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by

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ABSTRACT. – In this paper we study the structure of certain level set of the Ginzburg-Landau functional which has similar topology with the configuration space. As an application, we generalize Almeida-Bethuel’s result on multiplicity of solutions for the Ginzburg-Landau equation. 
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RÉSUMÉ. – On étudie la structure de certains ensembles de niveau de la fonctionnelle du type Ginzburg-Landau qui ont des topologies similaires à celles de l’espace de configuration. Comme application, on généralise le résultat d’Almeida-Bethuel sur la multiplicité des solutions des équations de G-L. © Elsevier, Paris

1. INTRODUCTION

Let \( \Omega \subset \mathbb{C} \) be a smooth, bounded and simply connected domain. Let
\( g : \partial \Omega \to \mathbb{C} \) be a prescribed smooth map with \( |g(x)| = 1 \), for all \( x \in \partial \Omega \).

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The Ginzburg-Landau functional, for any $\varepsilon > 0$, is given by

$$E_\varepsilon(u) = \frac{1}{2} \int_\Omega \left\{ |\nabla u|^2 + \frac{1}{2\varepsilon^2} (1 - |u|^2)^2 \right\}$$

which is defined on the Hilbert space

$$H^1_g(\Omega, \mathbb{C}) = \{ u \in H^1(\Omega, \mathbb{C}); u = g \text{ on } \partial\Omega \}.$$

It is easy to verify that $E_\varepsilon$ is a positive, $C^2$-functional satisfying the Palais-Smale condition. So

$$\mu_\varepsilon = \min_{u \in H^1_g(\Omega, \mathbb{C})} E_\varepsilon(u)$$

is achieved by some $u_\varepsilon \in H^1_g(\Omega, \mathbb{C})$ and these minimizers satisfy the following Ginzburg-Landau equation:

$$\begin{cases}
    -\Delta u = \frac{1}{\varepsilon^2} u(1 - |u|^2) & \text{in } \Omega \\
    u = g & \text{on } \partial\Omega.
\end{cases}
$$

The Ginzburg-Landau equation (1.2) has been extensively studied by F. Bethuel, H. Brezis and F. Hélein [BBH1, 2] and many others. A complete characterization of asymptotic behavior (as $\varepsilon \to 0^+$) for minimizing solutions of (1.2) is given. It has been shown that the degree of $g$, denoted by $k = \text{deg}(g, \partial\Omega)$, plays a crucial role in the asymptotic analysis of the minimizers. Without loss of generality, we will always assume $k \geq 0$ throughout this paper.

In this paper, we will study the multiplicity of the solutions for the Ginzburg-Landau equation (1.2), many such results have been given for special domains and/or boundary values (see for instance Almeida and Bethuel [AB1], Felmer and Del Pino [FP], F.H. Lin [Li]). The motivation of our paper comes from the recent work of Almeida-Bethuel [AB2, 3] concerning the existence of non-minimizing solutions of (1.2). They showed that if $k \geq 2$, the Ginzburg-Landau equation (1.2) has at least three distinct solutions, among which at least one is not minimizing. Based on topological arguments directly inspired by Almeida-Bethuel’s work, we obtain our main result as follows

**Theorem 1.** Assume that $k \geq 2$, there is some $\varepsilon_0 > 0$ (depending on $\Omega$ and $g$ only) such that if $\varepsilon < \varepsilon_0$, the equation (1.2) has at least $k + 1$ distinct solutions.
To prove Theorem 1, we will apply the standard Ljusternik-Schnirelman theory to a suitable covering space of a level set
\[ E_\varepsilon^a = \{ u \in H^1_g(\Omega, \mathbb{C}) ; E_\varepsilon(u) < a \}, \]
for an \( a \) of the form
\[ a = \mu_\varepsilon + \lambda \]
where \( \lambda \) is a fixed positive constant to be determined later. The proof is strongly related to the topological similarities between \( E_\varepsilon^a \) and the configuration space \( \Sigma_k(\Omega) \) of \( k \) distinct points in \( \Omega \). As in [AB3], we need to use a map \( \hat{\Phi} \) from \( E_\varepsilon^a \) into \( \Sigma_k(\Omega) \). More precisely, We may assign to each function \( u \) in \( E_\varepsilon^a \), a set of \( k \) distinct points \( \{a_1, \ldots, a_k\} \), called the vortices of \( u \), where each vortex has the topological degree +1. The map \( \hat{\Phi} : E_\varepsilon^a \rightarrow \Sigma_k(\Omega) \) is not continuous. However this difficulty can been overcome by applying the notion of \( \eta \)-almost continuity given in [AB3]. The topological similarity between \( E_\varepsilon^a \) and \( \Sigma_k(\Omega) \) allows us to define a covering space \( \tilde{E}_\varepsilon^a \) of \( E_\varepsilon^a \) corresponding to the covering \( F_k(\Omega) \rightarrow \Sigma_k(\Omega) \), where \( F_k(\Omega) \) is the configuration space of ordered \( k \) distinct points in \( \Omega \). Again we have topological similarity between these two spaces, and we than can prove that the category of \( \tilde{E}_\varepsilon^a \) is at least \( k \). The Ljusternik-Schnirelman minimax theorem concludes that the functional \( \tilde{E}_\varepsilon^a \) on \( \tilde{E}_\varepsilon^a \), which is the composition of \( E_\varepsilon^a \) and the covering projection either has at least \( k \) distinct critical values or the dimension of the critical set is at least 1. These imply that \( E_\varepsilon^a \) has at least \( k \) critical points on \( E_\varepsilon^a \). Finally, the fact that \( E_\varepsilon^\infty = H^1_g(\Omega, \mathbb{C}) \) is an affine space guarantee that \( E_\varepsilon^a \) has at least another critical point outside of \( E_\varepsilon^a \), if \( k \geq 2 \).

