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On the Landau approximation in plasma physics

Sur l'approximation de Landau en physique des plasmas

R. Alexandre [a], C. Villani [b]

[a] *MAPMO, Université d'Orléans, B.P. 6759, 45067 Orléans cedex 2, France*
[b] *École normale supérieure de Lyon, UMPA, 46, allée d'Italie, 60364 Lyon cedex 07, France*

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Abstract

This paper studies the approximation of the Boltzmann equation by the Landau equation in a regime when grazing collisions prevail. While all previous results in the subject were limited to the spatially homogeneous case, here we manage to cover the general, space-dependent situation, assuming only basic physical estimates of finite mass, energy, entropy and entropy production. The proofs are based on the recent results and methods introduced previously in [R. Alexandre, C. Villani, Comm. Pure Appl. Math. 55 (1) (2002) 30–70] by both authors, and the entropy production smoothing effects established in [R. Alexandre et al., Arch. Rational Mech. Anal. 152 (4) (2000) 327–355]. We are able to treat realistic singularities of Coulomb type, and approximations of the Debye cut. However, our method only works for finite-time intervals, while the Landau equation is supposed to describe long-time corrections to the Vlasov–Poisson equation. If the mean-field interaction is neglected, then our results apply to physically relevant situations after a time rescaling.

Résumé

Nous étudions l'approximation de l'équation de Boltzmann par l'équation de Landau quand les collisions rasantes sont dominantes. Alors que tous les résultats connus auparavant en la matière concernaient le cas spatialement homogène, ici nous parvenons à couvrir le cas général, spatialement inhomogène, supposant seulement des estimations a priori physiquement réalistes portant sur la masse, l'énergie, l'entropie et la production d'entropie. Les preuves reposent sur les résultats et méthodes mis au point récemment par les auteurs dans [R. Alexandre, C. Villani, Comm. Pure Appl. Math. 55 (1) (2002) 30–70], et sur les effets de régularisation par production d'entropie établis dans [R. Alexandre et al., Arch. Rational Mech. Anal. 152 (4) (2000) 327–355]. Nos résultats couvrent certaines singularités physiquement réalistes de type coulombien, et des approximations de la coupure de Debye. Cependant, nos résultats s'appliquent sur un intervalle de temps fini, alors que l'équation de Landau est censée décrire les corrections en temps grand de l'équation de Vlasov–Poisson. Si le terme d'interaction de champ moyen est négligé, nos résultats s'appliquent à des situations physiquement réalistes après un changement d'échelle de temps.

E-mail addresses: alexandre.radja@labomath.univ-orleans.fr (R. Alexandre), cvillani@umpa.ens-lyon.fr (C. Villani).

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1. Introduction: binary collisions in plasmas

In 1936, Landau, as part of his important works in plasma physics, established the kinetic equation which is now called after him, modelling the behavior of a dilute plasma interacting through binary collisions. Since then, this equation has been widely in use in plasma physics, see for instance [5,8,10,20,27] and references therein. In this paper we shall present what we believe to be an important advance in the problem of rigorously justifying Landau's approximation. Before we describe the results, let us explain their physical context and motivation.

The unknown in Landau's equation is the time-dependent distribution function $f(t,x,v)$ of the plasma in the phase space (time t, position $x \in \mathbb{R}^3$, velocity $v \in \mathbb{R}^3$), and the Landau equation reads

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f + F(x) \cdot \nabla_v f = Q_L(f,f). \tag{1}$$

Here $F(x)$ is the self-consistent force created by the plasma,

$$2F = -\nabla V * \rho, \qquad V(x) = \frac{K}{4\pi|x|}, \qquad \rho(t,x) = \int_{\mathbb{R}^3} f(t,x,v)\,dv, \tag{2}$$

where K is a physical constant. Moreover, Q_L is the Landau collision operator, acting only on the velocity dependence of f,

$$Q_L(f,f) = \nabla_v \cdot \left(\int_{\mathbb{R}^3} dv_* a(v-v_*)\big[f_* \nabla_v f - f(\nabla_v f)_*\big]\right), \tag{3}$$

$$a_{ij}(z) = \frac{L}{|z|}\left[\delta_{ij} - \frac{z_i z_j}{|z|^2}\right]. \tag{4}$$

Here the notation $\nabla\cdot$ stands for the divergence operator. In the expression of the collision operator we have used the shorthand $f_* = f(v_*)$ and we have omitted the dependence of f on t and x, since these variables are only parameters in (3). This fact reflects the physical assumption that collisions are *localized*: particles which are not located at the same (mesoscopic) position interact only via the mean-field force F. Finally, for simplicity we have written the equation for a single species of particles, say electrons, while plasma phenomena usually involve at least two species (typically, ions and electrons). The values of the physical constants K and L in (2) and (4) will be discussed later on.

The novelty of Landau's equation resided in the collision operator $Q_L(f,f)$, which had been obtained as an approximation of the well-known Boltzmann collision operator,

$$Q_B(f,f) = \int_{\mathbb{R}^3} dv_* \int_{S^2} d\sigma\, B(v-v_*,\sigma)(f'f'_* - ff_*). \tag{5}$$

Here $f' = f(v')$ and so on (again, t and x are only parameters in (5)), and the formulae

$$\begin{cases} v' = \dfrac{v+v_*}{2} + \dfrac{|v-v_*|}{2}\sigma \qquad \big(\sigma \in S^2\big), \\[2mm] v'_* = \dfrac{v+v_*}{2} - \dfrac{|v-v_*|}{2}\sigma \end{cases} \tag{6}$$

parameterize the set of all solutions to the laws of elastic collision, namely $v' + v'_* = v + v_*$, $|v'|^2 + |v'_*|^2 = |v|^2 + |v_*|^2$. We shall think of (v, v_*) as the velocities of two typical particles before collision, and (v', v'_*) as their velocities after collision (actually we should do the reverse, but this has no importance).

The collision kernel $B(v-v_*,\sigma)$, which only depends on $|v-v_*|$ (modulus of the relative velocity) and $\langle (v-v_*)/|v-v_*|, \sigma\rangle$ (cosine of the deviation angle), contains all the necessary information about the interaction.

For a given interaction potential $\phi(r)$ (r is the distance between two interacting particles), this kernel can be computed implicitly via the solution of a classical scattering problem. In all the sequel, we shall use the notation θ for the deviation angle, i.e., the angle between $v - v_*$ and $v' - v'_*$, so that

$$\left\langle \frac{v - v_*}{|v - v_*|}, \sigma \right\rangle = \cos\theta.$$

We shall also abuse notations by recalling explicitly the dependence of B upon $|v - v_*|$ and $\cos\theta$:

$$B(v - v_*, \sigma) = B\big(|v - v_*|, \cos\theta\big).$$

Even if we take into account only elastic collisions, there are several types of electrostatic interactions in plasmas: Coulomb interaction between two charged particles, Van der Waals interaction between two neutral particles, or Maxwellian interaction between one neutral and one charged particle. Usually, interactions between charged particles are prevailing; moreover the mathematical analysis of the Boltzmann equation is much simpler for Van der Waals or Maxwellian interaction, than for Coulomb interaction. Therefore we restrict to this last case.

When the interaction between particles is governed by the Coulomb potential,

$$\phi(r) = \frac{e^2}{4\pi\varepsilon_0 r}, \tag{7}$$

then B is given by the well-known Rutherford formula,

$$B^C(v - v_*, \sigma) = \frac{(e^2/(4\pi\varepsilon_0 m))^2}{|v - v_*|^3 \sin^4(\theta/2)}. \tag{8}$$

In the above formulae, ε_0 is the permittivity of vacuum, m is the mass of the electron and e its charge.

Even though the Boltzmann collision operator is widely accepted as a model for describing binary interactions in dilute gases, it is meaningless for Coulomb interactions. The mathematical reason of this failure is that B^C is extremely singular as $\theta \to 0$. This singularity for zero deviation angle reflects the great abundance of *grazing collisions*, i.e. collisions in which interacting particles are hardly deviated. From the physical point of view, these collisions correspond to encounters between particles which are *microscopically very far apart*, and this abundance is a consequence of the long range of Coulomb interaction.

Since grazing collisions hardly have any effect, one may a priori not be convinced that they are a serious problem for handling the Boltzmann operator (5). In fact, the Boltzmann equation can be used only if the *mean transfer of momentum* between two colliding particles of velocities v, v_* is well-defined. One can compute that the typical amount of momentum which is communicated to a particle of velocity v by collisions with particles of velocity v_* is

$$\int_{S^2} B(v - v_*, \sigma)(v' - v)\, d\sigma = -\frac{|S^1|}{2} \left(\int_0^\pi B\big(|v - v_*|, \cos\theta\big)(1 - \cos\theta)\sin\theta\, d\theta \right)(v - v_*) \tag{9}$$

(of course $|S^1| = 2\pi$). In the case of the cross-section (8), the integral in the right-hand side of (9) does not converge since

$$\frac{\cos(\theta/2)(1 - \cos\theta)}{\sin^3(\theta/2)}\, d\theta \sim 4\frac{d\theta}{\theta}$$

defines a logarithmically divergent integral as $\theta \to 0$.

A physical consequence of this divergence is that when particles interact by Coulomb interaction, *grazing collisions are so frequent as to be the only ones to count*, in some sense: the mechanism of momentum transfer is dominated by small-angle deviations, and a given particle is extremely sensitive to the numerous particles which are very far apart. It is widely admitted, though not quite clear a priori, that these collective effects can still be described by binary collisions, because corresponding deflections are very small.

The resulting model is not tractable: the divergence of the integral (9) makes the Boltzmann operator (5) meaningless, as was certainly guessed by Landau, and recently checked from the mathematical point of view [32, Part I, Appendix 1].

On the other hand, physicists usually agree that the physical phenomenon of the *screening* tends to tame the Coulomb interaction at large distances, i.e. when particles are separated by distances much larger than the so-called Debye (or screening) length. The screening effect may be induced by the presence of two species of particles with opposite charges: typically, the presence of ions constitutes a background of positive charge which screens the interaction of electrons at large distances [5,10]. Some half-heuristic, half-rigorous arguments suggest to model the interaction between charged particles by the so-called Debye (or Yukawa) potential, $e^{-r/\lambda_D}/(4\pi\varepsilon_0 r)$, where λ_D is the Debye length, rather than by the "bare" Coulomb potential $1/(4\pi\varepsilon_0 r)$. For the Debye interaction the Boltzmann operator can in principle be used. However, it is not very interesting, because the corresponding collision kernel is horribly complicated (and not explicit), and because in most physical applications the Debye length is *very large* with respect to the characteristic length r_0 for collisions (Landau length), so that the potential is approximately Coulomb after all. Hence it is desirable to search for an approximation at very large values of λ_D.

Of course, in the limit $\lambda_D/r_0 \to \infty$, the Boltzmann operator Q_B diverges. But by heuristic physical arguments, Landau was able to show that in this limit, it is to leading order proportional to the operator Q_L. The proportionality factor is the so-called *Coulomb logarithm*: essentially, it is $\log(\lambda_D/r_0)$. We refer to [20] for Landau's original argument, to [5,10,28] for more physical background, and to [9] for a slightly more mathematically oriented presentation. Also a variant of Landau's argument can be found in [33].

While the derivation of the Boltzmann equation with screened interaction is still in need of a precise mathematical discussion, the approximation in which it reduces to the Landau equation has been the object of several mathematical works in the nineties [3,9,11,16,30]. Before discussing them briefly, we mention that this approximation procedure (called the Landau approximation in the sequel) is one of the main theoretical justifications for the Landau equation, but not the only one. As was observed by Balescu [5], the Landau equation can also be recovered as an approximation of the so-called Balescu–Lenard equation. This equation (called the ring equation in [5]) has been established independently in various forms by several authors in the sixties: Balescu [4], Bogoljubov [6], Lenard [21], Guernsey [17], and (in some particular cases) Rostoker and Rosenbluth [26]. As a good modern source for the Balescu–Lenard equation and its Landau approximation, we recommend the contribution of Decoster in [8].

Some physicists would recommend the Balescu–Lenard equation as a more reliable starting point for the derivation of the Landau equation. However, at present the mathematical theory of the Balescu–Lenard equation is exactly void: to the best of our knowledge, no mathematically oriented paper has ever discussed it. Its complexity is just frightening for a mathematician, and a discussion of it would first require a good understanding of the influence of the permittivity of a plasma on the collisional mechanism, via the so-called "dynamical screening".... For the moment, we shall be content with a rigorous derivation of the Landau equation as an approximation of the Boltzmann equation, and this will already turn out to be an extremely technical matter.

Let us review the existing mathematical literature on the Landau approximation. In his well-known treatise, Cercignani [7] had shown, without a precise mathematical formulation, that for a fixed, smooth f, the contribution of grazing collisions to $Q_B(f,f)$ can be modelled by a Landau-type operator. Degond and Lucquin [9], and Desvillettes [11] gave a more precise discussion. These works were only concerned with the stationary approximation: the problem discussed was to show that

$$Q_B(f,f) \simeq Q_L(f,f)$$

in a certain asymptotic procedure, for a fixed, smooth f depending only on the velocity variable. It is interesting to note [11] that these asymptotic procedures are not necessarily limited to a Coulomb interaction, but can be performed for a whole range of interaction parameters. But the limit in the Coulomb case is the most troublesome, because of the high singularity of the cross-section (8) in the relative velocity variable.

Then, emphasis was put on the much more delicate (and of course much more relevant) problem of justification at the level of solutions. More precisely, the following problem was investigated: *prove that solutions of the Boltzmann equation are well approximated, in certain asymptotics, by solutions of the Landau equation*. Under various assumptions, this problem was solved by Arsen'ev and Buryak [3], Goudon [16], Villani [30] in the framework of *spatially homogeneous solutions*, i.e. when the distribution function f is assumed to depend only on t, v and not on the position x. The assumption of spatial homogeneity is an enormous mathematical simplification, but even taking this into account, the singularity in the relative velocity variable was still a significant source of difficulties. It is only in [30] that the Landau approximation for Coulomb-type collision kernels was first handled. In the present paper, we shall considerably generalize these previous works, by solving the problem of the Landau approximation in the general, spatially inhomogeneous setting. However this will be achieved only for the model in which the mean-field term is absent. This restriction looks strange, because the mean-field term is usually not such a serious problem; but the point is that the natural time scale for this term is not the same as for the collision operator! To include the mean-field term in a physically satisfactory way, we should look at the Landau approximation like a *long time correction* to the Vlasov–Poisson equation, and our methods do not apply for this problem. More explanations about these subtleties will be given in the next section.

In all previous works, the true Debye cross-section was never considered. The second author mentioned in [30] an application to true Debye, but this is because he had been abused by an ambiguous physical reference; in fact he was treating the close approximation which is presented in the Appendix. In the present work we shall introduce a very general framework, which in principle makes it possible to cover the true Debye potential; this however requires to check a few technical assumptions about the precise form of this cross-section, a task which we did not find enough courage to accomplish, due to the extremely intricate nature of the Debye cross-section. Here we shall be content with the treatment of approximations of the Debye cut like the one described in the Appendix, or others presented in the sequel.

