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NON-COMPACT LAMINATION CONVEX HULLS

ENVELOPPE LAMINEUSEMENT CONVEXE NON COMPACT

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Dedicated to my friends Renáta and Ivan Zahrádka

ABSTRACT. – For K a compact set of $m \times n$ matrices, let $\mathrm{L}(K)$ denote the lamination convex hull of K.

We give an example of a compact set K of symmetric two by two matrices such that $\mathrm{L}(K)$ is not compact, and similar examples for separate convexity in $\mathbb{R}^3$ and bi-convexity in $\mathbb{R}^2 \times \mathbb{R}$. Furthermore we show that function $\tilde{\mathrm{L}}$, where $\tilde{\mathrm{L}}(K) = \overline{\mathrm{L}(K)}$, is not upper semi-continuous with respect to Hausdorff metric on the space of all compact sets K of diagonal 3×3 matrices.

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RÉSUMÉ. – Si K est un ensemble compact des matrices du type $m \times n$, $\mathrm{L}(K)$ signifie le plus petit ensemble lamineusement convexe contenant K. (Un ensemble K est lamineusement convexe si $[a, b] \subset K$ pour tous $a, b \in K$ tels que $a - b$ est une matrice de rang 1.)

Nous démontrons qu'il y a K, un ensemble compact des matrices symétriques d'ordre 2 tel que $\mathrm{L}(K)$ ne soit pas compact. Nous présentons aussi des exemples similaires pour convexité séparée dans $\mathbb{R}^3$ et bi-convexité dans $\mathbb{R}^2 \times \mathbb{R}$. En plus, nous démontrons que l'application $\tilde{\mathrm{L}} : K \mapsto \overline{\mathrm{L}(K)}$ n'est pas semi-continue superieurement sur l'espace des ensembles compacts de matrices diagonales d'ordre 3 muni de la métrique de Hausdorff.

Mots Clés: Enveloppe lamineusement convexe; Bi-convexité; Convexité separée; Enveloppe convexe de rang un

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1. Introduction

We denote by $\mathbb{M}^{m\times n}$ the set of all real $m \times n$ matrices with the $\mathbb{R}^{mn}$ norm; $\mathbb{M}^{n\times n}_{\mathrm{sym}}$, $\mathbb{M}^{n\times n}_{\mathrm{diag}}$ are the sets of symmetric and diagonal $n \times n$ matrices, respectively. A set $K \subset \mathbb{M}^{m\times n}$ is called *lamination convex* [4] if for all $A, B \in K$, which satisfy $\operatorname{rank}(A - B) = 1$, one has $(1-\lambda)A + \lambda B \in K$ for all $\lambda \in (0, 1)$. For a given $K \subset \mathbb{M}^{m\times n}$, the *lamination convex hull* $\mathrm{L}(K)$ is defined as the smallest lamination convex set which contains K [4].

Zhang [6] writes that "it is not clear in general whether for a compact set, the lamination convex hull is closed". In fact, it is easy to obtain a counter-example in $\mathbb{M}^{2\times 4}$ from a paper of Aumann and Hart [1], see Example 2.4. The main purpose of this paper is to give an example of a compact set $K \subset \mathbb{M}^{2\times 2}_{\mathrm{sym}}$ such that $\mathrm{L}(K)$ is not compact.

For convenience, we identify $\mathbb{M}^{2\times 2}_{\mathrm{sym}}$ with $\mathbb{R}^3$ by the linear bijection $\phi(x, y, z) = \begin{pmatrix} z+x & y \\ y & z-x \end{pmatrix}$. We say that $(x, y, z) \in \mathbb{R}^3$ is a *rank-one direction* if $\det\phi(x, y, z) = z^2 - x^2 - y^2 = 0$, that points A, B are *rank-one connected* if $B - A$ is a rank-one direction and that a set $K \subset \mathbb{R}^3$ is *lamination convex* if $(1-\lambda)A + \lambda B \in K$ whenever $A, B \in K$ are rank-one connected and $\lambda \in (0, 1)$. Again, the *lamination convex hull* $\mathrm{L}(K)$ of a set $K \subset \mathbb{R}^3$ is the smallest lamination convex set containing K. Obviously, $K \subset \mathbb{R}^3$ is lamination convex if and only if $\phi(K) \subset \mathbb{M}^{2\times 2}_{\mathrm{sym}}$ is lamination convex, and $\mathrm{L}(\phi(K)) = \phi(\mathrm{L}(K))$ for every $K \subset \mathbb{R}^3$.

THEOREM 1.1. – *There is a compact set $K \subset \mathbb{M}^{2\times 2}_{\mathrm{sym}}$ such that $\mathrm{L}(K)$ is not compact.*

Before proving the theorem for the symmetric two by two matrices in Section 3 we would like to consider the easier case of $\mathbb{M}^{m\times n}$ with $\max(m, n) > 2$ where examples can be constructed using related notions of separate convexity and bi-convexity. In Section 4 we explain consequences to upper semi-continuity of the mapping $K \mapsto \mathrm{L}(K)$.

2. Examples

The diagonal matrix $\begin{pmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{pmatrix}$ is of rank one if and only if exactly one of the numbers x, y, z is non-zero. Let us say that $K \subset R^n$ is *separately lamination convex* if K contains every segment with end-points in K which is parallel to one of the coordinate axes. This is equivalent to lamination convexity of the corresponding set of diagonal matrices. The *separately lamination convex hull* $\mathrm{L}_{\mathrm{sc}}(K)$ is defined to be the smallest separately lamination convex set in $\mathbb{R}^n$ that contains K.

Example 2.1 (*Separate convexity in $\mathbb{R}^3$ and diagonal* 3×3 *matrices*). – Let

$$K = \{(1,1,1)\} \cup \{(-1,0,0), (0,-1,0), (0,0,-1)\} \\ \cup \bigcup_{n\in\mathbb{N}} \left\{ \left(-1, \frac{1}{n}, \frac{1}{n}\right), \left(\frac{1}{n+1}, -1, \frac{1}{n}\right), \left(\frac{1}{n+1}, \frac{1}{n+1}, -1\right) \right\}. \tag{1}$$

By induction, $\mathrm{L}_{\mathrm{sc}}(K)$ contains each of the segments

$$\left[\left(\frac{1}{n}, \frac{1}{n}, \frac{1}{n}\right), \left(-1, \frac{1}{n}, \frac{1}{n}\right)\right] \ni \left(\frac{1}{n+1}, \frac{1}{n}, \frac{1}{n}\right),$$

$$\left[\left(\frac{1}{n+1},\frac{1}{n},\frac{1}{n}\right),\left(\frac{1}{n+1},-1,\frac{1}{n}\right)\right]\ni\left(\frac{1}{n+1},\frac{1}{n+1},\frac{1}{n}\right),$$
$$\left[\left(\frac{1}{n+1},\frac{1}{n+1},\frac{1}{n}\right),\left(\frac{1}{n+1},\frac{1}{n+1},-1\right)\right]\ni\left(\frac{1}{n+1},\frac{1}{n+1},\frac{1}{n+1}\right)$$

for every $n\in\mathbb{N}$. Consequently, $(0,0,0)$ belongs to the closure of $\mathrm{L}_{\mathrm{sc}}(K)$. On the other hand, it does not belong to $\mathrm{L}_{\mathrm{sc}}(K)$ since the set

$$\bigl\{(-1,0,0),(0,-1,0),(0,0,-1)\bigr\}$$
$$\cup\bigl\{A\in\mathbb{R}^3\colon \text{ at least two coordinates of } A \text{ are strictly positive}\bigr\}$$

is separately lamination convex and contains K. Thus $K\subset\mathbb{R}^3$ is compact, but $\mathrm{L}_{\mathrm{sc}}(K)$ is not and the same is true for the lamination convex hull of the compact set

$$\left\{\begin{pmatrix} x & 0 & 0\\ 0 & y & 0\\ 0 & 0 & z\end{pmatrix}\colon (x,y,z)\in K\right\}.$$

Example 2.2 (*Separate convexity in* $\mathbb{R}^2$ *and diagonal* 2×2 *matrices*). – The lamination convex hull of a compact subset of $\mathbb{M}^{2\times 2}_{\mathrm{diag}}$ is always compact. This follows by the next result which is due to Kirchheim [3].

