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Existence results for mean field equations


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by

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ABSTRACT. – Let \( \Omega \) be an annulus. We prove that the mean field equation

\[
-\Delta \psi = \frac{e^{-\beta \psi}}{\int_{\Omega} e^{-\beta \psi}} \quad \text{in } \Omega
\]

\[
\psi = 0 \quad \text{on } \partial \Omega
\]

admits a solution for \( \beta \in (-16\pi, -8\pi) \). This is a supercritical case for the Moser-Trudinger inequality. © Elsevier, Paris

RÉSUMÉ. – On montre que l’équation de champ moyen

\[
-\Delta \psi = \frac{e^{-\beta \psi}}{\int_{\Omega} e^{-\beta \psi}} \quad \text{dans } \Omega
\]

\[
\psi = 0 \quad \text{sur } \partial \Omega,
\]

pour \( \Omega \) étant un anneau, admet une solution pour \( \beta \in (-16\pi, -8\pi) \). Celà représente un cas supercritique pour l’inegalité de Moser-Trudinger. © Elsevier, Paris
1. INTRODUCTION

Let \( \Omega \) be a smooth bounded domain in \( \mathbb{R}^2 \). In this paper, we consider the following mean field equation

\[
-\Delta \psi = \frac{e^{-\beta \psi}}{\int_{\Omega} e^{-\beta \psi}}, \quad \text{in } \Omega,
\]

\[
\psi = 0, \quad \text{on } \partial \Omega,
\]

for \( \beta \in (-\infty, +\infty) \). (1.1) is the Euler-Lagrange equation of the following functional

\[
J_{\beta}(\psi) = \frac{1}{2} \int_{\Omega} |\nabla \psi|^2 + \frac{1}{\beta} \log \int_{\Omega} e^{-\beta \psi}
\]

in \( H^{1,2}_0(\Omega) \). This variational problem arises from Onsager’s vortex model for turbulent Euler flows. In that interpretation, \( \psi \) is the stream function in the infinite vortex limit, see [12,p256ff]. The corresponding canonical Gibbs measure and partition function are finite precisely if \( \beta > -8\pi \). In that situation, Caglioti et al. [4] and Kiessling [9] showed the existence of a minimizer of \( J_{\beta} \). This is based on the Moser-Trudinger inequality

\[
\frac{1}{2} \int_{\Omega} |\nabla \psi|^2 \geq \frac{1}{8\pi} \log \int_{\Omega} e^{-8\pi \psi}, \quad \text{for any } \psi \in H^{1,2}_0(\Omega),
\]

which implies the relevant compactness and coercivity condition for \( J_{\beta} \) in case \( \beta > -8\pi \). For \( \beta \leq -8\pi \), the situation becomes different as described in [4]. On the unit disk, solutions blow up if one approaches \( \beta = -8\pi \) -the critical case for (1.3)-(see also [5] and [19]), and more generally, on starshaped domains, the Pohozaev identity yields a lower bound on the possible values of \( \beta \) for which solutions exist. On the other hand, for an annulus, [4] constructed radially symmetric solutions for any \( \beta \), and the construction of Bahri-Coron [2] makes it plausible that solutions on domains with non-trivial topology exist below \(-8\pi\). Thus, for \( \beta \leq -8\pi \), \( J_{\beta} \) is no longer compact and coercive in general, and the existence of solution depends on the geometry of the domain.

In the present paper, we thus consider the supercritical case \( \beta < -8\pi \) on domains with non-trivial topology.

**Theorem 1.1.** – Let \( \Omega \subset \mathbb{R}^2 \) be a smooth, bounded domain whose complement contains a bounded region, e.g. \( \Omega \) an annulus. Then (1.1) has a solution for all \( \beta \in (-16\pi, -8\pi) \).
The solutions we find, however, are not minimizers of $J_\beta$—those do not exist in case $\beta < 8\pi$, since $J_\beta$ has no lower bound—but unstable critical points. Thus, these solutions might not be relevant to the turbulence problem that was at the basis of [4] and [9].