This paper is organized as follows: In the next section we will recall some preliminary results about the configuration space and the construction of the map \( \hat{\Phi} \) in [AB3] and Theorem 1 will been proved in Section 3.

### 2. PRELIMINARIES

Our proof of Theorem 1 relies essentially on the properties of the map \( \hat{\Phi} : E_\varepsilon^a \rightarrow \Sigma_k(\Omega) \) described by Almeida and Bethuel [AB3]. With a such map, they showed that the fundamental group \( \pi_1(E_\varepsilon^a) \) is non trivial for some suitable value \( a \) of the form (1.3) when \( \varepsilon \) is sufficiently small. We review here some basic facts about the configuration space and the construction of the map \( \hat{\Phi} \).
We study the configuration space and renormalized energy first. Let the metric on $C^k$ be defined by the following norm

$$\| (z_1, \ldots, z_k) \| = \sum_{i=1}^{k} |z_i|. \quad (2.1)$$

The configuration space of the ordered $k$ distinct points in $\Omega$

$$F_k(\Omega) = \{ (a_1, \ldots, a_k) \in \Omega^k; a_i \neq a_j \text{ for all } i \neq j \} \subset C^k$$

with the inherited metric (2.1) on $C^k$ is a smooth manifold. The cohomology ring $H^*(F_k(\Omega)) = H^*(F_k(\Omega), \mathbb{R})$ of the space $F_k(\Omega)$ has been determined by Arnol'd in 1969 (see [Ar]), which is generated by elements $\omega_{ij} \in H^1(F_k(\Omega)), 1 \leq i < j \leq k$ and subject to the following defining relations

$$\omega_{ij}\omega_{jl} + \omega_{jl}\omega_{il} + \omega_{il}\omega_{ij} = 0.$$ 

Arnol'd also showed that the $p$th Betti number $B_p$ of $F_k(\Omega)$ is the coefficient of $t^p$ in the polynomial

$$(1 + t)(1 + 2t) \cdots (1 + (k - 1)t).$$

In particular, $B_{k-1} = (k - 1)! \neq 0$, and this concludes that

**Lemma 2.** The cuplength of $F_k(\Omega)$ is $k - 1$.

The cuplength of a space $X$ is the largest integer $n$ such that there are $n$ elements $\varphi_j \in H^{p_j}(X), p_j > 0, 1 \leq j \leq n$ and $\varphi_1 \cup \cdots \cup \varphi_n \neq 0$.

The symmetric group $S_k$ on $\{1, \ldots, k\}$ acts isometrically on $F_k(\Omega)$ by permuting coordinates, i.e., for all $\sigma \in S_k$,

$$\sigma(a_1, \ldots, a_k) = (a_{\sigma(1)}, \ldots, a_{\sigma(k)}).$$

This action is free, and the quotient space $F_k(\Omega)/S_k$ is called the configuration space of $k$ distinct point in $\Omega$ and it will be denoted by $\Sigma_k(\Omega)$.

On $\Sigma_k(\Omega)$, we have a natural metric such that the quotient map $\pi : F_k(\Omega) \to \Sigma_k(\Omega)$ is a Riemannian regular covering. This metric on $\Sigma_k(\Omega)$ is the same as the length of minimal connection introduced by Brezis, Coron and Lieb in [BCL], i.e., for $a = \{a_1, \ldots, a_k\}, a' = \{a'_1, \ldots, a'_k\} \in \Sigma_k(\Omega)$,

$$\| a - a' \| = L(a, a') = \inf_{\sigma \in S_k} \sum_{i=1}^{k} |a_i - a'_{\sigma(i)}|.\)
We now define the renormalized energy $W_g$ on $\Sigma_k(\Omega)$ which is introduced by Bethuel-Brezis-Hélein in [BBH2] as follows, for $a = \{a_1, \ldots, a_k\} \in \Sigma_k(\Omega),$

$$W_g(a_1, \ldots, a_k) = -\pi \sum_{i \neq j} \log |a_i - a_j| + \frac{1}{2} \int_{\partial\Omega} \phi \cdot (g \times g_\tau) - \pi \sum_{j=1}^k R(a_j)$$

where $\phi$ is the solution of

$$\begin{cases}
\Delta \phi = 2\pi \sum_{i=1}^k \delta_{a_i} & \text{in } \Omega \\
\frac{\partial \phi}{\partial \nu} = g \times g_\tau & \text{on } \partial\Omega \\
\int_{\partial\Omega} \phi = 0.
\end{cases}$$