The present study rests on recent developments in the study of grazing collisions for the Boltzmann equation:

1) a precise understanding of regularizing effects associated with the entropy production, which were studied in our joint work [2] with Desvillettes and Wennberg (what we call entropy production here, is called entropy dissipation there; this is just a matter of conventions). In short, the mechanism of production of the entropy, combined with the fact that collisions tend to be grazing in the Landau approximation, prevents the distribution function from "wildly" oscillating in the Landau approximation. This stabilizing effect is a key ingredient for passing to the limit in the nonlinear collision operator;

2) a notion of weak solution for the Boltzmann equation with singular cross-section, which was introduced in our previous work [1]. There we showed how to give sense to a so-called renormalized formulation of the Boltzmann collision operator, allowing singularities in the cross-section, both in the angular and the relative velocity variables. The most important features of this formulation is that (a) the basic a priori estimates of finite mass and energy are sufficient to make sense of it, (b) contrary to all previous works, it does not rely on the finiteness of the total angular cross-section, but rather on the finiteness of the total angular cross-section for momentum transfer. The fact that it allows strong kinetic singularities will turn out to be important here.

The plan of the paper is as follows. In Section 2, we give a preliminary discussion of the Landau approximation, which, so we hope, will help the reader to understand its physical and mathematical content. Also the physical relevance of our contribution is discussed precisely. In Section 3, we state our main result, the proof of which is performed in the rest of the paper. Sections 4 and 5 are devoted to some preliminary mathematical considerations on the Boltzmann and Landau equations; in particular we shall recall those results of [1] which will be useful here. Then Section 6 is devoted to the discussion of the role of the entropy production in our theorem, and there we shall exploit some results established in [2]. Finally, in Section 7, we complete the proof of the Landau approximation. In the Appendix, we discuss a relevant approximation of the Coulomb potential that can be found in the physical literature.

2. Preliminary discussion

In this section, we recall some facts from plasma physics, and discuss the asymptotics leading from the Boltzmann to the Landau collision operator, taking into account the physical scales and the most obvious mathematical difficulties.

As already mentioned, the Rutherford collision kernel can be written

$$B(v-v_*,\sigma)=\left(\frac{e^2}{4\pi\varepsilon_0 m}\right)^2\frac{b^C(\cos\theta)}{|v-v_*|^3}, \tag{10}$$

where the angular kernel b^C is defined by

$$b^C(\cos\theta)\sin\theta=\frac{2\cos(\theta/2)}{\sin^3(\theta/2)}. \tag{11}$$

The Rutherford cross-section is just $B(v-v_*,\sigma)/|v-v_*|$. The factor $\sin\theta$ in (11) is proportional to the Jacobian of the integration in spherical coordinates, and should be kept in mind when discussing integrability properties of the kernel.

In a Boltzmann description, the natural scale of velocity is given by the thermal velocity,

$$v_{th}=\sqrt{\frac{k\Theta}{m}}, \tag{12}$$

where k is Boltzmann's constant, m the mass of the electron, and Θ the mean temperature of the plasma. Only this velocity scale will be considered in the sequel.

On the other hand, when the interaction is Coulomb, the natural length scale is the so-called *Landau length* r_0, which is the distance between two particles having an interaction energy of the same order as their kinetic energy:

$$r_0=\frac{e^2}{4\pi\varepsilon_0 k\Theta}. \tag{13}$$

Note that the dimensional constant in front of (10) is $(r_0 v_{th}^2)^2$.

The Rutherford cross-section (10) presents *two* singularities of very different nature:

– first, the kinetic collision kernel, $|v-v_*|^{-3}$, is singular as $v-v_*\to 0$. It is classical that kinetic singularities are well tractable from the mathematical point of view if they are *locally integrable*, see for instance the well-known paper of DiPerna and Lions [14]. Of course, this is not the case here, since $|z|^{-3}\notin L^1_{\mathrm{loc}}(\mathbb{R}^3,\,dz)$. This will be a considerable source of complications (but presumably, only of mathematical nature);

– next, the angular kernel, $b^C(\cos\theta)$, is so singular that

$$b^C(\cos\theta)(1-\cos\theta)\notin L^1(\sin\theta\,d\theta).$$

This second singularity reflects a true physical obstruction, and is directly linked to the divergence of the Boltzmann collision operator (too many grazing collisions!). It is this angular singularity which makes the Boltzmann operator meaningless for this kernel.

Since the Coulomb interaction cannot be handled, consider now a model in which particles interact via the Debye potential

$$\phi_D(r)=\frac{e^2}{4\pi\varepsilon_0 r}e^{-r/\lambda_D}.$$

Justifications for this screening assumption can be found in many physical textbooks on plasma physics, as [5,10]; from the mathematical point of view we are still far from understanding it. The screening will result in a less important role of collisions with a high impact parameter, therefore a less important role of grazing collisions. In a rough approximation, we can consider that it amounts to a sheer truncation of the Rutherford kernel, yielding a new kernel proportional to

$$\frac{b^C(\cos\theta)1_{\theta\geqslant\theta_D}}{|v-v_*|^3}, \tag{14}$$

where [10]

$$\sin\frac{\theta_D}{2} = \frac{r_0}{2\lambda_D}. \tag{15}$$

Let us use the notation

$$\Lambda = \frac{2\lambda_D}{r_0}, \tag{16}$$

which is customary in plasma physics, up to numerical conventions (ours is from [10, Eq. (3.111)]). When Λ is large but finite, the cross-section for momentum transfer is finite too:

$$M_\Lambda(v-v_*) = \frac{|S^1|}{|v-v_*|^3}\int_{\theta_D}^{\pi} \frac{2(\cos\theta/2)}{\sin^3(\theta/2)}(1-\cos\theta)\,d\theta = \frac{8|S^1|}{|v-v_*|^3}\log\Lambda.$$

And because this integral is finite, the corresponding Boltzmann equation is expected to make sense.[1] It can be written

$$\frac{\partial f}{\partial t} + v\cdot\nabla_x f = \big(r_0 v_{th}^2\big)^2 Q_\Lambda(f,f), \tag{17}$$

where Q_Λ is the Boltzmann collision operator with nondimensional cross-section given by (14), as an approximation to the true Debye cross-section. As Λ goes to infinity, this equation diverges, and therefore we shall look for new physical scales on which there is a meaningful limit equation. Note that for the moment, we neglect the effect of mean-field interaction in (17).

Thus, let us now change the scales of time, length and density, and consider as unknown the new distribution function $\tilde f$ in nondimensional variables,

$$\tilde f(t,x,v) = \frac{v_{th}^3}{N} f(Tt, Xx, v_{th}v), \tag{18}$$

where N is a typical density, T is a typical time and X is a typical length of the system under study. We have set the velocity scale to be v_{th}, and to be consistent we impose $X = v_{th}T$. Thanks to the density scaling, we may assume the mass of $\tilde f$ to be of order 1 without physical inconsistency (recall that the Boltzmann equation is established in a regime when the density is small enough that only binary collisions should be taken into account). The same holds true for the kinetic energy because of our choice of the velocity scale.

Plugging (18) into the Boltzmann equation (17), and denoting $\tilde f$ by f again, we arrive at the *rescaled Boltzmann equation*

$$\frac{\partial f}{\partial t} + v\cdot\nabla_x f = r_0^2 v_{th} N T\, Q_\Lambda(f,f).$$

In order to make the limit $\Lambda\to\infty$ meaningful, we consider a time scale such that (say)

$$T = \frac{1}{(\log\Lambda) r_0^2 v_{th} N}. \tag{19}$$

[1] The equation does make sense under known a priori estimates, but it is still not known whether the Cauchy problem admits solutions; see the discussion in [1, Section 5]. This problem is however an artifact due to a too wild cut-off; if the Rutherford kernel had been replaced by a less singular, but still singular kernel, then it would not arise.

With this system of units the rescaled Boltzmann equation now reads

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = \frac{Q_\Lambda(f,f)}{\log \Lambda}, \tag{20}$$

and the total angular cross-section for momentum transfer of the rescaled Boltzmann collision operator converges to a finite limit as $\Lambda \to \infty$, namely $8|S^1||v - v_*|^{-3}$.

Moreover, for any $\theta_0 > 0$, the contribution of deviation angles $\theta \geqslant \theta_0$ in this total cross-section goes to 0 as $\Lambda \to \infty$, because of the division by $\log \Lambda$. In this sense, *only grazing collisions have an influence in the limit.*

It is precisely the combination of these two points:

- the total angular cross-section for momentum transfer stays finite;
- only grazing collisions count in the limit,

which ensures that, in the limit $\Lambda \to \infty$, the Boltzmann collision operator reduces to the Landau collision operator. A general mathematical framework for this was introduced in [30]: a family $(b_n)_{n\in\mathbb{N}}$ of angular collision kernels was said to *concentrate on grazing collisions* if

$$\begin{cases} \left|S^1\right| \displaystyle\int_0^\pi b_n(\cos\theta)(1-\cos\theta)\sin\theta\,d\theta \xrightarrow[n\to\infty]{} \mu_\infty \in (0,+\infty) \\ \forall\theta_0 > 0, \quad \displaystyle\sup_{\theta\geqslant\theta_0} b_n(\cos\theta) \xrightarrow[n\to\infty]{} 0. \end{cases} \tag{21}$$

From this mathematical point of view (contrary to what is often believed), the scalings considered in [9] and in [11] are just the same, and correspond respectively to the two model cases below:

Case 1: $\int_0^\pi b(\cos\theta)(1-\cos\theta)\sin\theta\,d\theta = +\infty$. Then, define

$$b_n(\cos\theta) = \frac{b(\cos\theta)1_{\theta\geqslant n^{-1}}}{|S^1|\int_{1/n}^\pi b(\cos\theta)(1-\cos\theta)\sin\theta\,d\theta};$$

Case 2: $\int_0^\pi b(\cos\theta)(1-\cos\theta)\sin\theta\,d\theta < +\infty$. Then, with the notation $\zeta_n(\theta) = b(\cos\theta)\sin\theta$, define $b_n(\cos\theta)$ in such a way that

$$\zeta_n(\theta) = n^3\zeta(n\theta)$$

(by convention ζ vanishes for angles greater than π).

However, this definition makes sense only when the collision kernels factor into the product of a fixed kinetic kernel and a variable angular kernel. This has no physical basis; in the discussion above such a situation occurred only because we resorted to a crude approximation of the kernel for Debye potential, wildly truncating small-deviation angles. For the "true" kernel associated with a Debye approximation, this factorization property does not hold. Moreover, this "true" kernel is known only via implicit formulas, apparently never used by physicists! This provides strong motivation for introducing a very general definition of "concentration on grazing collisions", with a view to cover realistic situations.

In short, our main result on the Landau approximation can be stated as follows. Consider a family of solutions (f^n) to the Boltzmann equation with respective collision kernel B_n, where $(B_n)_{n\in\mathbb{N}}$ concentrates on grazing collisions as $n \to \infty$ (in a sense which is made clear in next section), and (f_n) satisfies the basic a priori estimates of finite mass, energy, entropy and entropy production. Then the sequence (f^n) converges strongly, up to extraction of a subsequence, to the solution of a certain Landau-type equation.

Before turning to precise statements, we would like to discuss the physical relevance of these results. The reader who would only care about mathematics may skip all the rest of this section.

For the Boltzmann equation to be relevant, it is usually required that

$$N r_0^3 \ll 1, \qquad N\lambda_D^3 \gg 1.$$

Taking into account the definition of Λ, this is satisfied if

$$\frac{r_0}{v_{th}(\log\Lambda)} \ll T \ll \frac{r_0}{v_{th}}\frac{\Lambda^3}{\log\Lambda}. \tag{22}$$

The dimensionless quantity

$$g = \frac{1}{N\lambda_D^3} \tag{23}$$

is called the *plasma parameter*. Classical plasmas are often defined as those in which g is very small.

In the classical theory of plasmas, the Debye length can be computed in terms of the mean density N and the Landau length r_0:

$$\lambda_D = \sqrt{\frac{\varepsilon_0 k\Theta}{Ne^2}} = \frac{1}{\sqrt{4\pi N r_0}}, \tag{24}$$

and as a consequence

$$\Lambda = \frac{1}{\sqrt{\pi N r_0^3}}.$$

If we use the law (24), we find from (19)

$$T = \frac{r_0}{v_{th}\log\Lambda}\left(\frac{1}{N r_0^3}\right) = \frac{\pi r_0 \Lambda^2}{v_{th}\log\Lambda},$$

and the validity of our limit is ensured if $1 \ll \Lambda^2 \ll \Lambda^3$, which is of course consistent with the asymptotics $\Lambda \to \infty$. We note that in plasma physics, Λ ranges from 10^2 to 10^{30}, so that Λ is actually very large, but its logarithm is not so large.

However, it would be dishonest to claim that our results are fully satisfactory. In the above discussion we have assumed that the interaction between particles can be modelled by binary collisions. The resulting Boltzmann and Landau equations, as we wrote them down, can be encountered in many physical textbooks (e.g. [5,10]). However, most physicists would agree that a more precise description of a plasma is obtained when one also takes into account collective effects modelled by a mean-field self-consistent force term of Poisson type, as the one appearing in (1).

It would not be difficult to add such a term at the level of (20), and treat the Landau approximation for the model

$$\frac{\partial f}{\partial t} + v\cdot\nabla_x f + F\cdot\nabla_v f = \frac{Q_\Lambda(f,f)}{\log\Lambda}, \tag{25}$$

where

$$F(x) = -\nabla V(x), \qquad V(x) = \frac{1}{4\pi|x|} * \rho, \qquad \rho(t,x) = \int_{\mathbb{R}^3} f(t,x,v)\,dv. \tag{26}$$

Then, the self-consistent coupling can be handled in exactly the same way as in Lions [22,23], and our main result would apply. But this mathematical problem would not be consistent with physical scales.... Actually, writing down physical constants explicitly, the Boltzmann equation with a mean field term should be

$$\frac{\partial f}{\partial t} + v\cdot\nabla_x f + 4\pi\big(r_0 v_{th}^2\big)F\cdot\nabla_v f = \big(r_0 v_{th}^2\big)^2 Q_\Lambda(f,f),$$

with F defined as in (26), or

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f + 4\pi \big(r_0 v_{th}^2\big) F_\Lambda \cdot \nabla_v f = \big(r_0 v_{th}^2\big)^2 Q_\Lambda(f,f),$$

with F_Λ defined as F but with a Debye potential. It is not clear to the authors which equation is the right one, and neither has received a convincing mathematical derivation (they should be very similar as $\Lambda \to \infty$). These questions are certainly outstanding problems for future research in kinetic theory.

