|u\|_2 \\
&\geqslant \frac{1}{2}\gamma_0 C_2\|u\|_2^2 - C_4\|u\|_1^2 - C_5\|u\|_2^2 - (2\gamma_0 C_2)^{-1}C_6^2\|u\|_1^2 \\
&= \big\{(1/2)\gamma_0 C_2 - C_5\big\}\|u\|_2^2 - \left\{C_4 + \frac{C_6^2}{2\gamma_0 C_2}\right\}\|u\|_1^2 \\
&:= \psi\big(\|u\|_1, \|u\|_2\big).
\end{aligned}$$

Moreover $\psi(\sqrt{\mathcal{K}(u_*)/C_1}, R_2)$ is bounded from below by a positive number that depends only on C_1 to C_6, γ_0 and R_2.

The existence of w is now a direct consequence of Theorem 1 in [6], the proof of which we briefly recall because we need it to show the last assertion on the size of γ. Let $R_1, R_{\min} > 0$ be finite numbers such that

$$\mathcal{K}(u_*) < R_1^2, \quad \|u_*\|_2 \leqslant R_{\min} < R_2, \quad \psi\left(\frac{R_1}{\sqrt{C_1}}, R_{\min}\right) > 0. \tag{5}$$

These numbers can be chosen so that $R_1^2 - \mathcal{K}(u_*)$ depends only on C_1 to C_6, γ_0 and R_2. We then consider two smooth, non-decreasing penalisation functions $\rho_i : [0, R_i^2) \to \mathbb{R}$ such that

$$\rho_i(s) \to \infty \quad \text{as } s \nearrow R_i^2,\ i = 1, 2,$$

$$0 \leqslant s \leqslant \mathcal{K}(u_*) \Rightarrow \rho_1(s) = 0, \quad 0 \leqslant s \leqslant R_{\min}^2 \Rightarrow \rho_2(s) = 0,$$

and the functional defined by

$$\mathcal{J}(w) = \gamma_0\mathcal{K}(w) + \mathcal{L}(w) + \rho_2\big(\|w\|_2^2\big) + \rho_1\big(\mathcal{K}(w)\big)$$

in the domain $V := \{w \in U\colon \mathcal{K}(w) < R_1^2\}$. Observe that $\mathcal{J}$ is bounded from below on V, with

$$\mathcal{J}(w) \to \infty \quad \text{as } \|w\|_2 \nearrow R_2 \quad \text{and} \quad \mathcal{J}(w) \to \infty \quad \text{as } \mathcal{K}(w) \nearrow R_1^2$$

by the existence of the constants C_1 and C_3. From (1f) it follows that $\mathcal{J}$ has a minimiser $w \in V$ that satisfies $\mathcal{K}(w) < R_1^2$, $\|w\|_2 < R_2$ and

$$\big\{\gamma_0 + \rho_1'\big(\mathcal{K}(w)\big)\big\}\mathcal{K}'(w) + \mathcal{L}'(w) + 2\rho_2'\big(\|w\|_2^2\big)A^2 w = 0. \tag{6}$$

Assume, by contradiction, that $\varepsilon := 2\rho_2'(\|w\|_2^2) > 0$. Then, by (1e), $w \in X_4$,

$$0 = \mathcal{J}'(w)Aw \geqslant \psi\big(\|w\|_1, \|w\|_2\big) + \rho_1'\big(\mathcal{K}(w)\big)\mathcal{K}'(w)Aw + \varepsilon\|w\|_3^2$$

and

$$\psi\big(\|w\|_1, \|w\|_2\big) < 0. \tag{7}$$

Since $\|w\|_1 < R_1/\sqrt{C_1}$ (which follows from $\mathcal{K}(w) < R_1^2$), we get $\|w\|_2 < R_{\min}$ by (5) and (7), which leads to the contradiction $\rho_2'(\|w\|_2^2) = 0$.

Hence, we have proved that $w \in U$ satisfies

$$\gamma\mathcal{K}'(w) + \mathcal{L}'(w) = 0 \quad \text{with } \gamma := \gamma_0 + \rho_1'\big(\mathcal{K}(w)\big),$$
$$\gamma_0\mathcal{K}(w) + \mathcal{L}(w) \leqslant \mathcal{J}(w) \leqslant \mathcal{J}(u_*) = \gamma_0\mathcal{K}(u_*) + \mathcal{L}(u_*) < 0$$

and $w \neq 0$ by (2a). The inequalities

$$-C_3R_2^2 + \rho_1\big(\mathcal{K}(w)\big) \leqslant \mathcal{L}(w) + \rho_1\big(\mathcal{K}(w)\big) \leqslant \mathcal{J}(w) < 0$$

lead to an upper bound on $\rho_1(\mathcal{K}(w))$ that only depends on C_3 and R_2. As the function $s \to \rho_1(s + \mathcal{K}(u_*))$ can be chosen for $s \geqslant 0$ in a way that depends only on C_1 to C_6, γ_0 and R_2, the uniform bound on $\gamma = \gamma_0 + \rho_1'(\mathcal{K}(w))$ follows.

