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On the non-locality of quasiconvexity

by

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ABSTRACT. – It is shown that in the class of smooth real-valued functions on $n \times m$ matrices ($n \geq 3$, $m \geq 2$) there can be no "local condition" which is equivalent to quasiconvexity. © Elsevier, Paris.

Key words: Quasiconvexity, rank-one convexity.

RÉSUMÉ. – On démontre qu'il n'existe pas de condition locale qui dans l'espace des fonctions regulières est equivalente à celle de quasiconvexité. © Elsevier, Paris.

A continuous function $f: \mathbb{R}^{n\times m} \mapsto \mathbb{R}$ is called locally quasiconvex if at every point $X \in \mathbb{R}^{n\times m}$ there exists a neighborhood in which it coincides with a quasiconvex function. In this note we show that a C^2-function satisfying a strict Legendre-Hadamard condition at every point is locally quasiconvex. Using Šverák's (cf. [21]) example of a rank-one convex function which is not quasiconvex we show that in dimensions $n \geq 3$, $m \geq 2$ there are locally quasiconvex functions that are not quasiconvex. Indeed, for any positive number $r > 0$ we give an example of a smooth function, which equals a quasiconvex function on any ball of radius r, but which is not itself quasiconvex. As a consequence of this we obtain that in dimensions $n \geq 3$, $m \geq 2$ there is no "local condition" which

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for $\mathcal{C}^\infty$-functions is equivalent to quasiconvexity. In particular, we confirm the conjecture of Morrey (cf. [12]) saying that in general there is no condition involving only f and a finite number of its derivatives, which is both necessary and sufficient for quasiconvexity. However, it might still be possible to find a "local condition" which is equivalent to quasiconvexity in e.g. the class of polynomials.

The proof relies heavily on Šverák's example of a rank-one convex function which is not quasiconvex, and the main contribution here is contained in Lemma 2. Lemma 2 provides an extension result for quasiconvex functions, and is proved by use of Taylor's formula, a slight extension of Dacorogna's quasiconvexification formula and the equivalence of rank-one convexity and quasiconvexity for quadratic forms.

In the last part of this note we consider rank-one convexity and quasiconvexity in an abstract setting. We hereby prove that in the class of $\mathcal{C}^\infty$-functions, any convexity concept between rank-one convexity and quasiconvexity, which is equivalent to a "local condition" is in fact rank-one convexity.

For convenience of the reader and to fix the notation we recall some definitions. The space of (real) $n \times m$ matrices is denoted by $\mathbb{R}^{n\times m}$. We use the usual Hilbert-Schmidt norm for matrices.

A continuous real-valued function $f\colon \mathbb{R}^{n\times m} \mapsto \mathbb{R}$ is said to be rank-one convex at $X \in \mathbb{R}^{n\times m}$ if the inequality

$$f(X) \leq tf(Y) + (1-t)f(Z) \tag{1}$$

holds for all $t \in [0,1]$, $Y,\ Z \in \mathbb{R}^{n\times m}$ satisfying $\operatorname{rank}(Y-Z) \leq 1$ and $X = tY + (1-t)Z$. The function f is rank-one convex if it is rank-one convex at each point.

The space of compactly supported $\mathcal{C}^\infty$-functions $\varphi : \mathbb{R}^m \mapsto \mathbb{R}^n$ is denoted by $\mathcal{D}(\mathbb{R}^m;\mathbb{R}^n)$, or briefly, by $\mathcal{D}$. The support of φ is denoted by $\operatorname{spt}\varphi$, and the gradient of φ at x, $D\varphi(x)$, is identified in the usual way with a $n \times m$ matrix.

A continuous real-valued function $f : \mathbb{R}^{n\times m} \mapsto \mathbb{R}$ is said to be quasiconvex at $X \in \mathbb{R}^{n\times m}$ if the inequality

$$\int_{\mathbb{R}^m} (f(X + D\varphi(x)) - f(X))\,dx \geq 0 \tag{2}$$

holds for all $\varphi \in \mathcal{D}$. The function f is quasiconvex if it is quasiconvex at each point.

If for $X \in \mathbb{R}^{n\times m}$ there exists a positive number $\delta = \delta(X) > 0$, such that the inequality (2) holds for all $\varphi \in \mathcal{D}$ satisfying $\sup_x |D\varphi(x)| \leq \delta$, then f is said to be weakly quasiconvex at X. As above, f is weakly quasiconvex if it is weakly quasiconvex at each point.

