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On the non-locality of quasiconvexity

by

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ABSTRACT. – It is shown that in the class of smooth real-valued functions on $n \times m$ matrices ($n \geq 3$, $m \geq 2$) there can be no “local condition” which is equivalent to quasiconvexity. © Elsevier, Paris.

Key words: Quasiconvexity, rank-one convexity.

A continuous function $f : \mathbb{R}^{n \times m} \rightarrow \mathbb{R}$ is called locally quasiconvex if at every point $X \in \mathbb{R}^{n \times m}$ there exists a neighborhood in which it coincides with a quasiconvex function. In this note we show that a $C^2$-function satisfying a strict Legendre-Hadamard condition at every point is locally quasiconvex. Using Šverák’s (cf. [21]) example of a rank-one convex function which is not quasiconvex we show that in dimensions $n \geq 3$, $m \geq 2$ there are locally quasiconvex functions that are not quasiconvex. Indeed, for any positive number $r > 0$ we give an example of a smooth function, which equals a quasiconvex function on any ball of radius $r$, but which is not itself quasiconvex. As a consequence of this we obtain that in dimensions $n \geq 3$, $m \geq 2$ there is no “local condition” which

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for $C^\infty$-functions is equivalent to quasiconvexity. In particular, we confirm the conjecture of Morrey (cf. [12]) saying that in general there is no condition involving only $f$ and a finite number of its derivatives, which is both necessary and sufficient for quasiconvexity. However, it might still be possible to find a “local condition” which is equivalent to quasiconvexity in e.g. the class of polynomials.

The proof relies heavily on Šverák’s example of a rank-one convex function which is not quasiconvex, and the main contribution here is contained in Lemma 2. Lemma 2 provides an extension result for quasiconvex functions, and is proved by use of Taylor’s formula, a slight extension of Dacorogna’s quasiconvexification formula and the equivalence of rank-one convexity and quasiconvexity for quadratic forms.

In the last part of this note we consider rank-one convexity and quasiconvexity in an abstract setting. We hereby prove that in the class of $C^\infty$-functions, any convexity concept between rank-one convexity and quasiconvexity, which is equivalent to a “local condition” is in fact rank-one convexity.

For convenience of the reader and to fix the notation we recall some definitions. The space of (real) $n \times m$ matrices is denoted by $\mathbb{R}^{n \times m}$. We use the usual Hilbert-Schmidt norm for matrices.

A continuous real-valued function $f : \mathbb{R}^{n \times m} \to \mathbb{R}$ is said to be rank-one convex at $X \in \mathbb{R}^{n \times m}$ if the inequality

$$f(X) \leq tf(Y) + (1-t)f(Z)$$

holds for all $t \in [0,1]$, $Y, Z \in \mathbb{R}^{n \times m}$ satisfying $\text{rank}(Y - Z) \leq 1$ and $X = tY + (1-t)Z$. The function $f$ is rank-one convex if it is rank-one convex at each point.

The space of compactly supported $C^\infty$-functions $\varphi : \mathbb{R}^m \to \mathbb{R}^n$ is denoted by $\mathcal{D}(\mathbb{R}^m; \mathbb{R}^n)$, or briefly, by $\mathcal{D}$. The support of $\varphi$ is denoted by $\text{spt}\varphi$, and the gradient of $\varphi$ at $x$, $D\varphi(x)$, is identified in the usual way with a $n \times m$ matrix.

A continuous real-valued function $f : \mathbb{R}^{n \times m} \to \mathbb{R}$ is said to be quasiconvex at $X \in \mathbb{R}^{n \times m}$ if the inequality

$$\int_{\mathbb{R}^m} (f(X + D\varphi(x)) - f(X)) \, dx \geq 0$$

holds for all $\varphi \in \mathcal{D}$. The function $f$ is quasiconvex if it is quasiconvex at each point.
If for $X \in \mathbb{R}^{n \times m}$ there exists a positive number $\delta = \delta(X) > 0$, such that the inequality (2) holds for all $\varphi \in \mathcal{D}$ satisfying $\sup_x |D\varphi(x)| \leq \delta$, then $f$ is said to be weakly quasiconvex at $X$. As above, $f$ is weakly quasiconvex if it is weakly quasiconvex at each point.

The concepts of quasiconvexity and weak quasiconvexity are due to Morrey [12]. A concept of quasiconvexity relevant for higher order problems has been introduced by Meyers [11] (see also [5]).