It remains to prove (4):

$$C_9\|w\|_1^2 \leqslant \frac{\gamma_0}{2}\mathcal{K}''(0)(w,w) + \frac{1}{2}\mathcal{L}''(0)(w,w) = \gamma_0\mathcal{K}(w) + \mathcal{L}(w) - \gamma_0\mathcal{M}(w) - \mathcal{N}(w)$$
$$< -\gamma_0\mathcal{M}(w) - \mathcal{N}(w) \leqslant \gamma_0C_7\|w\|_1^p + C_8\|w\|_1^p,$$
$$\|w\|_1^{p-2} > \frac{C_9}{\gamma_0C_7 + C_8}$$

and (4) follows. □

Remark (*see Theorem* 2 *of* [6]). If Theorem 1 holds for some $R_2 > 0$, it also holds for any smaller R_2 for which $\|u_*\|_2 < R_2$ and (2b) are still satisfied. This leads to the following additional bound on the critical point w in the theorem:

$$\|w\|_2^2 \leqslant 2\max\left\{\|u_*\|_2^2, 2\frac{2\gamma_0C_2C_4 + C_6^2}{(\gamma_0C_2)^2 - 2\gamma_0C_2C_5}\,C_1^{-1}\mathcal{K}(u_*)\right\}. \tag{8}$$

3. Gravity-capillary water waves

For $P \geqslant 2$, let L_P^2 denote the usual real Banach space of P-periodic, real-valued, locally square-integrable measurable functions on $\mathbb{R}$ and let L_P^∞ denote the analogous space of essentially bounded functions. We denote by C_P^k (resp. C_P^∞), the space of P-periodic functions u which are k-times continuously differentiable (resp. infinitely differentiable).

With respect to the orthonormal basis $\{P^{-1/2}e^{2\pi ikt/P}\colon k \in \mathbb{Z}\}$, let the Fourier coefficients of $u \in L_P^2$ be denoted by $\hat{u}_k$ for $k \in \mathbb{Z}$. Then $\hat{u}_{-n} = \overline{\hat{u}_n}$ and L_P^2 is a real Hilbert space with inner product

$$\langle u, v\rangle = \sum_{n\in\mathbb{Z}} \hat{u}_n\overline{\hat{v}_n}.$$

The fractional order Sobolev space H_P^s is the Hilbert space of functions $u \in L_P^2$ with norm given by

$$\|u\|_s^2 = \sum_{k\in\mathbb{Z}} \bigl(1 + |2\pi k/P|^2\bigr)^s |\hat{u}_k|^2 < \infty. \tag{9}$$

Note that

$$\|u\|_1^2 = \|u\|_{L_P^2}^2 + \|u'\|_{L_P^2}^2 \quad \text{if } u \in H_P^1$$

and

$$\|u\|_2 = \|u - u''\|_{L_P^2} \quad \text{if } u \in H_P^2.$$

The conjugation operation [13] on L_P^2 is defined by

$$(\widehat{\mathcal{C}_P u})_0 = 0 \quad \text{and} \quad (\widehat{\mathcal{C}_P u})_k = -i\,\operatorname{sgn}(k)\hat{u}_k \quad \text{for } k \in \mathbb{Z}\setminus\{0\},\ \text{when } u \in L_P^2; \tag{10}$$

equivalently,

$$\mathcal{C}_P\bigl(\cos(2\pi nt/P)\bigr) = \sin(2\pi nt/P), \quad n \geqslant 0,$$

and

$$\mathcal{C}_P\bigl(\sin(2\pi nt/P)\bigr) = -\cos(2\pi nt/P), \quad n \geqslant 1.$$

Clearly $\mathcal{C}_P : L_P^2 \to L_P^2$ is a bounded linear operator and $u \mapsto \mathcal{C}_P u'$ is non-negative symmetric in the sense that

$$0 \leqslant \langle u, \mathcal{C}_P v'\rangle = \langle \mathcal{C}_P u', v\rangle \quad \text{for all } u, v \in C_P^\infty.$$

For any function $u \in H_P^1$,

$$\begin{aligned}\max|u| &\leqslant \frac{1}{\sqrt{P}}\sum_{k\in\mathbb{Z}}|\hat{u}_k| \leqslant \frac{1}{\sqrt{P}}\Bigl\{\sum_{k\in\mathbb{Z}}\bigl(1+(2\pi k/P)^2\bigr)|\hat{u}_k|^2\Bigr\}^{1/2}\Bigl\{\sum_{k\in\mathbb{Z}}\bigl(1+(2\pi k/P)^2\bigr)^{-1}\Bigr\}^{1/2}\\ &\leqslant \frac{1}{\sqrt{P}}\|u\|_1\Bigl\{1 + P(2\pi)^{-1}\int_{\mathbb{R}}\bigl(1+s^2\bigr)^{-1}\,ds\Bigr\}^{1/2} = \frac{1}{\sqrt{P}}\{1+P/2\}^{1/2}\|u\|_1 \leqslant \|u\|_1\end{aligned} \tag{11}$$

because $P \geqslant 2$.

When surface-tension effects are included, the P-periodic steady water-wave problem can be formulated as follows [4]: find w such that

$$\frac{\nu^2}{2}\bigl\{w'^2 + (1+\mathcal{C}_P w')^2\bigr\}^{-1} + \lambda w - \beta\frac{(1+\mathcal{C}_P w')w'' - w'(1+\mathcal{C}_P w')'}{\{w'^2 + (1+\mathcal{C}_P w')^2\}^{3/2}} = \frac{1}{2}\nu^2$$

almost everywhere, (12a)

$$w'^2 + (1+\mathcal{C}_P w')^2 > 0 \quad \text{on } \mathbb{R},\ w \in H_P^2\setminus\{0\},\ \lambda, \beta, \nu > 0. \tag{12b}$$

The wave is then given in parametric form by

$$t \to \bigl(t + (\mathcal{C}_P w)(t),\ w(t)\bigr)$$

or, using the function η of the introduction, by

$$x = t + (\mathcal{C}_P w)(t), \quad \eta(x) = w(t).$$

This is a generalisation of Babenko's formulation in absence of surface tension [1,2]. The parameters λ, β and ν^2 are dimensionless measures of gravity, the surface tension coefficient and the square of the wave velocity. Since we can divide (12a) by any one of these, there are effectively only two parameters in the problem. After rescaling, we are even left with only one independent parameter (here the fact that the depth is infinite is crucial).