Certainly we can generalize Theorem 1.1 to the following equation

$$-\Delta \psi = \frac{K e^{-\beta \psi}}{\int_\Omega K e^{-\beta \psi}}, \quad \text{in } \Omega,$$

$$\psi = 0, \quad \text{on } \partial\Omega,$$

which was studied in [5]. Here $K$ is a positive function on $\Omega$.

With the same method, we may also handle the equation

$$\Delta u - c + cK e^u = 0, \quad \text{for } 0 \leq c < \infty$$

(1.4)

on a compact Riemann surface $\Sigma$ of genus at least 1, where $K$ is a positive function. (1.4) can also be considered as a mean field equation because it is the Euler-Lagrange equation of the functional

$$J_c(u) = \frac{1}{2} \int_\Sigma |\nabla u|^2 + c \int_\Sigma u - c \log \int_\Sigma K e^u.$$ 

(1.5)

Because of the term $c \int_\Sigma u$, $J_c$ remains invariant under adding a constant to $u$, and therefore we may normalize $u$ by the condition

$$\int_\Sigma K e^u = 1$$

which explains the absence of the factor $(\int K e^u)^{-1}$ in (1.4). $c < 8\pi$ again is a subcritical case that can easily be handled with the Moser-Trudinger inequality. The critical case $c = 8\pi$ yields the so-called Kazdan-Warner equation [8] and was treated in [7] and [14] by giving sufficient conditions for the existence of a minimizer of $J_{8\pi}$. Here, we construct again saddle point type critical points to show

**Theorem 1.2.** — Let $\Sigma$ be a compact Riemann surface of positive genus. Then (1.4) admits a non-minimal solution for $8\pi < c < 16\pi$.

Now we give a outline of the proof of the Theorems. First from the non-trivial topology of the domain, we can define a minimax value $\alpha_\beta$, which is bounded below by an improved Moser-Trudinger inequality, for $\beta \in (-16\pi, -8\pi)$. Using a trick introduced by Struwe in [16] and [17], for a certain dense subset $\Lambda \subset (-16\pi, -8\pi)$ we can overcome the lack of a
coercivity condition and show that $\alpha_\beta$ is achieved by some $u_\beta$ for $\beta \in \Lambda$. Next, for any fixed $\beta \in (-16\pi, -8\pi)$, considering a sequence $\beta_k \subset \Lambda$ tending to $\beta$, with the help of results in [3] and [11] we show that $u_{\beta_k}$ subconverges strongly to some $u_\beta$ which achieves $\alpha_\beta$.

After completing our paper, we were informed that Struwe and Tarantello [18] obtained a non-constant solution of (1.4), when $\Sigma$ is a flat torus with fundamental cell domain $[-\frac{1}{2}, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}]$, $K \equiv 1$ and $c \in (8\pi, 4\pi^2)$. In this case, it is easy to check that our solution obtained in Theorem 1.2 is non-constant.

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2. MINIMAX VALUES

Let $\rho = -\beta$ and $u = -\beta \psi$. We rewrite (1.1) as

\begin{equation}
-\Delta u = \rho \frac{e^u}{\int e^u}, \quad \text{in } \Omega,
\end{equation}

\begin{equation}
u = 0, \quad \text{on } \partial \Omega,
\end{equation}

and (1.2) as

\begin{equation}
J_\rho(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 - \rho \log \int e^u
\end{equation}

for $u \in H^{1,2}_0(\Omega)$.