Then, going back to (18), with the notations of (26) we obtain the following rescaled Boltzmann equation:

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f + 4\pi r_0 v_{th}^2 N T^2 F_{(\Lambda)} \cdot \nabla_v f = r_0^2 v_{th} N T\, Q_\Lambda(f,f). \tag{27}$$

Up to numerical constants, the quantity

$$\omega = \sqrt{4\pi v_{th}^2 r_0 N} \tag{28}$$

is called the *plasma oscillation frequency*. It is believed to measure the inverse time scale for oscillations due to the Poisson coupling, and plays a major role in plasma physics.

If we now wish to consider physical scales N, T on which the relevant equation is the Landau equation with a mean-field term, say

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f + F_{(\Lambda)} \cdot \nabla_v f = Q_L(f,f) \quad \left(\simeq \frac{Q_\Lambda(f,f)}{\log \Lambda}\right),$$

we have to impose

$$T = \omega^{-1} = \frac{1}{4\pi \sqrt{v_{th}^2 r_0 N}}$$

Identification of this formula with (19) implies that $\log \Lambda = \sqrt{4\pi}(N r_0^3)^{-1/2}$, or

$$\frac{2(\log \Lambda)^2}{\pi \Lambda^3} = g. \tag{29}$$

Under this assumption (which implies that the typical length for oscillations is much smaller than the Debye length) we are able to recover the “full” Landau equation (with a mean-field term) from the Boltzmann equation. Unfortunately, Eq. (29) is *not* satisfied in the classical theory of plasmas, since it is incompatible with (24). Instead, one should have

$$\lambda_D = \omega^{-1} v_{th}, \qquad \Lambda = \frac{8\pi}{g}. \tag{30}$$

In particular, up to numerical constants, *the typical length for oscillations should coincide with the Debye length.* We refer to [10,8] for a more precise discussion, and much more on these scale problems.

The physical content of this obstruction is the following: strictly speaking, in the classical theory of plasmas, the Landau equation with a mean-field term,

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f + F \cdot \nabla_v f = Q_L(f,f) \tag{31}$$

is relevant on *no* physical scale! Indeed, as the parameter Λ goes to infinity, or equivalently as $g \to 0$, the Boltzmann equation with a mean-field term should converge to the “pure” (collisionless) Vlasov–Poisson equation, and the effect of collisions should only be felt as *large-time corrections* to the Vlasov–Poisson equation.

Eq. (31) is however of great importance in physics, and it would be stupid to dismiss it. Let us try to sketch what could be a mathematical justification of the Landau approximation when a mean-field term is present and when the

Debye length satisfies (24), (30). There are two natural candidate statements; for each of them we are aware of no mathematical discussion, even formal.

1) *First possibility*: *adopt the time scale for the Landau equation*,

$$T = \frac{2\pi\Lambda}{\log\Lambda}\omega^{-1}.$$

Then the rescaled Boltzmann equation is

$$\frac{\partial f}{\partial t} + v\cdot\nabla_x f + \left(\frac{2\pi\Lambda}{\log\Lambda}\right)^2 F_{(\Lambda)}\cdot\nabla_v f = \frac{Q_\Lambda(f,f)}{\log\Lambda}.$$

Problem: prove that on a fixed time interval, as $\Lambda\to\infty$, solutions to this equation are close to solutions of

$$\frac{\partial f}{\partial t} + v\cdot\nabla_x f + \left(\frac{2\pi\Lambda}{\log\Lambda}\right)^2 F_{(\Lambda)}\cdot\nabla_v f = Q_L(f,f).$$

It is not a priori clear if such a statement has a chance to hold true. As $\Lambda\to\infty$, the very large mean-field term is expected to induce very fast oscillations, and the strong compactification effects induced by the entropy production mechanism will be lost. Passing to the limit in the colllision operator when such oscillations are present seems a desperate task, and apparently the only hope would be to prove that solutions to both equations are wildly oscillating in exactly the same way, but asymptotically close to each other in strong sense. Moreover, this problem should be replaced in the context of a quasi-neutral limit, with a subsequent increase in complexity.

2) *Second possibility*: *adopt the time scale for the Vlasov–Poisson equation*,

$$T = \omega^{-1}.$$

Note that this is consistent with (22) since

$$1 \ll \frac{\Lambda}{\log\Lambda} \ll \Lambda^3.$$

Then the rescaled Boltzmann equation is

$$\frac{\partial f}{\partial t} + v\cdot\nabla_x f + F_{(\Lambda)}\cdot\nabla_v f = \frac{1}{2\pi\Lambda}Q_\Lambda(f,f) = \frac{\log\Lambda}{2\pi\Lambda}\frac{Q_\Lambda(f,f)}{\log\Lambda}. \tag{32}$$

Problem: prove that, as $\Lambda\to\infty$, *on a large time interval of size* $\mathrm{O}(\Lambda/\log\Lambda)$, solutions of this equation are close to solutions of

$$\frac{\partial f}{\partial t} + v\cdot\nabla_x f + F\cdot\nabla_v f = \frac{\log\Lambda}{2\pi\Lambda}Q_L(f,f) \tag{33}$$

(compare with [9]). Note that on any fixed time interval, solutions of both systems converge to solutions of the collisionless Vlasov–Poisson equation (because $\log\Lambda/\Lambda\to 0$), so that this problem can only be expressed in terms of long-time corrections. Note also that formally, $F_\Lambda - F = \mathrm{O}(1/\Lambda)$, so that the error due to the screening at the level of the mean field should be at most $\mathrm{O}(1/\log\Lambda)$. Finally note that the coefficient in front of the dissipative collision operator is vanishing as $\Lambda\to\infty$, which suggests a considerable weakening of the regularizing properties associated with collisions.

It is very difficult to imagine how the techniques once developed by DiPerna and Lions [14] for the Boltzmann equation, generalized in [1] and extended for the needs of the present paper, may handle such a statement. Indeed, they are mainly based on compactness arguments and convergence of approximate solutions; but, as we just discussed, we should be looking for an asymptotic result, not convergence. A difficulty of the same kind is encountered when trying to retrieve the compressible Navier–Stokes equations from the Boltzmann equation. Progress on the Landau approximation from this point of view will certainly require the development of completely new estimates. It seems likely that the recent theorems proven by Guo [18,19] are a plausible starting point for a complete treatment of the close-to-equilibrium setting.

3. Main result

This section is devoted to definitions and a precise statement of our main result. In order to make the mathematical features of the Landau approximation more transparent, we shall consider general Boltzmann operators. We set the problem on $[0,T]\times\mathbb{R}^N_x\times\mathbb{R}^N_v$: here $[0,T]$ is an arbitrary time interval fixed once for all, and both the position and the velocity take values in $\mathbb{R}^N$, $N\geqslant 2$. Thus the equations under consideration will be of the form

$$\frac{\partial f}{\partial t}+v\cdot\nabla_x f=Q(f,f), \tag{34}$$

where Q is either a general Boltzmann-type operator,

$$Q_B(f,f)=\int_{\mathbb{R}^N} dv_* \int_{S^{N-1}} d\sigma\, B(v-v_*,\sigma)(f'f'_*-ff_*), \tag{35}$$

with the notations (6) still in use, or a general Landau-type operator,

$$Q_L(f,f)=\nabla_v\biggl(\int_{\mathbb{R}^N} dv_* a(v-v_*)[f_*\nabla_v f-f(\nabla_v f)_*]\biggr). \tag{36}$$

In the Boltzmann case the collision kernel will satisfy the usual assumption of dependence upon the modulus of the relative velocity and on the cosine of the deviation angle, and we shall write freely

$$B(v-v_*,\sigma)=B\bigl(|v-v_*|,\cos\theta\bigr),\qquad \cos\theta=\biggl\langle\frac{v-v_*}{|v-v_*|},\sigma\biggr\rangle.$$

Moreover, we shall assume that θ *ranges only from* 0 *to* $\pi/2$. Not only is this sometimes considered as physically realistic, but it is actually always possible to reduce to this case upon replacing the kernel B by

$$\bigl[B(v-v_*,\sigma)+B(v-v_*,-\sigma)\bigr]1_{\cos\theta\geqslant 0}. \tag{37}$$

Indeed, the product $f'f'_*$ is invariant under this operation (from the physical point of view, this means that due to the undiscernability of particles, one may freely rename particles after collision). Finally, we shall denote by k the unit vector parallel to $v-v_*$, so that

$$k=\frac{v-v_*}{|v-v_*|},\qquad \cos\theta=k\cdot\sigma.$$

More precisions will be given later on the assumptions that B has to satisfy for us to be able to handle the corresponding Boltzmann equation.

On the other hand, in the Landau case, the matrix-valued function $a(z)$ will be of the form

$$a(z)=\Psi\bigl(|z|\bigr)\Pi(z),\qquad \Pi_{ij}(z)=\delta_{ij}-\frac{z_iz_j}{|z|^2},$$

where Ψ is a nonnegative measurable function. Of course $\Pi(z)$ is the orthogonal projection upon $z^\perp$.

In our previous study of the Boltzmann equation [1], we found that two mathematical objects play a central role for the Boltzmann equation:

1) *The cross-section for momentum transfer* (at a given relative velocity),

$$M\bigl(|v-v_*|\bigr)=\int_{S^{N-1}} B(v-v_*,\sigma)(1-k\cdot\sigma)\,d\sigma \tag{38}$$

$$=\bigl|S^{N-2}\bigr|\int_0^{\pi/2} B\bigl(|v-v_*|,\cos\theta\bigr)(1-\cos\theta)\sin^{N-2}\theta\,d\theta. \tag{39}$$

We shall always think of M as a function of $v - v_*$, but we abuse notations by writing $M(|v - v_*|)$ to recall that it is a radially symmetric function.

2) *The compensated integral kernel for* $Q_B(\cdot, 1)$,

$$S\big(|v - v_*|\big) = \big|S^{N-2}\big| \int_0^{\pi/2} \left[\frac{1}{\cos^N(\theta/2)} B\left(\frac{|v - v_*|}{\cos(\theta/2)}, \cos\theta\right) - B\big(|v - v_*|, \cos\theta\big)\right] \sin^{N-2}\theta \, d\theta. \tag{40}$$

Here

$$Q_B(g, f) = \int_{\mathbb{R}^N \times S^{N-1}} B(v - v_*, \sigma)(g'_* f' - g_* f) \, dv_* \, d\sigma$$

stands for the bilinear Boltzmann operator. The kernel S is linked to the bilinear operator by the relation that we established in [1]: if $f(v)$ is a distribution function, then

$$\mathcal{S} f \equiv S * f = Q_B(f, 1) = \int_{\mathbb{R}^N \times S^{N-1}} B(v - v_*, \sigma)(f'_* - f_*) \, dv_* \, d\sigma \equiv \mathcal{S} f. \tag{41}$$

Remark. By the change of variable $z \to z/\cos(\theta/2)$,

$$\int_A S(|z|) \, dz = \int_0^{\pi/2} d\theta \sin^{N-2}\theta \left[\int_{A/\cos(\theta/2)} B\big(|z|, \cos\theta\big) \, dz - \int_A B\big(|z|, \cos\theta\big) \, dz\right].$$

This is an alternative way of defining S as a measure (note that the integral of S on a domain which is starshaped with respect to the origin is always nonnegative).

It was shown in [1,2] that many properties of the Boltzmann collision operator were governed by M and S. Our new definition of the Landau asymptotics is based on these two objects.

Definition 1. Let $(B_n)_{n \in \mathbb{N}}$ be a sequence of admissible collision kernels for the Boltzmann equation, and let the corresponding kernels M_n, S_n be defined as in formulas (38), (40). We say that (B_n) concentrates on grazing collisions if

(i) $S_n(|z|)$, $|z| M_n(|z|)$ define sequences of measures which are bounded in total variation on compact sets, uniformly in n;

(ii) there exists a nonnegative, radially symmetric measurable function M_∞ such that

$$z M_n\big(|z|\big) \xrightarrow[n \to \infty]{} z M_\infty\big(|z|\big) \quad \text{locally weakly in the sense of measures;}$$

(iii) for all $\theta_0 > 0$,

$$S_n^{\theta_0}\big(|z|\big), |z| M_n^{\theta_0}\big(|z|\big) \xrightarrow[n \to \infty]{} 0 \quad \text{locally weakly in the sense of measures,}$$

where $S_n^{\theta_0}$, $M_n^{\theta_0}$ are associated to the truncated cross-sections $B_n^{\theta_0} = B_n 1_{\theta \geqslant \theta_0}$ via formulas similar to (38), (40).

Remarks.

(1) The class of admissible cross-sections for the Boltzmann equation will be defined in Section 4. Essentially, admissible cross-sections are those such that M and S define meaningful mathematical objects.

(2) Assumption (ii) is formally equivalent to the convergence of M_n to M_∞, but our formulation is less sensitive to the behavior of M_n at the origin. This is important because of the nonintegrable singularity of the Rutherford collision kernel.
(3) Condition (iii) means that only grazing collisions count in the limit.

Example. In view of Proposition 5 in Section 4, the family

$$B_n(z,\sigma) = \Phi_n\big(|z|\big)b_n(\cos\theta) + \frac{\beta_n(\cos\theta)}{|z|^N}$$

concentrates on grazing collisions as soon as $\Phi_n \to \Phi$ in $L^1_{\mathrm{loc}}(\mathbb{R}^N)$,

$$\sup_{1<\lambda\leqslant\sqrt{2}} \frac{\Phi_n(\lambda|z|) - \Phi_n(|z|)}{\lambda - 1} \in L^1_{\mathrm{loc}}\big(\mathbb{R}^N_z\big), \quad \text{uniformly in } n,$$

and b_n, β_n concentrate on grazing collisions in the sense of (21).

Actually we shall also be led to introduce a few additional technical conditions. The first one is a decay condition at infinity (this is to control large velocities). The second one ensures that "on the whole", the sequence of cross-sections is truly long-range (this is to ensure the immediate damping of oscillations via the entropy production). The third one is a smoothness technical condition used in the passage to the limit.

Condition of decay at infinity.

$$M_n\big(|z|\big) = \mathrm{o}(1) \quad \text{as } |z| \to \infty, \text{ uniformly in } n. \tag{42}$$

Overall singularity condition. We require that

$$B_n(z,\sigma) \geqslant \Phi_0(|z|)b_{0,n}(k\cdot\sigma), \tag{43}$$

where $\Phi_0(|z|)$ is a continuous nonnegative function, nonvanishing for $|z| \neq 0$, and

$$\big|S^{N-2}\big| \int_0^{\pi/2} b_{0,n}(\cos\theta)(1-\cos\theta)\sin^{N-2}\theta\, d\theta \xrightarrow[n\to\infty]{} \mu > 0. \tag{44}$$

Smoothness of the approximation out of the origin. We require that

$$\frac{1}{\sqrt{M_n}} \xrightarrow[n\to\infty]{} \frac{1}{\sqrt{M_\infty}} \tag{45}$$

locally uniformly on $\mathbb{R}^N \setminus \{0\}$.