It is known [4] that (12a) is satisfied by any $w \in H_P^2$ such that $w'^2 + (1+\mathcal{C}_P w')^2 > 0$ and such that, almost everywhere,

$$0 = -\nu^2 \mathcal{C}_P w' + \lambda\{w + w\,\mathcal{C}_P w' + \mathcal{C}_P(ww')\} - \beta\left\{\frac{w'}{\sqrt{w'^2+(1+\mathcal{C}_P w')^2}}\right\}' + \beta\mathcal{C}_P\left\{\frac{1+\mathcal{C}_P w'}{\sqrt{w'^2+(1+\mathcal{C}_P w')^2}}\right\}'. \tag{13}$$

Eq. (13) is the Euler equation of the functional

$$J(w) = \int_0^P \left\{-\frac{1}{2}\nu^2 w\,\mathcal{C}_P w' + \frac{1}{2}\lambda w^2(1+\mathcal{C}_P w') + \beta\sqrt{w'^2+(1+\mathcal{C}_P w')^2} - \beta(1+\mathcal{C}_P w')\right\} dt.$$

For all w, the integral of the last term is $-\beta P$ and it is included here to ensure that the constant and linear parts of the integrand vanish when $w = 0$.

Henceforth we assume $\beta = \lambda$ by considering $\widetilde{w}(t) := \sqrt{\lambda/\beta}\; w(\sqrt{\beta/\lambda}\; t)$ instead of $w(t)$. Moreover let $\lambda = 1$ or, equivalently, divide (13) and J by λ, so that ν^2 is replaced by ν^2/λ (that we still denote by ν^2). We now apply the abstract result of Section 1 to J in such a way that the various constants $p > 2$ and C_1 to C_9 do not depend on $P \geqslant 2$. To put the functional J in the context of Section 2, let

$$X_0 = L_P^2, \quad Aw = -w'' + w, \quad X_k = H_P^k \ (k \geqslant 1). \tag{14}$$

If $R_2 \leqslant 1/2$ then (11) implies that

$$\sup|\mathcal{C}_P w'| < 1/2 \quad \text{and} \quad \sup|w'| < 1/2 \quad \text{when } \|w\|_{X_2}^2 < R_2^2. \tag{15}$$

For $w \in U$, the ball of radius R_2 centred at the origin in X_2, let

$$\mathcal{K}_P(w) = \int_0^P \sqrt{w'^2+(1+\mathcal{C}_P w')^2} - (1+\mathcal{C}_P w')\,dt + \frac{1}{2}\int_0^P w^2(1+\mathcal{C}_P w')\,dt = \int_0^P \frac{w'^2\,dt}{\sqrt{w'^2+(1+\mathcal{C}_P w')^2}+(1+\mathcal{C}_P w')} + \frac{1}{2}\int_0^P w^2(1+\mathcal{C}_P w')\,dt,$$

$$\mathcal{L}_P(w) = -\frac{1}{2}\int_0^P w\,\mathcal{C}_P w'\,dt.$$

With ν^2 represented from now on by γ^{-1}, we check the hypotheses of Theorem 1. In (1a) the constant C_1 can be chosen, when (15) holds, to equal $\min\{1/4, 2/7\} = 1/4$:

$$\frac{2}{7}\int_0^P w'^2\,dt + \frac{1}{4}\int_0^P w^2\,dt \leqslant \mathcal{K}_P(w) = \int_0^P \frac{w'^2}{\sqrt{w'^2+(1+\mathcal{C}_P w')^2}+(1+\mathcal{C}_P w')}\,dt + \frac{1}{2}\int_0^P w^2(1+\mathcal{C}_P w')\,dt \leqslant 2\|w\|_1^2. \tag{16}$$

In the same way we can prove the existence of the other constants needed for the abstract theorem, independently of $P \geqslant 2$. In particular we can take $C_4 = C_5 = C_8 = 0$ (note that $\mathcal{N}_P \equiv 0$ because $\mathcal{L}_P$ is quadratic). We thus get

Theorem 2. *Let $R_2 > 0$ be small enough, $P \geqslant 2$, $\gamma_0 > 1/2$ and suppose that there exists $u_P \in H_P^2$ such that*

$$\int_0^P |u_P - u_P''|^2\,dt < R_2^2,$$

$$\gamma_0 \mathcal{K}_P(u_P) + \mathcal{L}_P(u_P) < 0$$

and

$$\mathcal{K}_P(u_P) = \int_0^P \sqrt{u_P'^2 + (1 + \mathcal{C}_P u_P')^2} - (1 + \mathcal{C}_P u_P')\,dt + \frac{1}{2}\int_0^P u_P^2(1 + \mathcal{C}_P u_P')\,dt < C_1 \frac{\gamma_0^2 C_2^2}{2C_6^2} R_2^2.$$

Then there exists $w_P \in H_P^2$ such that

$$0 < \int_0^P |w_P - w_P''|^2\,dt < R_2^2, \tag{17}$$

$$\gamma_0 \mathcal{K}_P(w_P) + \mathcal{L}_P(w_P) \leqslant \gamma_0 \mathcal{K}_P(u_P) + \mathcal{L}_P(u_P)$$

and (13) *holds with $0 < \nu^2 \leqslant \gamma_0^{-1}$ and $\lambda = 1$. Hence* (12a) *is satisfied. Moreover, for some $C_9 = C_9(\gamma_0) > 0$,*

$$-\gamma_0 \mathcal{M}_P(w_P) > C_9 \left\{ \frac{C_9}{\gamma_0 C_7 + C_8} \right\}^{2/(p-2)} \tag{18}$$

and ν^2 is bounded from below by a positive constant that depends only on C_1 to C_6, γ_0 and R_2.