PROPOSITION 2.3. – *If $C\subset\mathbb{R}^2$ is compact, then* $\mathrm{L}_{\mathrm{sc}}(C)$ *is compact as well.*

Proof (B. Kirchheim). – By x_1,x_2 we denote the two coordinates of $x\in\mathbb{R}^2$, and $e_1=(1,0)$, $e_2=(0,1)$. Let $\mathrm{L}^{(0)}_{\mathrm{sc}}(C)=C$ and for $k\in N$ let

$$\mathrm{L}^{(k)}_{\mathrm{sc}}(C)=\bigcup\bigl\{[y,z]\colon\ y,z\in\mathrm{L}^{(k-1)}_{\mathrm{sc}}(C),\ y_1=z_1 \text{ or } y_2=z_2\bigr\}.$$

Then $\mathrm{L}^{(k)}_{\mathrm{sc}}(C)$ are compact and $\mathrm{L}_{\mathrm{sc}}(C)=\bigcup_k\mathrm{L}^{(k)}_{\mathrm{sc}}(C)$. We say that $\mathrm{gen}_C(x)=k$ provided $x\in\mathrm{L}^{(k)}_{\mathrm{sc}}(C)\setminus\mathrm{L}^{(k-1)}_{\mathrm{sc}}(C)$. Suppose the claim fails. Then we can find a compact set $C\subset\mathbb{R}^2\setminus[-1,1]^2$ such that

$$0\in\overline{\mathrm{L}_{\mathrm{sc}}(C)}\setminus\mathrm{L}_{\mathrm{sc}}(C).$$

Obviously, for $i=1,2$ we find $\sigma_i\in\{-1,1\}$ such that

$$t\cdot\sigma_i e_i\notin\mathrm{L}_{\mathrm{sc}}(C)\quad\text{whenever } t\geqslant 0. \tag{2}$$

Moreover, we find $\varepsilon>0$ such that

$$\sigma_i x_i<-\varepsilon\quad\text{or}\quad |x_{3-i}|>\varepsilon\quad\text{for all } x\in C,\ i\in\{1,2\}. \tag{3}$$

Now we set

$$M_i=\bigl\{x\colon\ |x_{3-i}|\leqslant\varepsilon \text{ and } \sigma_i x_i\geqslant-\varepsilon\bigr\},\qquad M_i^+=\{x\in M_i\colon\ \sigma_i x_i\geqslant 0\}$$

and claim that

$$\mathrm{L}_{\mathrm{sc}}(C)\cap M_i^+=\emptyset. \tag{4}$$

Let us assume that (4) is not true for an $i \in \{1,2\}$. Let $g = \min\{\mathrm{gen}_C(x)\colon x \in \mathrm{L_{sc}}(C) \cap M_i^+\}$. Due to (3) we know $g \geqslant 1$ and find x in the compact set $M_i^+ \cap \mathrm{L}_{\mathrm{sc}}^{(g)}(C)$ maximizing the non-negative function $x \mapsto \sigma_i x_i$ over this set. By the definition of $\mathrm{L}_{\mathrm{sc}}^{(g)}(C)$ there are $y, z \in \mathrm{L}_{\mathrm{sc}}^{(g-1)}(C)$ such that $x \in \mathrm{L}_{\mathrm{sc}}^{(1)}(\{y,z\})$. From the maximality of $\sigma_i x_i$ we conclude that $\sigma_i y_i = \sigma_i z_i = \sigma_i x_i \geqslant 0$. The definition of g implies that $y, z \notin M_i^+$, hence $|y_{3-i}|, |z_{3-i}| > \varepsilon$ and $y_{3-i} z_{3-i} < 0$. Consequently,

$$\mathrm{L_{sc}}(C) \cap \{t \cdot \sigma_i e_i\colon t \geqslant 0\} \supset [y,z] \cap \{t \cdot \sigma_i e_i\colon t \geqslant 0\} = \{x_i e_i\},$$

a contradiction to (2) establishing (4).

Finally, denote by $g' \geqslant 1$ the minimum of gen_C over the nonvoid set $\mathrm{L_{sc}}(C) \cap M_1 \cap M_2$. Again, let x' maximize $\sigma_1 x_1'$ over $\mathrm{L}_{\mathrm{sc}}^{(g')}(C) \cap M_1 \cap M_2$ and suppose $x' \in \mathrm{L}_{\mathrm{sc}}^{(1)}(\{y', z'\})$ for $y', z' \in \mathrm{L}_{\mathrm{sc}}^{(g'-1)}(C)$. As before, we infer that $y_1' = z_1' = x_1'$, $|y_2'|, |z_2'| > \varepsilon$ and $y_2' z_2' < 0$. So

$$\mathrm{L_{sc}}(C) \cap M_2^+ \supset [y', z'] \cap M_2^+ \neq \emptyset,$$

which together with (4) finishes the proof. □

Example 2.4 (*Bi-convexity in* $\mathbb{R}^2 \times \mathbb{R}$ *and* 2×3 *matrices*). – A set $A \subset \mathbb{R}^k \times \mathbb{R}^l$ is *bi-convex* [1] if the sections A_x, A^y are convex for every $x \in \mathbb{R}^k$ and $y \in \mathbb{R}^l$. The *bi-convex hull* $\mathrm{L}_{(k,l)}(A)$ is defined accordingly. Obviously, A is bi-convex if and only if the set

$$\left\{ \begin{pmatrix} x_1 & x_2 & \dots & x_k & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & y_1 & y_2 & \dots & y_l \end{pmatrix} \in \mathbb{M}^{2 \times (k+l)}\colon \right.$$
$$\left. (x_1, x_2, \dots, x_k; y_1, y_2, \dots, y_l) \in A \right\}$$

is lamination convex. Aumann and Hart [1] constructed a compact set $K \subset \mathbb{R}^2 \times \mathbb{R}^2$ such that $\mathrm{L}_{(2,2)}(K)$ is not compact. We will show that this is possible in $\mathbb{R}^2 \times \mathbb{R}$ and hence also for matrices of the form $\left(\begin{smallmatrix} a & b & 0 \\ 0 & 0 & c \end{smallmatrix}\right)$.

