MULTIPLE POSITIVE SOLUTIONS FOR SINGULARLY PERTURBED ELLIPTIC PROBLEMS IN EXTERIOR DOMAINS ✪

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ABSTRACT. – The equation $-\varepsilon^2 \Delta u + a_\varepsilon(x)u = u^{p-1}$ with boundary Dirichlet zero data is considered in an exterior domain $\Omega = \mathbb{R}^N \setminus \bar{\omega}$ ($\omega$ bounded and $N \geq 2$). Under the assumption that $a_\varepsilon \geq a_0 > 0$ concentrates round a point of $\Omega$ as $\varepsilon \to 0$, that $p > 2$ and $p < 2N/(N-2)$ when $N \geq 3$, the existence of at least three positive distinct solutions is proved.

MSC: 35J20; 35J60

Keywords: Exterior domains; Lack of compactness; Multiplicity of solutions

1. Introduction

In this paper we consider the problem

$$
(P_\varepsilon) \begin{cases} 
-\varepsilon^2 \Delta u + a_\varepsilon(x)u = u^{p-1} & \text{in } \Omega, \\
u > 0 & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
$$

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where $\Omega = \mathbb{R}^N \setminus \tilde{\omega}$, $\omega$ being a nonempty, bounded domain having smooth boundary $\partial \omega = \partial \Omega$, $N \geq 2$, $\varepsilon \in \mathbb{R}^+ \setminus \{0\}$, $p > 2$ and $p < 2N/(N - 2)$ when $N \geq 3$. $a_\varepsilon$ is a given nonnegative function that, as $\varepsilon \to 0$, concentrates round a point $x_0 \in \Omega$, namely $a_\varepsilon$ has the form

$$a_\varepsilon(x) = a_0 + \alpha \left( \frac{x - x_0}{\varepsilon} \right)$$

and satisfies

(A1) $a_0 \in \mathbb{R}^+ \setminus \{0\}$, $x_0 \in \Omega$, $\alpha(x) \geq 0$, $\alpha \in L^{N/2}(\mathbb{R}^N)$, $|\alpha|_{L^{N/2}(\mathbb{R}^N)} \neq 0$,

(A2) $\int_{\mathbb{R}^N} \alpha(x) e^{2|\varepsilon|} \left( 1 + |x|^{\frac{N}{2} - 1} \right) dx < \infty$ for some $\sigma \in (1, 2]$.

Problem $(P_\varepsilon)$ has a variational structure: the solutions of $(P_\varepsilon)$ can be characterized as the nonnegative functions that are critical points of the functional $I_\varepsilon : H_0^1(\Omega) \rightarrow \mathbb{R}$

$$I_\varepsilon(u) = \int_{\Omega} \left( \varepsilon^2 |\nabla u|^2 + a_\varepsilon(x) u^2 \right) dx$$

constrained to lie on the manifold

$$\mathcal{M} = \{ u \in H_0^1(\Omega) \mid |u|_{L^p(\Omega)} = 1 \}.$$

However, it is well known that the unboundedness of the domain gives rise to a lack of compactness, not allowing a straight application of the usual variational techniques. In particular $(P_\varepsilon)$ cannot be solved by minimization, in fact (see Section 2), the infimum of $I_\varepsilon$ on $\mathcal{M}$ is not achieved, moreover the functional $I_\varepsilon$ does not satisfy the Palais-Smale condition in every energy level (see [1] and [3] for a careful analysis of the compactness question). The study of $(P_\varepsilon)$ needs subtle tools as the minimax theory together with topological arguments.

In recent years problems like $(P_\varepsilon)$ have been object of several researches, here we only recall that, without any symmetry assumption on $\omega$, the existence of one solution for $(P_\varepsilon)$ has been proved, first, in [3], in the case $a_\varepsilon(x) \equiv a_0$, then in [1], under more general assumptions; multiplicity results have been obtained, when $a_\varepsilon(x) \equiv a_0$, in domains having several holes [7,8,11,15] relating the number of solutions of $(P_\varepsilon)$ to the metric and/or topological properties of $\Omega$. We also remark that, for equations in $\mathbb{R}^N$ having nonconstant, nonsymmetric coefficients, the existence of one positive solution has been stated in [2,4], while multiple solutions have been found in [13].

In this work, motivated by former results, [6,9], that emphasize the role that a concentrating potential $a_\varepsilon$ can play in obtaining multiplicity of solutions for problems like $(P_\varepsilon)$ in bounded domains, we investigate the effect of such a potential when $\Omega$ is an unbounded exterior domain.

The result we obtain is stated in the following

**Theorem 1.1.** Let $a_\varepsilon$ be as in (1.1) and let the assumptions $(A_1)$ and $(A_2)$ be satisfied. Then there exists $\tilde{\varepsilon} > 0$ such that for every $\varepsilon \in (0, \tilde{\varepsilon})$ Problem $(P_\varepsilon)$ has at least three distinct solutions $u_{1,\varepsilon}, u_{2,\varepsilon}, u_{3,\varepsilon}$. Moreover
\[ \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^{N(1-2/p)}} \mathcal{I}_\varepsilon \left( \frac{u_{1,\varepsilon}}{|u_{1,\varepsilon}|_{L^p(\Omega)}} \right) = m, \quad (1.2) \]

\[ \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^{N(1-2/p)}} \mathcal{I}_\varepsilon \left( \frac{u_{2,\varepsilon}}{|u_{2,\varepsilon}|_{L^p(\Omega)}} \right) \in (m, 2^{1-2/p}m), \quad (1.3) \]

\[ \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^{N(1-2/p)}} \mathcal{I}_\varepsilon \left( \frac{u_{3,\varepsilon}}{|u_{3,\varepsilon}|_{L^p(\Omega)}} \right) = 2^{1-2/p}m, \quad (1.4) \]

where

\[ m = \inf \left\{ \int_{\mathbb{R}^N} [|
abla u|^2 + a_0 u^2] \, dx \left| u \in H^1(\mathbb{R}^N), \ |u|_{L^p(\mathbb{R}^N)} = 1 \right. \right\}. \]

We remark that the above theorem gives the existence of at least three solutions whatever \( \Omega \) is, even the complement of a convex domain.

It is worth observing, also, that the asymptotic energy estimates give some information about the shape of the solutions. Indeed \( u_{1,\varepsilon} \) is a “single peak” solution, that is a function that, suitably translated and scaled, tends, as \( \varepsilon \to 0 \), to a solution of the limit problem

\[ (P_\infty) \begin{cases} -\Delta u + a_0 u = u^{p-1} & \text{in } \mathbb{R}^N, \\ u > 0 & \text{in } \mathbb{R}^N, \\ u(x) \to 0 & \text{as } |x| \to \infty, \end{cases} \]

and, on the other hand, \( u_{3,\varepsilon} \) must be a “two-peaks” solution, in fact its energy, suitably scaled, tends to the energy of a pairs of not interacting solutions of \((P_\infty)\). About the last solution, \( u_{2,\varepsilon} \), we can guess (but we have not a rigorous proof) that it, suitably scaled in \( x_0 \), as \( \varepsilon \to 0 \), tends to a solution of

\[ (P_\alpha) \begin{cases} -\Delta u + (a_0 + \alpha(x)) u = u^{p-1} & \text{in } \mathbb{R}^N, \\ u > 0 & \text{in } \mathbb{R}^N, \\ u(x) \to 0 & \text{as } |x| \to \infty, \end{cases} \]

whose shape depends on \( \alpha \) (see [13]).

Finally, we point out that we can look at problem \((P_\varepsilon)\) in a “dual” way: an equation not depending on \( \varepsilon \), considered in an exterior domain whose complement, as \( \varepsilon \to 0 \), widens and becomes far and far from the relevant part (in the sense of \( L^{N/2}(\mathbb{R}^N) \)) of \( \alpha \).