Proof. It remains to verify hypothesis (3) of Theorem 1:

$$\begin{aligned}
&\frac{\gamma_0}{2}\mathcal{K}_P''(0)(u,u) + \frac{1}{2}\mathcal{L}_P''(0)(u,u)\\
&\quad = \frac{1}{2}\int_0^P \{\gamma_0 u'^2 + \gamma_0 u^2 - u\mathcal{C}_p u'\}\,dt\\
&\quad = \frac{1}{2}\sum_{k\in\mathbb{Z}} \{\gamma_0 k^2 |\hat{u}_k|^2 + \gamma_0 |\hat{u}_k|^2 - |k||\hat{u}_k|^2\}\\
&\quad = \frac{1}{2}\sum_{k\in\mathbb{Z}} |\hat{u}_k|^2 \left\{\frac{1}{2\gamma_0}\gamma_0 k^2 + \frac{1}{2\gamma_0}\gamma_0 - |k|\right\} + \frac{1}{2}\left\{1 - \frac{1}{2\gamma_0}\right\}\sum_{k\in\mathbb{Z}} |\hat{u}_k|^2 \{\gamma_0 k^2 + \gamma_0\}\\
&\quad \geqslant \frac{1}{2}\left\{1 - \frac{1}{2\gamma_0}\right\}\sum_{k\in\mathbb{Z}} |\hat{u}_k|^2 \{\gamma_0 k^2 + \gamma_0\}\\
&\quad \geqslant C_9 \|u\|_1^2.
\end{aligned}$$

Note that, for the critical value $\gamma_0 = 1/2$, the minimum of the function

$$\mathbb{R} \ni k \to \gamma_0 k^2 - |k| + \gamma_0$$

is 0 and it is attained at $k = (2\gamma_0)^{-1} = 1$. This explains why $\cos t$ appears in the test function considered below. □

4. Periodic solutions of large periods

To apply Theorem 2 for large P, we need to find u_P.

For $v \in L^2(\mathbb{R})$, we denote by $\mathcal{H}v$ its Hilbert transform:

$$\widehat{\mathcal{H}v}(s) = -i \,\mathrm{sgn}(s)\hat{v}(s),$$

where

$$\hat{v}(s) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ist} v(t)\,dt.$$

For almost all $t \in \mathbb{R}$,

$$\mathcal{H}v(t) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{v(s)}{t-s}\,ds$$

where the integral exists in the sense of Cauchy's principal value (around $s = t$) for almost all t. When $v \in L^2(\mathbb{R}) \cap C^1(\mathbb{R})$, the integral exists in the sense of Cauchy's principal value for all t. Clearly $\{\mathcal{H}v(\alpha\cdot)\}(t) = \mathcal{H}v(\alpha t)$ almost everywhere, $\mathcal{H}$ commutes with differentiation in $W^{1,2}(\mathbb{R})$, and $\mathcal{H}\frac{d}{dt}$ is self-adjoint in $L^2(\mathbb{R})$ and positive definite. Moreover, similarly to [9], we get

$$\begin{aligned}\pi\mathcal{H}v'(t) &= \int_{t-1}^{t+1} \frac{v'(s)}{t-s}\,ds + \int_{-\infty}^{t-1} \frac{v'(s)}{t-s}\,ds + \int_{t+1}^{\infty} \frac{v'(s)}{t-s}\,ds \\ &= \int_{t-1}^{t+1} v''(s)\ln|t-s|\,ds - \int_{-\infty}^{t-1} \frac{v(s)}{(t-s)^2}\,ds + v(t-1) - \int_{t+1}^{\infty} \frac{v(s)}{(t-s)^2}\,ds + v(t+1)\end{aligned}$$

for all $v \in W^{1,2}(\mathbb{R}) \cap C^2(\mathbb{R})$ and, as a consequence, for all $v \in W^{2,2}(\mathbb{R})$. Therefore there exists a constant $C > 0$ such that

$$\left|\mathcal{H}v'(t)\right| \leqslant C\left\{1 + \mathrm{dist}\big(t, \mathrm{supp}(v)\big)\right\}^{-3/2} \|v\|_{W^{2,2}(\mathbb{R})} \tag{19}$$

for all $v \in W^{2,2}(\mathbb{R})$ with compact support (C being independent of the size of the support). If $v \in W^{2,2}(\mathbb{R}) \cap L^1(\mathbb{R})$, this can be improved to lead to

$$\left|\mathcal{H}v'(t)\right| \leqslant C\left\{1 + \mathrm{dist}\big(t, \mathrm{supp}(v)\big)\right\}^{-2} \left\{\|v\|_{W^{2,2}(\mathbb{R})} + \int_{\mathbb{R}} |v|\,dt\right\}. \tag{20}$$