It is easy to see that $J_\rho$ has no lower bound for $\rho \in (8\pi, 16\pi)$. Hence, to get a solution of (1.1) for $\rho \in (8\pi, 16\pi)$, we have to use a minimax method. First, we define a center of mass of $u$ by

\begin{equation}
m_c(u) = \frac{\int x e^u}{\int e^u}.
\end{equation}

Let $B$ be the bounded component of $\mathbb{R}^2 \setminus \Omega$. For simplicity, we assume that $B$ is the unit disk centered at the origin. Then we define a family of functions

\begin{equation}
h : D \to H^{1,2}_0(\Omega)
\end{equation}
satisfying

\begin{equation}
(2.3) \quad \lim_{r \to 1} J_\rho(h(r, \theta)) \to -\infty
\end{equation}

and

\begin{equation}
(2.4) \quad \lim_{r \to 1} m_c(h(r, \theta)) \text{ is a continuous curve enclosing } B.
\end{equation}

Here \( D = \{(r, \theta) | 0 \leq r < 1, \theta \in [0, 2\pi)\} \) is the open unit disk. We denote the set of all such families by \( \mathcal{D}_\rho \). It is easy to check that \( \mathcal{D}_\rho \neq \emptyset \). Now we can define a minimax value

\[ \alpha_\rho := \inf_{h \in \mathcal{D}_\rho} \sup_{u \in h(D)} J_\rho(u). \]

The following lemma will make crucial use of the non-trivial topology of \( \Omega \), more precisely of the fact that the complement of \( \Omega \) has a bounded component.

**Lemma 2.1.** – For any \( \rho \in (8\pi, 16\pi) \) \( \alpha_\rho > -\infty \).

**Remark.** – It is an interesting question whether \( \alpha_{16\pi} = -\infty \).

To prove Lemma 2.1, we use the improved Moser-Trudinger inequality of [6] (see also [1]). Here we have to modify a little bit.

**Lemma 2.2.** – Let \( S_1 \) and \( S_2 \) be two subsets of \( \tilde{\Omega} \) satisfying \( \text{dist}(S_1, S_2) \geq \delta_0 > 0 \) and \( \gamma_0 \in (0, 1/2) \). For any \( \epsilon > 0 \), there exists a constant \( c = c(\epsilon, \delta_0, \gamma_0) > 0 \) such that

\[ \int_{\Omega} e^u \leq c \exp \left\{ \frac{1}{32\pi - \epsilon} \int_{\Omega} |\nabla u|^2 + c \right\} \]

holds for all \( u \in H^{1,2}_0(\Omega) \) satisfying

\begin{equation}
(2.5) \quad \frac{\int_{S_1} e^u}{\int_{\Omega} e^u} \geq \gamma_0 \quad \text{and} \quad \frac{\int_{S_2} e^u}{\int_{\Omega} e^u} \geq \gamma_0.
\end{equation}

**Proof.** – The Lemma follows from the argument in [6] and the following Moser-Trudinger inequality

\begin{equation}
(*) \quad \frac{1}{2} \int_{\Omega} |\nabla u|^2 - 8\pi \log \int_{\Omega} e^u \geq c
\end{equation}

for any \( u \in H^{1,2}_0(\Omega) \), where \( c \) is a constant independent of \( u \in H^{1,2}_0(\Omega) \). \( \square \)
We will discuss the inequality (*) and its application in another paper.

**Proof of Lemma 2.1.** – For fixed $\rho \in (8\pi, 16\pi)$ we claim that there exists a constant $c_\rho$ such that

$$\sup_{u \in h(D)} J_{\rho}(u) \geq c_\rho, \quad \text{for any } h \in \mathcal{D}_\rho. \quad (2.6)$$

Clearly (2.6) implies the Lemma. By the definition of $h$, for any $h \in \mathcal{D}_\rho$, there exists $u \in h(D)$ such that

$$m_c(u) = 0.$$ 

We choose $\epsilon > 0$ so small that $\rho < 16\pi - 2\epsilon$. Assume (2.6) does not hold. Then we have sequences $\{h_i\} \subset \mathcal{D}_\rho$ and $\{u_i\} \subset H^1_0(\Omega)$ such that $u_i \in h_i(D)$ and

$$m_c(u_i) = 0 \quad (2.7)$$

$$\lim_{i \to \infty} J(u_i) = -\infty. \quad (2.8)$$

We have the following Lemma.