Let $v_1 = (0,2)$, $v_2 = (-1,0)$, $v_3 = (1,-1)$, $v_4 = (2,1)$ be the usual four-point configuration. Let $w_1 = (0,1)$, $w_2 = (0,0)$, $w_3 = (1,0)$, $w_4 = (1,1)$ and

$$\begin{aligned} L_0 = {} & \big([0,1] \times [0,1]\big) \cup \big(\{0\} \times [0,2]\big) \cup \big([-1,1] \times \{0\}\big) \cup \big(\{1\} \times [-1,1]\big) \\ & \cup \big([0,2] \times \{1\}\big) = \mathrm{L_{sc}}\big(\{v_i, w_i\}\big). \end{aligned}$$

Finally, let $\tilde{K} = \mathcal{I}(([0,1] \times \{v_1, v_2, v_3, v_4\}) \cup (\{1\} \times \{w_1, w_2, w_3, w_4\}))$ and $L = \mathcal{I}(((0,1] \times L_0) \cup (\{0\} \times \{v_1, v_2, v_3, v_4\}))$, where $\mathcal{I}(x; y, z) = (x, y; z)$ identifies $\mathbb{R} \times \mathbb{R}^2$ with $\mathbb{R}^2 \times \mathbb{R}$. We claim that $\mathrm{L}_{(2,1)}(\tilde{K}) = L$ and this is not compact.

Let $w_i(t) = \mathcal{I}(t, w_i)$, $v_i(t) = \mathcal{I}(t, v_i)$. We have $w_i(1) \in \tilde{K}$ and then inductively $w_i(2^{-k}) \in \mathrm{L}_{(2,1)}(\tilde{K})$ for every $i \in \{4,3,2,1\}$ and $k \in \mathbb{N}$, because the following convex combinations are compatible with the definition of bi-convexity: $w_4(t/2) = \frac{1}{2} w_1(t) + \frac{1}{2} v_4(0)$ and $w_i(t) = \frac{1}{2} w_{i+1}(t) + \frac{1}{2} v_i(t)$ for $i = 3,2,1$. Now it is easy to see that $w_i(t) \in \mathrm{L}_{(2,1)}(\tilde{K})$ for every $t \in (0,1]$ and hence $L \subset \mathrm{L}_{(2,1)}(\tilde{K})$. On the other hand, L is bi-convex, so that $\mathrm{L}_{(2,1)}(\tilde{K}) \subset L$.

3. The proof of Theorem 1.1

Notation. – For $\alpha \in [0, \frac{\pi}{2}]$ let $e_i(\alpha) = (\sin\alpha + \cos\alpha, (-1)^i \sin\alpha, \alpha + 1)$ and $\gamma(\alpha) = (\sin\alpha, 0, \alpha)$. Let $E_0 = \{e_i(\alpha)\colon\ \alpha \in [0, \frac{\pi}{2}],\ i = 1, 2\}$.

LEMMA 3.1. – *For every* $0 < \alpha_2 < \alpha_1 < \frac{\pi}{2}$, $\gamma(\alpha_1) \in \overline{\mathrm{L}(E_0 \cup \{\gamma(\alpha_2)\})}$.

Proof. – For $i = 1, 2$, let $\Phi_i : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$\begin{aligned}\Phi_i\big((a,b,c),(x,y,z)\big) = \big(&(a-x)^2 + (b-y)^2 - (c-z)^2,\\ &\sin(z-1) + \cos(z-1) - x,\\ &(-1)^i \sin(z-1) - y\big).\end{aligned}$$

For every $\alpha \in [\alpha_2, \alpha_1]$ and $i = 1, 2$ we have $\Phi_i(\gamma(\alpha), e_i(\alpha)) = 0$, as well as $\det(\frac{\partial \Phi_i}{\partial(x,y,z)}(\gamma(\alpha), e_i(\alpha))) = 2\cos^2\alpha - 2 \neq 0$. By the implicit function theorem, there is $\delta_0 > 0$ and two smooth functions $\varphi_1, \varphi_2 : \mathcal{U}_{\delta_0} \to R^3$ defined on the δ_0-neighborhood $\mathcal{U}_{\delta_0}$ of $\{\gamma(\alpha)\colon\ \alpha \in [\alpha_2, \alpha_1]\}$ such that $\Phi_i(w, \varphi_i(w)) = 0$ for $w \in \mathcal{U}_{\delta_0}$ and $\varphi_i(\gamma(\alpha)) = e_i(\alpha)$ for $\alpha \in [\alpha_2, \alpha_1]$. Note that by the definition of Φ_i, $\varphi_i(w) - w$ is a rank-one direction and $\varphi_i(w) = e_i(\alpha)$ for all $w \in \mathcal{U}_{\delta_0}$, where $\alpha + 1$ is the third coordinate of $\varphi_i(w)$. By making δ_0 smaller, we may ensure that $\varphi_i(w) \in E_0$ for $w \in \mathcal{U}_{\delta_0}$. Let $u_i(w) = \varphi_i(w) - w$. Replacing δ_0 by a smaller number again, there is $K > 0$ such that the functions u_1, u_2 are K-Lipschitz on $\mathcal{U}_{\delta_0}$ and $\|u_1(w)\|, \|u_2(w)\| \leqslant K$ for $w \in \mathcal{U}_{\delta_0}$.

It is easy to check that γ satisfies the equation

$$\dot\gamma(\alpha) = \frac{u_1(\gamma(\alpha)) + u_2(\gamma(\alpha))}{2}. \tag{5}$$

Next, we will approximate the solution γ by a piecewise linear curve with derivatives given by u_1 on odd and by u_2 on even segments. We do an easy error estimate usual in numerical analysis.

Let $\delta > 0$ be given. Find $n \in \mathbb{N}$ such that, for $h = (\alpha_1 - \alpha_2)/n$, $Kh < \delta_0$ and

$$\frac{h}{2}(2\,\mathrm{Lip}\,\gamma + K)\big((1 + hK)^n - 1\big) < \min\left(\delta, \frac{\delta_0}{2}\right).$$

For $k = 1, \ldots, n$, define

$$w_0 = \gamma(\alpha_2), \qquad w_{k-\frac{1}{2}} = w_{k-1} + \frac{h}{2}u_1(w_{k-1}), \qquad w_k = w_{k-\frac{1}{2}} + \frac{h}{2}u_2(w_{k-\frac{1}{2}}). \tag{6}$$

Let $\varepsilon_k = \|w_k - \gamma(\alpha_2 + kh)\|$, $k = 0, 1, \ldots, n$. Then

$$\begin{aligned}\varepsilon_{k+1} &= \big\|w_{k+1} - \gamma\big(\alpha_2 + (k+1)h\big)\big\|\\ &= \left\| w_k - \gamma(\alpha_2 + kh) + \int\limits_{\alpha_2+kh}^{\alpha_2+(k+1)h} \frac{u_1(w_k) + u_2(w_{k+\frac{1}{2}})}{2} - \dot\gamma(\alpha)\,\mathrm{d}\alpha \right\|\end{aligned}$$

and hence, by (5),

$$\begin{aligned}
\varepsilon_{k+1} &\leqslant \varepsilon_k + \frac{1}{2} \int_{\alpha_2+kh}^{\alpha_2+(k+1)h} \|u_1(w_k) - u_1(\gamma(\alpha))\| + \|u_2(w_{k+\frac{1}{2}}) - u_2(\gamma(\alpha))\| \,\mathrm{d}\alpha \\
&\leqslant \varepsilon_k + \frac{h}{2}\big(\operatorname{Lip} u_1(h \operatorname{Lip}\gamma + \varepsilon_k) + \operatorname{Lip} u_2\big(h \operatorname{Lip}\gamma + h\|u_1(w_k)\| + \varepsilon_k\big)\big) \\
&\leqslant A\varepsilon_k + B
\end{aligned}$$

where $A = (1 + hK)$ and $B = \frac{h^2K}{2}(2\operatorname{Lip}\gamma + K)$. We have $\varepsilon_0 = 0$ and, by induction,

$$\begin{aligned}
\varepsilon_k \leqslant B\big(1 + A + A^2 + \cdots + A^{k-1}\big) &= B\big(A^k - 1\big)/(A - 1) \\
&= \frac{h}{2}(2\operatorname{Lip}\gamma + K)\big((1 + hK)^k - 1\big) \\
&< \min(\delta, \delta_0/2).
\end{aligned}$$

Hence $w_k, w_{k+\frac{1}{2}} \in \mathcal{U}_{\delta_0}$ (so that the sequence is well defined) and $\|\gamma(\alpha_1) - w_n\| < \delta$.