For all $\phi \in C_0^\infty(\mathbb{R})$, it follows from the definitions of $\mathcal{H}$ and $\mathcal{C}_P$ that

$$\int_{\mathbb{R}} e^{-int/P} \mathcal{H}\phi'(t)\,dt = \sqrt{2\pi}\left|\frac{n}{P}\right| \hat{\phi}\left(\frac{n}{P}\right) = \int_{\mathbb{R}} \phi(t)\left(\mathcal{C}_P \frac{d}{dt} e^{-in\cdot/P}\right)(t)\,dt$$

for all $n \in \mathbb{Z}$. By continuity

$$\int_{\mathbb{R}} w\mathcal{H}v'\,dt = \int_{\mathbb{R}} vC_P w'\,dt \tag{21}$$

for all $w \in H_P^1$ and $v \in W^{2,2}(\mathbb{R})$ such that $\mathrm{supp}(v)$ is compact. For $v \in W^{2,2}(\mathbb{R})$ with compact support and $P > 0$, we define

$$v_P(t) = \sum_{k \in \mathbb{Z}} v(t - kP)$$

(for P large enough, v_P is the periodic extension of v) and note that the series $\sum_{k\in\mathbb{Z}} \mathcal{H}v'(\cdot - kP)$ is absolutely convergent in $L^\infty_{\mathrm{loc}}(\mathbb{R})$. By (21),

$$\mathcal{C}_P v'_P = \sum_{k\in\mathbb{Z}} \mathcal{H}v'(\cdot - kP)$$

in $L^\infty_{\mathrm{loc}}(\mathbb{R})$ because

$$\int_{\mathbb{R}} \phi \mathcal{C}_P v'_P \, dt = \int_{\mathbb{R}} v_P \mathcal{H}\phi' \, dt = \int_{\mathbb{R}} \sum_{k\in\mathbb{Z}} v(\cdot - kP)\mathcal{H}\phi' \, dt = \int_{\mathbb{R}} \phi \sum_{k\in\mathbb{Z}} \mathcal{H}v'(\cdot - kP)\, dt$$

for all $\phi \in C_0^\infty(\mathbb{R})$. Therefore

$$\mathcal{H}v' = \lim_{P\to\infty} \sum_{k\in\mathbb{Z}} \mathcal{H}v'(\cdot - kP) = \lim_{P\to\infty} \mathcal{C}_P v'_P \quad \text{in } L^\infty_{\mathrm{loc}}(\mathbb{R}) \tag{22}$$

(and also $\sup_{P\geqslant 2} \|\mathcal{C}_P v'_P\|_{L^\infty(\mathbb{R})} < \infty$). Another consequence of (21) is that if $P_n \to \infty$, $v \in W^{1,2}(\mathbb{R})$,

$$w_n \in H^1_{P_n}, \quad \sup_n \|w_n\|_{L^\infty_{P_n}} < \infty, \quad w_n \to v \quad \text{a.e. in } \mathbb{R},$$

then

$$\int_{\mathbb{R}} \phi \mathcal{C}_{P_n} w'_n \, dt = \int_{\mathbb{R}} w_n \mathcal{H}\phi' \, dt \to \int_{\mathbb{R}} v\mathcal{H}\phi' \, dt = \int_{\mathbb{R}} \phi \mathcal{H}v' \, dt \tag{23}$$

for all $\phi \in W^{2,2}(\mathbb{R})$ with compact support.

To check the assumptions of Theorem 2, we first modify them by replacing everywhere $\mathcal{C}_P$ by $\mathcal{H}$ and integration over $(0, P)$ by integration over $\mathbb{R}$. We are thus looking for $u \in W^{2,2}(\mathbb{R})$ such that

$$\int_{\mathbb{R}} |u - u''|^2 \, dt < R_2^2, \quad \gamma_0 \mathcal{K}_\infty(u) + \mathcal{L}_\infty(u) < 0 \quad \text{and} \quad \mathcal{K}_\infty(u) < C_1 \frac{\gamma_0^2 C_2^2}{2C_6^2} R_2^2, \tag{24}$$

where

$$\begin{aligned}
\mathcal{K}_\infty(u) &= \int_{\mathbb{R}} \sqrt{u'^2 + (1+\mathcal{H}u')^2} - (1+\mathcal{H}u')\, dt + \frac{1}{2}\int_{\mathbb{R}} u^2(1+\mathcal{H}u')\, dt \\
&= \int_{\mathbb{R}} \frac{u'^2\, dt}{\sqrt{u'^2 + (1+\mathcal{H}u')^2} + (1+\mathcal{H}u')} + \frac{1}{2}\int_{\mathbb{R}} u^2(1+\mathcal{H}u')\, dt, \\
\mathcal{L}_\infty(u) &= -\frac{1}{2}\int_{\mathbb{R}} u\mathcal{H}u'\, dt.
\end{aligned}$$

Since, for $|s| < 1$,

$$\sqrt{1+s} = 1 + \frac{1}{2}s - \frac{1}{8}s^2 + \frac{1}{16}s^3 - \frac{5}{128}s^4 + \cdots,$$

we get

$$\sqrt{u'^2 + (1+\mathcal{H}u')^2} - (1+\mathcal{H}u') = \frac{1}{2}u'^2 - \frac{1}{2}u'^2\mathcal{H}u' - \frac{1}{8}u'^4 + \frac{1}{2}u'^2(\mathcal{H}u')^2 + \cdots.$$