**Lemma 2.3.** – There exists $x_0 \in \Omega$ such that

$$\lim_{i \to \infty} \frac{\int_{B_{1/4}(x_0) \cap \Omega} e^{u_i}}{\int_{\Omega} e^{u_i}} \to 1. \quad (2.9)$$

**Proof.** – Set

$$A(x) := \lim_{i \to \infty} \frac{\int_{B_{1/4}(x) \cap \Omega} e^{u_i}}{\int_{\Omega} e^{u_i}}.$$ 

Assume that the Lemma were false, then there exists $x_0 \in \Omega$ such that

$$A(x_0) < 1 \quad \text{and} \quad A(x_0) \geq A(x) \quad \text{for any } x \in \Omega.$$

It is easy to check $A(x_0) > 0$, since $\Omega$ can be covered by finite many balls of radius $1/4$. Let $\gamma_0 = A(x_0)/2$. Recalling (2.8) and applying lemma 2.2, we obtain

$$\frac{\int_{\Omega \setminus B_{1/2}(x_0)} e^{u_i}}{\int_{\Omega} e^{u_i}} \to 0 \quad (2.10)$$

as $i \to \infty$, which implies (2.9). \qed
Now we continue to prove Lemma 2.1. (2.9) implies
\[
\frac{\int_{\Omega} xe^{u_i}}{\int_{\Omega} e^{u_i}} - x_0 = \frac{\int_{\Omega} (x - x_0)e^{u_i}}{\int_{\Omega} e^{u_i}} = \frac{\int_{B_{1/2}(x_0)} (x - x_0)e^{u_i}}{\int_{\Omega} e^{u_i}} + o(1)
\]
which, in turn, implies that \(|m_c(u_i) - x_0| < 2/3\). This contradicts (2.7). □

**Lemma 2.4.** \(\alpha_p/\rho\) is non-increasing in \((8\pi, 16\pi)\).

**Proof.** We first observe that if \(J(u) \leq 0\), then \(\log \int_{\Omega} e^{u} > 0\) which implies that
\[
J_{\rho}(u) \geq J_{\rho'}(u) \quad \text{for } \rho' \geq \rho.
\]
Hence \(D_{\rho} \subset D_{\rho'}\) for any \(16\pi > \rho' \geq \rho > 8\pi\). On the other hand, it is clear that
\[
\frac{J_{\rho}}{\rho} - \frac{J_{\rho'}}{\rho'} = \frac{1}{2} \left( \frac{1}{\rho} - \frac{1}{\rho'} \right) \int_{\Omega} |\nabla u|^2 \geq 0,
\]
if \(\rho' \geq \rho\). Hence we have
\[
\frac{\alpha_{\rho}}{\rho} \geq \frac{\alpha_{\rho'}}{\rho'}
\]
for \(16\pi > \rho' \geq \rho > 8\pi\). □

### 3. Existence for a Dense Set

In this section we show that \(\alpha_{\rho}\) is achieved if \(\rho\) belongs to a certain dense subset of \((8\pi, 16\pi)\) defined below.

The crucial problem for our functional is the lack of a coercivity condition, \(i.e.\) for a Palais-Smale sequence \(u_i\) for \(J_{\rho}\), we do not know whether \(\int_{\Omega} |\nabla u_i|^2\) is bounded.

We first have the following lemma.