In realistic cases, this assumption is always satisfied, because M_n is uniformly smooth away from the origin. Certainly this technical condition could be significantly relaxed, but we do not see any real motivation for this. The overall singularity condition could be relaxed by only requiring (43) and (44) to hold for $|z| \in (r,R)$, where r and R are arbitrary positive real numbers, and $b_{0,n}$ could be allowed to depend on r and R.

The three conditions above will be sufficient to prove the Landau approximation. Thanks to the generality of our definition, we shall be able to treat more complicated cases than the ones previously seen; for instance,

$$B_n\big(|v-v_*|,\cos\theta\big)\sin\theta = \frac{|v-v_*|\sin\theta}{(|v-v_*|^2\sin^2(\theta/2) + 1/n)^2}, \tag{46}$$

This family of collision kernels, used in quantum physics, is discussed in the Appendix.

Another kind of kernels that we should now be able to handle is the one associated with the true Debye cross-section. Even if all the tools seem there to perform this study, we were discouraged by the extremely tedious computations involved (although there does not seem to be any real conceptual difficulty). Therefore we did not check that the true Debye cross-section satisfies all our technical assumptions; for any interested reader we can provide the partial results that we obtained.

Before formulating a statement we have to make our notion of solutions precise. We shall use the weak notion introduced in [1] as a generalization of the well-known renormalized solutions of DiPerna and Lions [14]. Some motivations for this definition are given in the next section.

Definition 2. Let $f \in C(\mathbb{R}^+; \mathcal{D}'(\mathbb{R}^N_x \times \mathbb{R}^N_v))$ be a nonnegative function with finite mass and energy, in the sense

$$\sup_{t\geqslant 0} \int f(t,x,v)\big(1+|v|^2\big)\,dx\,dv < +\infty. \tag{47}$$

Let $\mathcal{B}$ be the set of all functions of the form $\beta(f) = f/(1+\delta f)$, $\delta > 0$. We say that f is a *renormalized solution* of Eq. (34) *with a defect measure* if

$$\begin{cases} \forall \beta \in \mathcal{B}, & \dfrac{\partial \beta(f)}{\partial t} + v\cdot\nabla_x \beta(f) \geqslant \beta'(f)Q(f,f), \\ \forall t \geqslant 0, & \displaystyle\int_{\mathbb{R}^{2N}} f(t,x,v)\,dx\,dv = \int_{\mathbb{R}^{2N}} f(0,x,v)\,dx\,dv. \end{cases} \tag{48}$$

In the rest of the paper, we shall simply say that f is a weak solution of Eq. (34).

Let us make some comments on this definition:

1. The class $\mathcal{B}$ can be considerably enlarged, but the interest of this extension is not clear. The important properties of the functions β in $\mathcal{B}$ are that they are strictly concave, vanish at the origin and that $\beta'(f)$ decreases at infinity faster than $1/(1+f)$.

2. If the inequality sign in the first line of (48) was replaced by an equality sign, then (formally, after use of the chain-rule formula) this equation would just be Eq. (34) multiplied by $\beta'(f)$. Thus definition 2 would coincide with the usual definition of renormalized solutions.

3. The mass-conservation condition in the second line of (48) ensures that any smooth weak solution has to satisfy the equality in the first line of (48). If not, f would have to lose some mass for positive times (to see this, multiply the equation by $1/\beta'(f)$ and integrate). Thus, this mass-conservation condition implies that *any smooth weak solution is a strong, classical solution*.

4. The precise meaning of $\beta'(f)Q(f,f)$, i.e., the *renormalized formulation* of the collision operator, will be given in Section 4 when Q is the Boltzmann collision operator, in Section 5 when Q is the Landau collision operator. The important point is that we shall be able to define it using only the physical assumptions of bounded mass and energy.

5. This notion may seem quite weak, and it is! These solutions cannot be considered as completely satisfactory answers to the Cauchy problem for Boltzmann or Landau equations. But in this work, we do not really want to construct well-behaved solutions to these equations: our goal is to prove that solutions to the Boltzmann equation converge to solutions of the Landau equation in a certain asymptotic regime, using only the basic physical estimates. The weakness of the concept of solutions is a price that we pay for such a generality. Note also that our main result is nontrivial even for smooth solutions. We however believe that developing a complete theory of smooth solutions is the way towards further progress in the field.

We are now ready to state our main theorem.

Theorem 3. *Let (f^n) be a sequence of weak solutions to the Boltzmann equation with respective collision kernel B_n, on $[0,T]\times\mathbb{R}^N_x\times\mathbb{R}^N_v$. Assume that (B_n) concentrates on grazing collisions, in the sense of Definition* 1, *and satisfies the conditions* (42), (43), (45). *Further assume that the sequence* (f^n) *satisfies the physical assumptions of finite mass, energy and entropy, together with a localization condition in the space variable*:

$$\sup_n \sup_{t\in[0,T]} \int_{\mathbb{R}_x^N\times\mathbb{R}_v^N} f^n(t,x,v)\big[1+|v|^2+|x|^2+\log f^n(t,x,v)\big]\,dx\,dv < +\infty, \tag{49}$$

as well as the assumption of finite entropy production, i.e.

$$\int_0^T dt \int_{\mathbb{R}_x^N} dx\, D_n\big(f^n(t,x,\cdot)\big) < +\infty, \tag{50}$$

where D_n *is the entropy production functional associated with the cross-section* B_n*, as defined in Section* 6. *Assume, without loss of generality, that for all* $p\in(1,+\infty)$, $f^n \to f$ *weakly in w-*$L^p([0,T];L^1(\mathbb{R}_x^N\times\mathbb{R}_v^N))$ *as* $n\to\infty$.

Then, f *is a weak solution of the Landau equation with collision kernel*

$$\Psi(|z|) = \frac{|z|^2 M_\infty(|z|)}{4(N-1)}. \tag{51}$$

Moreover, the convergence of f^n *to* f *is automatically strong.*

The proof of this theorem is performed in Sections 6 and 7. Before undertaking it, we recall some facts about the Boltzmann equation and about the Landau equation (just what we will need in the sequel).

4. Reminders from the theory of the Boltzmann equation

The lack of strong enough a priori estimates for the Boltzmann equation has hindered the development of the theory for a very long time. To overcome this problem, DiPerna and Lions [13,14] suggested to write the equation *in renormalized form*, i.e., as in Definition 2. The main idea is that the renormalized operator $\beta'(f)Q_B(f,f)$ is expected to be sublinear if $\beta'(f)\leqslant C/(1+f)$, and therefore should make sense under the assumptions of finite mass and energy. The problem of the *renormalized formulation of the collision operator* is to give a meaningful definition of $\beta'(f)Q_B(f,f)$ for all β in a large enough class $\mathcal{B}$ of nonlinearities, under the basic physical a priori estimates.

When the Boltzmann collision kernel is locally integrable, then it is quite easy to find a renormalized formulation [14]. But this formulation does not cover the case of angular singularities. Moreover, it is definitely not well-suited for the limit of the Landau approximation. A more general renormalized formulation, suitable for our purposes, was introduced in our recent work [1].

Definition 4 (Renormalized formulation). By convention,

$$\beta'(f)Q_B(f,f) \equiv (\mathcal{R}_1)+(\mathcal{R}_2)+(\mathcal{R}_3), \tag{52}$$

where

$$(\mathcal{R}_1) = \big[f\beta'(f)-\beta(f)\big] \int_{\mathbb{R}^N\times S^{N-1}} dv_*\,d\sigma\, B(f'_* - f_*), \tag{53}$$

$$(\mathcal{R}_2) = \int_{\mathbb{R}^N\times S^{N-1}} dv_*\,d\sigma\, B\big[f'_*\beta(f') - f_*\beta(f)\big] = Q_B\big(f,\beta(f)\big), \tag{54}$$

$$(\mathcal{R}_3) = -\int_{\mathbb{R}^N \times S^{N-1}} dv_*\, d\sigma\, B f'_* \Gamma(f, f'), \tag{55}$$
$$\Gamma(f, f') = \beta(f') - \beta(f) - \beta'(f)(f' - f).$$

In the case which we shall consider, $\beta(f) = f/(1 + \delta f)$, then

$$\Gamma(f, f') = \frac{\delta(f' - f)^2}{(1 + \delta f)^2 (1 + \delta f')}. \tag{56}$$

Let us examine successively each of the three terms $(\mathcal{R}_1)$, $(\mathcal{R}_2)$, $(\mathcal{R}_3)$.

1) First of all, by (41),

$$(\mathcal{R}_1) = \big[f\beta'(f) - \beta(f)\big]\mathcal{S} f,$$

where $\mathcal{S}$ is the linear convolution operator with kernel S defined in (40). This should be taken as a definition of $(\mathcal{R}_1)$.

2) As for the second term $(\mathcal{R}_2)$, it is defined by duality as follows,

$$\int (\mathcal{R}_2)\varphi(v)\, dv = \int_{\mathbb{R}^{2N} \times S^{N-1}} dv\, dv_*\, d\sigma\, B\big[f'_*\beta(f') - f_*\beta(f)\big]\varphi$$
$$= \int_{\mathbb{R}^{2N}} dv\, dv_*\, f_*\beta(f)\bigg[\int_{S^{N-1}} B(v - v_*, \sigma)(\varphi' - \varphi)\, d\sigma, \bigg].$$

For future use we introduce the linear adjoint Boltzmann operator: for given v_*, it is defined by

$$\mathcal{T} : \varphi \mapsto \int_{S^{N-1}} B(v - v_*, \sigma)(\varphi' - \varphi)\, d\sigma. \tag{57}$$

Equivalently, for a given $v_* \in \mathbb{R}^N$, $\mathcal{T}\varphi(v, v_*)$ is the adjoint of $f \mapsto Q(\delta_{v_*}, f)$. And, of course,

$$\int (\mathcal{R}_2)\varphi(v)\, dv = \int_{\mathbb{R}^{2N}} dv\, dv_*\, f_*\beta(f)\mathcal{T}\varphi(v). \tag{58}$$

3) Finally, the third term is *nonnegative*, and can be given a sense as a locally integrable function.

In [1], we gave some sufficient conditions on the collision kernel B for getting satisfactory estimates on $(\mathcal{R}_1), (\mathcal{R}_2), (\mathcal{R}_3)$. Assumption B.1 below ensures that B is not too singular in the angular variable (this means essentially that the cross-section for momentum transfer is finite, with an additional very slight regularity assumption), and also not too singular in the velocity variable (but borderline nonintegrable singularities are allowed). As for Assumption B.2 below, it controls the large-velocity behavior.

Assumption B.1 (At most borderline kinetic singularity). Assume that

$$B(z, \sigma) = \frac{\beta_0(k \cdot \sigma)}{|z|^N} + B_1(z, \sigma), \quad k = \frac{z}{|z|}, \tag{59}$$

for some nonnegative measurable functions β_0 and B_1, and define

$$\mu_0 = \int_{S^{N-1}} \beta_0(k \cdot \sigma)(1 - k \cdot \sigma)\, d\sigma, \tag{60}$$

$$M_1\big(|z|\big) = \int_{S^{N-1}} B_1(z,\sigma)(1-k\cdot\sigma)\,d\sigma, \tag{61}$$

$$M_1'\big(|z|\big) = \int_{S^{N-1}} B_1'(z,\sigma)(1-k\cdot\sigma)\,d\sigma, \tag{62}$$

where

$$B_1'(z,\sigma) = \sup_{1<\lambda\leqslant\sqrt{2}} \frac{|B_1(\lambda z,\sigma) - B_1(z,\sigma)|}{(\lambda-1)|z|}.$$

We require that

$$\mu_0 < +\infty, \quad \text{and} \quad M_1\big(|z|\big),\ |z|M_1'\big(|z|\big) \in L^1_{\mathrm{loc}}\big(\mathbb{R}^N\big).$$

Assumption B.2 (*Behavior at infinity*). As $|z| \to \infty$,

$$M\big(|z|\big) = \mathrm{o}(1), \qquad |z|M'\big(|z|\big) = \mathrm{o}\big(|z|^2\big). \tag{63}$$

Example. Consider the model case where

$$B(v-v_*,\sigma) = |v-v_*|^\gamma b(\cos\theta), \qquad \sin^{N-2}\theta\, b(\cos\theta) \sim K\theta^{-1-\nu},$$

$\nu > 0$, $K > 0$. Then Assumptions B.1 and B.2 allow $0 > \gamma \geqslant -N$, $0 \leqslant \nu < 2$.

Remarks.

(1) Assumption B.1 is just a “simple” sufficient condition for the kernels S and $|z|M(|z|)$ to be bounded, locally in the sense of measures. In fact this last property would suffice.
(2) Assumption B.2 could be relaxed to consider positive values of γ such that $\gamma + \nu < 2$, but when dealing with the Landau approximation this would not be interesting.

The following estimates were established in [1].

Proposition 5. *If B satisfies Assumptions* B.1 *and* B.2*, then*
(i) *S is bounded in the sense of measures, and more precisely,*

$$S\big(|z|\big) = \lambda\delta_0 + S_1\big(|z|\big), \tag{64}$$

where δ_0 is the Dirac mass at the origin,

$$\lambda = -\big|S^{N-2}\big|\big|S^{N-1}\big| \int_0^{\pi/2} \beta_0(\cos\theta)\log\cos(\theta/2)\sin^{N-2}\theta\,d\theta,$$

and S_1 is a locally integrable function,

$$\big|S_1\big(|z|\big)\big| \leqslant \frac{2^{(N-4)/2}}{\cos^2(\pi/8)}\big[NM_1\big(|z|\big) + |z|M_1'\big(|z|\big)\big].$$

(ii) *The linear operator $\mathcal{T}$ is bounded from $W^{2,\infty}$ to L^∞, in the sense*

$$\big|\mathcal{T}\varphi(v)\big| \leqslant \frac{1}{2}\|\varphi\|_{W^{2,\infty}}|v-v_*|\left(1+\frac{|v-v_*|}{2}\right)M\big(|v-v_*|\big).$$

Note that λ in (i) satisfies $\lambda \leqslant |S^{N-1}|/(4\cos^2(\pi/8))\mu_0$. As a consequence of this proposition, we obtained the following a priori control of $(\mathcal{R}_1)$, $(\mathcal{R}_2)$, $(\mathcal{R}_3)$:

Proposition 6. *Assume that the cross-section B satisfies Assumptions* B.1 *and* B.2*, and let f be a weak solution of the Boltzmann equation. Let $B_R(v) = \{v \in \mathbb{R}^N, |v| \leqslant R\}$. Then, for all $R > 0$, the following a priori estimates hold*:

(i) $(\mathcal{R}_1) \in L^\infty([0,T]; L^1(\mathbb{R}^N_x \times B_R(v)))$;
(ii) $(\mathcal{R}_2) \in L^\infty([0,T]; L^1(\mathbb{R}^N_x; W^{-2,1}(B_R(v))))$;
(iii) $(\mathcal{R}_3) \in L^1([0,T]; L^1(\mathbb{R}^N_x \times B_R(v)))$.