We set

$$u(t) = \alpha\phi(\alpha t)\cos t + \alpha^2\psi(\alpha t)\cos(2t), \tag{25}$$

where $\phi, \psi \in C_0^\infty(\mathbb{R})$ will be chosen later and $\alpha > 0$ is small. We get

$$u'(t) = -\alpha\phi(\alpha t)\sin t + \alpha^2\phi'(\alpha t)\cos t - 2\alpha^2\psi(\alpha t)\sin(2t) + \alpha^3\psi'(\alpha t)\cos(2t)$$

and, for all $n \in \mathbb{N}$,

$$\mathcal{H}u'(t) = \alpha\phi(\alpha t)\cos t + \alpha^2\phi'(\alpha t)\sin t + 2\alpha^2\psi(\alpha t)\cos(2t) + \alpha^3\psi'(\alpha t)\sin(2t) + \alpha^n \mathrm{O}(1+t^{-2}),$$

where $\mathrm{O}(1+t^{-2})$ depends on n too. Indeed if $\psi \equiv 0$ (for simplicity), we have

$$\{\phi(\alpha\cdot)e^{\pm i\cdot}\}^{\widehat{}}(s) = \alpha^{-1}\hat{\phi}\left(\frac{s \mp 1}{\alpha}\right),$$

$$\begin{aligned}\widehat{\mathcal{H}u'}(s) &= \frac{|s|}{2}\left\{\hat{\phi}\left(\frac{s-1}{\alpha}\right) + \hat{\phi}\left(\frac{s+1}{\alpha}\right)\right\} \\ &= \frac{s}{2}\left\{\hat{\phi}\left(\frac{s-1}{\alpha}\right) - \hat{\phi}\left(\frac{s+1}{\alpha}\right)\right\} + \frac{|s|-s}{2}\hat{\phi}\left(\frac{s-1}{\alpha}\right) + \frac{|s|+s}{2}\hat{\phi}\left(\frac{s+1}{\alpha}\right)\end{aligned}$$

and

$$\mathcal{H}u'(t) = \{\alpha\phi(\alpha t)\sin t\}' + \frac{\alpha^2}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{i\alpha ts}\left\{\frac{|s|-s}{2}\hat{\phi}(s-\alpha^{-1}) + \frac{|s|+s}{2}\hat{\phi}(s+\alpha^{-1})\right\} ds.$$

Two integrations by parts show that, for all $n \in \mathbb{N}$, the second term is of the type $\alpha^n \mathrm{O}(1+t^{-2})$ because the map $s \to |s| - s$ vanishes on $\mathbb{R}_+$, the map $s \to |s| + s$ vanishes on $\mathbb{R}_-$ and $|\hat{\phi}(s)| + |\hat{\phi}'(s)| + |\hat{\phi}''(s)|$ decreases at $\pm\infty$ faster than any $|s|^{-n}$. Going back to (25), we get, thanks to the fact that

$$\widehat{\phi^2},\quad \widehat{\phi'^2},\quad \widehat{\psi^2},\quad \widehat{\phi\phi'},\quad \widehat{\phi\psi},\quad \widehat{\phi\psi'},\quad \widehat{\phi'\psi},\quad \widehat{\phi^3},\quad \widehat{\phi^2\phi'},\quad \widehat{\phi^2\psi}\quad \text{and}\quad \widehat{\phi^4}$$

decrease at $\pm\infty$ faster than any $|s|^{-n}$, the following estimates:

$$\begin{aligned}\int_{\mathbb{R}} u'^2\,dt &= \alpha\int_{\mathbb{R}} \phi^2(t)\sin^2(t/\alpha)\,dt + \frac{\alpha^3}{2}\int_{\mathbb{R}} \phi'^2\,dt + 2\alpha^3\int_{\mathbb{R}} \psi^2\,dt + \mathrm{O}(\alpha^4) \\ &= \frac{\alpha}{2}\int_{\mathbb{R}} \phi^2(t)\{1-\cos(2t/\alpha)\}\,dt + \frac{\alpha^3}{2}\int_{\mathbb{R}} \phi'^2\,dt + 2\alpha^3\int_{\mathbb{R}} \psi^2\,dt + \mathrm{O}(\alpha^4) \\ &= \frac{\alpha}{2}\int_{\mathbb{R}} \phi^2\,dt + \frac{\alpha^3}{2}\int_{\mathbb{R}} \phi'^2\,dt + 2\alpha^3\int_{\mathbb{R}} \psi^2\,dt + \mathrm{O}(\alpha^4),\end{aligned}$$

$$\int_{\mathbb{R}} u^2\,dt = \frac{\alpha}{2}\int_{\mathbb{R}} \phi^2\,dt + \frac{\alpha^3}{2}\int_{\mathbb{R}} \psi^2\,dt + \mathrm{O}(\alpha^4),$$

$$\int_{\mathbb{R}} u\mathcal{H}u'\,dt = \frac{\alpha}{2}\int_{\mathbb{R}} \phi^2\,dt + \alpha^3\int_{\mathbb{R}} \psi^2\,dt + \mathrm{O}(\alpha^4),$$