**Lemma 3.1.** Let \(u_i\) be a Palais-Smale sequence for \(J_{\rho}\), \(i.e.\) \(u_i\) satisfies
\[
|J_{\rho}(u_i)| \leq c < \infty
\]
(3.1)
and

\begin{equation}
(3.2) \quad dJ_\rho(u_i) \to 0 \text{ strongly in } H^{-1,2}(\Omega).
\end{equation}

If, in addition, we have

\begin{equation}
(3.3) \quad \int_\Omega |\nabla u_i|^2 \leq c_0, \quad \text{for } i = 1, 2, \ldots
\end{equation}

for a constant \(c_0\) independent of \(i\), then \(u_i\) subconverges to a critical point \(u_0\) for \(J_\rho\) strongly in \(H_0^{1,2}(\Omega)\).

**Proof.** – The proof is standard, but we provide it here for convenience of the reader.

Since \(\int_\Omega |\nabla u_i|^2\) is bounded, there exists \(u_0 \in H_0^{1,2}(\Omega)\) such that

(i) \(u_i\) converges to \(u_0\) weakly in \(H_0^{1,2}(\Omega)\),

(ii) \(u_i\) converges to \(u_0\) strongly in \(L^p(\Omega)\) for any \(p > 1\) and almost everywhere,

(iii) \(e^{u_i}\) converges to \(e^{u_0}\) strongly in \(L^p(\Omega)\) for any \(p \geq 1\).

From (i)-(iii), we can show that \(dJ(u_0) = 0\), i.e. \(u_0\) satisfies

\[-\Delta u_0 = \rho \frac{e^{u_0}}{\int_\Omega e^{u_0}}.\]

Testing \(dJ_\rho\) with \(u_i - u_0\), we obtain

\[o(1) = \langle dJ_\rho(u_i) - dJ_\rho(u), u_i - u_0 \rangle = \int_\Omega |\nabla(u_i - u_0)|^2 - \rho \int_\Omega \frac{e^{u_i}}{\int_\Omega e^{u_i}} - \frac{e^{u_0}}{\int_\Omega e^{u_0}}(u_i - u_0)
\]

\[= \int_\Omega |\nabla(u_i - u_0)|^2 + o(1),\]

by (i)-(iii). Hence \(u_i\) converges to \(u_0\) strongly in \(H_0^{1,2}(\Omega)\).

Since by Lemma 2.4 \(\rho \to \alpha_\rho/\rho\) is non-increasing in \((8\pi, 16\pi)\), \(\rho \to \alpha_\rho/\rho\) is a.e. differentiable. Set

\begin{equation}
(3.4) \quad \Lambda := \{ \rho \in (8\pi, 16\pi) | \alpha_\rho/\rho \text{ is differentiable at } \rho \}.
\end{equation}

\(\bar{\Lambda} = [8\pi, 16\pi]\), see [16]. Let \(\rho \in \Lambda\) and choose \(\rho_k \not\to \rho\) such that

\begin{equation}
(3.5) \quad 0 \leq \lim_{k \to \infty} - \frac{1}{(\rho - \rho_k)} \left( \frac{\alpha_\rho}{\rho} - \frac{\alpha_{\rho_k}}{\rho_k} \right) \leq c_1
\end{equation}

for some constant \(c_1\) independent of \(k\).
**Lemma 3.2.** \( \alpha_{\rho} \) is achieved by a critical point \( u_{\rho} \) for \( J_{\rho} \) provided that \( \rho \in \Lambda \).

**Proof.** Assume, by contradiction, that the Lemma were false. From Lemma 3.1, there exists \( \delta > 0 \) such that

\[
|dJ_{\rho}(u)|_{H^{-1,2}(\Omega)} \geq 2\delta
\]

in

\[
N_{\delta} := \{ u \in H^{1,2}_{0}(\Omega) \mid \int_{\Omega} |\nabla u|^2 \leq c_2, |J_{\rho}(u) - \alpha_{\rho}| < \delta \}.
\]

Here, \( c_2 \) is any fixed constant such that \( N_{\delta} \neq \emptyset \). Let \( X_{\rho} : N_{\delta} \to H^{1,2}_{0}(\Omega) \) be a pseudo-gradient vector field for \( J_{\rho} \) in \( N_{\delta} \), i.e. a locally Lipschitz vector field of norm \( \|X_{\rho}\|_{H^{1,2}_{0}} \leq 1 \) with

\[
\langle dJ_{\rho}(u), X_{\rho}(u) \rangle < -\delta.
\]

See [15] for the construction of \( X_{\rho} \).