Furthermore, $w_{k+\frac{1}{2}} \in [w_k, \varphi_1(w_k)]$, $w_{k+1} \in [w_{k+\frac{1}{2}}, \varphi_2(w_{k+\frac{1}{2}})]$ and the two segments have rank-one directions, so that $w_0, w_{\frac{1}{2}}, \ldots, w_n$ belong to the lamination convex hull of $E_0 \cup \{w_0\} = E_0 \cup \{\gamma(\alpha_2)\}$. Since $\delta > 0$ was arbitrarily small, $\gamma(\alpha_1)$ lies in its closure. □

Remark 3.2. – Under the assumption of Lemma 3.1 we have that $\gamma(\alpha_1)$ belongs to the rank-one convex hull of $E_0 \cup \{\gamma(\alpha_2)\}$. Also, the corresponding laminate μ with barycentre in $\gamma(\alpha_1)$ can be given explicitly:

$$\mu(A) = \mathrm{e}^{-(\alpha_1-\alpha_2)}\delta_{\gamma(\alpha_2)}(A) + \frac{1}{2}\sum_{i=1}^{2} \int_{(\alpha_2,\alpha_1)\cap \mathrm{e}_i^{-1}(A)} \mathrm{e}^{-(\alpha_1-\alpha)}\,\mathrm{d}\alpha, \tag{7}$$

where $\delta_{\gamma(\alpha_2)}$ is the Dirac measure at $\gamma(\alpha_2)$.

Indeed, (6) determinates prelaminate μ_n with barycentre $w_n^{(n)}$ supported by finite set $\{\gamma(\alpha_2); \varphi_1(w_{k-1}^{(n)}), \varphi_2(w_{k-\frac{1}{2}}^{(n)}),\ k = 1, \ldots, n\} \subset K$, recall that $u_i(w) = \varphi_i(w) - w$ is a rank-one direction. We added indices (n) to emphasize that points w_s depend on n as well. A calculation shows that the weak limit of μ_n is μ; the barycentre of μ is $\lim w_n^{(n)} = \gamma(\alpha_1)$.

Notation. – Let

$$\begin{aligned}
x(\alpha, t) &= \sin\alpha + t\cos\alpha, \\
y(\alpha, t) &= t\sin\alpha, \\
z(\alpha, t) &= \alpha + t, \\
\varphi(\alpha, t) &= \big(x(\alpha, t), z(\alpha, t)\big).
\end{aligned}$$

Also let $P = [0, \frac{\pi}{2}] \times [0, 1]$ and $D = \varphi(P) = \{(x, z)\colon\ z \in [0, \frac{\pi}{2}],\ \sin z \leqslant x \leqslant \min(1, z)\} \cup \{(x, z)\colon\ z \in [1, 1 + \frac{\pi}{2}],\ 1 \leqslant x \leqslant \sqrt{2}\sin(z + \frac{\pi}{4} - 1)\}$. The function $Y : D \to [0, \infty)$ is going to be defined by

$$Y\big(\varphi(\alpha, t)\big) = y(\alpha, t) \quad (\alpha, t) \in P. \tag{8}$$

LEMMA 3.3. – *Let $\alpha_1, \alpha_2 \in [0, \frac{\pi}{2}]$ and $\alpha_1 \neq \alpha_2$. Then $\varphi(\alpha_1, t_1) = \varphi(\alpha_2, t_2)$ if and only if*

$$t_1 = t_1(\alpha_1, \alpha_2) = \frac{\sin\alpha_1 - \sin\alpha_2 - (\alpha_1 - \alpha_2)\cos\alpha_2}{\cos\alpha_2 - \cos\alpha_1},$$
$$t_2 = t_2(\alpha_1, \alpha_2) = \frac{\sin\alpha_1 - \sin\alpha_2 - (\alpha_1 - \alpha_2)\cos\alpha_1}{\cos\alpha_2 - \cos\alpha_1}. \tag{9}$$

If $\alpha_1 > \alpha_2$ then $t_1 < 0$ and $t_2 > 0$.

Proof. – Formulae (9) are obvious. Assume $\alpha_1 > \alpha_2$. Let $f(x) = \sin x - \sin\alpha_2 - (x - \alpha_2)\cos\alpha_2$. Then $f(\alpha_2) = 0$ and $f'(x) = \cos x - \cos\alpha_2 < 0$ for $\alpha_2 < x \leqslant \frac{\pi}{2}$, hence $f(\alpha_1) < 0$ and $t_1 = f(\alpha_1)/(\cos\alpha_2 - \cos\alpha_1) < 0$. Similarly, for $g(x) = \sin\alpha_1 - \sin x - (\alpha_1 - x)\cos\alpha_1$ we have $g(\alpha_1) = 0$ and $g'(x) = -\cos x + \cos\alpha_1 < 0$ for $0 \leqslant x < \alpha_1$. Thus $g(\alpha_2) > 0$ and $t_2 > 0$. □

LEMMA 3.4. – *Let the function t_2 be defined by formula* (9) *for $\alpha_2 < \alpha_1$ and by $t_2(\alpha_1, \alpha_2) = 0$ if $\alpha_1 = \alpha_2$. Let $\alpha_1 \in (0, \frac{\pi}{2}]$ be fixed. Then*

$$D_{\alpha_1} = \{\varphi(\alpha_2, t)\colon\ \alpha_2 \in [0, \alpha_1], t \in [t_2(\alpha_1, \alpha_2), 1]\}$$

is a convex subset of D.