Here $\nu$ denotes the unit outer normal to $\partial\Omega$ and $\tau$ is unit tangent to $\partial\Omega$ oriented so that $\nu \times \tau = 1$. And the function $R$ is the regular part of $\phi$, i.e.,

$$R(z) = \phi(z) - \sum_{i=1}^k \log |z - a_i|.$$ 

It is clear that $W_g(a) \to +\infty$ if $\text{dist}(a_j, \partial\Omega) \to 0$ for some $i$ or if $|a_i - a_j| \to 0$ for some $i \neq j$. It has been proved in [BBH2] that, as $\varepsilon \to 0$, we have

$$\mu \varepsilon = k\pi |\log \varepsilon| + W_g(a_1^*, \ldots, a_k^*) + k\nu_0 + o(1),$$

where $o(1) \to 0$ as $\varepsilon \to 0$, $\nu_0$ is a universal constant, and $(a_1^*, \ldots, a_k^*)$ is a global minimum of the function $W_g$.

Next we will turn to the construction of the map $\bar{\Phi}$. We will use a regularization technique, that is, for any $u \in E_\varepsilon^a$, we can associate a map $u^h$, which is a minimizer (not necessarily to be unique) of the following minimization problem

$$\inf_{v \in H^1_g(\Omega, \mathbb{C})} \left\{ E_\varepsilon(v) + \int_\Omega |u - v|^2 \right\}$$

where $h = \varepsilon \frac{\varepsilon^2 + 1}{4}$. We denote $u^h = T(u)$ where $T : H^1_g(\Omega, \mathbb{C}) \to H^1_g(\Omega, \mathbb{C})$. Clearly we have $u^h \in E_\varepsilon^a$ and it satisfies an equation similar to the Ginzburg-Landau equation (1.2). One of the main observations in [AB3]
is that we can describe the “vortex structure” not only for the solutions of the Ginzburg-Landau equation, but also for such maps $u^h$. To be more precise, let us collect some of results of [AB3].

**Theorem 3** [AB3]. Assume that $a$ is of the form (1.3) for some constant $\lambda > 0$. Then there is a constant $0 < \varepsilon_0' < 1$ depending only on $\Omega$, $g$ and $\lambda$, such that if $\varepsilon < \varepsilon_0'$, then for $u \in E^a_\varepsilon$, $|u| \leq 1$ on $\Omega$, there is a point $a = \{a_1, \ldots, a_k\}$ in $\Sigma_k(\Omega)$ such that

$$
|u^h(x)| \geq \frac{1}{2}, \quad \forall x \in \Omega \setminus \bigcup_{i=1}^{k} B(a_i, \rho)
$$

where $\rho$ satisfies $\varepsilon^{\chi} \leq \rho \leq \varepsilon^{\bar{\chi}}$, for some constants $\chi, \bar{\chi} \in ]0, 1[$ independent of $\varepsilon$.

Moreover, there exists some constant $\beta > 0$ depending only on $\Omega$, $g$ and $\lambda$ such that $\text{dist}(a_i, \partial \Omega) \geq \beta$, for all $1 \leq i \leq k$ and $|a_i - a_j| \geq \beta$, for all $1 \leq i \neq j \leq k$.

Thus we can see that the properties of maps $u^h$ are very close to that of minimizers of (1.2) as in [BBH], and it allows us to define vortices $\{a_1, \ldots, a_k\}$ for $u^h$ and each of the vortices has topological degree $+1$. That defines a map $\Psi$ from $\text{Im}(T(P(E^a_\varepsilon)))$ to $\Sigma_k(\Omega)$, by $\Psi(u^h) = \{a_1, \ldots, a_k\}$, where the map $P : H^1_g(\Omega, \mathbb{C}) \rightarrow H^1_g(\Omega, \mathbb{C})$ defined by

$$
\begin{cases}
Pu(x) = u(x) & \text{if } |u(x)| \leq 1 \\
Pu(x) = \frac{u(x)}{|u(x)|} & \text{if } |u(x)| \geq 1
\end{cases}
$$

is continuous. Composing $P, T$ and $\Psi$, we define $\tilde{\Phi} : E^a_\varepsilon \rightarrow \Sigma_k(\Omega)$;