Actually, considering, for instance \( \Omega_{\varepsilon,x_0} = \{ x \in \mathbb{R}^N | \varepsilon x + x_0 \in \Omega \} \) an easy scale change shows that to any solution of \((P_\varepsilon)\) there corresponds, in a one to one way, a solution of

\[ \begin{cases} -\Delta u + (a_0 + \alpha(x)) u = u^{p-1} & \text{in } \Omega_{\varepsilon,x_0}, \\ u > 0 & \text{in } \Omega_{\varepsilon,x_0}, \\ u = 0 & \text{on } \partial \Omega_{\varepsilon,x_0}. \end{cases} \]

Thus the conclusion of Theorem 1.1 can be expressed equivalently as follows:

**Theorem 1.2.** Let \( a_0 \) and \( \alpha \) satisfy \((A_1)\) and \((A_2)\). Let \( \Omega_n \subset \mathbb{R}^N \) be a sequence of exterior domains such that for some \( y_n \in \mathbb{R}^N \) and \( r_n \to \infty \)

\[ B(y_n, r_n) \subset \mathbb{R}^N \setminus \Omega_n, \quad B(x_0, r_n) \subset \Omega_n. \]
Then there exists \( \bar{n} \in \mathbb{N} \) such that for all \( n > \bar{n} \) the equation \(-\Delta u + (a_0 + \alpha(x))u = u^{n-1}\) with zero Dirichlet boundary data in \( \Omega_n \) has at least three positive solutions, \( \bar{u}_{1,n}, \bar{u}_{2,n}, \bar{u}_{3,n}. \) Moreover

\[
\lim_{n \to +\infty} \frac{\int_{\Omega_n} ((\nabla \bar{u}_{1,n}(x))^2 + (a_0 + \alpha(x))\bar{u}_{1,n}^2(x)) \, dx}{|\bar{u}_{1,n}|_{L^p(\Omega_n)}} = m,
\]

\[
\lim_{n \to +\infty} \frac{\int_{\Omega_n} ((\nabla \bar{u}_{2,n}(x))^2 + (a_0 + \alpha(x))\bar{u}_{2,n}^2(x)) \, dx}{|\bar{u}_{2,n}|_{L^p(\Omega_n)}} \in (m, 2^{1-2/p}m),
\]

\[
\lim_{n \to +\infty} \frac{\int_{\Omega_n} ((\nabla \bar{u}_{3,n}(x))^2 + (a_0 + \alpha(x))\bar{u}_{3,n}^2(x)) \, dx}{|\bar{u}_{3,n}|_{L^p(\Omega_n)}} = 2^{1-2/p}m.
\]

The paper is organized as follows: Section 2 is devoted to introducing some notations and recalling some known results and useful relations; in Section 3 some useful tools are introduced and some basic asymptotic estimates are proved, Section 4 contains the proof of Theorem 1.1. Arguing as in proving Theorem 1.1, it is a simple matter to get the proof of Theorem 1.2.

2. Notations, known facts and useful remarks

Throughout the paper we make use of the following notations.

- \( L^p(D), 1 \leq p < +\infty, D \subseteq \mathbb{R}^N \), denotes a Lebesgue space; the norm in \( L^p(D) \) is denoted by \( |\cdot|_{L^p(D)} \).
- \( H^1_0(D), D \subseteq \mathbb{R}^N \) and \( H^1(\mathbb{R}^N) \) denote the Sobolev spaces obtained, respectively, as closure of \( C_0^{\infty}(D) \) and \( C_0^{\infty}(\mathbb{R}^N) \) with respect to the norms
- \( \|u\|_D = \left[ \int_D \left( |\nabla u|^2 + a_0 u^2 \right) \, dx \right]^{1/2}, \quad \|u\|_{\mathbb{R}^N} = \left[ \int_{\mathbb{R}^N} \left( |\nabla u|^2 + a_0 u^2 \right) \, dx \right]^{1/2} \).

- If \( D_1 \subseteq D_2 \subseteq \mathbb{R}^N \) and \( u \in H^1_0(D_1) \), we denote also by \( u \) its extension to \( D_2 \) obtained setting \( u \equiv 0 \) outside \( D_1 \).
- \( D_\varepsilon \) denotes the subset of \( \mathbb{R}^N \) \( \{ y \in \mathbb{R}^N \mid \varepsilon y \in D \} \), \( D \subseteq \mathbb{R}^N \).
- \( B(y, \rho) \) denotes the open ball, of \( \mathbb{R}^N \), having radius \( \rho \) and centered at \( y \).

In what follows, without any loss of generality, we assume \( a_0 = 1 \) and \( x_0 = 0 \).

Setting

\[
u_\varepsilon(x) = \varepsilon^{N/p} u(\varepsilon x)
\]

an easy computation shows that for every \( u \in H^1_0(\Omega) \) \( u_\varepsilon \in H^1_0(\Omega_\varepsilon) \), \( u \in \mathcal{M} \) if and only if \( |u_\varepsilon|_{L^p(\Omega_\varepsilon)} = 1 \) and

\[
\mathcal{I}_\varepsilon(u) = \int_{\Omega_\varepsilon} \left[ \varepsilon^2 |\nabla u|^2 + \left( 1 + \alpha \left( \frac{x}{\varepsilon} \right) \right) u^2 \right] \, dx = \varepsilon^{(1-2/p)N} \int_{\Omega_\varepsilon} \left[ |\nabla u_\varepsilon|^2 + (1 + \alpha(x))u_\varepsilon^2 \right] \, dx.
\]
Thus looking for critical points of $I_\varepsilon$ on $\mathcal{M}$ is equivalent to searching for critical points of the “rescaled” energy functional

$$E_\varepsilon(u) = \int_{\Omega_\varepsilon} \left[ |\nabla u|^2 + (1 + \alpha(x))u^2 \right] \, dx$$

on the manifold

$$M_\varepsilon = \{ u \in H^1(\Omega_\varepsilon) \mid |u|_{p,\Omega_\varepsilon} = 1 \}.$$

Let us set

$$m_\varepsilon = \inf \{ E_\varepsilon(u) \mid u \in M_\varepsilon \}$$

and

$$m = \inf \{ \|u\|^2_{p,R^N} \mid u \in H^1(R^N), \|u\|_{p,R^N} = 1 \}. \quad (2.3)$$

The infimum in (2.3) is achieved (see [16] or [5]) by a positive function $w$, that is unique modulo translations (see [12]) and radially symmetric about the origin, decreasing when the radial co-ordinate increases and such that

$$\lim_{|x| \to +\infty} |D^j w(x)| |x|^{-\frac{j}{2} + \frac{1}{p}} e^{\frac{|x|}{dj}} = d_j > 0, \quad d_j \in \mathbb{R}, \ j = 0, 1 \quad (2.4)$$

(see [5] and [10]).

On the contrary we have

**Proposition 2.1.** Let $\alpha$ satisfy $(A_1)$. Then

$$m_\varepsilon = m$$

and the minimization problem (2.2) has no solution.