Even if we do not write it explicitly, all these estimates are quantitative and depend only on the estimates on S and $\mathcal{T}$, and on the a priori estimates of mass and energy. Therefore, they will be *uniform* in n when we perform the limit leading to the Landau approximation.

To conclude this section, we recall the existence results established in [14] and in [1]. They depend on whether the collision kernel is singular or not.

Definition 7. The collision kernel B is said to be *nonsingular* if $B(z,\sigma) \in L^1_{\mathrm{loc}}(\mathbb{R}^N \times S^{N-1})$. On the other hand, it is said to present an *angular singularity* if, for all $r, R > 0$ there exists a function $b_0(k\cdot\sigma)$ such that

$$r \leqslant |z| \leqslant R \quad \Longrightarrow \quad B(z,\sigma) \geqslant b_0(k\cdot\sigma),$$

where

$$\int_{S^{N-1}} b_0(k\cdot\sigma)\,d\sigma = +\infty.$$

Of course, for singularity to hold true, it is sufficient that $B(z,\sigma) \geqslant \Phi_0(|z|)b_0(k\cdot\sigma)$, where b_0 is as above, and Φ_0 is a nonnegative continuous function, $\Phi_0 > 0$ if $|z| \neq 0$.

Proposition 8. *Let B satisfy conditions* B.1 *and* B.2*. If in addition, either B is nonsingular, or presents an angular singularity, then for all initial datum f_0 satisfying*

$$\int_{\mathbb{R}^{2N}} f_0(x,v)\big[1 + |v|^2 + |x|^2 + \log f_0(x,v)\big]\,dx\,dv < +\infty,$$

there exists a weak solution of the Boltzmann equation with initial datum f_0.

We note that this proposition leaves open the case where B does not present an angular singularity, but is still nonintegrable (due to a kinetic singularity). Such cases are not supposed to be realistic, but they are often encountered in the physical literature as approximations of more realistic kernels: see Section 2.

5. Reminders from the theory of the Landau equation

Most of the considerations in the previous section easily adapt to the Landau equation. The renormalized formulation of the Landau collision operator was already given several years ago by Lions [24], and further studied in Villani [29]. Our presentation here differs only by details. Let $a = a_{ij}(z)$ be the matrix appearing in the definition of the Landau collision operator. We define

$$b = \nabla\cdot a, \qquad c = \nabla\cdot b,$$

where $\nabla\cdot$ denotes the divergence operator. More explicitly,

$$b_j = \sum_i \partial_i a_{ij}, \qquad c = \sum_{ij} \partial_{ij} a_{ij}.$$

Because the matrix a is of the form $a(z) = \Psi(|z|)\Pi(z)$, with Π a projection operator,

$$b(z) = -(N-1)\frac{z\Psi(|z|)}{|z|^2}, \qquad c(z) = -(N-1)\nabla\cdot\left[\frac{z\Psi(|z|)}{|z|^2}\right]. \tag{65}$$

We also define

$$\bar{a} = a * f, \qquad \bar{b} = b * f, \qquad \bar{c} = c * f,$$

and we immediately note that the Landau operator can be rewritten as

$$Q_L(f,f) = \nabla_v \cdot (\bar{a}\nabla_v f - \bar{b} f).$$

In order to define a renormalized formulation of the Landau equation, it is sufficient to make the following assumptions.

Assumption L.1 (Integrability). We require that $|b(z)| \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$, and $|c(z)|$ be a locally bounded measure.

Assumption L.2. $\Psi(|z|) = \mathrm{o}(|z|^2)$ as $|z| \to \infty$.

Remark. When $N \geqslant 3$ and $\Psi(|z|) = |z|^{-(N-2)}$, then

$$c(z) = -(N-1)\nabla\cdot\left(\frac{z}{|z|^N}\right) = -(N-1)\left|S^{N-1}\right|\delta_0(z).$$

Definition 9. Let f be a distribution function with finite mass and energy, satisfying the additional a priori estimate

$$\bar{a}\nabla_v\beta(f)\nabla_v\beta(f) \in L^1_{\mathrm{loc}}\big([0,T];\mathbb{R}^N_x \times \mathbb{R}^N_v\big). \tag{66}$$

Then, by convention, the renormalized Landau collision operator is given by

$$\begin{aligned}
\beta'(f)Q_L(f,f) = {} & -\bar{c}\big[f\beta'(f) - \beta(f)\big] && (\mathcal{R}^L_1)\\
& + \nabla_v\cdot\big[\nabla_v\cdot(\bar{a}\beta(f)) - 2\bar{b}\beta(f)\big] && (\mathcal{R}^L_2)\\
& - \frac{\beta''(f)}{\beta'(f)^2}\bar{a}\nabla_v\beta(f)\nabla\beta(f) && (\mathcal{R}^L_3).
\end{aligned} \tag{67}$$

Remarks. (1) Expressions $(\mathcal{R}^L_1)$ and $(\mathcal{R}^L_2)$ in (67) are well-defined since $\bar{a}, \bar{b}, \bar{c} \in L^1_{\mathrm{loc}}$. In view of assumption (66), the expression $(\mathcal{R}^L_3)$ is also well-defined as an almost everywhere finite function. Moreover, $(\mathcal{R}^L_3)$ is nonnegative since β is concave. For $\delta = 1$, we find $-\beta''(f)/\beta'(f)^2 = 2(1+f)$.

(2) Formally,

$$\nabla_v\cdot\big(\bar{a}\beta(f)\big) - 2\bar{b}\beta(f) = \bar{a}\nabla_v\beta(f) - \bar{b}\beta(f),$$

so that $(\mathcal{R}^L_2)$ can be rewritten as

$$Q_L\big(f,\beta(f)\big),$$

where Q_L is now the bilinear Landau operator, defined by duality:

$$\int_{\mathbb{R}^N} Q_L\big(f,\beta(f)\big)\,dv = \int_{\mathbb{R}^{2N}} f_*\beta(f)\mathcal{T}_L\varphi\,dv\,dv_*,$$

where

$$[\mathcal{T}_L\varphi](v,v_*) = 2b(v-v_*)\cdot\nabla\varphi(v) + a(v-v_*) : D^2\varphi(v). \tag{68}$$

Here we used the notation $A : B = \sum A_{ij}B_{ij}$. Thus there is an excellent analogy between $(\mathcal{R}_2)$ and $(\mathcal{R}_2^L)$. And since $(\mathcal{R}_1^L)$ has the form $-[f\beta'(f) - \beta(f)]\mathcal{S}f$, where $\mathcal{S}$ is a convolution operator, there is a complete analogy between (52) and (67).

(3) The exact analogues of estimates (i)–(iii) of Proposition 6 hold for the Landau equation under Assumptions L.1 and L.2. In particular, this includes the additional estimate (66) which is required in the definition of the renormalized formulation. Details (under somewhat different assumptions) can be found in Lions [24].

(4) One could be curious about a precise a priori definition of $\bar{a}\nabla_v\beta(f)\nabla_v\beta(f)$, for a function f which is only assumed to have finite mass and energy. A simple (and natural) way to define it is by

$$\liminf_{\varepsilon\to 0}\ \bar{a}_\varepsilon\nabla_v\beta(f)\nabla_v\beta(f),$$

where $\bar{a}_\varepsilon$ is defined as $\bar{a}$, except that $\Psi(|z|)$ is replaced by a cutoffed version $\Psi_\varepsilon(|z|) = \Psi(|z|)\chi_\varepsilon(|z|)$, with χ_ε smooth, identically vanishing for $|z| \leqslant \varepsilon$ and identically equal to 1 for $|z| \geqslant 2\varepsilon$ (this definition does not depend on the particular choice of χ_ε). Then we can say that $\bar{a}_\varepsilon\nabla_v\beta(f)\nabla_v\beta(f)$ lies in $L^1_{\mathrm{loc}}([0,T]\times\mathbb{R}^N_x\times\mathbb{R}^N_v)$ if

$$\sqrt{\Psi_\varepsilon(|v-v_*|)}\,\Pi(v-v_*)\sqrt{f_*}\,\nabla\beta(f) \in L^2\big([0,T]\times\mathbb{R}^N_{v_*}; L^2_{\mathrm{loc}}\big(\mathbb{R}^N_x\times\mathbb{R}^N_v\big)\big).$$

This last object always makes sense as a distribution since $\beta(f)\in L^\infty$, $\sqrt{f_*}\in L^2$, and $\nabla\cdot(\Pi\sqrt{\Psi_\varepsilon}) = \sqrt{\Psi_\varepsilon}(\nabla\cdot\Pi)\in L^2$ (because Ψ_ε vanishes near the origin). Note that the identity $\nabla\cdot(\Pi\sqrt{\Psi_\varepsilon}) = \sqrt{\Psi_\varepsilon}(\nabla\cdot\Pi)$ follows from $\Pi\nabla\sqrt{\Psi_\varepsilon} = 0$. This remark was already used in the end of [12] for a similar problem.

(5) In the proof of our main result, we shall show that Assumptions L.1 and L.2 are automatically satisfied for the limit Ψ, and that the a priori estimate (66) automatically holds for the limit distribution function.

6. Damping of oscillations via entropy production

Our main result asserts that the limit of a sequence of solutions of particular Boltzmann equations is itself the solution of a Landau equation. As is well-known, passing to the limit in such nonlinear equations is most of the time impossible if the sequence of solutions *oscillates* too much, in the sense that the convergence is only weak. A noticeable exception to this rule is the case of the Boltzmann equation with cutoff, for which one can pass to the limit using only weak compactness [14].

The point which we want to discuss in this section, and which is a crucial first step towards our main result, is that in the limit that we are investigating, *there are no such oscillations*.

If we consider a sequence (f^n) of solutions to a given Boltzmann equation, satisfying the usual a priori bounds and converging to a weak limit f, it is known from [1] (see [25] for preliminary results) that the convergence is automatically strong if the collision kernel is singular (and this is not true if the collision kernel is not singular [24]). The situation that we are investigating now is almost the same: even if the collision kernels that we consider are not necessarily singular, they become singular in the limit of grazing collisions, as ensured by the "overall singularity condition" (43). This is why *strong compactness will appear in the limit of grazing collisions*.

The basic ingredient towards the proof of this immediate damping of oscillations is the following proposition, extracted from [2].

Proposition 10 (Control of velocity oscillations). *Assume that* (B_n) *concentrates on grazing collisions, and satisfies the overall singularity condition* (43)*. Then, there exists a sequence* $\alpha(n) \to 0$*, such that*

$$\begin{cases} \displaystyle\int_0^{\alpha(n)} b_{0,n}(\cos\theta)(1-\cos\theta)\sin^{N-2}\theta\,d\theta \xrightarrow[n\to\infty]{} \mu > 0, \\ \displaystyle\int_{\alpha(n)}^{\pi/2} b_{0,n}(\cos\theta)\sin^{N-2}\theta\,d\theta \equiv \psi(n) \xrightarrow[n\to\infty]{} +\infty. \end{cases} \tag{69}$$

Let $\chi_R(v)$ *be a smooth cutoff function, identically* 1 *for* $|v| \leqslant R$ *and identically 0 for* $|v| \geqslant R+1$*, and* $f^n(v)$ *be a distribution function depending on the* v *variable. Let* $f_R^n = \chi_R f^n$ *be the localized distribution function, and let* $\mathcal{F}\sqrt{f_R^n}$ *be the Fourier transform of its square root. Then for all* $A > 0$*, and* n *large enough,*

$$\int_{|\xi|\geqslant A} \left|\mathcal{F}\sqrt{f_R^n}(\xi)\right|^2 d\xi \leqslant \frac{C(N,\chi_R,f^n)}{\min[\psi(n),\mu A^2]}\left[D_n\left(f^n\right) + \int_{\mathbb{R}_v^N} f^n(v)\left(1+|v|^2\right)dv\right]. \tag{70}$$

Moreover, the constant C *depends on* f^n *only via an upper and a lower bound on the density* $\int f^n\,dv$*, and upper bounds on the energy* $\int f^n|v|^2\,dv$ *and the entropy* $\int f^n \log f^n\,dv$*.*

Combining Proposition 10 with the a priori estimate (50), the renormalized formulation, velocity-averaging lemmas [15,22] and some work, one concludes to the statement in Theorem 3 that *the convergence is automatically strong*. The complete proof is not so short, but it is exactly similar to the one given in [1] for sequences of solutions to the Boltzmann equation; so we skip it and refer to this work for details.

We conclude this section by displaying two basic examples for the sequences $\alpha(n)$, $\psi(n)$ which control the oscillations in Proposition 10.

Examples. (1) Consider the case $\sin^{N-2}\theta b_n(\cos\theta) = \zeta_n(\theta)$, and let $\zeta_n(\theta) = n^3\zeta(n\theta)$, where ζ has compact support in $[0,\pi/2]$. This framework is equivalent to the one considered in Desvillettes [11]. Then if a is any point such that $\int_0^a \zeta$ and $\int_a^{\pi/2}\zeta$ are positive, we can let $\alpha(n) = an^{-1}$, and it follows that

$$\int_0^{\alpha(n)} \zeta_n(\theta)(1-\cos\theta)\,d\theta = \int_0^a n^2\left(1-\cos\frac{\theta}{n}\right)\zeta(\theta)\,d\theta \xrightarrow[n\to\infty]{} \frac{1}{2}\int_0^a \theta^2\zeta(\theta)\,d\theta.$$

On the other hand,

$$\int_{an^{-1}}^{\pi/2} \zeta_n(\theta)\,d\theta = n^2\int_a^{\pi/2}\zeta(\theta)\,d\theta \xrightarrow[n\to\infty]{} +\infty.$$

(2) For the approximate Debye potential discussed in the Appendix, Eq. (A.1) below, we write

$$B_n\left(|z|,\cos\theta\right) \geqslant \left(\frac{|z|}{4\max(|z|,1)^4}\right)\left(\frac{1}{\log n}\frac{1_{\theta\geqslant(\pi/2)n^{-1}}}{\sin^4(\theta/2)}\right),$$

and we choose $\alpha(n) = n^{-1/2}$, so that on one hand

$$\frac{1}{\log n}\int_{(\pi/2)n^{-1}}^{n^{-1/2}} \frac{(1-\cos\theta)}{\sin^4(\theta/2)}\sin\theta\,d\theta \sim \frac{8}{\log n}\int_{(\pi/2)n^{-1}}^{n^{-1/2}} \frac{d\theta}{\theta} \xrightarrow[n\to\infty]{} 4,$$

while on the other hand

$$\frac{1}{\log n}\int_{n^{-1/2}}^{\pi/2} \frac{\sin\theta\,d\theta}{\sin^4(\theta/2)} \sim \frac{n}{4\log n} \xrightarrow[n\to\infty]{} +\infty.$$

7. Proof of the Landau approximation

We begin with two important lemmas.