It is obvious that quasiconvexity of $f$ implies weak quasiconvexity of $f$, and, as shown by Morrey [12], weak quasiconvexity of $f$ implies rank-one convexity of $f$. Hence it follows in particular that quasiconvexity of $f$ implies rank-one convexity of $f$.

In the special case where $f$ is a quadratic form the converse is also true. Hence for quadratic forms the notion of rank-one convexity is equivalent to the notion of quasiconvexity (cf. [13]). A famous conjecture of Morrey [12] is that in dimensions $n \geq 2$, $m \geq 2$ there are rank-one convex functions that are not quasiconvex. In dimensions $n \geq 3$, $m \geq 2$ this was confirmed by Šverák in [21] giving a remarkable example of a polynomial of degree four which is rank-one convex, but not quasiconvex. In the remaining non-trivial cases, i.e. $n = 2$, $m \geq 2$, the question remains open. The problem is discussed in [3], [4], and more recently, in [15], [17], [26], [27].

It is not hard to see that for a $C^2$-function $f : \mathbb{R}^{n \times m} \mapsto \mathbb{R}$ rank-one convexity is equivalent to satisfaction of the Legendre-Hadamard (or ellipticity) condition at every $X \in \mathbb{R}^{n \times m}$, i.e. for each $X \in \mathbb{R}^{n \times m}$

$$D^2f(X)(a \otimes b, a \otimes b) \geq 0$$

for all $a \in \mathbb{R}^n$, $b \in \mathbb{R}^m$.

If for some $X \in \mathbb{R}^{n \times m}$ the inequality (3) holds strictly for all $a \neq 0$, $b \neq 0$, then we say that $f$ satisfies a strict Legendre-Hadamard (or strong ellipticity) condition at $X$. This is equivalent to the existence of a positive number $c = c(X)$, such that

$$D^2f(X)(a \otimes b, a \otimes b) \geq c|a|^2|b|^2$$

for all $a \in \mathbb{R}^n$, $b \in \mathbb{R}^m$. By using the Fourier transformation and the Plancherel theorem it is easily seen that (4) is equivalent to

$$\int_B D^2f(X)(D\varphi(x), D\varphi(x)) \, dx \geq c \int_B |D\varphi(x)|^2 \, dx$$

for all $\varphi \in \mathcal{D}$ with $\text{spt}\varphi \subset B$, where $B := \{x \in \mathbb{R}^m : |x| < 1\}$. 

By using Taylor’s formula and the equivalence of rank-one convexity and quasiconvexity for quadratic forms it can be proved that a $C^2$-function $f$ satisfying a strict Legendre-Hadamard condition at every point is weakly quasiconvex. The same kind of reasoning was used by Tartar [22] in proving a local form of a conjecture in compensated compactness.

DEFINITION. – A continuous real-valued function $f : \mathbb{R}^{n \times m} \to \mathbb{R}$ is said to be locally quasiconvex at $X \in \mathbb{R}^{n \times m}$ if there exists a quasiconvex function $g : \mathbb{R}^{n \times m} \to \mathbb{R}$, such that $f = g$ in a neighborhood of $X$.

The function $f$ is locally quasiconvex if it is locally quasiconvex at each point.

One could define a similar concept of local rank-one convexity. However, by using a mollifier argument and the Legendre-Hadamard condition it is easily proved that this concept coincides with the usual concept of rank-one convexity. It is obvious that there is no need for a local concept of weak quasiconvexity.

If $f : \mathbb{R}^{n \times m} \to \mathbb{R}$ is a locally bounded Borel function, then we define its quasiconvexification, $Qf : \mathbb{R}^{n \times m} \to [-\infty, +\infty]$, as

$$Qf(X) := \sup \{g(X) : g \text{ quasiconvex and } g \leq f\}.$$  

Notice that if at some $X$, $Qf(X) > -\infty$, then $Qf$ is quasiconvex.

The following result is a slight extension of a similar result due to Dacorogna [6]. We refer to [8] for the proof of this and for some extensions along these lines.