**Proof.** Since we may consider $H^1(\Omega_\varepsilon)$ as a subspace of $H^1(R^N)$,

$$m_\varepsilon \geq m.$$

To prove that the equality holds, we consider the sequence

$$w_{\varepsilon,y_n}(x) := \frac{\phi_\varepsilon(x) w(x - y_n)}{[\phi_\varepsilon(x) w(x - y_n)]_{p,\Omega_\varepsilon}} \quad (2.6)$$

where $y_n \in \Omega_\varepsilon$, $\lim_{n \to +\infty} |y_n| = +\infty$, $w$ is the function realizing (2.3) and $\phi_\varepsilon(x) = \phi(\varepsilon x)$ with $\phi: \mathbb{R}^N \to [0, 1]$ a $C^\infty$-function such that: $\phi(x) = 0$ if $x \in \omega$, $0 \leq \phi(x) \leq 1$, supp$(1 - \phi)$ is compact, and we show that

$$\lim_{n \to +\infty} E_\varepsilon(w_{\varepsilon,y_n}) = m. \quad (2.7)$$

Indeed, using (2.4) it is not difficult to show that
On the other hand, for every fixed \( \eta > 0 \), we can find \( \rho = \rho(\eta) > 0 \) so that

\[
\| \phi_\varepsilon(x) w(x - y_n) \|_{\Omega_\varepsilon \setminus B(y_n, \rho)} < \eta
\]

and

\[
|\alpha|_{N/2, B(y_n, \rho)} < \eta,
\]

if \( n \) is large enough; hence

\[
\begin{align*}
\int_{\Omega_\varepsilon} \alpha(x) \left[ \phi_\varepsilon(x) w(x - y_n) \right]^2 \, dx \\
\equiv \int_{B(y_n, \rho)} \alpha(x) \left[ \phi_\varepsilon(x) w(x - y_n) \right]^2 \, dx + \int_{\Omega_\varepsilon \setminus B(y_n, \rho)} \alpha(x) \left[ \phi_\varepsilon(x) w(x - y_n) \right]^2 \, dx \\
\leq \eta \| \phi_\varepsilon(x) w(x - y_n) \|_{N, R^N} + \eta |\alpha|_{N/2, R^N}
\end{align*}
\]

from which

\[
\lim_{n \to +\infty} \int_{\Omega_\varepsilon} \alpha(x) \left[ \phi_\varepsilon(x) w(x - y_n) \right]^2 \, dx = 0
\]

follows.

Hence (2.8), (2.9) and (2.10) give (2.7).

Let us now assume that the minimization problem (2.2) has a solution \( u^* \geq 0 \). Then

\[
m \leq \| u^* \|^2_{R^N} = \| u^* \|^2_{\Omega_\varepsilon} \leq \| u^* \|^2_{\Omega_\varepsilon} + \int_{\Omega_\varepsilon} \alpha(x) (u^*(x))^2 \, dx = m.
\]

Thus we deduce

\[
u^*(x) = w(x - y^*) \quad \text{for some } y^* \in R^N
\]

and, by \((A_1)\) and \( w(x) > 0 \) \( \forall x \in R^N \),

\[
0 = \int_{\Omega_\varepsilon} \alpha(x) (u^*(x))^2 \, dx = \int_{\Omega_\varepsilon} \alpha(x) w^2(x - y^*) \, dx > 0,
\]

a contradiction. \( \square \)

The functional \( E_\varepsilon \) constrained on \( M_\varepsilon \) does not verify globally the Palais-Smale condition, however, as proved in [3], the compactness is preserved in some energy range.

**Lemma 2.2.** Let \((u_n)_n\) be a Palais-Smale sequence for \( E_\varepsilon \) constrained on \( M_\varepsilon \), i.e. \( u_n \in M_\varepsilon \)

\[
\begin{align*}
\lim_{n \to +\infty} E_\varepsilon(u_n) &= c, \\
\lim_{n \to +\infty} \nabla E_\varepsilon(u_n) &= 0.
\end{align*}
\]
If $c \in (m, 2^{1-2/p}m)$ then $(u_n)_n$ is relatively compact.

The following lemma states a lower bound for the energy of a critical point $u$ of $E_\varepsilon$ on $M_\varepsilon$ that changes sign; the proof, that can be easily deduced using the definition of $m$, can be found in [7].

**Lemma 2.3.** – Let $u \in H^1_0(\Omega_\varepsilon)$ be such that

$$|u|_{p, \Omega_\varepsilon} = 1, \quad E_\varepsilon(u) = c, \quad \nabla E_\varepsilon|_{M_\varepsilon}(u) = 0.$$  

Then $u^+ \not\equiv 0$ and $u^- \not\equiv 0$ implies $c > 2^{1-2/p}m$.

This lemma and the maximum principle ensure that critical points of $E_\varepsilon$ on $M_\varepsilon$ in the range $(m, 2^{1-2/p}m)$ give rise to positive solutions of problem $(P_\varepsilon)$.

### 3. Tools, preliminary remarks, basic estimates

For what follows we need to introduce some barycenter type function. For $u \in L^p(\mathbb{R}^N)$ we set

$$\tilde{u}(x) = \frac{1}{|B(x, 1)|} \int_{B(x, 1)} |u(y)| \, dy$$

$|B(x, 1)|$ being the Lebesgue measure of $B(x, 1)$, and

$$\hat{u}(x) = \left[ \tilde{u}(x) - \frac{1}{2} \max_{\mathbb{R}^N} \tilde{u}(x) \right]^+;$$

we then define $\beta : L^p(\mathbb{R}^N) \setminus \{0\} \to \mathbb{R}^N$ by

$$\beta(u) = \frac{1}{|\tilde{u}|_{p, \mathbb{R}^N}} \int_{\mathbb{R}^N} x [\hat{u}(x)]^p \, dx. \quad (3.1)$$

We remark that $\beta$ is well defined for all $u \in L^p(\mathbb{R}^N) \setminus \{0\}$, because $\hat{u} \not\equiv 0$ and has compact support, moreover $\beta$ is continuous.

We define also, for every $\varepsilon > 0$, another map $\beta_\varepsilon : L^p(\mathbb{R}^N) \setminus \{0\} \to \mathbb{R}^N$ by

$$\beta_\varepsilon(u) = \frac{1}{|u|_{p, \mathbb{R}^N}} \int_{\mathbb{R}^N} \chi(x - \tilde{x}_\varepsilon) |u(x)|^p \, dx \quad (3.2)$$

where $\tilde{x}_\varepsilon = \tilde{x}/\varepsilon$, $\tilde{x}$ being a fixed point in $\omega = \mathbb{R}^N \setminus \overline{\Omega}$ and $\chi$ is the function