Proof. – It is easily seen that

$$\chi(z) = \begin{cases} z, & z \in [0, 1], \\ \sqrt{2}\sin(z + \frac{\pi}{4} - 1), & z \in [1, 1 + \frac{\pi}{2}] \end{cases}$$

is a concave function on $I = [0, 1 + \frac{\pi}{2}]$ and that D_{α_1} is the part of its subgraph $\{(x, z)\colon z \in I,\ x \leqslant \chi(z)\}$ which lies above the segment $\{\varphi(\alpha_1, t_1)\colon\ t_1 \in [t_1(\alpha_1, 0), 0] \cup [0, 1]\} = \{\varphi(\alpha_2, t_2(\alpha_1, \alpha_2))\colon\ \alpha_2 \in [0, \alpha_1]\} \cup \{\varphi(\alpha_1, t)\colon\ t \in [0, 1]\}$. (Recall that the functions t_1, t_2 came from $\varphi(\alpha_1, t_1) = \varphi(\alpha_2, t_2)$.) □

LEMMA 3.5. – *The function $Y : D \to [0, \infty)$ is well defined by* (8)*. Y is a $\mathcal{C}^\infty$-smooth function on the interior of D.*

Proof. – By Lemma 3.3, $\varphi : P \to D$ is a bijection. The Jacobi determinant of φ is $-t\sin\alpha \neq 0$ on int P, so that φ is a $\mathcal{C}^\infty$-diffeomorphism of int P onto int D. □

DEFINITION 3.6. – *Let $T = \{(x, y, z)\colon\ (x, z) \in D, |y| \leqslant Y(x, z)\}$ and let $F_i(\alpha, t) = (x(\alpha, t), (-1)^i y(\alpha, t), z(\alpha, t))$ so that $F_2(P)$ is the "front" surface of T. Assume $(\alpha, t) \in$ int P, $S = F_2(\alpha, t)$ and $v = A\,\partial_\alpha F_2(\alpha, t) + B\,\partial_t F_2(\alpha, t)$ where $(A, B) \neq (0, 0)$. The line $L = S + \mathbb{R}\,v$ will be called a* tangent *at the point S. It is said to be an* outer *or* inner *or* surface *tangent if there is $\varepsilon > 0$ such that, for every $r \in (-\varepsilon, 0) \cup (0, \varepsilon)$, $S + rv \notin T$ or $S + rv \in T$ or $S + rv \in F_2(P)$, respectively. Tangent L is said to be* rank-one *if v is a rank-one direction. The same terminology will be used for any segment $L = S + [r_1, r_2]v$, $r_1 < 0 < r_2$.*

Remark. – In order to give an interpretation of what follows, let us recall that if $\tilde{Y} : \tilde{D} \to \mathbb{R}$ is a function which has the second differential $\mathrm{D}^2\tilde{Y}$ negatively semi-definite everywhere on a convex set $\tilde{D}$, then the set $\tilde{T} = \{(x, y, z)\colon\ (x, z) \in \tilde{D}, |y| \leqslant \tilde{Y}(x, z)\}$ is convex.

In our case, $\mathrm{D}^2 Y$ is "negatively semi-definite with respect to a set of directions" (see Lemma 3.7) and we are going to prove that T is lamination convex (Proposition 3.11). Note that the set of directions is defined in terms of *all* variables including the dependent one and therefore it depends on the gradient of Y. Lemma 3.9 says that D is "sufficiently convex" (which is a property of the pair D, Y).

LEMMA 3.7. – *With the above notation, assume L is a rank-one tangent. Then either it is an outer tangent, or it is a surface tangent with the direction $v = \partial_t F_2(\alpha, t)$.*

Proof. – Let

$$u_1 = \partial_\alpha F_2(\alpha, t) = (\cos\alpha - t\sin\alpha, t\cos\alpha, 1),$$
$$u_2 = \partial_t F_2(\alpha, t) = (\cos\alpha, \sin\alpha, 1).$$

A simple calculation shows that $v = Au_1 + Bu_2$ is a rank-one direction if and only if

$$(A, B) = k\bigl(2\sin^2\alpha, t^2 - \sin^2\alpha - 2t\cos\alpha\sin\alpha\bigr) \quad (k \in \mathbb{R}) \tag{10}$$

or $A = 0$. In the second case, v is a multiple of u_2 and L is a surface tangent because $F_2(\alpha, t)$ is a linear function of t.

Assume (10) holds true. Let us write $\mathrm{D}f$ and $\mathrm{D}^2 f$ for the first and second differential of a function f at the point $S_0 = \varphi(\alpha, t)$, respectively. ($\mathrm{D}^2 f$ is a quadratic form.) We will write $\mathrm{D}f(w) = \langle \mathrm{D}f, w\rangle$ and $\mathrm{D}^2 f(w)$ when they are applied to a direction w. The set $F_2(P)$ can be viewed as the graph of the function Y (with interchanged second and third coordinates) and T is contained in the subgraph. To show that the tangent L is outer it is enough to verify that the second derivative of Y at S_0 in the direction $v_0 = A\partial_\alpha\varphi(S_0) + B\,\partial_t\varphi(S_0)$ equals

$$\mathrm{D}^2 Y(v_0) = -8k^2\sin^4\alpha\cos\alpha < 0. \tag{11}$$

Although this could be done directly, we suggest the following way which reduces the size of expressions involved. Let $\omega(s) = \varphi(\alpha + As, t + Bs)$. Then

$$\frac{\partial^2}{\partial s^2} Y\bigl(\omega(s)\bigr)\bigg|_{s=0} = \mathrm{D}^2 Y(v_0) + \bigl\langle \mathrm{D}Y, \bigl(\mathrm{D}^2 x(A, B), \mathrm{D}^2 z(A, B)\bigr)\bigr\rangle. \tag{12}$$

On the other hand, $Y(\omega(s)) = y(\alpha + As, t + Bs) = (t + Bs)\sin(\alpha + As)$, so that

$$\frac{\partial^2}{\partial s^2} Y\bigl(\omega(s)\bigr)\bigg|_{s=0} = 2AB\cos\alpha - A^2 t\sin\alpha. \tag{13}$$

Differentiating (8) and solving the resulting equation we easily obtain

$$\mathrm{D}Y = \left(\frac{t\cos\alpha - \sin\alpha}{-t\sin\alpha}, \frac{\cos\alpha\sin\alpha - t}{-t\sin\alpha}\right). \tag{14}$$

The calculation of D^2x and D^2z is straightforward and gives

$$D^2x(A,B) = -A^2(\sin\alpha + t\cos\alpha) - 2AB\sin\alpha, \quad D^2z(A,B) = 0. \tag{15}$$

Eqs. (12)–(15) imply

$$D^2Y(v_0) = -A\left(\frac{At}{\sin\alpha} - \frac{(A+2B)\sin\alpha}{t}\right).$$

Using (10) we get (11). □

LEMMA 3.8. – *Let $(\alpha_1, t_1) \in P$, $\alpha_2 \in [0, \frac{\pi}{2}]$ and $t_2 < 0$. Let $\varphi(\alpha_1, t_1) = \varphi(\alpha_2, t_2)$. Then $0 \leqslant y(\alpha_1, t_1) < -y(\alpha_2, t_2)$.*

Proof. – By Lemma 3.3, $\alpha_1 \leqslant \alpha_2$. If $\alpha_1 = \alpha_2$ then $0 \leqslant t_1 = t_2 < 0$. Thus $\alpha_1 < \alpha_2$ and by (9)

$$y(\alpha_2, t_2) + y(\alpha_1, t_1) = \frac{\sin^2\alpha_2 - \sin^2\alpha_1 - (\alpha_2 - \alpha_1)\sin(\alpha_2 + \alpha_1)}{\cos\alpha_1 - \cos\alpha_2} < 0$$

where the inequality comes from

$$\begin{aligned}(\alpha_2 - \alpha_1)\sin(\alpha_2 + \alpha_1) &> \sin(\alpha_2 - \alpha_1)\sin(\alpha_2 + \alpha_1)\\ &= \frac{1}{2}(\cos 2\alpha_1 - \cos 2\alpha_2)\\ &= \frac{1}{2}\left(1 - 2\sin^2\alpha_1 - 1 + 2\sin^2\alpha_2\right).\end{aligned}$$