$$\begin{aligned}\int_{\mathbb{R}} u'^2\mathcal{H}u'\,dt &= 2\alpha^3\int_{\mathbb{R}} \phi(t)^2\psi(t)\sin^2(t/\alpha)\cos(2t/\alpha)\,dt \\ &\quad + 4\alpha^3\int_{\mathbb{R}} \phi(t)^2\psi(t)\sin(t/\alpha)\cos(t/\alpha)\sin(2t/\alpha)\,dt + \mathrm{O}(\alpha^4) \\ &= \frac{\alpha^3}{2}\int_{\mathbb{R}} \phi^2\psi\,dt + \mathrm{O}(\alpha^4),\end{aligned}$$

$$\int_{\mathbb{R}} u^2\mathcal{H}u'\,dt = 2\times 2\alpha^3\int_{\mathbb{R}}\phi(t)^2\psi(t)\cos^2(t/\alpha)\cos(2t/\alpha)\,dt + \mathrm{O}(\alpha^4) = \alpha^3\int_{\mathbb{R}}\phi^2\psi\,dt + \mathrm{O}(\alpha^4),$$

$$\int_{\mathbb{R}} u'^4\,dt = \alpha^3\int_{\mathbb{R}}\phi(t)^4\sin^4(t/\alpha)\,dt + \mathrm{O}(\alpha^4) = \frac{3\alpha^3}{8}\int_{\mathbb{R}}\phi^4 dt + \mathrm{O}(\alpha^4)$$

and

$$\int_{\mathbb{R}} u'^2(\mathcal{H}u')^2\,dt = \frac{\alpha^3}{8}\int_{\mathbb{R}}\phi^4\,dt + \mathrm{O}(\alpha^4).$$

Setting $\gamma_0^{-1} = 2-\alpha^2$ and $\psi = -A\phi^2$ for some constant A, we obtain

$$\begin{aligned}\mathcal{K}_\infty(u) + \gamma_0^{-1}\mathcal{L}_\infty(u) &= \alpha\{1/4+1/4-1/2\}\int_{\mathbb{R}}\phi^2\,dt + \frac{\alpha^3}{4}\int_{\mathbb{R}}\phi^2\,dt + \alpha^3\{1+(1/4)-1\}\int_{\mathbb{R}}\psi^2\,dt\\ &\quad + \frac{\alpha^3}{4}\int_{\mathbb{R}}\phi'^2\,dt + \left(-\frac{1}{4}+\frac{1}{2}\right)\alpha^3\int_{\mathbb{R}}\phi^2\psi\,dt + \frac{\alpha^3}{16}\int_{\mathbb{R}}\phi^4\,dt - \frac{3\alpha^3}{64}\int_{\mathbb{R}}\phi^4\,dt + \mathrm{o}(\alpha^3)\\ &= \alpha^3\left\{\frac{1}{4}\int_{\mathbb{R}}\phi^2\,dt + \frac{1}{4}\int_{\mathbb{R}}\phi'^2\,dt\right\} + \alpha^3\left(\frac{1}{4}A^2 - \frac{1}{4}A + \frac{1}{64}\right)\int_{\mathbb{R}}\phi^4\,dt + \mathrm{o}(\alpha^3).\end{aligned}$$

We can now choose $A>0$ such that

$$\frac{1}{4}A^2 - \frac{1}{4}A + \frac{1}{64} < 0$$

and then ϕ such that $\mathcal{K}_\infty(u) + \gamma_0^{-1}\mathcal{L}_\infty(u) < 0$ for small $\alpha>0$. The other conditions in (24) are clearly satisfied if $\alpha>0$ is small enough.

For such a small $\alpha>0$ and for large enough $P>0$, we now check the assumptions of Theorem 2 with u_P defined by

$$u_P := \sum_{k\in\mathbb{Z}} u(\cdot - kP).$$

In fact they follow from

$$\int_0^P |u_P - u_P''|^2\,dt = \int_{\mathbb{R}} |u-u''|^2\,dt \quad \text{for large } P>0,$$

$$\mathcal{H}u' = \lim_{P\to\infty}\mathcal{C}_P u_P' \quad \text{in } L^\infty_{\mathrm{loc}}(\mathbb{R})$$

by (22), $\sup_{P\geqslant 2}\|\mathcal{C}_P u_P'\|_{L^\infty(\mathbb{R})} < 1/2$ if $\alpha>0$ is small enough,

$$\lim_{P\to\infty}\mathcal{K}_P(u_P) = \mathcal{K}_\infty(u) \quad\text{and}\quad \lim_{P\to\infty}\mathcal{L}_P(u_P) = \mathcal{L}_\infty(u)$$

by Lebesgue's dominated convergence theorem.

Remark. We could have sought u in a larger class of functions, namely

$$u(t) = \alpha\phi(\alpha t)\cos t + \alpha^2\psi(\alpha t)\cos(2t) + \alpha^2\xi(\alpha t)$$

with $\phi,\psi,\xi\in C_0^\infty(\mathbb{R})$ and $\alpha>0$ small.