**Lemma 1.** – Let $f : \mathbb{R}^{n \times m} \to \mathbb{R}$ be a locally bounded Borel function. Then

$$Qf(X) = \inf \left\{ \int_B f(X + D\varphi) \, dx : \varphi \in \mathcal{D} \text{ with spt}\varphi \subset B \right\}.$$  

For a $C^2$-function $f : \mathbb{R}^{n \times m} \to \mathbb{R}$ we have by Taylor’s formula

$$f(X + Y) = f(X) + Df(X)Y + \frac{1}{2}D^2f(X)(Y;Y) + R(X;Y),$$

where the remainder term $R(X;Y)$ is given by

$$R(X;Y) = \int_0^1 (1-t)(D^2f(X+tY) - D^2f(X))(Y;Y) \, dt.$$  

For notational reasons it is convenient to introduce an auxiliary function, which essentially is a continuity modulus for the second derivative of $f$.  

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For each \( r \in (0, +\infty) \) define \( \Omega_r : (0, +\infty) \mapsto [0, +\infty) \) as (the norm being the usual one for bilinear mappings)

\[
\Omega_r(t) := \sup \left\{ |D^2 f(X + Y) - D^2 f(X)| : |X| \leq r, |Y| < t \right\}.
\]

Obviously, \( \Omega_r \) is non-decreasing and continuous, and since \( D^2 f \) is uniformly continuous on compact sets, \( \Omega_r(t) \to 0 \) as \( t \to 0^+ \). Furthermore we notice that if \( |X| \leq r \), then

\[
|R(X; Y)| \leq \frac{1}{2} \Omega_r(|Y|)|Y|^2 \tag{6}
\]

for all \( Y \in \mathbb{R}^{n \times m} \).

**Lemma 2.** - Let \( f : \mathbb{R}^{n \times m} \mapsto \mathbb{R} \) be a \( C^2 \)-function, and assume that there exist numbers \( c, r > 0 \), such that

\[
\int_B D^2 f(X)(D\varphi, D\varphi) \, dx \geq c \int_B |D\varphi|^2 \, dx \tag{7}
\]

for \( |X| \leq r \) and \( \varphi \in \mathcal{D} \) with \( \text{spt} \varphi \subseteq B \). Put \( \delta := (1/2) \sup \{ t \in (0, r) : c \geq \Omega_r(t) \} \). Then there exists a quasiconvex function \( g : \mathbb{R}^{n \times m} \mapsto \mathbb{R} \) of at most quadratic growth, such that

\[
f(X) = g(X) \quad \text{whenever} \quad |X| \leq \delta.
\]

**Remark.** - Being quasiconvex \( g \) is necessarily locally Lipschitz continuous (cf. [6]), however, I do not know whether it is possible to obtain a quasiconvex extension \( g \) of \( f \) which is as regular as \( f \) is.

**Proof.** - Define the function \( g := QG \), where

\[
G(X) := \begin{cases} 
  f(X) & \text{if } |X| \leq \delta, \\
  \sup_{|Y| \leq \delta} \left( f(Y) + Df(Y)(X - Y) \right) \\
  + \frac{1}{2} D^2 f(Y)(X - Y, X - Y) & \text{otherwise}.
\end{cases}
\]

Then obviously \( g \) is quasiconvex, of at most quadratic growth and \( g(X) \leq f(X) \) for \( |X| \leq \delta \). We claim that \( g(X) = f(X) \) for \( |X| \leq \delta \). Fix \( X \) with \( |X| < \delta \). Let \( \varepsilon > 0 \) and find \( \varphi = \varphi_\varepsilon \in \mathcal{D} \), such that

\[
|\mathcal{B}|(g(X) + \varepsilon) > \int_B G(X + D\varphi) \, dx.
\]
Using Taylor’s formula, (6) and (7) we obtain

\[
|\mathcal{B}|(g(X) + \varepsilon) > \int_{B \cap \{|X + D\varphi| \leq \delta\}} f(X + D\varphi) \, dx \\
+ \int_{B \cap \{|X + D\varphi| > \delta\}} \left( f(X) + Df(X)D\varphi + \frac{1}{2} D^2 f(X)(D\varphi, D\varphi) \right) \, dx \\
= \int_{B \cap \{|X + D\varphi| \leq \delta\}} R(X, D\varphi) \, dx \\
+ \int_{B} \left( f(X) + Df(X)(D\varphi) + \frac{1}{2} D^2 f(X)(D\varphi, D\varphi) \right) \, dx \\
\geq |\mathcal{B}|f(X) + \frac{1}{2} \int_{B \cap \{|X + D\varphi| \leq \delta\}} |D\varphi|^2 (c - \Omega_r(|D\varphi|)) \, dx \geq |\mathcal{B}|f(X),
\]

where the last inequality follows from the definition of \( \delta \).