Thus $y(\alpha_1, t_1) < -y(\alpha_2, t_2)$. □

LEMMA 3.9. – *Let $A = (a_1, a_2, a_3) \in T$ and $B = (b_1, b_2, b_3) \in T$ be such that $B - A$ is a rank-one direction. Then $[(a_1, a_3), (b_1, b_3)] \subset D$.*

Proof. – By assumptions, $A_0 = (a_1, a_3) \in D$ and $B_0 = (b_1, b_3) \in D$, thus there exist $(\alpha_1, \tau_1), (\alpha_2, \tau_2) \in P$ such that $A_0 = \varphi(\alpha_1, \tau_1)$, $B_0 = \varphi(\alpha_2, \tau_2)$. Furthermore, $|a_2| \leqslant y(\alpha_1, \tau_1)$, $|b_2| \leqslant y(\alpha_2, \tau_2)$. If $\alpha_2 = \alpha_1$ then obviously $[A_0, B_0] \subset D$. We may assume e.g. $\alpha_2 < \alpha_1$.

Let $V_0 = \{(x, y, z)\colon\ x^2 + y^2 - z^2 < 0,\ z < 0\}$. V_0 is an open convex cone. A point X is rank-one connected to $(0, 0, 0)$ if and only if it belongs to ∂V_0 when it is below $(0, 0, 0)$ or $X \in -\partial V_0$ when X is above $(0, 0, 0)$ (“below” and “above” refers to the value of the third coordinate). It is easily seen that if L is a line with rank-one direction and $(0, 0, 0) \notin L$ then L intersects ∂V_0 in at most one point and, therefore, $L \cap V_0$ is either an open half-line directed “downwards” or empty.

Let $V_A = A + V_0$ and $V_1 = \gamma(\alpha_1) + V_0$. The point $\gamma(\alpha_1)$ is rank-one connected to $F_i(\alpha_1, \tau_1)$ and $A \in [F_1(\alpha_1, \tau_1), F_2(\alpha_1, \tau_1)]$ hence $A \in -\overline{V_0} + \gamma(\alpha_1)$, $\gamma(\alpha_1) \in \overline{V_A}$ and $V_1 \subset V_A$.

Let $t_1 < 0$, $t_2 > 0$ solve the equation $\varphi(\alpha_1, t_1) = \varphi(\alpha_2, t_2)$, cf. Lemma 3.3. Since $\gamma(\alpha_1)$ is also rank-one connected to the two points $F_i(\alpha_1, t_1)$, $i = 1, 2$, we have $F_i(\alpha_1, t_1) \in \overline{V_1} \subset \overline{V_A}$. By Lemma 3.8, with indices 1,2 interchanged, $0 \leqslant y(\alpha_2, t_2) < -y(\alpha_1, t_1)$. Thus $F_1(\alpha_2, t_2)$, $F_2(\alpha_2, t_2)$ are in the open segment $(F_1(\alpha_1, t_1), F_2(\alpha_1, t_1)) \subset V_A$.

Since the direction $\partial_t F_i(\alpha_1, t)$ of the line $\{F_i(\alpha_2, t)\colon t \in \mathbb{R}\}$ is a rank-one vector directed upwards, we have $F_i(\alpha_2, t) \in V_A$ for every $t \leqslant t_2$. Now, $B \in [F_1(\alpha_2, \tau_2), F_2(\alpha_2, \tau_2)]$ is not in V_A since it is rank-one connected to A. Therefore $\tau_2 > t_2 = t_2(\alpha_1, \alpha_2)$ and hence $B_0 = \varphi(\alpha_2, \tau_2) \in D_{\alpha_1}$.

By Lemma 3.4, it follows that $[A_0, B_0] \subset D_{\alpha_1} \subset D$. □

LEMMA 3.10. – *Let $A = (a_1, a_2, a_3) \in T$, $B = (b_1, b_2, b_3) \in T$, $A_0 = (a_1, a_3)$, $B_0 = (b_1, b_3)$. Assume A and B are rank-one connected. Then the open segment (A_0, B_0) does not contain any point $\varphi(\alpha, 0)$, $\alpha \in [0, \frac{\pi}{2}]$. Furthermore (A_0, B_0) contains no point $\varphi(0, t)$, $t \in [0, 1]$, unless $[A, B] \subset [(0, 0, 0), (1, 0, 1)] \subset T$.*

Proof. – Let $v = (v_1, v_2, v_3) = B - A$. Assume there is $\alpha \in [0, \frac{\pi}{2}]$ such that $S_0 = \varphi(\alpha, 0) \in (A_0, B_0)$. Clearly $\alpha \neq 0$, because $D \subset \mathbb{R} \times \mathbb{R}^+ \cup \{(0, 0)\}$. Since S_0 is a smooth point of the boundary of D and $[A_0, B_0] \subset D$ by Lemma 3.9, we have $(v_1, v_3) = k\partial_\alpha \varphi(\alpha, 0) = k(\cos\alpha, 1)$ for some k. Thus $v_2 = \pm k \sin\alpha$ because v is assumed to be a rank-one direction. There is no loss of generality in assuming $v_2 > 0$, so that $v = k.(\cos\alpha, \sin\alpha, 1)$.

Note that $v = k\partial_t F_2(\alpha, 0)$ and F_2 is linear in t. Thus $F_2(\alpha, t) = A$ or $F_2(\alpha, t) = B$ for some $t < 0$. However, Lemma 3.8 immediately implies that $F_2(\alpha, t) \notin T$ for every $t < 0$ which is a contradiction.

The second assertion is obvious since segment $M = [(0, 0), (1, 1)]$ is extremal in $D \subset \{(x, z)\colon z \geqslant x\}$ and $Y = 0$ on M. □

PROPOSITION 3.11. – *The set T is lamination convex. Any set $\tilde{T}$ such that $T \setminus \{\gamma(\alpha)\colon \alpha \in (0, \frac{\pi}{2})\} \subset \tilde{T} \subset T$ is lamination convex, too.*

Proof. – Assume that T is not lamination convex. Then there is $A = (a_1, a_2, a_3) \in T$, $B = (b_1, b_2, b_3) \in T$ such that segment $[A, B]$ is not a subset of T and $B - A$ is a rank-one direction. We will gradually change the segment with the goal to find an inner tangent parallel to the original $[A, B]$.

Let $A_0 = (a_1, a_3)$, $B_0 = (b_1, b_3)$ and $A_0' = (a_1, 0, a_3)$, $B_0' = (b_1, 0, b_3)$. Obviously $A_0 \neq B_0$. By Lemma 3.9, $[A_0, B_0] \subset D$.