The concepts of quasiconvexity and weak quasiconvexity are due to Morrey [12]. A concept of quasiconvexity relevant for higher order problems has been introduced by Meyers [11] (see also [5]).

It is obvious that quasiconvexity of f implies weak quasiconvexity of f, and, as shown by Morrey [12], weak quasiconvexity of f implies rank-one convexity of f. Hence it follows in particular that quasiconvexity of f implies rank-one convexity of f.

In the special case where f is a quadratic form the converse is also true. Hence for quadratic forms the notion of rank-one convexity is equivalent to the notion of quasiconvexity (cf. [13]). A famous conjecture of Morrey [12] is that in dimensions $n \geq 2$, $m \geq 2$ there are rank-one convex functions that are not quasiconvex. In dimensions $n \geq 3$, $m \geq 2$ this was confirmed by Šverák in [21] giving a remarkable example of a polynomial of degree four which is rank-one convex, but not quasiconvex. In the remaining non-trivial cases, *i.e.* $n = 2$, $m \geq 2$, the question remains open. The problem is discussed in [3], [4], and more recently, in [15], [17], [26], [27].

It is not hard to see that for a $\mathcal{C}^2$-function $f : \mathbb{R}^{n\times m} \mapsto \mathbb{R}$ rank-one convexity is equivalent to satisfaction of the Legendre-Hadamard (or ellipticity) condition at every $X \in \mathbb{R}^{n\times m}$, *i.e.* for each $X \in \mathbb{R}^{n\times m}$

$$D^2 f(X)(a \otimes b, a \otimes b) \geq 0 \tag{3}$$

for all $a \in \mathbb{R}^n$, $b \in \mathbb{R}^m$.

If for some $X \in \mathbb{R}^{n\times m}$ the inequality (3) holds strictly for all $a \neq 0$, $b \neq 0$, then we say that f satisfies a strict Legendre-Hadamard (or strong ellipticity) condition at X. This is equivalent to the existence of a positive number $c = c(X)$, such that

$$D^2 f(X)(a \otimes b, a \otimes b) \geq c|a|^2|b|^2 \tag{4}$$

for all $a \in \mathbb{R}^n$, $b \in \mathbb{R}^m$. By using the Fourier transformation and the Plancherel theorem it is easily seen that (4) is equivalent to

$$\int_{\mathcal{B}} D^2 f(X)(D\varphi(x), D\varphi(x))\, dx \geq c \int_{\mathcal{B}} |D\varphi(x)|^2\, dx \tag{5}$$

for all $\varphi \in \mathcal{D}$ with spt$\varphi \subset \mathcal{B}$, where $\mathcal{B} := \{x \in \mathbb{R}^m : |x| < 1\}$.

By using Taylor's formula and the equivalence of rank-one convexity and quasiconvexity for quadratic forms it can be proved that a $\mathcal{C}^2$-function f satisfying a strict Legendre-Hadamard condition at every point is weakly quasiconvex. The same kind of reasoning was used by Tartar [22] in proving a local form of a conjecture in compensated compactness.

DEFINITION. – *A continuous real-valued function $f : \mathbb{R}^{n\times m} \mapsto \mathbb{R}$ is said to be locally quasiconvex at $X \in \mathbb{R}^{n\times m}$ if there exists a quasiconvex function $g : \mathbb{R}^{n\times m} \mapsto \mathbb{R}$, such that $f = g$ in a neighborhood of X.*

The function f is locally quasiconvex if it is locally quasiconvex at each point.

One could define a similar concept of local rank-one convexity. However, by using a mollifier argument and the Legendre-Hadamard condition it is easily proved that this concept coincides with the usual concept of rank-one convexity. It is obvious that there is no need for a local concept of weak quasiconvexity.

If $f: \mathbb{R}^{n\times m} \mapsto \mathbb{R}$ is a locally bounded Borel function, then we define its quasiconvexification, $Qf : \mathbb{R}^{n\times m} \mapsto [-\infty, +\infty]$, as

$$Qf(X) := \sup\{g(X) \; : \; g \text{ quasiconvex and } g \leq f\}.$$

Notice that if at some X, $Qf(X) > -\infty$, then Qf is quasiconvex.

The following result is a slight extension of a similar result due to Dacorogna [6]. We refer to [8] for the proof of this and for some extensions along these lines.