**Proposition 1.** – Let \( f : \mathbb{R}^{n \times m} \rightarrow \mathbb{R} \) be a \( C^2 \)-function satisfying a strict Legendre-Hadamard condition at every point. Then \( f \) is locally quasiconvex.

**Proof.** – This follows easily by applying Lemma 2 to the functions \( f_X(Y) := f(X + Y) \), \( Y \in \mathbb{R}^{n \times m} \), where \( X \in \mathbb{R}^{n \times m} \) is fixed.

According to Šverák [21] there exists a polynomial \( p \) of degree four on \( \mathbb{R}^{3 \times 2} \), which is rank-one convex but not quasiconvex. A closer inspection of the proof in [21] reveals that we may take \( p \) so that it additionally satisfies a strict Legendre-Hadamard condition at every point, hence by the above result \( p \) is locally quasiconvex.

Recall that a continuous function \( f \) is polyconvex if \( f(X) \) can be written as a convex function of the minors of \( X \). A polyconvex function is quasiconvex, but not conversely (cf. Ball [2], and [1], [20], [24], [25]). If one defines a concept of local polyconvexity as done above for quasiconvexity it is possible to prove that there are locally polyconvex functions on \( \mathbb{R}^{n \times m} (n, m \geq 2) \) that are not polyconvex. In higher dimensions, i.e. \( n \geq 3 \), \( m \geq 2 \), there are locally polyconvex functions on \( \mathbb{R}^{n \times m} \) that are not quasiconvex (cf. [9]).

**Proposition 2.** – Assume that \( n \geq 3 \), \( m \geq 2 \). For any \( r > 0 \) there exists a \( C^\infty \)-function \( f_r : \mathbb{R}^{n \times m} \rightarrow \mathbb{R} \) with the following two properties:

(I) \( f_r \) is not quasiconvex;

(II) for all \( X \in \mathbb{R}^{n \times m} \) there exists a quasiconvex function \( g_X \), such that \( g_X(Y) = f_r(Y) \) holds for \( |Y - X| < r \).

In particular, local quasiconvexity does not imply quasiconvexity.
Proof. – Let \( p: \mathbb{R}^{n \times m} \to \mathbb{R} \) be a polynomial of degree four which is rank-one convex, but not quasiconvex (cf. Šverák [21]). Take for each \( s > 1 \) two auxiliary functions \( \zeta_s, \xi_s \in C^\infty(\mathbb{R}) \) verifying

\[
\zeta_s(t) = \begin{cases} 
1 & \text{if } t < s \\
0 & \text{if } t > s + 1,
\end{cases}
\]

\[
\xi_s(t) = \begin{cases} 
0 & \text{if } t < s - 1 \\
t^2 & \text{if } t > s + 1,
\end{cases}
\]

and \( \xi_s \) non-decreasing, convex and \( \xi''_s(t) > 0 \) for \( t \in (s - 1, s + 1) \).

It is not hard to see that we may find \( s > 1 \) and \( k > 0 \), such that

\[
p(X)\zeta_s(|X|) + k\xi_s(|X|)
\]

is rank-one convex, but not quasiconvex (cf. Šverák [19] remark 3.4 and [20]). Next take \( \varepsilon > 0 \), so that

\[
g(X) := p(X)\zeta_s(|X|) + k\xi_s(|X|) + \varepsilon|X|^2
\]

is not quasiconvex. Notice that \( g \) satisfies a uniform Legendre-Hadamard condition:

\[
\int_B D^2g(X)(D\varphi, D\varphi) \, dx \geq \varepsilon \int_B |D\varphi|^2 \, dx
\]

for all \( X \in \mathbb{R}^{n \times m} \) and all \( \varphi \in \mathcal{D} \) with \( \text{spt} \varphi \subset B \).

Notice also that if \( R(X, Y) \) denotes the remainder term in the Taylor expansion of \( g \) about \( X \), then for some constant \( C > 0 \)

\[
|R(X, Y)| \leq 3 \int_0^1 (1 - t)^2 \sum_{|\alpha|=3} |\partial^\alpha g(X + tY)\frac{Y^\alpha}{\alpha!}| \, dt \leq C|Y|^3
\]

for all \( X, Y \in \mathbb{R}^{n \times m} \). In the notation of Lemma 2 (see (6)) this corresponds to \( \Omega_r(t) = 2Ct, t > 0 \), independent of \( r > 0 \).