We claim that $(A_0, B_0) \subset \operatorname{int} D$ and thus $(A_0', B_0') \subset \operatorname{int} T$. If not, then there is a point $(c_1, c_2, c_3) \in (A, B)$ such that $(c_1, c_2) = \varphi(\alpha_3, t_3) \in \partial D$. Hence $(\alpha_3, t_3) \in \partial P$. The shape of domain D rules out that $t_3 = 1$. By Lemma 3.10, $t_3 \neq 0$ and $\alpha_3 \neq 0$. Thus $\alpha_3 = \frac{\pi}{2}$ and $a_1 = b_1 = c_1 = 1$, $c_3 \geqslant \frac{\pi}{2}$. Assume $a_3 \geqslant c_3 \geqslant \frac{\pi}{2}$ (otherwise $b_3 \geqslant c_3 \geqslant \frac{\pi}{2}$ which is similar). Then $|a_2| \leqslant Y(1, a_3) = y(\frac{\pi}{2}, a_3 - \frac{\pi}{2}) = a_3 - \frac{\pi}{2}$. If $b_3 < \frac{\pi}{2}$ then, by Lemma 3.8, $|b_2| \leqslant Y(1, b_3) < -y(\frac{\pi}{2}, b_3 - \frac{\pi}{2}) = \frac{\pi}{2} - b_3$, hence $|a_2 - b_2| \leqslant |a_2| + |b_2| < a_3 - b_3$ and, in consequence, A and B are not rank-one connected. If $b_3 \geqslant \frac{\pi}{2}$ then $Y(1, z) = z - \frac{\pi}{2}$ is linear on $[b_3, a_3]$ and $[A, B] \subset T$. Since any case leads to a contradiction, we see that, indeed, $(A_0, B_0) \subset \operatorname{int} D$.

Eventually truncating the segment at a point $(x, 0, z) \in T$, with $(x, z) \in D$, we may assume $a_2 b_2 \geqslant 0$. We lose no generality assuming $0 \leqslant a_2$, $0 \leqslant b_2$ because T is symmetric. Finally, we can exchange A, B to have $0 \leqslant a_2 \leqslant b_2$.

Now, we will shift the segment $[A, B]$. For $\tau \in [0, b_2]$, let $A_\tau = (a_1, a_2 - \tau, a_3)$, $B_\tau = (b_1, b_2 - \tau, b_3)$, and $L_\tau = [A_\tau, B_\tau] \cap \{(x, y, z)\colon y \geqslant 0\}$. That means $L_\tau = [\tilde{A}_\tau, B_\tau]$ where

$\tilde{A}_\tau = A_\tau$ for $\tau \leqslant a_2$ and $\tilde{A}_\tau \in (A'_0, B'_0)$ for $a_2 < \tau < b_2$. Recall that $(A'_0, B'_0) \subset \operatorname{int} T$. Let $\operatorname{int}_D T$ be the interior of T relative to $\{(x,y,z)\colon (x,z) \in D\}$. For $\tau > 0$, $\tilde{A}_\tau, B_\tau \in \operatorname{int}_D T$.

Let $\tau_1 = \sup\{\tau \in [0, b_2]\colon L_\tau \setminus T \neq \emptyset\}$. Obviously $L_{b_2} \subset T$ and hence $\tau_1 \leqslant b_2$. Since T is closed we have $L_{\tau_1} \subset T$ and $\tau_1 > 0$. Since the endpoints of L_{τ_1} are in $\operatorname{int}_D T$, L_{τ_1} must have an interior point $S = (s_1, s_2, s_3)$ which belongs to the boundary of T, i.e. $s_2 = Y(s_1, s_3)$. Since $(s_1, s_3) \in \operatorname{int} D$ and Y is a smooth function on $\operatorname{int} D$, L_{τ_1} is a rank-one inner tangent.

By Lemma 3.7, we know that L_{τ_1} must be a surface tangent with the direction $\partial_t F_2(\varphi^{-1}(s_1, s_3))$. Since F_2 is linear in t, L_{τ_1} is in the surface $F_2(P)$. However, $\tilde{A}_{\tau_1}, B_{\tau_1} \in \operatorname{int}_D T$. Thus there exists no segment $[A, B]$ as above and T is lamination convex.

As regards points $\gamma(\alpha)$, $\alpha \in (0, \frac{\pi}{2})$, the first part of Lemma 3.10 says that they may be freely removed from T and the set remains lamination convex. □

Remark. – For $\alpha \in (0, \frac{\pi}{2})$, not only the set $T \setminus \{\gamma(\alpha)\}$ is lamination convex. Also for $\hat{T} = T \setminus \{F_i(\alpha, t)\colon t \in [0,1), i = 1, 2\}$ the same is true. Indeed if $t \in (0,1)$ and $F_2(\alpha, t) \in (A, B)$ where the segment (A, B) has rank-one direction and $A, B \in \hat{T}$, then by Lemma 3.7, (A, B) is a surface tangent with the direction $\partial_t F_2(\alpha, t)$. Hence A, B are in the segment we removed from T, a contradiction.

Proof of Theorem 1.1. – Let $0 < \alpha_2 < \alpha_1 < \frac{\pi}{2}$ and

$$\begin{aligned} K &= E_0 \cup \{\gamma(\alpha_2)\} \\ &= \left\{ \big(\sin\alpha + \cos\alpha, (-1)^i \sin\alpha, \alpha + 1\big)\colon \alpha \in \left[0, \frac{\pi}{2}\right],\ i = 1, 2 \right\} \cup \{(\sin\alpha_2, 0, \alpha_2)\}. \end{aligned}$$

Then the point $(\sin\alpha_1, 0, \alpha_1)$ does not belong to the lamination convex hull of K (Proposition 3.11) but does belong to its closure (Lemma 3.1). For symmetric two by two matrices, the set

$$\left\{ \begin{pmatrix} z + x & y \\ y & z - x \end{pmatrix}\colon (x, y, z) \in K \right\}$$

serves as an example. □

Remarks. –

(1) It is very easy to see that for every compact set K, $\mathrm{L}(K)$ is an F_σ-set. Is it always a G_δ-set?

(2) We believe that in some classes of compact subsets of $\mathbb{M}^{2\times 2}_{\text{sym}}$ it is typical, in a sense, for a compact K to have non-closed $\mathrm{L}(K)$. For example if K consists of two curves (or segments) and a point which is rank-one connected to both curves, it is likely that the solution of an equation similar to (5) will move outside $\mathrm{L}(K)$ *unless* the critical area is covered by other rank-one connections (far from or closely related to the one in (5)). Note, however, that the convex combination coefficients on the right-hand side of (5) have to be properly chosen and, in general, they will depend on α. If the above works when the two curves are segments with rank-one directions, K could be replaced by a five-point set.

(3) The first compact $K \subset \mathbb{R}^3 \cong \mathbb{M}^{2\times 2}_{\text{sym}}$ for which we had proven non-compactness of $\mathrm{L}(K)$ was

$$K = \{(x, y, 0)\colon\ 4(x-1)^2 + y^2 \leqslant 4\} \cup \{(a_0, 0, \sqrt{8(a_0 - 2)}\,)\},$$

where $a_0 \in (2, 4]$. The lamination convex superset T of this compact is $\{r((1-t)x + t(4-x), \pm(1-t)\sqrt{4-4(x-1)^2}, t\sqrt{8(2-x)})\colon\ r \in [0,1],\ t \in [0,1],\ x \in [0,2]\}$. The method of the proof was quite similar: Contracting a "bad" segment towards point $(0,0,0)$, an inner rank-one tangent would be found, but none exists except "canonical" surface tangents. The sin-based curves in our example were chosen because they lead to much easier calculations at the cost of some additional reasoning.