$$
\tilde{\Phi}(u) = \Psi(T(Pu)).
$$

As already noticed in [AB3], the minimizer $u^h$ to the problem (2.2) may not be unique and moving slightly the points $a_i$’s, the new positions would still match the requirements of Theorem 2. Hence the assignment of $u^h$ and the vortices for $u^h$ require some choices, so we can not expect the map $\tilde{\Phi}$ to be continuous. However the freedom in these choices are not too wild, and we can say that $\tilde{\Phi}$ is “almost” a continuous map from $E^a_\varepsilon$ to $\Sigma_k(\Omega)$. More precisely, we have
PROPOSITION 4 [AB3]. – Assume that $a$, $\varepsilon'_0$, $\bar{\chi}$ are as in Theorem 3. Then for all $\varepsilon < \varepsilon'_0$, $u, v \in E^a_{\varepsilon}$ we have

$$\|\tilde{\Phi}(u) - \tilde{\Phi}(v)\| \leq C_1\left(\|\log \varepsilon\|_{H^{2,\gamma}_{\varepsilon}} + \varepsilon \bar{\chi} + \|u - v\|_{H^{\gamma}_{\varepsilon}(\Omega, \mathbb{C})}\right)$$

where $C_1$ is a constant depending only on $\Omega$ and $g$.

Remark. – In [AB3], Almeida-Bethuel studied the more general configuration space corresponding to the “vortices” of the map $u^h$ for $u \in E^a_{\varepsilon}$, where $a$ is of the form

$$\mu_{\varepsilon} \leq a \leq K_1(|\log \varepsilon| + 1),$$

and the map $\tilde{\Phi}$ from $E^a_{\varepsilon}$ to the configuration space. We refer reader to [AB3] for the details.

Here is the notion of $\eta$-almost continuity introduced in [AB3]: A map $\Phi : X \to Y$ from a metric space $X$ to a metric space $Y$ is said to be $\eta$-almost continuous, if for all $x \in X$ and $\varepsilon > 0$, there is a $\delta$, such that for all $x'$ with $d_X(x, x') < \delta$, we have $d_Y(\Phi(x), \Phi(x')) \leq \eta + \varepsilon$. Proposition 4 says that the map $\tilde{\Phi}$ is actually $\eta$-almost equi-continuous for $\eta = C_1\left(\|\log \varepsilon\|_{H^{2,\gamma}_{\varepsilon}} + \varepsilon \bar{\chi}\right)$.

By Theorem 3, the image of $\tilde{\Phi}$ lies in the set

$$\Sigma_{k,\beta}(\Omega)$$

$$= \{a_1, \ldots, a_k \in \Sigma_k(\Omega); \text{dist}(a_i, \partial \Omega) \geq \beta, \text{ and } |a_i - a_j| \geq \beta \text{ for } i \neq j\}$$

which is compact in $\Sigma_k$. So we have

PROPOSITION 5 [AB3]. – We have an $\eta_0$ which only depends on $\beta$, such that for any $\eta \leq \eta_0$ and compact set $W \in H^1_{\varepsilon}(\Omega, \mathbb{C})$, if $\tilde{\Phi}$ is $\eta$-almost continuous and $\tilde{\Phi}(W) \subset \Sigma_{k,\beta}(\Omega)$, then there exists a continuous map $\Phi : W \to \Sigma_k(\Omega)$ such that

$$\|\tilde{\Phi}(u) - \Phi(u)\| \leq 3\eta \quad \text{for all } u \in W.$$  

3. PROOF OF THEOREM 1

In this section, we are going to prove Theorem 1 which is stated in §1.

Let $K \subset \Sigma_k(\Omega)$ be a compact core, i.e., $K$ is compact and the natural inclusion $i : K \to \Sigma_k(\Omega)$ is a homotopy equivalence. Actually, $\Sigma_{k,\beta}(\Omega)$
is a compact core for sufficiently small $\beta$. We start with a construction of maps $f_\varepsilon: K \to E^a_\varepsilon$.

**Lemma 6.** There are constants $\varepsilon_0', \lambda$ and $C_2$ such that for all $\varepsilon \leq \varepsilon_0'$, we can define $f_\varepsilon: K \to E^a_\varepsilon$, where $a = k\pi|\log \varepsilon| + \lambda$ such that

$$\|\tilde{\Phi} \cdot f_\varepsilon - \text{id}\| \leq \eta$$

on $K$, where $\eta$ is given by

$$\eta = C_2\left(|\log \varepsilon| e^{\frac{2}{4\pi + 1}} + \varepsilon^\lambda\right).$$