Fix \( X_0 \in \mathbb{R}^{n \times m} \). We claim that there exists a quasiconvex extension of \( g \) from the closed ball \( |X - X_0| \leq \varepsilon/(4C) \). Indeed, define \( g_{X_0}(X) := g(X_0 + X) \) and notice that by Lemma 2 we may find a quasiconvex function \( G_{X_0} \), such that \( g(X + X_0) = g_{X_0}(X) = G_{X_0}(X) \) for \( |X| \leq \varepsilon/(4C) \), or equivalently, such that

\[
g(X) = G_{X_0}(X - X_0) \quad \text{for} \quad |X - X_0| \leq \frac{\varepsilon}{4C}.
\]
This proves the claim. Finally we define the function $f_r$ as

$$f_r(X) := g \left( \frac{4C}{\varepsilon r} X \right), \quad X \in \mathbb{R}^{n \times m}.$$ 

This finishes the proof.

Let $\mathcal{C}^\infty(\mathbb{R}^{n \times m})$ denote the space of all real-valued $\mathcal{C}^\infty$-functions $f : \mathbb{R}^{n \times m} \rightarrow \mathbb{R}$ and let $\mathcal{F}$ denote the space of all extended real-valued functions $F : \mathbb{R}^{n \times m} \mapsto [-\infty, +\infty]$.

If we define the operator $\mathcal{P}_{rc} : \mathcal{C}^\infty(\mathbb{R}^{n \times m}) \mapsto \mathcal{F}$ as

$$\mathcal{P}_{rc}(f)(X) := \inf \{ D^2 f(X)(a \otimes b, a \otimes b) : a \in \mathbb{R}^n, b \in \mathbb{R}^m \}, \quad X \in \mathbb{R}^{n \times m},$$

then $f \in \mathcal{C}^\infty(\mathbb{R}^{n \times m})$ is rank-one convex if and only if $\mathcal{P}_{rc}(f) = 0$. Furthermore, the operator $\mathcal{P}_{rc}$ is local in the sense that if $f, g \in \mathcal{C}^\infty(\mathbb{R}^{n \times m})$ are equal in a neighborhood of $X$, then also $\mathcal{P}_{rc}(f)$ equals $\mathcal{P}_{rc}(g)$ in a neighborhood of $X$. Thus:

$$f = g \text{ in a neighborhood of } X \Rightarrow \mathcal{P}_{rc}(f) = \mathcal{P}_{rc}(g) \text{ in a neighborhood of } X.$$

It would be interesting if one could find a similar condition for quasiconvexity. That is, a local operator $\mathcal{P}_{qc} : \mathcal{C}^\infty(\mathbb{R}^{n \times m}) \mapsto \mathcal{F}$ with the property

$$\mathcal{P}_{qc}(f) = 0 \iff f \text{ is quasiconvex} \quad (*)$$

for $f \in \mathcal{C}^\infty(\mathbb{R}^{n \times m})$.

**Theorem 1.** - In dimensions $n \geq 3, m \geq 2$ there does not exist a local operator

$$\mathcal{P} : \mathcal{C}^\infty(\mathbb{R}^{n \times m}) \mapsto \mathcal{F}$$

with the property $(*)$.

**Remark.** - The proof will show that the operator $\mathcal{P}$ cannot satisfy $(*)$ and the following locality-type condition: There exists a number $r > 0$, such that for $f, g \in \mathcal{C}^\infty(\mathbb{R}^{n \times m})$ and $X \in \mathbb{R}^{n \times m}$

$$f(Y) = g(Y) \text{ for } |Y - X| \leq r \Rightarrow \mathcal{P}(f)(X) = \mathcal{P}(g)(X).$$

**Proof.** - We argue by contradiction and assume that it is possible to find a local operator with the property $(*)$.
By Proposition 2 we may find a $C^\infty$-function $f : \mathbb{R}^{n \times m} \mapsto \mathbb{R}$ which is not quasiconvex, but agrees with quasiconvex functions on all balls of, say, radius one.

Let $\Phi_\varepsilon \in C^\infty$, $\varepsilon > 0$, be a non-negative mollifier with support contained in $\{ X : |X| \leq \varepsilon \}$. Put $f_\varepsilon := f * \Phi_\varepsilon$, i.e. the convolution of $f$ and $\Phi_\varepsilon$.