Since

\[
\|dJ_{\rho}(u) - dJ_{\rho_k}(u)\| = \|dJ_{\rho} - \frac{\rho}{\rho_k} dJ_{\rho_k}(u)\| + ||(1 - \frac{\rho}{\rho_k})dJ_{\rho_k}(u)\| \\
\leq \frac{1}{2}(1 - \frac{\rho}{\rho_k}) \int_{\Omega} |\nabla u|^2 + c(1 - \frac{\rho}{\rho_k}) \int_{\Omega} |\nabla u|^2 \to 0
\]

uniformly in \( \{ u \mid \int_{\Omega} |\nabla u|^2 \leq c_2 \} \), \( X_{\rho} \) is also a pseudo-gradient vector field for \( J_{\rho_k} \) in \( N_{\delta} \) with

\[
\langle dJ_{\rho_k}(u), X_{\rho}(u) \rangle < -\delta/2,
\]

for \( u \in N_{\delta} \), provided that \( k \) is sufficiently large.

For any sequence \( \{ h_k \} \), \( h_k \in D_{\rho_k} \subset D_{\rho} \) such that

\[
\sup_{u \in h_k(D)} J_{\rho_k}(u) \leq \alpha_{\rho_k} + \rho - \rho_k
\]

and all \( u \in h_k(D) \) such that

\[
J_{\rho}(u) \geq \alpha_{\rho} - (\rho - \rho_k),
\]

we have the following estimate

\[
\frac{1}{2} \int_{\Omega} |\nabla u|^2 = \rho \cdot \rho_k \frac{J_{\rho_k}(u) - J_{\rho}(u)}{\rho - \rho_k} \\
\leq \rho \cdot \rho_k \frac{\alpha_{\rho_k} - \alpha_{\rho}}{\rho - \rho_k} + (\rho + \rho_k)
\]

\[
\leq C
\]

by (3.5), (3.9) and (3.10), where \( C = (16\pi)^2 c_1 + 32\pi \).
Now we consider in $N_\delta$ the following pseudo-gradient flow for $J_\rho$. First choose a Lipschitz continuous cut-off function $\eta$ such that $0 \leq \eta \leq 1$, $\eta = 0$ outside $N_\delta$, $\eta = 1$ in $N_{\delta/2}$. Then consider the following flow in $H_0^{1,2}(\Omega)$ generated by $\eta X_\rho$

$$\frac{\partial \phi}{\partial t}(u, t) = \eta(\phi(u, t))X_\rho(\phi(u, t))$$
$$\phi(u, 0) = u.$$  

By (3.7) and (3.8), for $u \in N_{\delta/2}$, we have

$$\frac{d}{dt} J_\rho(\phi(u, t))|_{t=0} \leq -\delta$$  
(3.12)  

and

$$\frac{d}{dt} J_{\rho_k}(\phi(u, t))|_{t=0} \leq -\delta/2$$  
(3.13)  

for large $k$.

It is clear that for any $h \in D_{\rho_k}$, $h(r, \theta) \notin N_\delta$ for $r$ close to 1. Hence $\phi(h, t) \in D_{\rho_k}$ for any $t > 0$. In particular, $\phi(\cdot, t)$ preserves the class of $h_k \in D_{\rho_k}$ with condition (3.9). On the other hand, for any $h \in D_\rho$ by definition

$$\sup_{u \in h(D)} J_\rho(u) \geq \alpha_\rho.$$  

Hence for any $h_k \in D_{\rho_k}$ with condition (3.9), $\sup_{u \in \phi(h(D), t)} J_\rho(u)$ is achieved in $N_{\delta/2}$, provided that $k$ is large enough. Consequently, by (3.12), we have

$$\frac{d}{dt} \sup \{J_\rho(u) | u \in \phi(h(D), t)\} \leq -\delta$$

for all $t \geq 0$, which is a contradiction.  