5. Solitary water waves

Our aim is to find $w \in W^{2,2}(\mathbb{R})$ and $\nu > 0$ such that

$$\frac{\nu^2}{2}\left\{w'^2 + (1+\mathcal{H}w')^2\right\}^{-1} + w - \frac{(1+\mathcal{H}w')w'' - w'(1+\mathcal{H}w')'}{\{w'^2 + (1+\mathcal{H}w')^2\}^{3/2}} = \frac{1}{2}\nu^2 \quad \text{a.e.,} \tag{26a}$$

$$w'^2 + (1+\mathcal{H}w')^2 > 0. \tag{26b}$$

As in the periodic case, (26a) is satisfied by any $w \in W^{2,2}(\mathbb{R})$ such that (26b) holds and, almost everywhere,

$$\begin{aligned} 0 = {} & -\nu^2\mathcal{H}w' + w + w\mathcal{H}w' + \mathcal{H}(w^2/2)' \\ & - \left\{\frac{w'}{\sqrt{w'^2 + (1+\mathcal{H}w')^2}}\right\}' + \mathcal{H}\left\{\frac{1+\mathcal{H}w'}{\sqrt{w'^2 + (1+\mathcal{H}w')^2}} - 1\right\}'. \end{aligned} \tag{27}$$

We follow the method of Turner, explained in [12], consisting in taking the limit of periodic water waves. By (17), there exists a sequence $P_n \to \infty$ and $w_\infty \in W^{2,2}(\mathbb{R})$ such that $\{\nu_{P_n}\}$ converges to some $\nu_\infty \in (0, \gamma_0^{-1/2}]$ and $w_{P_n} \rightharpoonup w_\infty$ weakly in $W^{2,2}_{\mathrm{loc}}(\mathbb{R})$. Hence, for every bounded interval I, $w_{P_n} \to w_\infty$, $w'_{P_n} \to w'_\infty$ and $\mathcal{C}_P w'_{P_n} \to \mathcal{H}w'_\infty$ in $C(I)$ as $n \to \infty$ (this follows from (23)).

We now multiply (13) in which $w = w_{P_n}$ and $\lambda = \beta = 1$ by an arbitrary smooth function with compact support and take the limit $n \to \infty$, which shows that $w_\infty \in W^{2,2}(\mathbb{R})$ satisfies Eq. (27) (thanks to (21)). Moreover

$$\int_{\mathbb{R}} (-w''_\infty + w_\infty)^2\,dt \leqslant R_2^2.$$

It remains to discuss how this argument can be modified to yield that $w_\infty \not\equiv 0$.

From

$$\mathcal{M}_P(w_P) = \int_0^P w_P'^2\left(\frac{1}{\sqrt{w_P'^2 + (1+\mathcal{C}_P w'_P)^2} + (1+\mathcal{C}_P w'_P)} - \frac{1}{2}\right)dt + \frac{1}{2}\int_0^P w_P^2 \mathcal{C}_P w'_P\,dt,$$

we get

$$\begin{aligned} \left|\mathcal{M}_P(w_P)\right| &\leqslant K\int_0^P w_P'^2\left(|w'_P| + |\mathcal{C}_P w'_P|\right)dt + \frac{1}{4}\left\{\max_t |w_P(t)|\right\}\int_0^P \left\{w_P^2 + w_P'^2\right\}dt \\ &\leqslant \left\{\max_t\left|w'_P(t)\right| + \max_t\left|w_P(t)\right|\right\}(2K + 1/4)\int_0^P \left\{w_P^2 + w_P'^2\right\}dt \end{aligned}$$

for some $K > 0$ independent of P. From (18), we deduce that

$$\inf_{P\geqslant 2}\left(\max_t\left|w_P(t)\right| + \max_t\left|w'_P(t)\right|\right) > 0.$$

We now set $\widehat{w}_P(t) = w_P(t + t_P)$, where t_P is such that

$$\max\left\{\left|w_P(t_P)\right|, \left|w'_P(t_P)\right|\right\} = \max\left\{\max_t\left|w_P(t)\right|, \max_t\left|w'_P(t)\right|\right\}$$

and replace in the previous argument the family $\{w_P\colon\ P \geqslant 2\}$ by $\{\widehat{w}_P\colon\ P \geqslant 2\}$. The corresponding $w_\infty \in W^{2,2}(\mathbb{R})$ is then not identically 0 because $\max\{|w_\infty(0)|, |w'_\infty(0)|\} > 0$. Moreover, by (8),

$$\|w_\infty\|^2_{W^{2,2}(\mathbb{R})} \leqslant C\alpha$$

for some positive constant C. Thus we have proved the following theorem:

Theorem 3. *For all $\alpha > 0$ small enough, there exists*

$$(\nu_\alpha, w_\alpha) \subset (0, \infty) \times W^{2,2}(\mathbb{R})$$

such that

$$\nu_\alpha^2 \leqslant 2 - \alpha^2 \quad \textit{and} \quad 0 < \|w_\alpha\|^2_{W^{2,2}(\mathbb{R})} \leqslant C\alpha$$

for some positive constant C,

$$\sup_t \left|w'_\alpha(t)\right| \leqslant 1/2 \quad \textit{and} \quad \sup_t \left|\mathcal{H}w'_\alpha(t)\right| \leqslant 1/2,$$

and

$$\frac{\nu_\alpha^2}{2}\left\{w_\alpha'^2 + (1 + \mathcal{H}w'_\alpha)^2\right\}^{-1} + w_\alpha - \frac{(1 + \mathcal{H}w'_\alpha)w''_\alpha - w'_\alpha(1 + \mathcal{H}w'_\alpha)'}{\{w_\alpha'^2 + (1 + \mathcal{H}w'_\alpha)^2\}^{3/2}} = \frac{1}{2}\nu_\alpha^2$$

almost everywhere. Finally

$$\lim_{\alpha \to 0^+} \nu_\alpha = \sqrt{2}$$

because the linearisation of (27) *around $w = 0$ seen as an operator from $W^{2,2}(\mathbb{R})$ to $L^2(\mathbb{R})$ is invertible when $\nu^2 < 2$ and therefore, for such a value of ν, there is a neighbourhood of* 0 *such that* (27) *has no solution.*

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