We claim that if $\varepsilon \in (0, 1/2)$, then $f_\varepsilon$ is quasiconvex.

Fix $X \in \mathbb{R}^{n \times m}$. By the assumption on $f$ we may find a quasiconvex function $g_X : \mathbb{R}^{n \times m} \mapsto \mathbb{R}$, such that

$$f(Y) = g_X(Y) \text{ whenever } |Y - X| \leq 1.$$ 

Now if $g_{X,\varepsilon} := g_X * \Phi_\varepsilon$, then $g_{X,\varepsilon}$ is a quasiconvex $C^\infty$-function. Furthermore, if $|Y - X| < 1/2$, then

$$g_{X,\varepsilon}(Y) = \int_{|Z-Y|\leq\varepsilon} \Phi_\varepsilon(Y-Z) g_X(Z) \, dZ = f_\varepsilon(Y),$$

hence by the locality of $P$ and the quasiconvexity of $g_{X,\varepsilon}$

$$P(f_\varepsilon)(X) = P(g_{X,\varepsilon})(X) = 0.$$ 

Therefore it follows from the assumption that $f_\varepsilon$ is quasiconvex if $\varepsilon < 1/2$. If we let $\varepsilon$ tend to zero we get a contradiction. \( \square \)

Before we state the next result we need some additional terminology. Let $C^0(\mathbb{R}^{n \times m})$, the space of continuous real-valued functions, be endowed with the usual metric making it a Fréchet space. The dual space, $C(\mathbb{R}^{n \times m})'$, is identified with, $M_{\text{comp}}(\mathbb{R}^{n \times m})$, the space of compactly supported Radon measures. The space $M_{\text{comp}}(\mathbb{R}^{n \times m})$ is endowed with the weak* topology.

Let $\Lambda$ be a non-empty set of compactly supported probabilities on $\mathbb{R}^{n \times m}$ all of which have center of mass at 0. Then we say that a continuous real-valued function $f : \mathbb{R}^{n \times m} \mapsto \mathbb{R}$ is $\Lambda$-convex if

$$\int f(X + Y) \, d\mu(Y) \geq f(X)$$

for all $\mu \in \Lambda$ and all $X \in \mathbb{R}^{n \times m}$.

Obviously, $\Lambda$-convexity is equivalent to $\text{co} \Lambda$-convexity, where $\text{co} \Lambda$ denotes the closed convex hull of $\Lambda$ in $M_{\text{comp}}(\mathbb{R}^{n \times m})$.

This convexity concept also captures the concept of directional convexity (cf. [10], [14], [18], [23]).
Let $V$ be a non-empty subset of $C^0(\mathbb{R}^{n \times m})$. We say that the concept of $\Lambda$-convexity is local on $V$ if there exists a local operator $\mathcal{P} : V \mapsto \mathcal{F}$, such that for $f \in V$ we have

$$f \text{ is } \Lambda\text{-convex } \iff \mathcal{P}(f) = 0.$$ 

Let $\Lambda_{rc}$ denote the set of probabilities $\mu$ of the form

$$\int \Phi \, d\mu := \sum_{i=1}^{N} t_i \Phi(X_i), \Phi \in C^0(\mathbb{R}^{n \times m}),$$

where $t_i \in [0, 1]$, $X_i \in \mathbb{R}^{n \times m}$ satisfy the $(H_N)$ condition and $\sum_{i=1}^{N} t_i X_i = 0$. We refer to Dacorogna (cf. [6]) for the definition of the $(H_N)$ condition.

We notice that $\Lambda_{rc}$-convexity is rank-one convexity.

Let $\Lambda_{qc}$ be the set of probabilities $\nu$ of the form

$$\int \Phi \, d\nu := \int_{B} \Phi(D\varphi(x)) \, dx, \Phi \in C^0(\mathbb{R}^{n \times m}),$$

for some $\varphi \in \mathcal{D}$ with $\text{spt} \varphi \subset B$.

We notice that $\Lambda_{qc}$-convexity is quasiconvexity.

The probabilities in $\overline{\text{co}} \Lambda_{rc}$ and $\overline{\text{co}} \Lambda_{qc}$ can be interpreted as certain homogeneous Young measures (cf. Kinderlehrer and Pedregal [7] and [16]). However, we shall not use this viewpoint here.