\[ \square \]

4. PROOF OF THEOREM 1.1

From section 3, we know that for any $\bar{\rho} \in (8\pi, 16\pi)$ there exists a sequence $\rho_k \nearrow \bar{\rho}$ such that $\alpha_{\rho_k}$ is achieved by $u_k$. Consequently $u_k$ satisfies

$$-\Delta u_k = \rho_k \frac{e^{u_k}}{\int_{\Omega} e^{u_k}}, \quad \text{in } \Omega,$$

$$u_k = 0, \quad \text{on } \partial \Omega.$$  
(4.1)
From Lemma 2.4, we have

\[ J_\beta(u_k) = \alpha \rho_k \leq c_0, \]

for some constant \( c_0 > 0 \) which is independent of \( k \). Let \( v_k = u_k - \log \int_\Omega e^{u_k} \). Then \( v_k \) satisfies

\[ -\Delta v_k = \rho_k e^{v_k} \]

with

\[ \int_\Omega e^{v_k} = 1. \]

By results of Brezis-Merle [3] and Li-Shafrir [11] we have

**Lemma 4.1** ([3], [11]). There exists a subsequence (also denoted by \( v_k \)) satisfying one of the following alternatives:

(i) \( \{v_k\} \) is bounded in \( L^\infty_{\text{loc}}(\Omega) \);

(ii) \( v_k \to -\infty \) uniformly on any compact subset of \( \Omega \);

(iii) there exists a finite blow-up set \( \Sigma = \{a_1, \ldots, a_m\} \subset \Omega \) such that, for any \( 1 \leq i \leq m \), there exists \( \{x_k\} \subset \Omega \), \( x_k \to a_i \), \( u_k(x_k) \to \infty \), and \( v_k(x) \to -\infty \) uniformly on any compact subset of \( \Omega \setminus \Sigma \). Moreover,

\[ \rho_k \int_\Omega e^{v_k} \to \sum_{i=1}^m 8\pi n_i \]

where \( n_i \) is positive integer.

For our special functions \( v_k \), we can improve Lemma 4.1 as follows

**Lemma 4.2.** There exists a subsequence (also denoted by \( v_k \)) satisfying one of the following alternatives:

(i) \( \{v_k\} \) is bounded in \( L^\infty_{\text{loc}}(\Omega) \);

(ii) \( v_k \to -\infty \) uniformly on \( \Omega \);

(iii) there exists a finite blow-up set \( \Sigma = \{a_1, \ldots, a_m\} \subset \Omega \) such that, for any \( 1 \leq i \leq m \), there exists \( \{x_k\} \subset \Omega \), \( x_k \to a_i \), \( u_k(x_k) \to \infty \), and \( v_k(x) \to -\infty \) uniformly on any compact subset of \( \Omega \setminus \Sigma \). Moreover, \( (4.5) \) holds.

**Proof.** From Lemma 4.1, we only have to consider one more case in which blow-up points are in the boundary of \( \Omega \). There are two possibilities: One is bubbling too fast such that after rescaling we obtain a solution of \( -\Delta u = e^u \) in a half plane; Another is bubbling slow such that after...
rescaling we obtain a solution of $-\Delta u = e^u$ in $\mathbb{R}^2$. One can exclude the first case. In the second case, one can follow the idea in [11] to show that (4.5) holds. See also [10].

**Proof of Theorem 1.1.** (4.4), (4.5) and $\bar{\rho} \in (8\pi, 16\pi)$ imply that cases (ii) and (iii) in Lemma 4.2 does not occur. Consequently $\{v_k\}$ is bounded in $L^\infty_{loc}(\Omega)$. Now we can again apply Lemma 2.2 as follows.