**Proof.** Since $K$ is compact, we can pick $\eta_K > 0$ such that for any $\{a_1, \ldots, a_k\} \in K$, the balls $B(a_i, 4\eta_K) \subset \Omega$ and are pairwise disjoint. Now once $\varepsilon \leq 4\eta_K$, we can construct a map $f_\varepsilon: \Sigma_k(\Omega) \to H^1_g(\Omega, \mathbb{C})$ as follows: for any $a = \{a_1, \ldots, a_k\} \in \Sigma_k(\Omega)$, let

$$\Omega_{\varepsilon, a} = \Omega \setminus \bigcup_{i=1}^{k} B(a_i, \varepsilon),$$

then on $\Omega_{\varepsilon, a}$, $f_\varepsilon(a)$ is defined by

$$f_\varepsilon(a)(z) = e^{i\varphi_{\varepsilon, a}(z)} \prod_{j=1}^{k} \frac{z - a_j}{|z - a_j|}$$

where the function $\varphi_{\varepsilon, a}$ is defined on $\Omega$ by the following equation

$$\begin{cases}
\Delta \varphi_{\varepsilon, a}(z) = 0 & \text{in } \Omega \\
e^{i\varphi_{\varepsilon, a}(z)} \prod_{j=1}^{k} \frac{z - a_j}{|z - a_j|} = g & \text{on } \partial \Omega.
\end{cases}$$

Notice that for a given $a$ the map $\varphi_{\varepsilon, a}$ is uniquely defined, up to an integer multiple of $2\pi$. In fact, we can choose this constant such that the map $a \to e^{i\varphi_{\varepsilon, a}}$ is continuous by the standard lifting argument. On each $B(a_i, \varepsilon)$, $f_\varepsilon(a)$ is defined by

$$\begin{cases}
\Delta f_\varepsilon(a) = 0 & \text{in } B(a_i, \varepsilon) \\
f_\varepsilon(a)(z) = e^{i\varphi_{\varepsilon, a}(z)} \prod_{j=1}^{k} \frac{z - a_j}{|z - a_j|} & \text{on } \partial B(a_i, \varepsilon).
\end{cases}$$

It is then easy to check that $f_\varepsilon$ is a continuous map from $\Sigma_k(\Omega)$ to $H^1_g(\Omega, \mathbb{C})$. 

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Moreover we can estimate the energy $E_\varepsilon(f_\varepsilon(a))$. Using the same analysis as in [Section I, BBH2], we have a constant $C$ which depends on $\Omega$ and $g$ only, such that

$$E_\varepsilon(f_\varepsilon(a)) \leq W_g(a_1, \ldots, a_k) + k\pi|\log \varepsilon| + C.$$  

Let

$$\lambda' = \sup_{a \in K} W_g(a_1, \ldots, a_k),$$

it is finite by the compactness of $K$. So

$$E_\varepsilon(f_\varepsilon(a)) \leq k\pi|\log \varepsilon| + \lambda' + C.$$  

Hence there is an $\varepsilon_1 > 0$ such that for all $\varepsilon \leq \varepsilon_1$, $f_\varepsilon(a) \in E_\varepsilon^\alpha$, for $a = \mu_\varepsilon + \lambda$ provided $\lambda$ is chosen large enough (but independent of $\varepsilon$).

Now suppose that $\varepsilon \leq \varepsilon_0'' = \min\{\varepsilon_0', \varepsilon_1, 4\eta_K\}$, and denote $f_\varepsilon(a)$ by $f_{\varepsilon,a}$ for simplicity. Let $a = \{a_1, \ldots, a_k\}$ be given in $K$, and $a' = \{a'_1, \ldots, a'_k\}$ be the vortices for $(f_{\varepsilon,a})^h$, i.e., $\Phi(f_{\varepsilon,a}) = \{a'_1, \ldots, a'_k\}$. According to Theorem 3, on $\Omega_{\rho,a'} = \Omega \setminus \bigcup_{i=1}^k B(a'_i, \rho)$, we have

$$|f_{\varepsilon,a}^h(x)| \geq \frac{1}{2}, \text{ for all } x \in \Omega_{\rho,a'},$$  

where $\varepsilon^\chi \leq \rho \leq \varepsilon^\chi$. We may therefore consider on $\tilde{\Omega} = \Omega_{\rho,a'} \setminus \bigcup_{i=1}^k B(a_i, \varepsilon)$, the map $\xi = \frac{f_{\varepsilon,a}}{|f_{\varepsilon,a}|} f_{\varepsilon,a}^{-1}$. $\xi$ takes its values in $S^1$ and satisfies $\xi \equiv 1$ on $\partial \Omega$. Moreover we have

$$|\xi - 1| \leq 4|f_{\varepsilon,a}^h - f_{\varepsilon,a}|.$$  

This yields

$$\int_{\tilde{\Omega}} |\xi - 1|^2 \leq 16 \int_{\tilde{\Omega}} |f_{\varepsilon,a}^h - f_{\varepsilon,a}|^2 \leq 32\varepsilon^{\frac{4}{n+1}} (E_\varepsilon(f_{\varepsilon,a}) - E_\varepsilon(f_{\varepsilon,a}^h))$$

$$\leq C|\log \varepsilon|\varepsilon^{\frac{4}{n+1}}$$  

for some constant $C$ depending only on $g, K$ and $\Omega$.