**Theorem 2.** Let $\Lambda$ be a set of compactly supported probabilities with center of mass at 0. Assume that

$$\overline{\text{co}} \Lambda_{rc} \subseteq \overline{\text{co}} \Lambda \subseteq \overline{\text{co}} \Lambda_{qc}.$$ 

If $\Lambda$-convexity is local on $C^\infty(\mathbb{R}^{n \times m})$, then $\overline{\text{co}} \Lambda = \overline{\text{co}} \Lambda_{rc}$.

For the proof of Theorem 2 we need the following result which is essentially contained in [7], [16]. We outline the proof for the convenience of the reader.

**Lemma 3.** Let $\mu$ be a compactly supported probability measure on $\mathbb{R}^{n \times m}$ with center of mass $\overline{\mu} = 0$. If for all rank-one convex $C^\infty$-functions $f : \mathbb{R}^{n \times m} \mapsto \mathbb{R}$ with $\sup_{X} |D^3 f(X)| \leq 1$ the inequality

$$\int f \, d\mu \geq f(0)$$

holds, then $\mu \in \overline{\text{co}} \Lambda_{rc}$. 

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Proof. – It is easily seen that if \( f \) is a rank-one convex function, then it follows from (8) that also
\[
\int f \, d\mu \geq f(0).
\]
(9)

Let \( T \) be a weakly* continuous linear functional on \( \mathcal{M}_{\text{comp}}(\mathbb{R}^{n \times m}) \) satisfying
\[
T(\nu) \geq \alpha
\]
for all \( \nu \in \co \Lambda_{rc} \), where \( \alpha \in \mathbb{R} \). By Hahn-Banach’s separation theorem it is enough to show that also \( T(\mu) \geq \alpha \). A weakly* continuous linear functional is an evaluation functional. Hence
\[
T(\nu) = \int \Phi \, d\nu, \, \nu \in \mathcal{M}_{\text{comp}}(\mathbb{R}^{n \times m}),
\]
for some \( \Phi \in C^0(\mathbb{R}^{n \times m}) \). Now (10) gives that
\[
R\Phi(0) = \inf \left\{ \int \Phi \, d\nu : \nu \in \co \Lambda_{rc} \right\} \geq \alpha,
\]
where \( R\Phi \) is the rank-one convexification of \( \Phi \) (cf. Dacorogna [6] and [8]). We end the proof by applying (9) with \( f = R\Phi \).

Proof (of Theorem 2). – Let \( \mathcal{P} : C^\infty(\mathbb{R}^{n \times m}) \mapsto \mathcal{F} \) denote the local operator detecting \( \Lambda \)-convexity. Let \( \mu \in \Lambda \), and fix a rank-one convex \( C^\infty \)-function \( f \) with \( \sup_X |D^3 f(X)| \leq 1 \). For \( \gamma > 0 \), put \( f_\gamma(X) := f(X) + \gamma |X|^2 \), \( X \in \mathbb{R}^{n \times m} \). Notice that
\[
\int_B D^2 f(X)(D\varphi, D\varphi) \, dx \geq \gamma \int_B |D\varphi|^2 \, dx
\]
for all \( \varphi \in \mathcal{D} \) with \( \text{spt}\varphi \subset B \), and that \( \sup_X |D^3 f_\gamma(X)| \leq 1 \). Hence by Lemma 2 \( f_\gamma \) coincides with quasiconvex functions on balls of radius \( \gamma/4 \). Take \( \varepsilon \in (0, \gamma/8) \), put \( f_{\gamma,\varepsilon} := f_\gamma * \Phi_\varepsilon \). Here \( \Phi_\varepsilon \) is the mollifier from the proof of Theorem 2. Obviously, \( f_{\gamma,\varepsilon} \) equals quasiconvex \( C^\infty \)-functions on balls of radius \( \gamma/8 \). Consequently, by the locality of the operator \( \mathcal{P} \), \( \mathcal{P}(f_{\gamma,\varepsilon}) = 0 \), and therefore by the assumption, \( f_{\gamma,\varepsilon} \) is \( \Lambda \)-convex. In particular,
\[
\int f_{\gamma,\varepsilon} \, d\mu \geq f_{\gamma,\varepsilon}(0)
\]
for \( \gamma > 0, \varepsilon \in (0, \gamma/8) \). Now let \( \gamma \) tend to zero and apply Lemma 3 to finish the proof.
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