Lemma 11. *If (B_n) concentrates on grazing collisions, in the sense of Definition* 1 *in Section* 3, *then*

$$S_n\bigl(|z|\bigr) \xrightarrow[n\to\infty]{} \frac{1}{4}\nabla\cdot\bigl(zM_\infty\bigl(|z|\bigr)\bigr)$$

weakly in the sense of measures.

Remarks.

(1) Formally, $\nabla\cdot(zM_\infty(|z|)) = |z|M'_\infty(|z|) + NM_\infty(|z|)$, where M'_∞ denotes the (distributional) derivative of M_∞ on the real line.
(2) The limit coincides with the negative of the kernel c in (65) if $M_\infty(|z|) = 4(N-1)\Psi(|z|)/|z|^2$.
(3) As a particular case, if $B_n(|v-v_*|,\sigma) = b_n(k\cdot\sigma)/|z|^N$, with an total angular cross-section for momentum transfer $\mu_n \to \mu_\infty$, then $S_n = \lambda_n\delta_0$, where

$$\lambda_n \equiv -\bigl|S^{N-1}\bigr|\bigl|S^{N-2}\bigr|\int_0^{\frac{\pi}{2}} d\theta\,\sin^{N-2}\theta\, b_n(\cos\theta)\log\cos(\theta/2)\,d\theta \xrightarrow[n\to\infty]{} \frac{|S^{N-1}|}{4}\mu_\infty.$$

Proof of Lemma 11. From Definition 1, we know that, up to extraction of a subsequence, S_n converges to a signed measure, locally weakly in the sense of measures, and that, for all θ_0,

$$\int_{\theta_0}^{\pi/2}\left[\frac{1}{\cos^N(\theta/2)}B_n\left(\frac{|z|}{\cos(\theta/2)},\cos\theta\right) - B_n\bigl(|z|,\cos\theta\bigr)\right]\sin^{N-2}\theta\,d\theta \to 0$$

(weakly). Thus we may consider only the contribution of angles $\theta \leqslant \theta_0$, where θ_0 is small enough.

Next, let $\varphi(z)$ be a test-function with compact support. By the change of variables $z \to z/\cos(\theta/2)$,

$$\begin{aligned}
&\bigl|S^{N-2}\bigr|\int_{\mathbb{R}^N} dz\int_0^{\theta_0} d\theta\,\sin^{N-2}\theta\left[\frac{1}{\cos^N(\theta/2)}B_n\left(\frac{|z|}{\cos(\theta/2)},\cos\theta\right) - B_n\bigl(|z|,\cos\theta\bigr)\right]\varphi(z)\,dz\\
&\quad= \bigl|S^{N-2}\bigr|\int_{\mathbb{R}^N} dz\int_0^{\theta_0} d\theta\,\sin^{N-2}\theta\, B_n\bigl(|z|,\cos\theta\bigr)\bigl[\varphi\bigl(z\cos(\theta/2)\bigr) - \varphi(z)\bigr]. \qquad (71)
\end{aligned}$$

Since φ is smooth and compactly supported,

$$\varphi\big(z\cos(\theta/2)\big)-\varphi(z)=-\nabla\varphi(z)\cdot z\big(1-\cos(\theta/2)\big)+\mathrm{O}\big(|z|^2\theta^4\big).$$

The error term $O(|z|^2\theta^4)$ is negligible in the limit: indeed,

$$|z|^2\int_0^{\theta_0} B_n\big(|z|,\cos\theta\big)\theta^4\sin^{N-2}\theta\,d\theta=\mathrm{O}\big(\theta_0^2|z|^2\big)M_n(|z|)$$

converges to 0 locally in measure. Therefore, up to a small error $\mathrm{O}(\theta_0^2)$, the expression in (71) is well approximated by

$$-\frac{|S^{N-2}|}{4}\int_{\mathbb{R}^N} dz\left[\int_0^{\theta_0} d\theta\,\sin^{N-2}\theta\,B_n\big(|z|,\cos\theta\big)(1-\cos\theta)\right]z\cdot\nabla\varphi(z)\xrightarrow[n\to\infty]{}-\frac{1}{4}\int_{\mathbb{R}^N}\big[M_\infty\big(|z|\big)z\big]\cdot\nabla\varphi(z)\,dz.$$

This proves our claim. □

In the next lemma, we are interested in the behavior of the $\mathcal{T}_n$'s as defined in (57).

Lemma 12. *Let $\mathcal{T}_n$ be the operator associated with B_n as in formula* (57), *where (B_n) is a sequence of collision kernels concentrating on grazing collisions. Then,*

$$\mathcal{T}_n\to\mathcal{T}_\infty\quad\text{in distributional sense},$$

where

$$\mathcal{T}_\infty\varphi(v)=-\frac{1}{2}M_\infty\big(|v-v_*|\big)(v-v_*)\cdot\nabla\varphi(v)+\frac{M_\infty(|v-v_*|)|v-v_*|^2}{4(N-1)}\Pi(v-v_*):D^2\varphi(v).\tag{72}$$

Remarks.

(1) This property is underlying already existing proofs of the Landau approximation in a spatially homogeneous context. But the variants which had been used so far, were based on symmetrization with respect to v, v_*, which would be a bad idea in the present case.
(2) If we set $M_\infty(|z|)=4(N-1)\Psi(|z|)/|z|^2$, then the linear operator $\mathcal{T}_\infty$ coincides with $\mathcal{T}_L$ in formula (68).

Proof of Lemma 12. Let us recall from [1, proof of Proposition 4] that

$$\begin{aligned}\mathcal{T}_n\varphi=&-\int_{S^{N-1}} d\sigma\,B_n(v-v_*,\sigma)(1-k\cdot\sigma)\frac{v-v_*}{2}\cdot\nabla\varphi(v)\\&+\int_{S^{N-1}} d\sigma\,B_n(v-v_*,\sigma)|v-v_*|^2\int_0^1 ds(1-s)\left\langle D^2\varphi\big(v+s(v'-v)\big)\cdot\frac{v'-v}{|v-v_*|},\frac{v'-v}{|v-v_*|}\right\rangle.\end{aligned}\tag{73}$$

Here we have used the fact that $|v'-v|=|v-v_*|\sin(\theta/2)$.

Our goal is to show that $\mathcal{T}_n\varphi$ converges to $\mathcal{T}_\infty\varphi$ in $\mathcal{D}'(\mathbb{R}^N_v\times\mathbb{R}^N_{v_*})$; so we can assume that (v,v_*) stays within a bounded subset of $\mathbb{R}^N\times\mathbb{R}^N$. By using assumption (iii) of Definition 1 and estimate (ii) in Proposition 5, it is easy to show that the contribution of "large" deviation angles in (73) is negligible and therefore one can assume that $\theta\leqslant\theta_0$ where θ_0 is arbitrarily small.

We change variables to separate θ from the other coordinates, writing the spherical decomposition

$$\sigma = (\cos\theta, \sin\theta\phi), \quad \phi \in S^{N-2},$$

where the first component is the projection onto $\mathbb{R}^k$. We note that

$$\frac{v' - v}{|v - v_*|} = \frac{\sigma - k}{2}.$$

Since φ is smooth and compactly supported, we find that, up to an error $\varepsilon(\theta_0)$ which goes to 0 as $\theta_0 \to 0$, $\mathcal{T}_n\varphi$ is well approximated by

$$-M_n\big(|v - v_*|\big)\frac{v - v_*}{2} \cdot \nabla\varphi(v) + \int_0^{\pi/2} d\theta \sin^{N-2}\theta B_n\big(|v - v_*|, \cos\theta\big)|v - v_*|^2 \times \frac{1}{2}\left[\int_{S^{N-2}} \left\langle D^2\varphi(v) \cdot \frac{\sigma - k}{2}, \frac{\sigma - k}{2}\right\rangle d\phi\right].$$

We shall now study the behavior of the integral in square brackets. Let $(\lambda_{ij})_{1\leqslant i,j\leqslant N}$ be the components of the symmetric matrix $D^2\varphi(v)$, where the first component corresponds to the axis k. If we separate the first component from the other ones, the components of $\sigma - k$ are $((\cos\theta - 1), \sin\theta\phi)$. Therefore, in the term involving λ_{11} there is a factor $(\cos\theta - 1)^2$, of order 4 in θ, and this term disappears in the limit $n \to \infty$. By symmetry with respect to ϕ, the terms with λ_{jk}, $j \neq k$, also disappear. We are only left with the λ_{ii}, $i = 2, \ldots, N$, and they all appear with the same coefficient, which is

$$\frac{1}{4}\sin^2\theta \int_{S^{N-2}} (e \cdot \phi)^2 \, d\phi, \quad |e| = 1.$$

By a classical computation,

$$\int_{S^{N-2}} (e \cdot \phi)^2 \, d\phi = \big|S^{N-2}\big| \frac{\int_0^\pi \cos^2\alpha \sin^{N-3}\alpha \, d\alpha}{\int_0^\pi \sin^{N-3}\alpha \, d\alpha} = \frac{|S^{N-2}|}{N-1}.$$

On the whole, using $\sin^2\theta \simeq 2(1 - \cos\theta)$ as $\theta \to 0$, we find

$$\int_{S^{N-2}} d\phi \left\langle D^2\varphi(v) \cdot \frac{\sigma - k}{2}, \frac{\sigma - k}{2}\right\rangle \simeq \frac{|S^{N-2}|(1 - \cos\theta)}{2(N-1)} \Pi(v - v_*) : D^2\varphi(v).$$

Here, by "$\simeq$" we mean again "up to an error which goes to 0 as $\theta_0 \to 0$". The conclusion follows immediately. □

After these preparations, we finally turn to the proof of our main result.

Proof of Theorem 3. Let (f^n) be a sequence of weak solutions of the Boltzmann equation, satisfying the assumptions of Theorem 3. Without loss of generality we can assume that $f^n \to f$ weakly, and our goal is to pass to the limit in the renormalized equation satisfied by f^n. By the strong compactness discussed in Section 6, we have in fact strong convergence of f^n to f; therefore it is immediate that $(\partial_t + v \cdot \nabla_x)\beta(f^n) \to (\partial_t + v \cdot \nabla_x)\beta(f)$ in the sense of distributions. Next, combining Lemmas 11 and 12 above, the estimates recalled in Sections 4 and 5, the technical assumption (42) and the strong compactness again, it is an easy task to show that

$$(\mathcal{R}_1)^n \to (\mathcal{R}_1)^\infty, \quad (\mathcal{R}_2)^n \to (\mathcal{R}_2)^\infty \quad \text{in weak sense},$$

with obvious notations. The proofs follow the very same lines as in [1, Section 4] and we skip them to avoid repetition.

It remains to show that

$$(\mathcal{R}_3)^\infty \leqslant \liminf_{n\to\infty}(\mathcal{R}_3)^n. \tag{74}$$

This is considerably more difficult that the similar problem treated in [1] because we cannot rely on Fatou's lemma any more. All the rest of this section is devoted to the proof of (74), which will conclude the proof of Theorem 3. □

Without loss of generality, we only consider the case $\delta = 1$, i.e. $\beta(f) = f/(1+f)$. In view of (55) and (56),

$$(\mathcal{R}_3)^n = \int_{\mathbb{R}^N\times S^{N-1}} dv_*\,d\sigma\,(f^n)'_*(1+f^n)' B_n(v-v_*,\sigma)\big[\beta\big(f^n\big)' - \beta\big(f^n\big)\big]^2.$$

Our argument is somewhat lengthy and a little bit intricate, and we shall sometimes skip easy verifications. In a preliminary step, we establish a crucial a priori bound stemming out (once again) of the entropy production estimate.

Lemma 13 (Entropy production bounds).

$$\sup_{n\in\mathbb{N}} \int_0^T dt \int_{\mathbb{R}^N\times\mathbb{R}^{2N}\times S^{N-1}} dx\,dv\,dv_*\,d\sigma\,B_n(v-v_*,\sigma)\big[f^n_* + \big(f^n\big)'_*\big]\big[\beta\big(f^n\big) - \beta\big(f^n\big)'\big]^2 < +\infty.$$

Proof of Lemma 13. By the standard inequality $(x-y)\log(x/y) \geqslant 4(\sqrt{x}-\sqrt{y})^2$, we deduce from the entropy production bounds that

$$\sup_{n\in\mathbb{N}} \int dt\,dx\,dv\,dv_*\,d\sigma\,B_n\Big(\sqrt{\big(f^n\big)'\big(f^n\big)'_*} - \sqrt{f^n f^n_*}\Big)^2 < +\infty.$$

Let us introduce an increasing Lipschitz function P, to be precised later. Clearly,

$$\sup_{n\in\mathbb{N}} \int dt\,dx\,dv\,dv_*\,d\sigma\,B_n\Big(\sqrt{\big(f^n\big)'\big(f^n\big)'_*} - \sqrt{f^n f^n_*}\Big)^2 \left[\frac{P(f^n)' - P(f^n)}{(f^n)' - f^n}\right] < +\infty. \tag{75}$$

Just as in [31], we write

$$\sqrt{f' f'_*} - \sqrt{f f_*} = \frac{1}{2}\big(\sqrt{f'} + \sqrt{f}\,\big)\big(\sqrt{f'_*} - \sqrt{f_*}\,\big) + \frac{1}{2}\big(\sqrt{f'} - \sqrt{f}\,\big)\big(\sqrt{f'_*} + \sqrt{f_*}\,\big),$$

plug this inside (75) and expand the square, to find

$$\sup_n \left\{ \int B_n\Big[\sqrt{\big(f^n\big)'_*} + \sqrt{f^n_*}\,\Big]^2 \Big(\sqrt{f^n} - \sqrt{\big(f^n\big)'}\,\Big)^2 \frac{P(f^n)' - P(f^n)}{(f^n)' - f^n} + \int B_n\big[\big(f^n\big)'_* - f^n_*\big]\big[\big(f^n\big)' - f^n\big]\frac{P(f^n)' - P(f^n)}{(f^n)' - f^n} \right\} < +\infty \tag{76}$$

(there is also another nonnegative term, that we throw away). By pre-postcollisional change of variables and symmetry, the second integral can be rewritten as

$$\begin{aligned}\int B_n\big[\big(f^n\big)'_* - f^n_*\big]\big[P\big(f^n\big)' - P\big(f^n\big)\big] &= \int B_n P\big(f^n\big)\big[\big(f^n\big)'_* - f^n_*\big] \\ &= \int dt\,dx\,dv\,dv_* S_n\big(|v-v_*|\big) P\big(f^n\big) f^n_*,\end{aligned}$$

where we have used the Cancellation Lemma, formula (41). This integral is a priori bounded as soon as $P \in L^\infty$, since in that case $(S_n * P)(v_*) \leqslant C(1+|v_*|^2)$. Choosing $P(f) = \beta(f) = f/(1+f)$, from the preceding remark, formula (76) and the identity

$$\big(\sqrt{f'} - \sqrt{f}\big)^2 \frac{\beta(f') - \beta(f)}{f' - f} = \big[\beta(f') - \beta(f)\big]^2 \frac{(1+f)(1+f')}{(\sqrt{f} + \sqrt{f'})^2},$$

we get

$$\sup_n \int B_n \big[\sqrt{(f^n)'_*} + \sqrt{f^n_*}\,\big]^2 \big[\beta\big(f^n\big)' - \beta\big(f^n\big)\big]^2 \frac{(1+f^n)(1+f^n)'}{[\sqrt{f^n} + \sqrt{(f^n)'}]^2} < +\infty.$$

To conclude, it suffices to note that $(1+f)(1+f')/(\sqrt{f} + \sqrt{f'})^2 \geqslant 1/2$. $\square$

Remark. Lemma 13 will imply the estimate

$$\int_0^T dt \int_{\mathbb{R}^N \times \mathbb{R}^{2N}} dx\, dv\, dv_*\, \bar{a} \nabla_v \beta(f) \nabla_v \beta(f) < +\infty.$$

To see this, write $B_n(g'-g)^2 = B_n|v-v_*|^2 \sin^2(\theta/2)(g'-g)^2/|v'-v|^2$ (where $g = \beta(f)$) and pass to the limit as $n \to \infty$ in the same way as below. This remark shows that the estimate $\bar{a}\nabla_v\beta(f)\nabla_v\beta(f) \in L^1_{\mathrm{loc}}$, which is required in our definition of renormalized solutions, is satisfied. In fact, we could even remove the “loc”!