$$\chi(x) = \frac{x}{1 + |x|}.$$  

We remark that $\beta_\varepsilon$ is a continuous map in $L^p(\mathbb{R}^N) \setminus \{0\}$; we observe also that $\beta_\varepsilon(w(x - \tilde{x}_\varepsilon)) = 0.$
We put
\[ B_0 := \inf \left\{ \int_{\mathbb{R}^N} \left[ |\nabla u|^2 + (1 + \alpha(x)) u^2 \right] \, dx \mid u \in H^1(\mathbb{R}^N), \right. \]
\[ \left. |u|_{p,\mathbb{R}^N} = 1, \beta(u) = 0 \right\} \] (3.3)
and, for all \( \varepsilon > 0 \), we set
\[ B_{0,\varepsilon} := \inf \{ E_{\varepsilon}(u) \mid u \in M_{\varepsilon}, \beta(u) = 0 \}, \] (3.4)
\[ B_{\varepsilon} := \inf \{ E_{\varepsilon}(u) \mid u \in M_{\varepsilon}, \beta(u) = \bar{x}_\varepsilon \}, \] (3.5)
\[ B_{0,\beta_\varepsilon} := \inf \{ E_{\varepsilon}(u) \mid u \in M_{\varepsilon}, \beta_\varepsilon(u) = 0 \}. \] (3.6)
We denote by \( L_\varepsilon \) the segment joining 0 and \( \bar{x}_\varepsilon \), i.e.
\[ L_\varepsilon = \left\{ t \bar{x}_\varepsilon \mid t \in [0, 1] \right\} \]
and by
\[ A_\varepsilon := \inf \{ E_{\varepsilon}(u) \mid u \in M_{\varepsilon}, \beta(u) \in L_\varepsilon \}. \] (3.7)
Fixed a point \( \zeta \in \partial B(0, 1) \) we denote by \( \phi_{\Sigma \theta} = \partial B(\zeta, 2) \) i.e.
\[ \phi_{\Sigma \theta} = \{ z \in \mathbb{R}^N \mid |z - \zeta| = 2 \}. \] (3.8)
For every \( \varepsilon > 0 \) and \( \rho > 0 \) we define the operator
\[ \psi_{\varepsilon, \rho} : \Sigma \times [0, 1] \to M_{\varepsilon} \]
by
\[ \psi_{\varepsilon, \rho}[\zeta, t](x) = \frac{\phi_{\varepsilon}(x)[(1 - t)w(x - \rho z) + tw(x - \rho \zeta)]}{|\phi_{\varepsilon}(x)[(1 - t)w(x - \rho z) + tw(x - \rho \zeta)]|_{p, \Omega}} \] (3.9)
where \( \phi_{\varepsilon} \) is the cut-off function introduced in Proposition 2.1 to define the sequence (2.6).
We put for all \( z \in \mathbb{R}^N \)
\[ w_{\varepsilon,z}(x) = \frac{\phi_{\varepsilon}(x)w(x - z)}{|\phi_{\varepsilon}(x)w(x - z)|_{p, \Omega}} \] (3.10)
and we remark that \( \forall z \in \Sigma \)
\[ \psi_{\varepsilon, \rho}[\zeta, 0](x) = w_{\varepsilon, \rho z}(x), \quad \psi_{\varepsilon, \rho}[\zeta, 1](x) = w_{\varepsilon, \rho \zeta}(x). \]
We consider, also, for every \( \rho > 0 \), the operator
\[ \psi_{\rho} : \Sigma \times [0, 1] \to \{ u \in H^1(\mathbb{R}^N) \mid |u|_{p, \mathbb{R}^N} = 1 \} \]
defined by
\[ \psi_{\rho}[z, t](x) = \frac{(1-t)w(x - \rho z) + tw(x - \rho \zeta)}{|(1-t)w(x - \rho z) + tw(x - \rho \zeta)|_{p, \mathbb{R}^N}}. \] (3.11)

**Proposition 3.1.** Let \( \alpha \) satisfy (A1). Let \( B_0, B_{0, \varepsilon} \) and \( m \) as defined, respectively, in (3.3), (3.4), (2.3). Then the relation
\[ B_{0, \varepsilon} \geq B_0 > m \] (3.12)
holds for all \( \varepsilon > 0 \).

**Proof.** Clearly, \( \forall \varepsilon > 0, B_{0, \varepsilon} \geq B_0 \) and \( B_0 \geq m \), so, in order to prove (3.12), we have to show that the equality \( B_0 = m \) cannot be true.

Arguing by contradiction, we assume \( B_0 = m \). Hence a sequence of nonnegative functions \( (u_n)_n \) in \( H^1(\mathbb{R}^N) \) must exist so that
\[ \beta(u_n) = 0 \] (a)
\[ |u_n|_{p, \mathbb{R}^N} = 1, \int_{\mathbb{R}^N} [||\nabla u_n||^2 + (1 + \alpha(x))u_n^2] \, dx \to m \] (b).

Moreover (A1), (2.3) and (3.13)(b) imply \( \lim_{n \to +\infty} \|u_n\|_{\mathbb{R}^N}^2 = m \).

Then, by the uniqueness of the solution of (2.3), a sequence of points \( (z_n)_n \) in \( \mathbb{R}^N \) and a sequence of functions \( (\varphi_n)_n \) in \( H^1(\mathbb{R}^N) \) exist so that, up to a subsequence still denoted by \((u_n)_n\),
\[ u_n(x) = w(x - z_n) + \varphi_n(x), \quad x \in \mathbb{R}^N, \]
\[ \lim_{n \to +\infty} \varphi_n(x) = 0 \quad \text{in} \quad H^1(\mathbb{R}^N) \quad \text{and} \quad \text{in} \quad L^p(\mathbb{R}^N) \]
and, by the same arguments of Proposition 2.1, \( \lim_{n \to +\infty} |z_n| = +\infty \).

On the other hand
\[ \lim_{n \to +\infty} \sup_{x \in \mathbb{R}^N} |\tilde{u}_n(x + z_n) - \tilde{w}(x)| = 0, \]
and, as a consequence,
\[ |\beta(u_n(x)) - \beta(w(x - z_n))| \to 0 \quad \text{as} \quad n \to +\infty, \]
that is
\[ |\beta(u_n(x)) - z_n| \to 0 \quad \text{as} \quad n \to +\infty, \]
contradicting (3.13)(a). \( \square \)

**Lemma 3.2.** Let \( \Sigma, \psi_{\varepsilon, \rho}, B_{0, \varepsilon} \) be as defined, respectively, in (3.8), (3.9), (3.4). Then for every \( \rho > 0 \) there exists \( \varepsilon_{\rho} > 0 \) such that for all \( \varepsilon \in (0, \varepsilon_{\rho}) \)
\[ B_{0, \varepsilon} \leq \max_{\Sigma \times [0, 1]} E_\varepsilon(\psi_{\varepsilon, \rho}[z, t]). \] (3.14)
Proof. – In view of (2.4), of the radial symmetry round 0 of \( w(x) \) and of the fact that 
\[
\lim_{\varepsilon \to 0} \max_{\Sigma} |\beta \circ \psi_{\varepsilon,0} - \rho z| = 0.
\]
Thus, for all \( \varepsilon > 0 \) small enough, \( \beta \circ \psi_{\varepsilon,0}(\Sigma \times \{0\}) \) is homotopically equivalent in 
\( \mathbb{R}^N \setminus \{0\} \) to \( \rho \Sigma \) and, then, there exists \( (\hat{z}_\varepsilon, \hat{t}_\varepsilon) \in \Sigma \times [0,1] \) such that \( \beta \circ \psi_{\varepsilon,0}(\hat{z}_\varepsilon, \hat{t}_\varepsilon) = 0 \), hence 
\[
B_0, \varepsilon 
\[
\leq E_{\varepsilon}(\psi_{\varepsilon,0}(\hat{z}_\varepsilon, \hat{t}_\varepsilon)) = \max_{\psi_{\varepsilon,0}(\hat{z}_\varepsilon, \hat{t}_\varepsilon)} E_{\varepsilon}(\psi_{\varepsilon,0}(z, t)).
\]

Proposition 3.3. – Let \( \alpha \) satisfy \( (A_1), (A_2) \) then there exist constants \( \rho_\alpha > 0 \), \( \mu_\alpha > 0 \) and \( \varepsilon_1 > 0 \), such that for all \( \varepsilon \in (0, \varepsilon_1) \) 
\[
\max_{\Sigma \times [0,1]} E_{\varepsilon}(\psi_{\varepsilon,\rho_\alpha}(z, t)) < \mu_\alpha < \frac{2}{1-2/p} m,
\]
\[
\max_{\Sigma \times [0,1]} E_{\varepsilon}(\psi_{\varepsilon,\rho_\alpha}(z, 0)) < B_0.
\]

Proof. – The proof is carried out in three steps.

Step 1. There exists \( \rho_1 > 0 \) such that \( \forall \rho > \rho_1 \) 
\[
\max_{\Sigma \times [0,1]} \int_{\mathbb{R}^N} \left[ |\nabla \psi_{\rho}(z, t)|^2 + (1 + \alpha(x)) (\psi_{\rho}(z, t))^2 \right] dx := \mu_\rho < \frac{2}{1-2/p} m.
\]

The argument is very similar to that of Lemma 3.5 in [8] so we only sketch it for the reader’s convenience.