LEMMA 1. – *Let $f: \mathbb{R}^{n\times m} \mapsto \mathbb{R}$ be a locally bounded Borel function. Then*

$$Qf(X) = \inf\left\{ ⨍_{\mathcal{B}} f(X + D\varphi)\, dx \; : \; \varphi \in \mathcal{D} \textit{ with } spt\varphi \subset \mathcal{B} \right\}.$$

For a $\mathcal{C}^2$-function $f : \mathbb{R}^{n\times m} \mapsto \mathbb{R}$ we have by Taylor's formula

$$f(X+Y) = f(X) + Df(X)Y + \frac{1}{2}D^2 f(X)(Y;Y) + R(X;Y),$$

where the remainder term $R(X;Y)$ is given by

$$R(X;Y) = \int_0^1 (1-t)(D^2 f(X+tY) - D^2 f(X))(Y;Y)\, dt.$$

For notational reasons it is convenient to introduce an auxiliary function, which essentially is a continuity modulus for the second derivative of f.

For each $r \in (0, +\infty)$ define $\Omega_r : (0, +\infty) \mapsto [0, +\infty)$ as (the norm being the usual one for bilinear mappings)

$$\Omega_r(t) := \sup\{|D^2 f(X+Y) - D^2 f(X)| : |X| \leq r, |Y| < t\}.$$

Obviously, Ω_r is non-decreasing and continuous, and since $D^2 f$ is uniformly continuous on compact sets, $\Omega_r(t) \to 0$ as $t \to 0+$. Furthermore we notice that if $|X| \leq r$, then

$$|R(X;Y)| \leq \frac{1}{2}\Omega_r(|Y|)|Y|^2 \tag{6}$$

for all $Y \in \mathbb{R}^{n\times m}$.

LEMMA 2. – *Let $f : \mathbb{R}^{n\times m} \mapsto \mathbb{R}$ be a $\mathcal{C}^2$-function, and assume that there exist numbers c, $r > 0$, such that*

$$\int_{\mathcal{B}} D^2 f(X)(D\varphi, D\varphi)\, dx \geq c \int_{\mathcal{B}} |D\varphi|^2\, dx \tag{7}$$

for $|X| \leq r$ and $\varphi \in \mathcal{D}$ with spt$\varphi \subset \mathcal{B}$. Put $\delta := (1/2)\sup\{t \in (0, r) : c \geq \Omega_r(t)\}$. Then there exists a quasiconvex function $g : \mathbb{R}^{n\times m} \mapsto \mathbb{R}$ of at most quadratic growth, such that

$$f(X) = g(X) \text{ whenever } |X| \leq \delta.$$

Remark. – Being quasiconvex g is necessarily locally Lipschitz continuous (cf. [6]), however, I do not know whether it is possible to obtain a quasiconvex extension g of f which is as regular as f is.

Proof. – Define the function $g := QG$, where

$$G(X) := \begin{cases} f(X) & \text{if } |X| \leq \delta, \\ \sup_{|Y|\leq\delta} (f(Y) + Df(Y)(X-Y) \\ \qquad + \frac{1}{2}D^2 f(Y)(X-Y, X-Y)) & \text{otherwise.} \end{cases}$$

Then obviously g is quasiconvex, of at most quadratic growth and $g(X) \leq f(X)$ for $|X| \leq \delta$. We claim that $g(X) = f(X)$ for $|X| \leq \delta$. Fix X with $|X| < \delta$. Let $\varepsilon > 0$ and find $\varphi = \varphi_\varepsilon \in \mathcal{D}$, such that

$$|\mathcal{B}|(g(X) + \varepsilon) > \int_{\mathcal{B}} G(X + D\varphi)\, dx.$$

Using Taylor's formula, (6) and (7) we obtain

$$
\begin{aligned}
|\mathcal{B}|(g(X)+\varepsilon) > \int_{\mathcal{B}\cap\{|X+D\varphi|\leq\delta\}} & f(X+D\varphi)\,dx \\
&+\int_{\mathcal{B}\cap\{|X+D\varphi|>\delta\}}\Big(f(X)+Df(X)D\varphi+\frac{1}{2}D^2f(X)(D\varphi,D\varphi)\Big)\,dx \\
&=\int_{\mathcal{B}\cap\{|X+D\varphi|\leq\delta\}} R(X,D\varphi)\,dx \\
&+\int_{\mathcal{B}}\Big(f(X)+Df(X)(D\varphi)+\frac{1}{2}D^2f(X)(D\varphi,D\varphi)\Big)\,dx \\
&\geq |\mathcal{B}|f(X)+\frac{1}{2}\int_{\mathcal{B}\cap\{|X+D\varphi|\leq\delta\}}|D\varphi|^2(c-\Omega_r(|D\varphi|))\,dx \geq |\mathcal{B}|f(X),
\end{aligned}
$$