On the other hand, for any $1 \leq i \leq k$, we have

$$\deg(\xi, \partial B(a'_i, \rho)) = -\deg(\xi, \partial B(a_i, \varepsilon)) = 1.$$  

So for any regular value $y \in S^1$ of $\xi$ and $y \neq 1$, $\xi^{-1}(y)$ is a connection between balls $B(a_i, \varepsilon)$ and $B(a'_i, \rho)$. By the definition of length of minimal connection $L$ given in (2.2), we get

$$L(a'_i, a) - k(\rho + \varepsilon) \leq \mathcal{H}^1(\xi^{-1}(y))$$

for almost every $y \in S^1$.  

Let
\[ N = \left\{ y \in S^1, \frac{1}{8} \leq |y - 1| \leq \frac{1}{4} \right\} \]
and take \( A = \xi^{-1}(N) \), using the coarea formula of Federer-Fleming, we obtain
\[
\int_{N} \mathcal{H}^{1}(\xi^{-1}(y)) \, dy = \int_{A} |\nabla \xi| 
\leq \left( \int_{A} |\nabla \xi|^{2} \right)^{1/2} \left( \text{meas } A \right)^{1/2}.
\tag{3.2}
\]
By (3.1), we have
\[
(\text{meas } A) \leq 64 \int_{\Omega} |\xi - 1|^{2} \leq C|\log \varepsilon|^{\frac{4}{4+\varepsilon}};
\]
On the other hand
\[
\int_{\Omega} |\nabla \xi|^{2} \leq 8 \left( \int_{\Omega} |\nabla f_{\varepsilon,a}^{h}|^{2} + |\nabla f_{\varepsilon,a}|^{2} \right) \leq C|\log \varepsilon|.
\]
Together with (3.2) we get that
\[
L(a, a') - k(\rho + \varepsilon) \leq \frac{1}{(\text{meas } N)} \int_{N} \mathcal{H}^{1}(\xi^{-1}(y)) \, dy \leq C|\log \varepsilon|^{\frac{4}{4+\varepsilon}},
\]
that is the conclusion we required.

For any \( a \in K \), the ball \( B(a, 4\eta_{K}) \subset \Sigma_{k}(\Omega) \) with radius \( 4\eta_{K} \), where \( \eta_{K} \) is the constant in the proof of Lemma 6, is in fact isometric to a standard ball in \( C^{k} \). To see this, let \( K = \pi^{-1}(K) \subset F_{k}(\Omega) \), which is also a compact core of \( F_{k}(\Omega) \), and for any \( \hat{a} \in \pi^{-1}(a) \), the condition that \( B(a_{i}, 4\eta_{K})'s \) are pairwise disjoint implies that the ball \( B(\hat{a}, 4\eta_{K}) \subset C^{k} \) is contained in \( F_{k}(\Omega) \) entirely, and \( B(a, 4\eta_{K}) \) is isometric to \( B(\hat{a}, 4\eta_{K}) \).

**Lemma 7.** - There is an \( \varepsilon_{o} \), such that for any \( \varepsilon \leq \varepsilon_{o} \), the map \( f_{\varepsilon} \) induces an injection
\[
f_{\varepsilon} : \pi_{1}(K) \to \pi_{1}(E_{\varepsilon}^{a}),
\]
where \( a \) is chosen by Lemma 6.

**Proof.** - The constant \( \varepsilon_{0} \leq \varepsilon_{o}' \) is chosen such that
\[
\max(C_{1}, C_{2}) \left( |\log \varepsilon_{0}|^{\frac{4}{4+\varepsilon}} + \varepsilon_{0}^{\frac{4}{4+\varepsilon}} \right) \leq \min\{\eta_{0}, \eta_{K}\},
\]
where \( C_i \) is as in Proposition 4, \( \eta_0 \) as in Proposition 5 and \( \varepsilon_0', C_2 \) and \( \eta_K \) as in Lemma 6.

For each element \( \alpha \in \pi_1(K) \), we can choose a closed path \( c : S^1 \to K \) which representing \( \alpha \). Now for \( \varepsilon \leq \varepsilon_0 \), if \( f_\varepsilon \cdot c : S^1 \to E_\varepsilon \) is null homotopic, we get a map \( \tilde{f} : D^2 \to E_\varepsilon \), such that \( \tilde{f}|_{\partial D^2} = f_\varepsilon \cdot c \). By Proposition 5, on the compact set \( \tilde{f}(D^2) \subset E_\varepsilon \), we can define a continuous map \( \Phi : \tilde{f}(D^2) \to \Sigma_k(\Omega) \), such that for any \( u \in \tilde{f}(D^2) \), \( \|\Phi(u) - \tilde{\Phi}(u)\| < 3\eta_K \). The map \( \Phi \cdot \tilde{f}|_{\partial D^2} = \Phi \cdot f_\varepsilon \cdot c : S^1 \to \Sigma_k(\Omega) \) is null homotopic. On the other hand, by Lemma 6,

\[
\|\Phi \cdot f_\varepsilon \cdot c(t) - c(t)\| \leq \|\Phi \cdot f_\varepsilon \cdot c(t) - \tilde{\Phi} \cdot f_\varepsilon \cdot c(t)\| + \|\tilde{\Phi} \cdot f_\varepsilon \cdot c(t) - c(t)\| < 4\eta_K.
\]