We define 
\[
N_{\rho}(z, t) = \int_{\mathbb{R}^N} \left[ |\nabla (1-t)w(x-\rho z) + tw(x-\rho z)|^2 \right.
\]
\[
+ (1 + \alpha(x)) \left( (1-t)w(x-\rho z) + tw(x-\rho z) \right)^2 \right] dx,
\]
\[
D_{\rho}(z, t) = \left( (1-t)w(x-\rho z) + tw(x-\rho z) \right)^p_{\rho, \mathbb{R}^N}.
\]

To verify (3.17) we must prove that if \( \rho \) is large enough 
\[
\max_{\Sigma \times [0,1]} \frac{N_{\rho}(z, t)}{(D_{\rho}(z, t))^{2/p}} < \frac{2}{1-2/p} m.
\]

Taking into account that \( -\Delta w + w = mw^{p-1} \) in \( \mathbb{R}^N \) we obtain 
\[
N_{\rho}(z, t) = \left[ (1-t)^2 + t^2 \right] m + 2t(1-t)\eta_{\rho} + 2t^2 \theta_{\rho} + 2(1-t)^2 \delta_{\rho}
\]
where 
\[
\eta_{\rho} = \int_{\mathbb{R}^N} w(x-\rho z)^{p-1}w(x-\rho z) dx = \int_{\mathbb{R}^N} w(x-\rho z)^{p-1}w(x-\rho z) dx,
\]
\[
\theta_{\rho} = \int_{\mathbb{R}^N} w(x-\rho z)^{p-1}w(x-\rho z) dx = \int_{\mathbb{R}^N} w(x-\rho z)^{p-1}w(x-\rho z) dx,
\]
\[
\delta_{\rho} = \int_{\mathbb{R}^N} w(x-\rho z)^{p-1}w(x-\rho z) dx = \int_{\mathbb{R}^N} w(x-\rho z)^{p-1}w(x-\rho z) dx.
\]
\[ \theta_{\rho} = \int_{\mathbb{R}^N} \alpha(x) |w(x - \rho\xi)|^2 \, dx, \]

\[ \delta_{\rho} = \int_{\mathbb{R}^N} \alpha(x) |w(x - \rho\zeta)|^2 \, dx. \]

Using Lemma 2.2 of [1], (2.4) and condition (\(A_2\)) we then deduce

\[ \lim_{\rho \to +\infty} \eta_{\rho} \left[ 2\rho \frac{\alpha_1}{\alpha_2} e^{2\eta_0} \right] = C_1 > 0, \]

\[ \lim_{\rho \to +\infty} \theta_{\rho} \left[ \rho \frac{\alpha_1}{\alpha_2} e^{2\eta_0} \right] = C_2 \geq 0, \]

\[ \lim_{\rho \to +\infty} \delta_{\rho} \left[ \rho \frac{\alpha_1}{\alpha_2} e^{2\eta_0} \right] = C_3 \geq 0, \]

that allow to obtain

\[ N_{\rho}[z, t] = [(1 - t)^2 + t^2]m + 2t(1 - t)m\eta_{\rho} + g(\rho) \]

with \(g(\rho) = o(\eta_{\rho})\), because \(\sigma \in (1, 2]\).

On the other hand, using Lemma 2.7 of [8] we get

\[ D_{\rho}[z, t] \geq [(1 - t)^p + t^p] + (p - 1)[(1 - t)^{p-1} + t^{p-1}(1 - t)]\eta_{\rho}. \]

Hence

\[ \frac{N_{\rho}[z, t]}{(D_{\rho}[z, t])^{2/p}} \leq \frac{[(1 - t)^2 + t^2]}{[(1 - t)^p + t^p]^{2/p}} m + 2\gamma(t)m\eta_{\rho} + o(\eta_{\rho}) \]

where

\[ \gamma(t) = \frac{(1 - t)t}{[(1 - t)^p + t^p]^{2/p}} \left\{ 1 - \frac{p - 1}{p} \frac{(1 - t)^2 + t^2}{(1 - t)^p + t^p} \right\}. \]

Now \(\gamma(1/2) < 0\), so there exists a neighbourhood \(I(1/2)\) such that \(\gamma(t) < c < 0\) \(\forall t \in I(1/2)\) and

\[ \max \left\{ \frac{N_{\rho}[z, t]}{(D_{\rho}[z, t])^{2/p}} \mid z \in \Sigma, t \in I \left( \frac{1}{2} \right) \right\} \leq 2^{1 - 2/p} m + 2cm\eta_{\rho} + o(\eta_{\rho}) < 2^{1 - 2/p} m \]

(3.19) for \(\rho\) large enough. Moreover the relation

\[ \lim_{\rho \to +\infty} \max \left\{ \frac{N_{\rho}[z, t]}{(D_{\rho}[z, t])^{2/p}} \mid z \in \Sigma, t \in [0, 1] \setminus I(1/2) \right\} = m \max \left\{ \frac{[(1 - t)^2 + t^2]}{[(1 - t)^p + t^p]^{2/p}} \mid t \in [0, 1] \setminus I(1/2) \right\} < 2^{1 - 2/p} m \]

holds and together with (3.19) gives (3.18) as desired.
Step 2. There exists \( \hat{\rho} \geq \rho_1 \) such that \( \forall \rho \geq \hat{\rho} \)
\[
\max_{\mathbb{R}^N} \int \left[ \left| \nabla \psi_\rho[z,0] \right|^2 + (1 + \alpha(x)) (\psi_\rho[z,0])^2 \right] \, dx \leq B_0.
\] (3.20)

Since (3.12) holds and
\[
\int_{\mathbb{R}^N} \left[ \left| \nabla \psi_\rho[z,0] \right|^2 + (1 + \alpha(x)) (\psi_\rho[z,0])^2 \right] \, dx
= \int_{\mathbb{R}^N} \left[ \left| \nabla w(x - \rho z) \right|^2 + (1 + \alpha(x)) w(x - \rho z)^2 \right] \, dx
= m + \int_{\mathbb{R}^N} \alpha(x) w(x - \rho z)^2 \, dx,
\]
to prove (3.20) we only need the relation
\[
\lim_{|\xi| \to +\infty} \int_{\mathbb{R}^N} \alpha(x) w(x - \xi)^2 \, dx = 0
\]
that follows, easily, arguing as in Proposition 2.1 to prove relation (2.10).

Step 3. Let \( \rho_a \geq \hat{\rho} \) and \( \mu_a \in (\hat{\mu}_\rho(\hat{\rho}), \frac{2^{1-2/p}m}{p}) \) be fixed, then there exists \( \varepsilon_1 > 0 \) such that (3.15) and (3.16) hold for all \( \varepsilon \in (0, \varepsilon_1) \).

Because of the choice of \( \rho_a \), the inequalities (3.17) and (3.20) hold true when \( \rho = \rho_a \). Then in order to obtain (3.15) and (3.16) it is enough to observe that for all compact set \( K \subset \Sigma \times [0, 1] \)
\[
\lim_{\varepsilon \to 0} \max_{(z,t) \in K} E_\varepsilon(\psi_{\varepsilon, \rho_a}[z, t])
= \max_{(z,t) \in K} \int_{\mathbb{R}^N} \left( \left| \nabla \psi_{\rho_a}[z, t] \right|^2 + (1 + \alpha(x)) (\psi_{\rho_a}[z, t])^2 \right) \, dx.
\] (3.21)

In fact, let \( \varepsilon_\alpha \) and \( (z_\alpha, t_\alpha) \in K \) be such that \( \lim_{n \to +\infty} \varepsilon_\alpha = 0 \) and \( \lim_{n \to +\infty} (z_\alpha, t_\alpha) = (z_0, t_0) \in K \), then in view of (2.4) and of the fact that \( \text{dist}(\omega_\alpha, 0) \to +\infty \) it is not difficult to see that
\[
\lim_{n \to +\infty} \psi_{\varepsilon_\alpha, \rho_a}[z_\alpha, t_\alpha] = \psi_{\rho_a}[z_0, t_0] \quad \text{in} \ H^1(\mathbb{R}^N)
\]
hence
\[
\lim_{n \to +\infty} E_{\varepsilon_\alpha}(\psi_{\varepsilon_\alpha, \rho_a}[z_\alpha, t_\alpha]) = \int_{\mathbb{R}^N} \left( \left| \nabla \psi_{\rho_a}[z_0, t_0] \right|^2 + (1 + \alpha(x)) (\psi_{\rho_a}[z_0, t_0])^2 \right) \, dx
\]
so (3.21) and the claim easily follow. \( \square \)
PROPOSITION 3.4. — Let $B_{\bar{x}}$ be as defined in (3.5). Let $\alpha$ satisfy (A$_1$). Then there exists a constant $C_{\bar{x}} > m$ such that the relation
\[ B_{\bar{x}} \geq C_{\bar{x}} > m \] (3.22)
holds for all $\varepsilon > 0$.