where the last inequality follows from the definition of δ. □

PROPOSITION 1. – *Let $f:\mathbb{R}^{n\times m}\mapsto\mathbb{R}$ be a $\mathcal{C}^2$-function satisfying a strict Legendre-Hadamard condition at every point. Then f is locally quasiconvex.*

Proof. – This follows easily by applying Lemma 2 to the functions $f_X(Y):=f(X+Y)$, $Y\in\mathbb{R}^{n\times m}$, where $X\in\mathbb{R}^{n\times m}$ is fixed. □

According to Šverák [21] there exists a polynomial p of degree four on $\mathbb{R}^{3\times 2}$, which is rank-one convex but not quasiconvex. A closer inspection of the proof in [21] reveals that we may take p so that it additionally satisfies a strict Legendre-Hadamard condition at every point, hence by the above result p is locally quasiconvex.

Recall that a continuous function f is polyconvex if $f(X)$ can be written as a convex function of the minors of X. A polyconvex function is quasiconvex, but not conversely (cf. Ball [2], and [1], [20], [24], [25]). If one defines a concept of local polyconvexity as done above for quasiconvexity it is possible to prove that there are locally polyconvex functions on $\mathbb{R}^{n\times m}$ (n, $m\geq 2$) that are not polyconvex. In higher dimensions, *i.e.* $n\geq 3$, $m\geq 2$, there are locally polyconvex functions on $\mathbb{R}^{n\times m}$ that are not quasiconvex (cf. [9]).

PROPOSITION 2. – *Assume that $n\geq 3$, $m\geq 2$. For any $r>0$ there exists a $\mathcal{C}^\infty$-function $f_r:\mathbb{R}^{n\times m}\mapsto\mathbb{R}$ with the following two properties:*

(I) *f_r is not quasiconvex;*

(II) *for all $X\in\mathbb{R}^{n\times m}$ there exists a quasiconvex function g_X, such that $g_X(Y)=f_r(Y)$ holds for $|Y-X|<r$.*

In particular, local quasiconvexity does not imply quasiconvexity.

Proof. – Let $p: \mathbb{R}^{n\times m} \mapsto \mathbb{R}$ be a polynomial of degree four which is rank-one convex, but not quasiconvex (cf. Šverák [21]). Take for each $s > 1$ two auxiliary functions ζ_s, $\xi_s \in \mathcal{C}^\infty(\mathbb{R})$ verifying

$$\zeta_s(t) = \begin{cases} 1 & \text{if } t < s \\ 0 & \text{if } t > s+1, \end{cases}$$

$$\xi_s(t) = \begin{cases} 0 & \text{if } t < s-1 \\ t^2 & \text{if } t > s+1, \end{cases}$$

and ξ_s non-decreasing, convex and $\xi_s''(t) > 0$ for $t \in (s-1, s+1)$.

It is not hard to see that we may find $s > 1$ and $k > 0$, such that

$$p(X)\zeta_s(|X|) + k\xi_s(|X|)$$

is rank-one convex, but not quasiconvex (cf. Šverák [19] remark 3.4 and [20]). Next take $\varepsilon > 0$, so that

$$g(X) := p(X)\zeta_s(|X|) + k\xi_s(|X|) + \varepsilon|X|^2$$

is not quasiconvex. Notice that g satisfies a uniform Legendre-Hadamard condition:

$$\int_{\mathcal{B}} D^2 g(X)(D\varphi, D\varphi)\, dx \geq \varepsilon \int_{\mathcal{B}} |D\varphi|^2\, dx$$

for all $X \in \mathbb{R}^{n\times m}$ and all $\varphi \in \mathcal{D}$ with spt$\varphi \subset \mathcal{B}$.

Notice also that if $R(X,Y)$ denotes the remainder term in the Taylor expansion of g about X, then for some constant $C > 0$

$$|R(X,Y)| \leq 3 \int_0^1 (1-t)^2 \sum_{|\alpha|=3} |\partial^\alpha g(X+tY) \frac{Y^\alpha}{\alpha!}|\, dt \leq C|Y|^3$$

for all $X, Y \in \mathbb{R}^{n\times m}$. In the notation of Lemma 2 (see (6)) this corresponds to $\Omega_r(t) = 2Ct$, $t > 0$, independent of $r > 0$.