Then we can find a unique minimum geodesic in \( \Sigma_k(\Omega) \) connecting \( \Phi \cdot f_\varepsilon \cdot c(t) \) and \( c(t) \). This implies that \( \Phi \cdot f_\varepsilon \cdot c \) is homotopic to \( c \). So \( \alpha \) is a trivial element in \( \pi_1(K) \), and this means that \( f_\varepsilon \ast \) is injective. \( \square \)

Since \( \pi_\ast : \pi_1(K) \to \pi_1(K) \) and \( f_\varepsilon \ast : \pi_1(K) \to \pi_1(E_\varepsilon) \) are injective, so is \( f_\varepsilon \ast \cdot \pi_\ast : \pi_1(K) \to \pi_1(E_\varepsilon) \). Consider a covering space \( p : \tilde{E}_\varepsilon \to E_\varepsilon \) corresponding to the group \( f_\varepsilon \ast \cdot \pi_\ast (\pi_1(K)) \subset \pi_1(E_\varepsilon) \), the map \( f_\varepsilon \ast : \tilde{K} \to \tilde{E}_\varepsilon \) can be lift to a map \( \tilde{f} : K \to \tilde{E}_\varepsilon \) such that the following diagram commutes

\[
\begin{array}{ccc}
\tilde{K} & \longrightarrow & \tilde{E}_\varepsilon \\
\downarrow & & \downarrow \\
K & \longrightarrow & E_\varepsilon.
\end{array}
\]

**Lemma 8.** The map \( \tilde{f} \) induces maps \( \tilde{f}_\ast : H_p(\tilde{K}) \to H_p(\tilde{E}_\varepsilon) \) on the homology groups which are injective for all \( p \).

**Proof.** The argument here goes in the same fashion as the proof of Lemma 7. Consider a singular cycle \( c \in Z_p(\tilde{K}) \) such that \( \tilde{f}_\ast([c]) = 0 \) in \( H_p(\tilde{E}_\varepsilon) \). This means that we have a \( p + 1 \)-chain \( c' \in C_{p+1}(E_\varepsilon) \) and \( \partial c' = \tilde{f}_\ast(c) \). The set \( W = \tilde{f}(\tilde{K}) \cup \text{support}(c') \) is compact in \( \tilde{E}_\varepsilon \). Then we define a continues map \( \Phi_1 : p(W) \to \Sigma_k(\Omega) \) such that for any \( u \in p(W) \), \( \|\Phi_1(u) - \tilde{\Phi}(u)\| < 3\eta_K \).

Notice that \( \|\Phi_1 \cdot f_\varepsilon - \text{id}\| < 4\eta_K \), as before, we have \( \Phi_1 \ast \cdot f_\varepsilon \ast = \text{id} \). This implies that \( \Phi_1 \ast \cdot p_\ast (\pi_1(W)) \subset \Phi_1 \ast \cdot f_\varepsilon \ast \cdot \pi_\ast (\pi_1(\tilde{K})) = \pi_\ast (\pi_1(\tilde{K})) \). So we can lift \( \Phi_1 \ast \cdot p : W \to \Sigma_k(\Omega) \) to \( \tilde{\Phi}_1 : W \to \tilde{F}_k(\Omega) \).

In fact, we can make \( \|\tilde{\Phi}_1 \cdot \tilde{f} - \text{id}\| < 4\eta_K \). Since \( \|\Phi_1 \cdot p \cdot \tilde{f} - \pi\| < 4\eta_K \), there is a homotopy \( H_t \) such that \( H_0 = \pi \) and \( H_1 = \Phi_1 \cdot p \cdot \tilde{f} \). Lift this homotopy to a homotopy \( \tilde{H}_t \) with \( \|\tilde{H}_0 - \tilde{H}_1\| < 4\eta_K \) and \( \tilde{H}_0 = \text{id}_{\tilde{K}} \). Define \( \tilde{\Phi}_2 : \tilde{f}(\tilde{K}) \to F_k(\Omega) \) by \( \tilde{\Phi}_2(f(a)) = H_1(a) \). Note that

\[
\pi \cdot \tilde{\Phi}_2 = \Phi_1 \cdot p = \pi \cdot \tilde{\Phi}_1|_{\tilde{f}(\tilde{K})}.
\]
\( \tilde{\Phi}_2 \) and \( \tilde{\Phi}_1 \) differ by a deck transformation, i.e., there is an element \( \sigma \in S_k \), such that
\[
\tilde{\Phi}_2 = \sigma \cdot \tilde{\Phi}_1 |_{f(K)}.
\]
Replace \( \tilde{\Phi}_1 \) by \( \sigma \cdot \tilde{\Phi}_1 \), which is also a lifting \( p : W \to \Sigma_k(\Omega) \) and \( \| \sigma \cdot \tilde{\Phi}_1 \cdot \tilde{f} - \text{id} \| < 4\eta_K \). The new lifting will still denote \( \tilde{\Phi}_1 \).