Proof. — To prove the claim, we argue by contradiction; so, we assume that a sequence $(\varepsilon_n)_n$ exists such that $B_{\bar{x}_n} \rightarrow m$, as $n \rightarrow +\infty$. We can also assume $\varepsilon_n \rightarrow 0$, as $n \rightarrow +\infty$, otherwise we get a contradiction at once, observing that $\varepsilon_n \geq \lambda > 0$ for some $\lambda \in \mathbb{R}$ implies $\bar{x}_n \in \tilde{\omega}_\lambda := \bigcup_{\varepsilon \geq \lambda} \omega_{\varepsilon}$
and that, in view of the boundedness of $\tilde{\omega}_\lambda$, arguing as in Proposition 3.1, it is not difficult to conclude $C_{\lambda} > m$.

So a sequence of nonnegative functions $(u_n)_n$, $u_n \in H^1_0(\Omega_{\varepsilon_n})$, must exist, such that $E_{\varepsilon_n}(u_n) \rightarrow m$, $\varepsilon_n \rightarrow 0$ as $n \rightarrow +\infty$, $|u_n|_{\varepsilon_n, \Omega_{\varepsilon_n}} = 1$ and $\beta(u_n) = \bar{x}_n / \varepsilon_n$. Hence there exist sequences $(z_n)_n$ in $\mathbb{R}^N$ and $(\varphi_n)_n$ in $H^1(\mathbb{R}^N)$ such that, up to a subsequence,
\[ u_n(x) = w(x - z_n) + \varphi_n(x) \quad \forall x \in \mathbb{R}^N, \] (3.23)
and
\[ \lim_{n \rightarrow +\infty} \varphi_n(x) = 0 \quad \text{strongly in } H^1(\mathbb{R}^N) \quad \text{and in } L^p(\mathbb{R}^N). \]
So by the continuity of $\beta$, we infer
\[ \left| \frac{\bar{x}_n - z_n}{\varepsilon_n} \right| = \left| \beta(u_n) - z_n \right| \rightarrow 0 \quad \text{as } n \rightarrow +\infty \]
from which the relation
\[ \lim_{n \rightarrow +\infty} \text{dist}(\Omega_{\varepsilon_n}, z_n) = +\infty \]
follows. Thus, for any $R > 0$ and for $n$ large enough, $B(z_n, R) \cap \Omega_{\varepsilon_n} = \emptyset$ that implies
\[ \int_{B(z_n, R)} |u_n(x)| \, dx = 0. \]
The above relation contradicts the relation
\[ \lim_{n \rightarrow +\infty} \int_{B(z_n, R)} |u_n(x)| \, dx = \int_{B(0, R)} w(x) \, dx > 0 \]
that follows from the properties of $w$ and (3.23). \qed
**Proposition 3.5.** Let \( \alpha \) satisfy \( (A_1) \). Let \( A_\varepsilon, B_0, w_{\varepsilon, z}, C_{\bar{x}} \) be as defined respectively in (3.7), (3.3), (3.10) and in Proposition 3.4. Let \( R \in \mathbb{R}, R > 0 \) be chosen so that \( B(0, R) \subset \Omega \). Then there exists \( \varepsilon_2 > 0 \) such that
\[
m < A_\varepsilon \leq \max_{|z|=R/2\varepsilon} E_\varepsilon(w_{\varepsilon, z}) < \min(B_0, C_{\bar{x}})
\] (3.24)
for all \( \varepsilon \in (0, \varepsilon_2) \).

**Proof.** Clearly, for every fixed \( \varepsilon \), by the same arguments of Proposition 2.1, \( m < A_\varepsilon \).

Let us, now, observe that, in view of (2.4), of the radial symmetry of \( w \) and of the fact that \( \text{dist}(\partial B(0, R/2\varepsilon), \bar{\omega}_\varepsilon) \to +\infty \) as \( \varepsilon \to 0 \), we have
\[
\lim_{\varepsilon \to 0} \max_{|z|=R/2\varepsilon} \|w_{\varepsilon, z}(x) - w(x - z)\|_{\mathbb{R}^N} = 0
\] (3.25)
and
\[
\lim_{\varepsilon \to 0} \max_{|z|=R/2\varepsilon} |\beta(w_{\varepsilon, z}) - z| = 0.
\] (3.26)

(3.25) implies \( \lim_{\varepsilon \to 0} \max_{|z|=R/2\varepsilon} E_\varepsilon(w_{\varepsilon, z}) = m \) and this relation, with (3.12) and (3.22), gives the third inequality for small \( \varepsilon \).

As a consequence of (3.26), for small \( \varepsilon \), the map
\[
z \to \beta(w_{\varepsilon, z})
\]
is homotopic to the identity map \( i \) on \( \partial B(0, R/2\varepsilon) \) by the homotopy
\[
K(\theta, z) = \theta \beta(w_{\varepsilon, z}) + (1 - \theta)z, \quad 0 \leq \theta \leq 1,
\] (3.27)
and \( K(\theta, z) \notin \{0, \bar{x}_\varepsilon\}, \forall \theta \in [0, 1] \forall z \in \partial B(0, R/2\varepsilon) \).

Then there exists \( \tilde{z} \in \partial B(0, R/2\varepsilon) \) such that \( \beta(w_{\varepsilon, \tilde{z}}) \in L_{\varepsilon} \), hence the relation
\[
A_\varepsilon \leq E_\varepsilon(w_{\varepsilon, \tilde{z}}) \leq \max_{|z|=R/2\varepsilon} E_\varepsilon(w_{\varepsilon, z})
\]
gives the second inequality. \( \square \)

**Proposition 3.6.** Let \( \alpha \) satisfy \( (A_1) \). Let \( B_{0, \beta_\varepsilon} \) be as defined in (3.6). Let \( \mu \) be a constant such that \( \mu \in (m, 2^{1-2/p} m) \) then there exists \( \varepsilon_\mu > 0 \) such that
\[
B_{0, \beta_\varepsilon} > \mu
\] (3.28)
for all \( \varepsilon \in (0, \varepsilon_\mu) \).