(4) We do not know whether the set T from Definition 3.6 (considered as a subset of $\mathbb{M}^{2\times 2}$) is rank-one convex or even quasiconvex. Therefore we do not know what are rank-one convex and quasiconvex hulls of K. In the case T would be rank-one convex, the question Q1 of [2, p. 87 (§ 4.1.2)] would be answered negatively with an impact on understanding of rank-one extreme points.
The set T is not polyconvex. Indeed, taking three matrices $M = \{\gamma(0), e_1(\frac{\pi}{2}), e_2(\frac{\pi}{2})\}$ and $t = (\frac{\pi^2}{2} + 2\pi - 2)/(\pi^2 + 4\pi) \doteq 0.41$, the matrix $X = (1-2t)\gamma(0) + te_1(\frac{\pi}{2}) + te_2(\frac{\pi}{2})$ belongs to the polyconvex hull of M since the determinants of the three matrices are $d_0 = 0$, $d_1 = d_2 = \frac{\pi^2}{4} + \pi - 1$ and it is easy to check that determinant of the matrix X equals $(1-2t)d_0 + td_1 + td_2$. On the other hand, $X \notin T$ since it does not lie "above" D. Without giving any details we note that X can be separated from K by a translate of the quasiconvex function F_0 defined in [5], so that the quasiconvex and polyconvex hulls of K are different.

(5) In a future paper we plan to give another proof of Theorem 1.1 as well as some results related to rank-one convexity, namely a version of Krein-Milman type theorem and the proof that rank-one convex hull and quasiconvex hull in $\mathbb{M}^{2\times 2}_{\mathrm{sym}}$ have infinite Carathéodory number. Also, we will provide a proof for formula (7) "different" from direct calculation of the limit of corresponding prelaminates.

4. Upper semi-continuity

Let X be a metric space. For $\varepsilon > 0$, the ε-neighborhood of a set $A \subset X$ will be denoted by $\mathcal{U}_\varepsilon(A) = \{x \in X\colon\ \operatorname{dist}(x, A) < \varepsilon\}$.

On $\mathcal{K}(X)$, the set of all nonempty compact subsets of X, the Hausdorff metric is defined by $\varrho(K_1, K_2) = \inf\{\varepsilon\colon\ K_1 \subset \mathcal{U}_\varepsilon(K_2)$ and $K_2 \subset \mathcal{U}_\varepsilon(K_1)\}$. This definition can be extended for non-compact sets A_1, A_2, but it turns out that $\varrho(A_1, A_2) = \varrho(\bar{A}_1, \bar{A}_2)$.

We say that a function $f\colon \mathcal{K}(X) \to \mathcal{K}(X)$ is upper semi-continuous (with respect to Hausdorff metric) if for every $\varepsilon > 0$ and $K_0 \in \mathcal{K}(X)$ there is $\delta > 0$ such that $f(K) \subset \mathcal{U}_\varepsilon(f(K_0))$ whenever $K \in \mathcal{K}(X)$ and $\varrho(K, K_0) < \delta$.

Let $\mathrm{Q}(K)$ denote the quasiconvex hull of a set $K \subset \mathbb{M}^{m\times n}$. In [6], it is shown that the function $K \mapsto \mathrm{Q}(K)$ is upper semi-continuous with respect to Hausdorff metric on the space of all compact subsets of $\mathbb{M}^{m\times n}$. Lamination convex hull and separately lamination convex hull do not share this property.

PROPOSITION 4.1. – *Function $K \mapsto \overline{\mathrm{L}_{\mathrm{sc}}(K)}$ is not upper semi-continuous with respect to Hausdorff metric on $\mathcal{K}(\mathbb{R}^3)$. Function $K \mapsto \overline{\mathrm{L}(K)}$ is not upper semi-*

continuous on $\mathcal{K}(X)$ (*with respect to Hausdorff metric*) *where*

$$X = \mathbb{M}^{3\times 3}_{\mathrm{diag}} \quad or \quad X = \left\{ \begin{pmatrix} a & b & 0 \\ 0 & 0 & c \end{pmatrix} \right\}.$$

We do not know what the cases of $\mathbb{M}^{2\times 2}_{\mathrm{sym}}$ and $\mathbb{M}^{2\times 2}$ look like.

Proof sketch. – Let K be as in (1), $\varepsilon = \frac{1}{3}$, $J = (0, -1, 0)$, $x_n = (-\frac{1}{2}, \frac{1}{n}, \frac{1}{n}) \in \mathrm{L}_{\mathrm{sc}}(K)$, $x = (-\frac{1}{2}, 0, 0) \in \overline{\mathrm{L}_{\mathrm{sc}}(K)} \setminus \mathrm{L}_{\mathrm{sc}}(K)$, $K_0 = K \cup \{x + J\}$, $K_n = K \cup \{x_n + J\}$. Then $\varrho(K_n, K_0) \to 0$. On the other hand

$$\mathrm{L}_{\mathrm{sc}}(K_0) \subset \mathrm{L}_{\mathrm{sc}}(K) \cup [x + J, (0, -1, 0)] \quad \text{(a separately lamination convex set)}$$
$$\mathrm{L}_{\mathrm{sc}}(K_n) \supset [x_n, x_n + J],$$

hence $\mathrm{L}_{\mathrm{sc}}(K_n) \not\subset \mathcal{U}_\varepsilon(\mathrm{L}_{\mathrm{sc}}(K_0))$. Thus $K \mapsto \overline{\mathrm{L}_{\mathrm{sc}}(K)}$ is not upper semi-continuous on $\mathbb{R}^3$ and after a transformation we see that $K \mapsto \overline{\mathrm{L}(K)}$ is not upper semi-continuous on $\mathbb{M}^{3\times 3}_{\mathrm{diag}}$.

For the last case we start with $\tilde{K}$ and L from Example 2.4 and set $J = (0, 0; -2)$, $x_n = (\frac{1}{n}, \frac{1}{2}; 0) \in \mathrm{L}_{(2,1)}(\tilde{K})$, $x = (0, \frac{1}{2}; 0)$, $K_0 = \tilde{K} \cup \{x + J\}$, $K_n = \tilde{K} \cup \{x_n + J\}$. Again, the segment $[x_n, x_n + J]$ is contained in $\mathrm{L}_{(2,1)}(K_n)$ but $[x, x + J]$ does not belong to $\mathrm{L}_{(2,1)}(K_0)$ (nor to its closure) because K_0 is contained in the bi-convex set $L \cup \{x + J\}$. □

Remark. – Let $\mathrm{L}^{\mathrm{c}}(K)$ be the *closed lamination convex hull* of $K \subset \mathbb{M}^{m\times n}$, i.e., the smallest *closed* lamination convex set containing K. Similarly, the closed separately lamination convex hull $\mathrm{L}^{\mathrm{c}}_{\mathrm{sc}}(K)$ is defined for $K \subset \mathbb{R}^n$. There are compacta K such that $\mathrm{L}^{\mathrm{c}}(K) \neq \overline{\mathrm{L}(K)}$ and $\mathrm{L}^{\mathrm{c}}_{\mathrm{sc}}(K) \neq \overline{\mathrm{L}_{\mathrm{sc}}(K)}$, respectively. The two sets named K_0 above serve as an example. We do not know whether $\mathrm{L}^{\mathrm{c}}(K) = \overline{\mathrm{L}(K)}$ for every compact $K \subset \mathbb{M}^{2\times 2}_{\mathrm{sym}}$ or $K \subset \mathbb{M}^{2\times 2}$.

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