Now \( \tilde{\Phi}_1 \) maps the chain \( c' \) into a chain in \( C_{p+1}(F_k(\Omega)) \), and \( \partial \tilde{\Phi}_1(c') = \tilde{\Phi}_1(\partial c') = \tilde{\Phi}_1 \cdot f_*(c) \). We get that \( \tilde{\Phi}_1 \cdot f_*(c) \) is a boundary in \( C_p(F_k(\Omega)) \). On the other hand, \( \tilde{\Phi}_1 \cdot \tilde{f} \) is homotopic to the natural inclusion \( i : K \to F_k(\Omega) \). So \( c \) is homologous to \( \tilde{\Phi}_1 \cdot f_*(c) \), and \( c \) is null homologous as well. This shows that \( f_* \) is injective.

The lemma allows us to estimate the category of \( E_\varepsilon^a \).

**Corollary 9.** The category \( \text{cat}(E_\varepsilon^a) \) of \( E_\varepsilon^a \) is at least \( k \).

**Proof.** By Lemma 8, the map \( f^* : H^*(E_\varepsilon^a) \to H^*(\tilde{K}) \) between cohomology rings is surjective, and this implies that the cuplength of \( E_\varepsilon^a \) is at least the cuplength of \( \tilde{K} \), which is the same as the cuplength of \( F_k(\Omega) \). By Lemma 2, the cuplength of \( E_\varepsilon^a \) is at least \( k - 1 \). Finally, according to [BG], the category \( \text{cat}(E_\varepsilon^a) \) of \( E_\varepsilon^a \) is at least the cuplength of \( E_\varepsilon^a \) plus one. This completes the proof.

Now we are in the position to complete the proof of Theorem 1. The Lusternik-Schnirelman minimax theorem we will use is the following

**Theorem 10.** Suppose \( F \) is a \( C^2 \) non-negative functional defined on a smooth Hilbert manifold \( M \) such that

i) the backwards gradient flow is complete;

ii) \( F \) satisfies the following weak Palais-Smale condition: if we have a sequence \( \{u_n\} \) in \( M \) such that \( F(u_n) \to c \) and \( \| \nabla F(u_n) \| \to 0 \) as \( n \to \infty \), then \( c \) is a critical value;

iii) \( \text{cat} M = k \).

Then we have either \( F \) has at least \( k \) distinct critical values in \([0, a]\) or the dimension of the critical set of \( F \) is at least 1.

The proof is standard, we refer reader to [Pa].

**Proof of Theorem 1.** Now we want to apply Theorem 10 to the positive functional \( \tilde{E}_\varepsilon = E_\varepsilon \cdot p : \tilde{E}_\varepsilon^a \to \mathbb{R} \). Notice that \( \tilde{E}_\varepsilon \) and \( E_\varepsilon \) have the same critical values and critical sets of the two functionals have the same dimension. If all three conditions in the theorem hold, both conclusions will imply that \( E_\varepsilon \) has at least \( k \) critical points on \( E_\varepsilon^a \).

We now check the three conditions in Theorem 10. First, the backwards gradient flow of \( \tilde{E}_\varepsilon \) is a lift of the backwards flow of \( E_\varepsilon \), so it is
complete. Second, let \( \{u_n\} \) be a sequence in \( \tilde{E}_\varepsilon^a \) such that \( \tilde{E}_\varepsilon(u_n) \to c \) and \( \|\nabla \tilde{E}_\varepsilon(u_n)\| \to 0 \) as \( n \to \infty \), then \( E_\varepsilon(p(u_n)) \to c \) and \( \|\nabla E_\varepsilon(p(u_n))\| \to 0 \).

We know that \( E_\varepsilon \) satisfies Palais-Smale condition, so \( p(u_n) \) has a subsequence converges to a critical point. This shows that \( c \) is a critical value of \( E_\varepsilon \) and then it is a critical value of \( \tilde{E}_\varepsilon \) as well. Finally, \( \text{cat} \tilde{E}_\varepsilon^a \geq k \) is the conclusion of Corollary 9. So we now can conclude that \( E_\varepsilon \) has at least \( k \) critical points on \( E_\varepsilon^a \).

Outside of \( E_\varepsilon^a \), \( E_\varepsilon \) has at least another critical point, since \( H^1_{\text{ad}}(\Omega, \mathbb{C}) \) is contractible, but \( E_\varepsilon^a \) is not (if \( k \geq 2 \)). So totally \( E_\varepsilon \) will have at least \( k + 1 \) critical points on \( H^1_{\text{ad}}(\Omega, \mathbb{C}) \).

REFERENCES


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