**Proof.** The claim follows from the asymptotic estimate
\[
\lim_{\varepsilon \to 0} B_{0, \beta_\varepsilon} = 2^{1-2/p} m
\]
that can be obtained arguing exactly as in Lemma 3.3 and Remark 3.4 of [15]. \( \square \)
LEMMA 3.7. – Let $\Sigma$, $\psi_{\varepsilon, \rho}$, $B_{0, \beta_\varepsilon}$ be as defined respectively in (3.8), (3.9), (3.6). Then for every $\varepsilon > 0$ there exists $\hat{\rho}_\varepsilon > 0$ such that for all $\rho > \hat{\rho}_\varepsilon$

$$B_{0, \beta_\varepsilon} \leq \max_{\Sigma \times [0,1]} E_\varepsilon (\psi_{\varepsilon, \rho} [z, t]).$$

(3.29)

Proof. – In view of (2.4), of the radial symmetry of $w$ and by the definition (3.2) of $\beta_\varepsilon$, it is not difficult to verify that, for every fixed $\varepsilon > 0$,

$$\lim_{\rho \to +\infty} \max_{z \in \Sigma} |\beta_\varepsilon \circ \psi_{\varepsilon, \rho} [z, 0] - \chi(\rho \bar{z} - \bar{x}_\varepsilon)| = 0.$$

Hence, for all $\rho$ large enough, the set $\beta_\varepsilon \circ \psi_{\varepsilon, \rho} (\Sigma \times \{0\})$ is homotopically equivalent in $\mathbb{R}^N \setminus \{0\}$ to $\rho \Sigma$ and, then, there exists $(\bar{z}_\rho, \bar{t}_\rho) \in \Sigma \times [0,1]$ such that $\beta_\varepsilon \circ \psi_{\varepsilon, \rho} (\bar{z}_\rho, \bar{t}_\rho) = 0$, thus

$$B_{0, \beta_\varepsilon} \leq E_\varepsilon (\psi_{\varepsilon, \rho} (\bar{z}_\rho, \bar{t}_\rho)) \leq \max_{\Sigma \times [0,1]} E_\varepsilon (\psi_{\varepsilon, \rho} [z, t]).$$

\[\square\]

PROPOSITION 3.8. – Let $\alpha$ satisfy (A1) and let $\mu$ be so that $\mu \in (m, 2^{1 - 2/p}m)$. For every $\varepsilon > 0$ there exists $\bar{\rho}_{\varepsilon, \mu} > 0$ such that for all $\rho > \bar{\rho}_{\varepsilon, \mu}$

$$\max_{\Sigma \times [0,1]} E_\varepsilon (\psi_{\varepsilon, \rho} [z, t]) < 2^{1 - 2/p}m,$$

(3.30)

$$\max_{\Sigma} E_\varepsilon (\psi_{\varepsilon, \rho} [z, 0]) < \mu.$$  

(3.31)

Proof. – The proof is carried out in three steps.

Step 1. For every $\varepsilon > 0$ there exists $\bar{\rho}_{\varepsilon, 1} > 0$ such that for all $\rho > \bar{\rho}_{\varepsilon, 1}$

$$\max_{\Sigma \times [0,1]} \int_{\Omega_\varepsilon} \left[ \left( \nabla \psi_{\varepsilon, \rho} [z, t] \right)^2 + (\psi_{\varepsilon, \rho} [z, t])^2 \right] dx \leq 2^{1 - 2/p}m.$$  

(3.32)

The proof of this step is just Lemma 3.5 in [8].

Step 2. For every $\varepsilon > 0$ there exists $\bar{\rho}_{\varepsilon, 2} > \bar{\rho}_{\varepsilon, 1}$ such that

$$\max_{\Sigma} \int_{\Omega_\varepsilon} \left[ \left( \nabla \psi_{\varepsilon, \rho} [z, 0] \right)^2 + (\psi_{\varepsilon, \rho} [z, 0])^2 \right] dx \leq \mu$$

(3.33)

holds for all $\rho > \bar{\rho}_{\varepsilon, 2}$.

By (2.4), the shape of $w$ and the choice of $\phi_\varepsilon$ we have

$$\lim_{|z| \to +\infty} \|\phi_\varepsilon (x) w(x - z) - w(x - z)\|_{\mathbb{R}^N} = 0$$

from which

$$\lim_{\rho \to \infty} \max_{\Sigma} \left[ \|\psi_{\varepsilon, \rho} [z, 0]\|_{\mathbb{R}^N}^2 - \|w(x - \rho z)\|_{\mathbb{R}^N}^2 \right] = 0.$$
that implies
\[ \lim_{\rho \to +\infty} \max_{\Omega_c} \left[ \| \nabla \psi_{\varepsilon,\rho}[z,0] \|^2 + (\psi_{\varepsilon,\rho}[z,0])^2 \right] dx = m. \]

**Step 3.** For every \( \varepsilon > 0 \) there exists \( \bar{\rho}_{\varepsilon} > \bar{\rho}_{\varepsilon,2} \) such that (3.30) and (3.31) hold for all \( \rho > \bar{\rho}_{\varepsilon} \).

Taking into account that \( |\phi_{\varepsilon}(x)|[(1-t)w(x-\rho z) + tw(x-\rho \xi)]_{\rho,\Omega_c} \geq c > 0 \), arguing as in Proposition 2.1 to prove (2.10) it is not difficult to see that
\[ \lim_{\rho \to +\infty} \max_{\Omega_c} \left[ \| \nabla \psi_{\varepsilon,\rho}[z,0] \|^2 + (\psi_{\varepsilon,\rho}[z,0])^2 \right] dx = 0. \]

Hence
\[ \lim_{\rho \to +\infty} \max_{\Omega_c} \left[ E_{\varepsilon}(\psi_{\varepsilon,\rho}[z,t]) - \int_{\Omega_c} \left( \| \nabla \psi_{\varepsilon,\rho}[z,t] \|^2 + (\psi_{\varepsilon,\rho}[z,t])^2 \right) dx \right] = 0 \]
that, with (3.32) and (3.33), gives (3.30) and (3.31).

4. **Proof of Theorem 1.1**

To prove the theorem we show that, for small \( \varepsilon \), \( E_{\varepsilon} \) has on \( M_{\varepsilon} \) three distinct critical values, lying in the energy range \( (m, 2^{1-2/p}\mu) \), to which there correspond at least three distinct solutions of \( (P_{\varepsilon}) \), positive by Lemma 2.3.

In what follows \( \rho_{\alpha}, \mu_{\alpha} \) are the constants whose existence is stated in Proposition 3.3, moreover we choose \( \bar{\varepsilon} = \min(\varepsilon_{\rho_{\alpha}}, \varepsilon_1, \varepsilon_2, \varepsilon_{\mu_{\alpha}}) \) where \( \varepsilon_1, \varepsilon_2 \) are, respectively, the numbers found in Propositions 3.3 and 3.5 and \( \varepsilon_{\rho_{\alpha}}, \varepsilon_{\mu_{\alpha}} \) are as stated in Lemma 3.2 and Proposition 3.6.

We remark that, by the results of Section 3, for all \( \varepsilon \in (0, \bar{\varepsilon}) \) the following inequalities hold
\[ m < A_{\varepsilon} \leq \max_{|z|=R/2} E_{\varepsilon}(w_{\varepsilon,z}) < \min(B_0, C_\varepsilon), \]
\[ \max_{\Omega_c} E_{\varepsilon}(\psi_{\varepsilon,\rho_{\alpha}}[z,0]) < B_0 \leq B_{0,\varepsilon} \leq \max_{\Omega_c} E_{\varepsilon}(\psi_{\varepsilon,\rho_{\alpha}}[z,t]) \]
\[ < \mu_{\alpha} < B_{0,\bar{\rho}_{\varepsilon}} < 2^{1-2/p}m. \quad (4.1) \]
and, fixed \( \varepsilon \in (0, \bar{\varepsilon}) \), for all \( \rho > \max(\bar{\rho}_{\varepsilon}, \bar{\rho}_{\varepsilon,\mu_{\alpha}}, \rho_{\alpha}) \) (\( \bar{\rho}_{\varepsilon} \) and \( \bar{\rho}_{\varepsilon,\mu_{\alpha}} \) being the numbers whose existence is stated in Lemma 3.7 and Proposition 3.8, respectively)
\[ \max_{\Omega_c} E_{\varepsilon}(\psi_{\varepsilon,\rho}[z,0]) < \mu_{\alpha} < B_{0,\bar{\rho}_{\varepsilon}} \leq \max_{\Omega_c} E_{\varepsilon}(\psi_{\varepsilon,\rho}[z,t]) < 2^{1-2/p}m. \quad (4.2) \]