Let $S_1$ and $S_2$ be two disjoint compact subdomains of $\Omega$. Since $\{v_k\}$ is bounded in $L^\infty_{loc}(\Omega)$, we have

$$\frac{\int_{S_i} e^{u_k}}{\int_\Omega e^{u_k}} = \int_{S_i} e^{v_k} \geq c_0, \quad i = 1, 2$$

for a constant $c_0 = c_0(S_1, S_2, \Omega) > 0$ independent of $k$. Choosing $\epsilon$ such that $16\pi - \bar{\rho} > 2\epsilon$ and applying Lemma 2.2, with the help of (4.2), we obtain

$$c \geq J_{\rho_k}(u_k) = \frac{1}{2} \int_\Omega |\nabla u_k|^2 - \rho_k \log \int_\Omega e^{u_k}$$

$$\geq \frac{1}{2} (1 - \frac{\rho_k}{16\pi - \epsilon/2}) \int_\Omega |\nabla u|^2$$

$$\geq \frac{1}{2} (1 - \frac{\bar{\rho}}{16\pi - \epsilon/2}) \int_\Omega |\nabla u|^2$$

which implies that $\int_\Omega |\nabla u_k|^2$ is bounded. Now by the same argument in the proof of Lemma 3.1, $u_k$ subconverges to $u_{\bar{\rho}}$ strongly in $H^1_{0, \Omega}$ and $u_{\bar{\rho}}$ is a critical point of $J_{\bar{\rho}}$. Clearly, $u_{\bar{\rho}}$ achieves $\alpha_{\bar{\rho}}$. This finishes the proof of Theorem 1.1. \hfill \square

**Proof of Theorem 1.2.** Since the proof is very similar to one presented above, we only give a sketch of the proof of Theorem 1.2. Let $\Sigma$ be a Riemann surface of positive genus. We embed $X : \Sigma \to \mathbb{R}^N$ for some $N \geq 3$ and define the center of mass for a function $u \in H^{1,2}(\Sigma)$ by

$$m_c(u) = \frac{\int_\Sigma X e^u}{\int_\Sigma e^u}.$$ 

Since $\Sigma$ is of positive genus, we can choose a Jordan curve $\Gamma^1$ on $\Sigma$ and a closed curve $\Gamma^2$ in $\mathbb{R}^N \setminus \Sigma$ such that $\Gamma^1$ links $\Gamma^2$. We know that $\inf_{u \in H^{1,2}(\Sigma)} J_c(u)$ is finite if and only if $c \in [0, 8\pi]$ (see [7]). Now define a family of functions $h : D \to H^{1,2}(\Sigma)$ (as in section 2) satisfying

$$\lim_{r \to 1} J_{\rho}(h(r, \theta)) \to -\infty$$
and
\[
\lim_{r \to 1} m_c(h(r, \theta)) \text{ as a map from } S^1 \to \Gamma^1 \text{ is of degree 1.}
\]
Let \( \mathcal{D}_c \) denote the set of all such families. It is also easy to check that \( \mathcal{D}_c \neq \emptyset \). Set
\[
\alpha_c := \inf_{h \in \mathcal{D}_c} \sup_{u \in h(D)} J_c(u).
\]
We first have
\[
\alpha_c > -\infty,
\]
using the fact that \( \Gamma^1 \) links \( \Gamma^2 \) and Lemma 2.2. Then by the same method as presented above, we can prove that \( \alpha_c \) is achieved by some \( u_c \in H^{1,2}(\Sigma) \), which is a solution of (1.4), for \( c \in (8\pi, 16\pi) \).

REFERENCES

[10] YanYan Li, \( -\Delta u = \lambda \left( \int_M V e^u - W \right) \) on Riemann surfaces, preprint.

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