Fix $X_0 \in \mathbb{R}^{n\times m}$. We claim that there exists a quasiconvex extension of g from the closed ball $|X - X_0| \leq \varepsilon/(4C)$. Indeed, define $g_{X_0}(X) := g(X_0 + X)$ and notice that by Lemma 2 we may find a quasiconvex function G_{X_0}, such that $g(X + X_0) = g_{X_0}(X) = G_{X_0}(X)$ for $|X| \leq \varepsilon/(4C)$, or equivalently, such that

$$g(X) = G_{X_0}(X - X_0) \quad \text{for} \quad |X - X_0| \leq \frac{\varepsilon}{4C}.$$

This proves the claim. Finally we define the function f_r as

$$f_r(X) := g\left(\frac{4C}{\varepsilon r}X\right),\ X \in \mathbb{R}^{n\times m}.$$

This finishes the proof. □

Let $\mathcal{C}^\infty(\mathbb{R}^{n\times m})$ denote the space of all real-valued $\mathcal{C}^\infty$-functions $f : \mathbb{R}^{n\times m} \mapsto \mathbb{R}$ and let $\mathcal{F}$ denote the space of all extended real-valued functions $F : \mathbb{R}^{n\times m} \mapsto [-\infty, +\infty]$.

If we define the operator $\mathcal{P}_{rc} : \mathcal{C}^\infty(\mathbb{R}^{n\times m}) \mapsto \mathcal{F}$ as

$$\mathcal{P}_{rc}(f)(X) := \inf\left\{D^2 f(X)(a\otimes b, a\otimes b) \ :\ a\in\mathbb{R}^n,\ b\in\mathbb{R}^m\right\},\ X\in\mathbb{R}^{n\times m},$$

then $f \in \mathcal{C}^\infty(\mathbb{R}^{n\times m})$ is rank-one convex if and only if $\mathcal{P}_{rc}(f) = 0$. Furthermore, the operator $\mathcal{P}_{rc}$ is local in the sense that if $f, g \in \mathcal{C}^\infty(\mathbb{R}^{n\times m})$ are equal in a neighborhood of X, then also $\mathcal{P}_{rc}(f)$ equals $\mathcal{P}_{rc}(g)$ in a neighborhood of X. Thus:

$$f = g \text{ in a neighborhood of } X \Rightarrow \mathcal{P}_{rc}(f) = \mathcal{P}_{rc}(g) \text{ in a neighborhood of } X.$$

It would be interesting if one could find a similar condition for quasiconvexity. That is, a local operator $\mathcal{P}_{qc} : \mathcal{C}^\infty(\mathbb{R}^{n\times m}) \mapsto \mathcal{F}$ with the property

$$(*) \qquad \mathcal{P}_{qc}(f) = 0 \Leftrightarrow f \text{ is quasiconvex}$$

for $f \in \mathcal{C}^\infty(\mathbb{R}^{n\times m})$.

THEOREM 1. – *In dimensions $n \geq 3$, $m \geq 2$ there does not exist a local operator*

$$\mathcal{P} : \mathcal{C}^\infty(\mathbb{R}^{n\times m}) \mapsto \mathcal{F}$$

with the property $(*)$.

Remark. – The proof will show that the operator $\mathcal{P}$ cannot satisfy $(*)$ and the following locality-type condition: There exists a number $r > 0$, such that for f, $g \in \mathcal{C}^\infty(\mathbb{R}^{n\times m})$ and $X \in \mathbb{R}^{n\times m}$

$$f(Y) = g(Y) \text{ for } |Y - X| \leq r \Rightarrow \mathcal{P}(f)(X) = \mathcal{P}(g)(X).$$

Proof. – We argue by contradiction and assume that it is possible to find a local operator with the property $(*)$.

By Proposition 2 we may find a $\mathcal{C}^\infty$-function $f : \mathbb{R}^{n\times m} \mapsto \mathbb{R}$ which is not quasiconvex, but agrees with quasiconvex functions on all balls of, say, radius one.

Let $\Phi_\varepsilon \in \mathcal{C}^\infty$, $\varepsilon > 0$, be a non-negative mollifier with support contained in $\{X : |X| \leq \varepsilon\}$. Put $f_\varepsilon := f * \Phi_\varepsilon$, *i.e.* the convolution of f and Φ_ε.