We consider a fixed \( \varepsilon \in (0, \bar{\varepsilon}) \) and we carry out the proof in three steps: first we prove, in Step 1, the existence of a critical value \( c_{1,\varepsilon} \) satisfying
\[ A_{\varepsilon} \leq c_{1,\varepsilon} \leq \max_{|z|=R/2} E_{\varepsilon}(w_{\varepsilon,z}). \]
then, in Step 2, we show that another critical level \( c_{2, \epsilon} \) exists so that
\[
\mathcal{B}_{0, \epsilon} \leq c_{2, \epsilon} \leq \max_{\Sigma \times [0,1]} E_\epsilon(\psi_{\epsilon, \rho_0}[z, t]).
\]

finally, in Step 3, we state the existence of a third critical level \( c_{3, \epsilon} \geq \mathcal{B}_{0, \rho_0} \).
The above levels are distinct because, by (4.1), (4.2),
\[
m < c_{1, \epsilon} < \mathcal{B}_0 \leq c_{2, \epsilon} < \mu_\alpha < c_{3, \epsilon} < 2^{1-2/p} m.
\]

Moreover, since, by (3.25), \( \lim_{\epsilon \to 0} \max_{z \in [R/2, R]} E_\epsilon(w_\epsilon(z)) = m \), and, by Proposition 3.6, the asymptotic estimate \( \lim_{\epsilon \to 0} \mathcal{B}_{0, \rho_\epsilon} = 2^{1-2/p} m \) holds, using again (4.1), we deduce
\[
\lim_{\epsilon \to 0} c_{1, \epsilon} = m, \quad \lim_{\epsilon \to 0} c_{2, \epsilon} \in [\mathcal{B}_0, \mu_\alpha] \subset (m, 2^{1-2/p} m), \quad \lim_{\epsilon \to 0} c_{3, \epsilon} = 2^{1-2/p} m,
\]
that, with (2.1), imply (1.2)–(1.4).

In what follows, for a given \( \gamma \in \mathbb{R} \), we set \( E_\gamma^\epsilon = \{ u \in M_\epsilon \mid E_\epsilon(u) \leq \gamma \} \).

**Step 1.** Let us denote by \( S_{R, \epsilon} = \max_{z \in [R/2, R]} E_\epsilon(w_\epsilon(z)) \). We assume, by contradiction, that
\[
\{ u \in M_\epsilon \mid A_\epsilon(u) \leq S_{R, \epsilon}, \nabla E_\epsilon \mid M_\epsilon(u) = 0 \} = \emptyset.
\]
Since the pair \((E_\epsilon, M_\epsilon)\) satisfies the Palais-Smale condition, using a well known deformation lemma (see f.i. [17]), we find a positive number \( \delta_1 > 0 \) and a continuous map \( \eta : [0,1] \times E_\epsilon^{S_{R, \epsilon}} \to E_\epsilon^{S_{R, \epsilon}} \) such that
\[
\eta(0, u) = u, \quad \forall u \in E_\epsilon^{S_{R, \epsilon}}, \\
\eta(1, E_\epsilon^{S_{R, \epsilon}}) \subseteq E^{A_\epsilon, -\delta_1}_\epsilon.
\]

Then we define \( \forall \theta \in [0,1] \) and \( \forall z \in \partial B(0, R/2\epsilon) \) the continuous map
\[
G(\theta, z) = \begin{cases} 
K(2\theta, z) & 0 \leq \theta \leq 1/2, \\
\beta(\eta(2\theta - 1, w_\epsilon(z))) & 1/2 \leq \theta \leq 1,
\end{cases}
\]
\( K \) being the map defined in (3.27). By the definition of \( K \), \( G(\theta, z) \notin \{0, \bar{x}_\epsilon\} \forall \theta \in [0,1/2] \forall z \in \partial B(0, R/2\epsilon) \), moreover, by the relations (4.1) \( S_{R, \epsilon} < \min(B_0, C_1) \leq \min(B_0, B_\epsilon) \), \( G(\theta, z) \notin \{0, \bar{x}_\epsilon\} \forall \theta \in [1/2, 1], \forall z \in \partial B(0, R/2\epsilon) \). Hence, taking into account that \( K(0, z) = z \forall z \in \partial B(0, R/2\epsilon) \), we deduce the existence of \( \hat{z} \in \partial B(0, R/2\epsilon) \) such that
\[
G(1, \hat{z}) = \beta \circ \eta(1, w_\epsilon(\hat{z})) \in L_\epsilon.
\]

On the other hand by (4.3) and (3.7)
\[
G(1, \partial B(0, R/2\epsilon)) \cap L_\epsilon = \emptyset,
\]
that contradicts (4.4).

**Step 2.** Set \( Q_{\rho_\alpha, \epsilon} = \max_{\Sigma \times [0,1]} E_\epsilon(\psi_{\epsilon, \rho_\alpha}[z, t]) \). We assume, by contradiction, that
\[
\{ u \in M_\epsilon \mid \mathcal{B}_{0, \epsilon} \leq E_\epsilon(u) \leq Q_{\rho_\alpha, \epsilon}, \nabla E_\epsilon \mid M_\epsilon(u) = 0 \} = \emptyset.
\]
then, arguing as in the previous step, we find a number \( \delta_2 > 0 \) and a continuous function \( \sigma : E^{Q_{\rho,\epsilon}} \rightarrow E^{B_0,\epsilon - \delta_2} \) such that

\[
\sigma (u) = u \quad \forall u \in E^{B_0,\epsilon - \delta_2}.
\]

Furthermore, by (3.12) and (3.16), \( \delta_2 \) can be chosen in such a way that

\[
\max_{\Sigma} E_{\epsilon} (\psi_{\epsilon,\rho_\alpha}[z,0]) < B_0,\epsilon - \delta_2.
\]

Setting

\[
\tilde{\Sigma} = \frac{\Sigma \times [0,1]}{\sim}
\]

where \( \sim \) identifies the points \((z,1)\), we define a map \( \mathcal{H} \) on \( \tilde{\Sigma} \) by

\[
\mathcal{H}[z,t] = \beta (\sigma (\psi_{\epsilon,\rho_\alpha}[z,t])).
\]

Since \( \epsilon < \epsilon_{\rho_\alpha} \), by Lemma 3.2, (4.5) and (4.6), \( \mathcal{H} \) maps \( \partial \tilde{\Sigma} \) in a set homotopically equivalent to \( \rho_\alpha \Sigma \) (and then to \( \Sigma \)) in \( \mathbb{R}^N \setminus \{0\} \). Moreover \( \mathcal{H} \) is continuous, so a point \((\tilde{z},\tilde{t}) \in \Sigma \) must exist, for which

\[
0 = \mathcal{H}(\tilde{z},\tilde{t}) = \beta (\sigma (\psi_{\epsilon,\rho_\alpha}[\tilde{z},\tilde{t}])).
\]

This is impossible because \( \sigma (\tilde{\Sigma}) \subset \sigma (E^{Q_{\rho,\epsilon}}) \subset E^{B_0,\epsilon - \delta_2} \) and by (3.4), so we are in contradiction.

**Step 3.** Considering a fixed \( \rho > \max(\hat{\rho}_\epsilon, \bar{\rho}_\epsilon, \rho_\alpha) \), taking into account (4.2) and using the same argument displayed in Step 2, we deduce, as desired, that

\[
\{ u \in M_\epsilon | B_{0,\epsilon} \leq E_{\epsilon}(u) \leq \max_{\Sigma \times [0,1]} E_{\epsilon}(\psi_{\epsilon,\rho_\alpha}[z,t]), \nabla E_{\epsilon}|M_\epsilon}(u) = 0 \} \neq \emptyset.
\]

REFERENCES