We now come back to the main argument.

Proof of (74)**.** In view of our a priori bounds, we only need to show that for all smooth nonnegative test-function $\varphi(t,x,v)$ with compact support in $[0,T] \times \mathbb{R}^N_x \times \mathbb{R}^N_v$,

$$\int (\mathcal{R}_3)^\infty \varphi^2\, dv\, dx\, dt \leqslant \liminf_{n\to\infty} \int (\mathcal{R}_3)^n \varphi^2\, dv\, dx\, dt. \tag{77}$$

We divide the proof into six steps.

Step 1. By the usual pre-postcollisional change of variables $(v, v_*, \sigma) \to (v', v'_*, k)$, which has unit Jacobian, we rewrite $\int (\mathcal{R}_3)^n \varphi^2$ as

$$\int (\mathcal{R}_3)^n \varphi^2\, dv\, dx\, dt = \int f^n_*\big(1+f^n\big) B_n\big(|v-v_*|, \cos\theta\big)\big[\beta\big(f^n\big)' - \beta\big(f^n\big)\big]^2 (\varphi')^2\, d\sigma\, dv\, dv_*\, dx\, dt.$$

Step 2. By monotonicity, we only need to prove the result when B_n is replaced by $\chi_\varepsilon(|v-v_*|) B_n(|v-v_*|, \cos\theta)$, where $\varepsilon > 0$ and $\chi_\varepsilon(|z|)$ is a smooth cutoff function, identically vanishing for $|z| \leqslant \varepsilon$, $|z| \geqslant \varepsilon^{-1}$. This truncation will save us from considerable trouble associated with small relative velocities.

Step 3. Let us introduce

$$\delta_n\big(|v-v_*|, \theta\big) = \frac{|S^{N-2}| \sin^{N-2}\theta\, B_n(|v-v_*|, \cos\theta)|v-v_*|^2(1-\cos\theta)}{M_n(|v-v_*|)|v-v_*|^2}.$$

By construction, $\int_0^{\pi/2} \delta_n(|v-v_*|, \theta)\, d\theta = 1$. We again introduce a spherical system of coordinates (θ, ϕ) for σ. By Jensen's inequality, $\int (\mathcal{R}_3)^n \varphi^2$ is greater than

$$\frac{1}{2} \int f^n_*\big(1+f^n\big) M_n\big(|v-v_*|\big)|v-v_*|^2 \left[\int_0^{\pi/2} \delta_n\big(|v-v_*|, \theta\big) \varphi' \frac{\beta(f^n)' - \beta(f^n)}{|v'-v|}\, d\theta\right]^2 d\phi\, dv\, dv_*\, dx\, dt$$

$$= \frac{1}{2}\int_0^T dt \int_{\mathbb{R}^N} dx \int_{\mathbb{R}^{2N}} dv\, dv_* \int_{S^{N-2}} d\phi\, (a)_n^2, \tag{78}$$

where

$$(a)_n \equiv \sqrt{f_*^n}\sqrt{1+f^n}\int_0^{\pi/2} d\theta \sin^{N-2}\theta \frac{|S^{N-2}|B_n(|v-v_*|,\cos\theta)|v-v_*|^2(1-\cos\theta)}{\sqrt{M_n(|v-v_*|)|v-v_*|^2}}\varphi'\left[\frac{\beta(f^n)'-\beta(f^n)}{|v'-v|}\right]$$

$$= \sqrt{f_*^n}\sqrt{1+f^n}\int_0^{\pi/2} |v-v_*|\eta_n\big(|v-v_*|,\theta\big)\varphi'\left[\frac{\beta(f^n)'-\beta(f^n)}{|v'-v|}\right] d\theta.$$

The factor $1/2$ in (78) comes from the identity

$$\frac{1}{|v-v_*|^2} = \frac{1}{2}\left(\frac{1-\cos\theta}{|v'-v|^2}\right),$$

and we have introduced the quantity

$$\eta_n\big(|v-v_*|,\cos\theta\big) = \big|S^{N-2}\big|\sin^{N-2}\theta\frac{B_n(|v-v_*|,\cos\theta)(1-\cos\theta)}{\sqrt{M_n(|v-v_*|)}}. \tag{79}$$

From the fact that (B_n) concentrates on grazing collisions, we deduce that $\eta_n\sqrt{M_n}$ converges to $M_\infty(|z|)\delta_{\theta=0}$, locally in the sense of measures, on $\mathbb{R}^N \times [0,\pi/2]$. Combining this with (45), we deduce that for all bounded open set $\Omega \subset \mathbb{R}^N$ with $0 \notin \overline{\Omega}$,

$$\eta_n\big(|z|,\cos\theta\big) \xrightarrow[n\to\infty]{} \sqrt{M_\infty(|z|)}\delta_{\theta=0} \tag{80}$$

weakly in the sense of measures on $\Omega \times [0,\pi/2]$.

To conclude the proof it is sufficient to show that

$$(a)_n \to (a)_\infty \tag{81}$$

in the sense of distributions, where

$$(a)_\infty = \sqrt{f_*}\sqrt{1+f}\sqrt{M_\infty(|v-v_*|)|v-v_*|^2}\varphi\nabla_v\beta\big(f^n\big)\cdot e_\phi.$$

Here $e_\phi = e_\phi(v,v_*)$ is a unit vector orthogonal to $k = (v-v_*)/|v-v_*|$, with coordinate ϕ in our system of spherical coordinates. The convergence of $(a)_n$ to $(a)_\infty$ should hold in $\mathcal{D}'([0,T]\times\mathbb{R}^N_x\times\mathbb{R}^N_v\times\mathbb{R}^N_{v_*}\times S^{N-2}_\phi)$ (as in the other weak limits considered below). If (81) holds true, then the proof of (77) follows by Jensen's inequality (convexity of the square function) in the form

$$\int (a)_\infty^2 \leqslant \liminf_{n\to\infty}\int (a)_n^2,$$

and the formula

$$\forall \ell\in\mathbb{R}^N, \qquad \int_{S^{N-2}} (\ell\cdot e_\phi)^2\, d\phi = \frac{|S^{N-2}|}{N-1}|\Pi\ell|^2$$

(cf. the proof of Lemma 12).

We immediately note a mathematical subtlety: it is not always possible to define e_ϕ in a smooth way on the sphere S^{N-1} (think that there is no smooth field of unit tangent vectors on spheres of even dimension). So convergence in $\mathcal{D}'(\mathbb{R}^N_v\times\mathbb{R}^N_{v_*}\times S^{N-2}_\phi)$ seems meaningless. But what we really wish to prove is a local property, that is $\int_A (a)_\infty^2 \leqslant \liminf \int_A (a)_n^2$ for all (bounded) open set A in $[0,T]\times\mathbb{R}^N_x\times\mathbb{R}^N_v\times\mathbb{R}^N_{v_*}\times S^{N-2}_\phi$; so we only have to prove $(a)_n \to (a)_\infty$ on "sufficiently small" open sets; the general case can then be obtained by using partitions of unity, etc. And since we have already cut out small values of $v - v_*$, we may assume that the projection of A onto $\mathbb{R}^N_v \times \mathbb{R}^N_{v_*}$ lies in a neighborhood on which one can define spherical coordinates with axis $(v - v_*)/|v - v_*|$, in a smooth way (with respect to v, v_*). From now on, we work in such an open subset A of $[0,T] \times \mathbb{R}^N_x \times \mathbb{R}^N_v \times \mathbb{R}^N_{v_*} \times S^{N-2}_\phi$, which will also be assumed to be smooth and bounded.

Also, we shall not really prove that $(a)_n \to (a)_\infty$, because the lack of smoothness of the square root function in $\sqrt{f^n_*}$ might cause technical problems. Instead, we shall replace $\sqrt{f^n_*}$ by $s_\alpha(f^n_*)$, where s_α is a smooth (Lipschitz) bounded approximation of the square root, $s_\alpha(f) \leqslant \sqrt{f}$, $s_\alpha(f) \to \sqrt{f}$ as $\alpha \to 0$, the convergence being monotone. Denoting by $\widetilde{(a)}_{n,\alpha}$ this modification of $(a)_n$, we shall prove (with obvious notations) that

$$\widetilde{(a)}_{n,\alpha} \xrightarrow[n\to\infty]{} \widetilde{(a)}_{\infty,\alpha} \quad \text{in the sense of distributions.} \tag{82}$$

This will entail that, for each $\alpha > 0$,

$$\int \widetilde{(\mathcal{R}_3)^\infty_\alpha}\varphi^2 \leqslant \liminf_{n\to\infty} \int \widetilde{(\mathcal{R}_3)^n_\alpha}\varphi^2 \leqslant \liminf_{n\to\infty} \int (\mathcal{R}_3)^n\varphi^2, \tag{83}$$

where $\widetilde{(\mathcal{R}_3)^\infty_\alpha}$, $\widetilde{(\mathcal{R}_3)^n_\alpha}$ are just the same as $(\mathcal{R}_3)^\infty$, $(\mathcal{R}_3)^n$, but with f_* replaced by $s_\alpha(f_*)^2$. Once (83) is established, if we let $\alpha \to 0$, Beppo Levi's monotone convergence theorem will imply the same inequality with $(\mathcal{R}_3)^\infty$ in place of $\widetilde{(\mathcal{R}_3)^\infty_\alpha}$, and the conclusion will follow.

In the sequel, we shall drop the subscript α and write just $s(f)$, $(a)_n$, etc.

Step 4. From the strong convergence of the sequence f^n we know that $\sqrt{1+f^n} \to \sqrt{1+f}$, strongly in $L^2_{\mathrm{loc}}([0,T] \times \mathbb{R}^N_x \times \mathbb{R}^N_v)$. Therefore, to prove (82) it is sufficient to prove the convergence of the distribution

$$(b)_n \equiv s\big(f^n_*\big) \int_0^{\pi/2} \eta_n\big(|v - v_*|, \theta\big)|v - v_*| \left[\varphi' \frac{\beta(f^n)' - \beta(f^n)}{|v' - v|}\right] d\theta, \tag{84}$$

in weak $L^2([0,T] \times \mathbb{R}^N_x \times \mathbb{R}^N_v \times \mathbb{R}^N_{v_*} \times S^{N-2}_\phi)$, or more rigorously in $L^2(A)$, where η_n is defined in (79), to

$$(b)_\infty = s(f_*)\sqrt{M_\infty(|v - v_*|)|v - v_*|^2}\varphi\nabla_v\beta(f) \cdot e_\phi.$$

From (80) and the boundedness of A, we have

$$\eta_n|v - v_*| \to \delta_{\theta=0} \otimes \sqrt{M_\infty(|v - v_*|)|v - v_*|^2}$$

weakly in the sense of measures. By Lemma 13 and Jensen's inequality, we can see that

$$\sup_n \big\|(b)_n\big\|_{L^2} < +\infty;$$

therefore we only have to check that

$$(b)_n \xrightarrow[n\to\infty]{} (b)_\infty \quad \text{in distributional sense.} \tag{85}$$

Step 5. At this point, we want to partly symmetrize the integrand in (84). This will be done with the help of the following auxiliary lemma:

Lemma 14. *If $\chi(t,x,v,v_*,\phi,\theta)$ is a smooth function supported in $A \times [0,\pi/2]$, and if F^n and G^n converge to F and G respectively, locally in $L^p([0,T] \times \mathbb{R}^N_x \times \mathbb{R}^N_v)$ and in $L^q([0,T] \times \mathbb{R}^N_x \times \mathbb{R}^N_v)$ respectively, with $(1/p) + (1/q) = 1$, then*

$$\int_0^{\pi/2} d\theta\, \eta_n\big(|v - v_*|, \theta\big)\chi(t,x,v,v_*,\phi,\theta)F^n(v')G^n(v'_*) \to \sqrt{M_\infty(|v - v_*|)}\chi(t,x,v,v_*,\phi,0)F(v)G(v_*) \tag{86}$$

in $\mathcal{D}'(A)$.