We claim that if $\varepsilon \in (0, 1/2)$, then f_ε is quasiconvex.

Fix $X \in \mathbb{R}^{n\times m}$. By the assumption on f we may find a quasiconvex function $g_X : \mathbb{R}^{n\times m} \mapsto \mathbb{R}$, such that

$$f(Y) = g_X(Y) \text{ whenever } |Y - X| \leq 1.$$

Now if $g_{X,\varepsilon} := g_X * \Phi_\varepsilon$, then $g_{X,\varepsilon}$ is a quasiconvex $\mathcal{C}^\infty$-function. Furthermore, if $|Y - X| < 1/2$, then

$$g_{X,\varepsilon}(Y) = \int_{|Z-Y|\leq\varepsilon} \Phi_\varepsilon(Y - Z) g_X(Z)\, dZ = f_\varepsilon(Y),$$

hence by the locality of $\mathcal{P}$ and the quasiconvexity of $g_{X,\varepsilon}$

$$\mathcal{P}(f_\varepsilon)(X) = \mathcal{P}(g_{X,\varepsilon})(X) = 0.$$

Therefore it follows from the assumption that f_ε is quasiconvex if $\varepsilon < 1/2$. If we let ε tend to zero we get a contradiction. □

Before we state the next result we need some additional terminology. Let $\mathcal{C}^0(\mathbb{R}^{n\times m})$, the space of continuous real-valued functions, be endowed with the usual metric making it a Fréchet space. The dual space, $\mathcal{C}(\mathbb{R}^{n\times m})'$, is identified with, $\mathcal{M}_{comp}(\mathbb{R}^{n\times m})$, the space of compactly supported Radon measures. The space $\mathcal{M}_{comp}(\mathbb{R}^{n\times m})$ is endowed with the weak* topology.

Let Λ be a non-empty set of compactly supported probabilities on $\mathbb{R}^{n\times m}$ all of which have center of mass at 0. Then we say that a continuous real-valued function $f : \mathbb{R}^{n\times m} \mapsto \mathbb{R}$ is Λ-convex if

$$\int f(X + Y)\, d\mu(Y) \geq f(X)$$

for all $\mu \in \Lambda$ and all $X \in \mathbb{R}^{n\times m}$.

Obviously, Λ-convexity is equivalent to $\bar{\text{co}}\Lambda$-convexity, where $\bar{\text{co}}\Lambda$ denotes the closed convex hull of Λ in $\mathcal{M}_{comp}(\mathbb{R}^{n\times m})$.

This convexity concept also captures the concept of directional convexity (cf. [10], [14], [18], [23]).

Let $\mathcal{V}$ be a non-empty subset of $\mathcal{C}^0(\mathbb{R}^{n\times m})$. We say that the concept of Λ-convexity is local on $\mathcal{V}$ if there exists a local operator $\mathcal{P} : \mathcal{V} \mapsto \mathcal{F}$, such that for $f \in \mathcal{V}$ we have

$$f \text{ is } \Lambda\text{-convex} \quad \Leftrightarrow \mathcal{P}(f) = 0.$$

Let Λ_{rc} denote the set of probabilities μ of the form

$$\int \Phi \, d\mu := \sum_{i=1}^{N} t_i \Phi(X_i), \; \Phi \in \mathcal{C}^0(\mathbb{R}^{n\times m}),$$

where $t_i \in [0,1]$, $X_i \in \mathbb{R}^{n\times m}$ satisfy the (H_N) condition and $\sum_{i=1}^N t_i X_i = 0$. We refer to Dacorogna (cf. [6]) for the definition of the (H_N) condition.

We notice that Λ_{rc}-convexity is rank-one convexity.

Let Λ_{qc} be the set of probabilities ν of the form

$$\int \Phi \, d\nu := \fint_{\mathcal{B}} \Phi(D\varphi(x)) \, dx, \; \Phi \in \mathcal{C}^0(\mathbb{R}^{n\times m}),$$

for some $\varphi \in \mathcal{D}$ with $\mathrm{spt}\varphi \subset \mathcal{B}$.

We notice that Λ_{qc}-convexity is quasiconvexity.

The probabilities in $\bar{\mathrm{co}}\Lambda_{rc}$ and $\bar{\mathrm{co}}\Lambda_{qc}$ can be interpreted as certain homogeneous Young measures (cf. Kinderlehrer and Pedregal [7] and [16]). However, we shall not use this viewpoint here.

THEOREM 2. – *Let Λ be a set of compactly supported probabilities with center of mass at* 0. *Assume that*

$$\bar{co}\Lambda_{rc} \subseteq \bar{co}\Lambda \subseteq \bar{co}\Lambda_{qc}.$$

If Λ-convexity is local on $\mathcal{C}^\infty(\mathbb{R}^{n\times m})$, then $\bar{co}\Lambda = \bar{co}\Lambda_{rc}$.

For the proof of Theorem 2 we need the following result which is essentially contained in [7], [16]. We outline the proof for the convenience of the reader.

LEMMA 3. – *Let μ be a compactly supported probability measure on $\mathbb{R}^{n\times m}$ with center of mass $\bar{\mu} = 0$. If for all rank-one convex $\mathcal{C}^\infty$-functions $f : \mathbb{R}^{n\times m} \mapsto \mathbb{R}$ with $\sup_X |D^3 f(X)| \le 1$ the inequality*

$$\int f \, d\mu \ge f(0) \tag{8}$$

holds, then $\mu \in \bar{co}\Lambda_{rc}$.

Proof. – It is easily seen that if f is a rank-one convex function, then it follows from (8) that also

$$\int f\,d\mu \geq f(0). \tag{9}$$

Let T be a weakly* continuous linear functional on $\mathcal{M}_{comp}(\mathbb{R}^{n\times m})$ satisfying

$$T(\nu) \geq \alpha \tag{10}$$

for all $\nu \in \bar{\text{co}}\Lambda_{rc}$, where $\alpha \in \mathbb{R}$. By Hahn-Banach's separation theorem it is enough to show that also $T(\mu) \geq \alpha$. A weakly* continuous linear functional is an evaluation functional. Hence

$$T(\nu) = \int \Phi\,d\nu,\ \nu \in \mathcal{M}_{comp}(\mathbb{R}^{n\times m}),$$

for some $\Phi \in \mathcal{C}^0(\mathbb{R}^{n\times m})$. Now (10) gives that

$$R\Phi(0) = \inf\left\{\int \Phi\,d\nu\ :\ \nu \in \bar{\text{co}}\Lambda_{rc}\right\} \geq \alpha,$$

where $R\Phi$ is the rank-one convexification of Φ (cf. Dacorogna [6] and [8]). We end the proof by applying (9) with $f = R\Phi$. □

Proof (of Theorem 2). – Let $\mathcal{P} : \mathcal{C}^\infty(\mathbb{R}^{n\times m}) \mapsto \mathcal{F}$ denote the local operator detecting Λ-convexity. Let $\mu \in \Lambda$, and fix a rank-one convex $\mathcal{C}^\infty$-function f with $\sup_X |D^3 f(X)| \leq 1$. For $\gamma > 0$, put $f_\gamma(X) := f(X) + \gamma|X|^2$, $X \in \mathbb{R}^{n\times m}$. Notice that

$$\int_{\mathcal{B}} D^2 f(X)(D\varphi, D\varphi)\,dx \geq \gamma \int_{\mathcal{B}} |D\varphi|^2\,dx$$

for all $\varphi \in \mathcal{D}$ with $\text{spt}\varphi \subset \mathcal{B}$, and that $\sup_X |D^3 f_\gamma(X)| \leq 1$. Hence by Lemma 2 f_γ coincides with quasiconvex functions on balls of radius $\gamma/4$. Take $\varepsilon \in (0, \gamma/8)$, put $f_{\gamma,\varepsilon} := f_\gamma * \Phi_\varepsilon$. Here Φ_ε is the mollifier from the proof of Theorem 2. Obviously, $f_{\gamma,\varepsilon}$ equals quasiconvex $\mathcal{C}^\infty$-functions on balls of radius $\gamma/8$. Consequently, by the locality of the operator $\mathcal{P}$, $\mathcal{P}(f_{\gamma,\varepsilon}) = 0$, and therefore by the assumption, $f_{\gamma,\varepsilon}$ is Λ-convex. In particular,

$$\int f_{\gamma,\varepsilon}\,d\mu \geq f_{\gamma,\varepsilon}(0)$$

for $\gamma > 0$, $\varepsilon \in (0, \gamma/8)$. Now let γ tend to zero and apply Lemma 3 to finish the proof. □

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