Proof of Lemma 14. Let ψ be a test-function in the variables (t,x,v,v_*,ϕ). We have

$$\int dt\,dx\,dv\,dv_*\,d\phi\,d\theta\,\eta_n\big(|v-v_*|,\theta\big)\psi\chi\big(F^n\big)'\big(G^n\big)'_*$$
$$=\int dt\,dx\,dv\,dv_*\,d\sigma\,\frac{\eta_n(|v-v_*|,\theta)}{|S^{N-2}|\sin^{N-2}\theta}\psi\chi\big(F^n\big)'\big(G^n\big)'_*$$

where σ can be seen as a function of v, v_*, θ and ϕ via our choice of spherical coordinates; and similarly ϕ can be seen as a function of v, v_* and σ: $\phi=\phi(v,v_*,\sigma)$. We can now perform a pre-postcollisional change of variables, and find that the integral above coincides with

$$\int dt\,dx\,dv\,dv_*\,d\sigma\,\frac{\eta_n(|v-v_*|,\theta)}{|S^{N-2}|\sin^{N-2}\theta}\psi(t,x,v',v'_*,\phi')\chi(v',v'_*,\phi',\theta)\big(F^n\big)\big(G^n\big)_*,$$

where $\phi'=\phi(v',v'_*,k)$ (the notation $k=(v-v_*)/|v-v_*|$ still in use). Of course this integral can be rewritten as

$$\int dt\,dx\,dv\,dv_*\,d\theta\,d\phi\,\eta_n\big(|v-v_*|,\theta\big)\psi(t,x,v',v'_*,\phi')\chi(v',v'_*,\phi',\theta)\big(F^n\big)\big(G^n\big)_*.$$

As $\theta\to 0$, we have $v'\to v$, $v'_*\to v_*$, $\sigma\to k$ and $\phi'\to\phi$; and all of this is uniform on A. In particular,

$$\int d\theta\,d\phi\,\eta_n\big(|v-v_*|,\theta\big)\psi(t,x,v',v'_*,\phi')\chi(v',v'_*,\phi,\theta)$$
$$\to\sqrt{M_\infty(|v-v_*|)}\,\psi(t,x,v,v_*,\phi)\chi(v,v_*,\phi,0),$$

at least in weak sense. Combining this with the strong convergence of $F^n(t,x,v)G^n(t,x,v_*)$ to $F(t,x,v)G(t,x,v_*)$, we easily conclude the proof of the lemma. $\square$

Now, let us write $(b)_n$ as

$$s\big(f^n_*\big)\int\eta_n|v-v_*|\left(\frac{\varphi'-\varphi}{2}\right)\frac{\beta(f^n)'-\beta(f^n)}{|v'-v|}\,d\theta$$
$$+s\big(f^n_*\big)\int\eta_n|v-v_*|\left(\frac{\varphi'+\varphi}{2}\right)\frac{\beta(f^n)'-\beta(f^n)}{|v'-v|}\,d\theta. \tag{87}$$

Since φ is smooth, we have $|\varphi'-\varphi|\leqslant C|v'-v|$. It is easy to see that the first integral in (87) is bounded in L^2. We want to check that it actually goes weakly to 0, and for this we just have to check that it converges to 0 in $\mathcal{D}'(A)$. Since $\beta(f^n)$ converges to $\beta(f)$ in all L^p_{loc} spaces, $1\leqslant p<\infty$, this will be a consequence of Lemma 14, where we set $\chi(t,x,v,v_*,\phi)=(\varphi-\varphi')/|v-v'|$, $F^n=\beta(f^n)$ and $G^n=1$ (note that χ is not a smooth function of (v,v')!). Then we see that the first integral in (87) can be written as the difference of two integrals which have the same weak limit as $n\to\infty$.

Next, we wish to prove that

$$(e)_n\equiv\int d\theta\,\eta_n\big(|v-v_*|,\theta\big)\big[s\big(f^n\big)_*-s\big(f^n\big)'_*\big]\left(\frac{\varphi+\varphi'}{2}\right)\frac{\beta(f^n)'-\beta(f^n)}{|v'-v|}\xrightarrow[n\to\infty]{}0 \tag{88}$$

in weak sense.

Let $\kappa(v,v')$ be a smooth cut-off function with support in $(|v-v'|\leqslant\delta)$. By another application of Lemma 14, it is not difficult to show that $(e)_n\kappa$ converges weakly to 0; thus we only have to worry about small values of $|v'-v|$.

We can assume that s is chosen in such a way that $|s(x)-s(y)|\leqslant C|\beta(x)-\beta(y)|$; then, using the fact that $|v'_*-v_*|=|v'-v|$, we can bound the integral of $|(e)_n|$ over the region $(|v-v'|\leqslant\delta)$ by

$$C\delta \int \eta_n\big(|v-v_*|,\theta\big)\left|\frac{\beta(f^n)_* - \beta(f^n)'_*}{|v'_*-v_*|}\right|\left|\frac{\beta(f^n)-\beta(f^n)'}{|v'-v|}\right| d\sigma\, dv\, dv_*\, dx\, dt$$
$$\leqslant C\delta \int \eta_n\big(|v-v_*|,\theta\big)\left(\frac{\beta(f^n)_* - \beta(f^n)'_*}{|v'_*-v_*|}\right)^2 d\sigma\, dv\, dv_*\, dx\, dt,$$

where we have used the Cauchy–Schwarz inequality. By Lemma 13, this expression is bounded like $O(\delta)$ and hence negligible as $\delta \to 0$.

As a conclusion of Step 5, using (88) we see that (85) holds if

$$(c)_n \xrightarrow[n\to\infty]{} (c)_\infty \tag{89}$$

in distributional sense, where $(c)_n$ is the symmetric expression

$$\int d\theta\, \eta_n\big(|v-v_*|,\theta\big)\frac{[s(f^n)_* + s(f^n)'_*]}{2}\left(\frac{\varphi+\varphi'}{2}\right)\frac{\beta(f^n)'-\beta(f^n)}{|v'-v|},$$

and

$$(c)_\infty = s(f_*)\sqrt{M_\infty\big(|v-v_*|\big)}\,\varphi\nabla_v\beta(f)\cdot e_\phi.$$

Step 6. Now that we have obtained a symmetric enough expression, we can use a duality argument to prove (89). Let us multiply $(c)_n$ by a smooth, compactly supported test-function $\psi(v,v_*)$. Let also κ be a smooth test-function in the variables t, x, ϕ. The functions ψ and κ are chosen in such a way that $\psi\kappa$ is supported in A. We shall use the shorthand $\psi' = \psi(v', v'_*)$. What we have to prove is

$$\int (c)_n \psi\kappa\, d\theta\, d\phi\, dv\, dv_*\, dx\, dt \xrightarrow[n\to\infty]{} \int (c)_\infty \psi\kappa\, d\theta\, d\phi\, dv\, dv_*\, dx\, dt. \tag{90}$$

One should be careful in writing down the weak formulation for the integral on the right-hand side of (90). The formula

$$\psi(v,v_*) - \psi(v',v'_*) \simeq |v-v'|(\nabla_v - \nabla_{v_*})\psi(v,v_*)\cdot e_\phi \quad (\theta\to 0) \tag{91}$$

shows that this integral can be rewritten as

$$\frac{1}{2}\int dt\, dx\, d\phi\, dv\, dv_*\, d\theta\; \kappa s(f_*)\sqrt{M_\infty}\varphi\beta(f)(\nabla_v - \nabla_{v_*})\psi(v,v_*)\cdot e_\phi. \tag{92}$$

One should not be surprised by the non-appearance of derivatives of $\sqrt{M_\infty}$: this can be attributed to the fact that $\nabla\sqrt{M_\infty}$ is parallel to $v - v_*$.

By the pre-postcollisional change of variables, the integral on the left-hand side of (90) is

$$\frac{1}{2}\int dt\, dx\, d\phi\, dv\, dv_*\, d\theta\; \kappa\frac{[s(f^n)_* + s(f^n)'_*]}{2}\eta_n\big(|v-v_*|,\theta\big)\left(\frac{\varphi+\varphi'}{2}\right)\big[\beta\big(f^n\big)+\beta\big(f^n\big)'\big]\frac{\psi'-\psi}{|v'-v|}. \tag{93}$$

Taking into account (93) and (92), the convergence of (90) follows by a last application of Lemma 14. The proof of (74) is now complete. □

Appendix: An approximate Yukawa cross-section

In this appendix, we check our technical assumptions on a rather realistic model coming from physics. Computations are a little bit long and we shall present them in a slightly sketchy way. Most of the following physical discussion below is taken from [34].

The Yukawa potential is just the same as the Debye potential, except that it is usually considered in a quantum context (for instance, scattering of electrons or ions by neutrons). Up to a dimensional multiplicative factor, the Yukawa potential is given by

$$V(r) = \frac{e^{-r/\lambda}}{r},$$

where λ is the screening length. When dealing with quantum collision processes, one can "show" that (essentially)

$$\lambda = \frac{\hbar^2}{me^2}.$$

When the relative velocity z satisfies the *Born approximation*, i.e.

$$|z| \gg \frac{e^2}{\hbar},$$

then the scattering cross-section can be approximated in such a way that

$$B\big(|z|, \cos\theta\big)\sin\theta = \frac{1}{4}\frac{|z|(e^2/m)^2}{(|z|^2\sin^2(\theta/2) + (\hbar/(2m\lambda))^2)^2}.$$

This expression does not take into account exchange terms due to Pauli's exclusion principle. At the level of the Rutherford cross-section, these corrective terms can be computed explicitly, see for instance [10, Eq. (3.83)], and it is clear that they are negligible for small deviation angles. We shall admit that the same holds true here.

Note that

$$\left(\frac{\hbar}{2m\lambda}\right)^2 = \frac{e^2}{4m\lambda}.$$

Turning to nondimensional units, we denote this parameter by $1/n$; so, up to a multiplicative factor, n coincides with the screening length. Then we may rewrite the approximate cross-section as

$$B\big(|z|, \cos\theta\big)\sin\theta = \frac{|z|\sin\theta}{(|z|^2\sin^2(\theta/2) + 1/n)^2}.$$

Now, we rescale this expression by a factor $\log n$ (the Coulomb logarithm). This leads us to our final expression

$$B_n\big(|z|, \cos\theta\big)\sin\theta = \frac{1}{\log n}\frac{|z|\sin\theta}{(|z|^2\sin^2(\theta/2) + 1/n)^2}. \tag{A.1}$$

We claim that the sequence of collision kernels $(B_n)_{n\geqslant 1}$ concentrates on grazing collisions, in the sense of Definition 1, and satisfies the technical assumptions of Section 3 as well. Let us sketch the proof of this claim.

First of all, B_n is admissible (for fixed n), because it is just a nonsingular cross-section.

Next, let

$$M_n\big(|z|\big) = \frac{|S^1|}{\log n}\int_0^{\pi/2}\frac{|z|\sin\theta(1-\cos\theta)\,d\theta}{(|z|^2\sin^2(\theta/2) + 1/n)^2}. \tag{A.2}$$

For any $\theta_0 > 0$,

$$M_n^{\theta_0}\big(|z|\big) \leqslant \frac{C(\theta_0)}{|z|^3\log n},$$

and from Proposition 5 in Section 4 we see that $M_n^{\theta_0} \leqslant \lambda_n \delta_0$, with $\lambda_n \to 0$ as $n \to \infty$. Hence the contribution of large angles in (A.2) is asymptotically negligible, and we can assume that $\theta \leqslant \theta_0$ where θ_0 is very small. Then, locally in z, we can replace the integrand by its equivalent, and M_n by

$$\frac{|S^1||z|}{2\log n}\int_0^{\theta_0} \frac{\theta^3\,d\theta}{(|z|^2\theta^2/4+1/n)^2} = \frac{4|S^1|}{|z|^3}\left[1+\frac{1}{\log n}\log\left(\frac{|z|^2}{4}\theta_0^2+\frac{1}{n}\right)+\mathrm{O}\left(\frac{1}{\log n}\right)\right].$$

As a consequence, $zM_n(|z|)$ converges weakly (locally) in the sense of measures to $zM_\infty(|z|)$, with

$$M_\infty\bigl(|z|\bigr) = \frac{|S^1|}{|z|^3}.$$

Further note that, since $(2/\pi)\theta \leqslant \sin\theta \leqslant \theta$ and $1-\cos\theta \leqslant \theta^2/2$ for $\theta \in [0,\pi/2]$, we also have

$$\bigl|M_n\bigl(|z|\bigr)\bigr| \leqslant C|z| \int_0^{\pi/2} \frac{\theta^3\,d\theta}{(|z|^2\theta^2/2+1/n)^2} \xrightarrow[|z|\to\infty]{} 0,$$

uniformly in n, by the same estimate as above. Thus assumption (42) holds.

Next, we compute (see (40))

$$s_n\bigl(|z|,\theta\bigr) = \sin\theta\left[\frac{1}{\cos^3(\theta/2)}B_n\left(\frac{|z|}{\cos(\theta/2)},\cos\theta\right) - B_n\bigl(|z|,\cos\theta\bigr)\right], \tag{A.3}$$

and investigate the behavior of the θ-integral of this expression, which gives the kernel $S(|z|)$. Again, one can check that the contribution of large deviation angles is negligible in the limit $n \to \infty$, and we concentrate on small values of θ. For small θ we find that (A.3) is equivalent to

$$\frac{1}{\log n}\frac{4}{|z|^3\theta}\frac{1/(|z|^2(\theta^2 n/4))}{(1+1/(|z|^2(\theta^2 n/4)))^3}.$$

By the changes of variables $\theta \to (\sqrt{n}/2)\theta$ and $z \to z\theta$,

$$\frac{1}{\log n}\int_{|z|\leqslant A} dz \int_0^{\pi/2} \frac{d\theta}{|z|^3\theta}\frac{1/(|z|^2(\theta^2 n/4))}{(1+1/(|z|^2(\theta^2 n/4)))^3} = \frac{1}{\log n}\int_0^{(\sqrt{n}/2)\pi/2} \frac{d\theta}{\theta}\left(\int_{|z|\leqslant A\theta} \frac{dz}{|z|^3}\frac{1/|z|^2}{(1+1/|z|^2)^3}\right).$$

Since

$$\int_{|z|\leqslant A\theta} \frac{dz}{|z|^3}\frac{1/|z|^2}{(1+1/|z|^2)^3} \leqslant C\max\bigl(\theta^4,1\bigr),$$

we see that $\int_{|z|\leqslant A} S_n(|z|)\,dz = \int_{|z|\leqslant A}\int d\theta s_n(|z|,\theta)$ is bounded uniformly in n (this is the remaining part of assumption (i) in Definition 1).

Next, for any $\theta_0 > 0$,

$$s_n\bigl(|z|,\theta\bigr)1_{\theta\geqslant\theta_0} \leqslant \frac{C}{n\log n}\frac{|z|}{(|z|^2+1/n)^3}1_{\theta\geqslant\theta_0},$$

which easily leads to

$$\int_{|z|\leqslant A} dz \int_{\theta_0}^{\pi/2} s_n\bigl(|z|,\theta\bigr)\,d\theta \leqslant \frac{C(\theta_0)}{\log n}\int_{\mathbb{R}^3} \frac{|z|\,dz}{(|z|^2+1)^3} \xrightarrow[n\to\infty]{} 0.$$

Thus $S_n^{\theta_0}$ converges to 0 in $L^1_{\mathrm{loc}}(\mathbb{R}^3)$.

As for the technical conditions (43) and (80): the first one was established in Section 6, and the second one is a consequence of the uniform smoothness of B_n in the z variable away from the origin.

Remarks.

(1) We have not considered the part of the kernel B which comes from deviation angles larger than $\pi/2$ (recall formula (37)). This part has no influence on the estimates.

(2) If we denote by $a = 1/(4|z|^2)$, we have, again by a homogeneous change of variable,

$$\frac{1}{\log n}\int_0^{\pi/2}\frac{d\theta}{\theta}\frac{a/(\theta^2 n)}{(1+a/(n\theta^2))^3} = \frac{1}{\log n}\int_0^{\sqrt{n}\pi/2}\frac{d\theta}{\theta}\frac{a/\theta^2}{(1+a/\theta^2)^3} \simeq \frac{Ca}{n\log n}.$$

This expression goes to 0 for each z, and in fact uniformly for $|z| \geqslant \varepsilon$, which shows that S_n will converge weakly to a Dirac measure at the origin. This was expected in view of Lemma 12 and formula (64).

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