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Multi-peak bound states for nonlinear Schrödinger equations

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ABSTRACT. – In this paper we consider the study of *standing wave solutions* for a nonlinear Schrödinger equation. This problem reduces to that of finding nonnegative solutions of

$$\varepsilon^2 \Delta u - V(x) u + f(u) = 0 \quad \text{in} \quad \Omega,$$

with finite energy. Here ε is a small parameter, Ω is a smooth, possibly unbounded domain, f is an appropriate superlinear function, and V is a positive potential, bounded away from zero.

It is the purpose of this article to obtain multi-peak solutions in the "multiple well case". We find solutions exhibiting concentration at any prescribed finite set of local minima, possibly degenerate, of the potential.

The proof relies on variational arguments, where a penalization-type method is developed for the identification of the desired solutions. © Elsevier, Paris

RÉSUMÉ. – Dans cet article, on considère l'étude des solutions de *ondes permanentes* d'une équation de Schrödinger nonlinéaire. Ce problème se réduit à la recherche de solutions non négatives de

$$\varepsilon^2 \Delta u - V(x) u + f(u) = 0 \quad \text{dans} \quad \Omega,$$

avec une énergie finie. Ici ε est un paramètre petit, Ω est un domaine lisse qui peut être non borné, f est une fonction superlinéaire appropriée et V est un potentiel positif borné hors de zéro.

L'objectif de cet article consiste à obtenir des solutions à pics multiples dans le cas de puits multiples. Nous trouvons des solutions qui montrent une concentration pour tout ensemble choisi fini de minima locaux du potentiel, qui peuvent être dégénérés.

La démonstration se base sur des arguments variationnels, où une méthode de pénalisation est développée pour identifier les solutions cherchées. © Elsevier, Paris

0. INTRODUCTION

The nonlinear Schrödinger equation in $\mathbb{R}^N$

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi + W(x)\psi - \gamma |\psi|^{p-1}\psi \tag{0.1}$$

has been object of extensive research in recent years. In this paper we consider the study of *standing waves* of equation (0.1), namely of special solutions of the form $\psi(x,t) = \exp(-iEt/\hbar)v(x)$, where $v(x) > 0$. It is easily checked that a ψ of this form satisfies equation (0.1) if and only if the function $v(x)$ solves the elliptic equation

$$\frac{\hbar^2}{2m} \Delta v - (W(x) - E)v + \gamma v^p = 0. \tag{0.2}$$

The problem we will study in this paper is that of existence of positive solutions with finite energy to this equation when $W - E$ is strictly positive. away from zero, and the potential W exhibits multiple wells, namely several, possibly degenerate local minima. $\hbar$ will be regarded as a small parameter. After absorption of the parameters by scaling, the problem under consideration may be rewritten as

$$\varepsilon^2 \Delta u - V(x)u + u^p = 0. \tag{0.3}$$

where $V \geq \alpha > 0$ in $\mathbb{R}^N$, $N \geq 1$. In [5], Floer and Weinstein consider the case $N = 1$ and $p = 3$. For a given nondegenerate critical point of the potential V, assumed globally bounded, they construct a positive solution u_ε to (0.3), provided that ε is sufficiently small. This solution concentrates around the critical point as $\varepsilon \to 0$, in the sense that its shape is a sharp peak near that point, while it almost vanishes everywhere else.

Their method, based on an interesting Lyapunov-Schmidt finite dimensional reduction, was extended by Oh in [8], [9] to conclude a similar result in higher dimensions, provided that $1 < p < \frac{N+2}{N-2}$.

Oh restricts himself to potentials with "mild oscillation" at infinity, namely belonging to a Kato class. In case that V is bounded this restriction is not necessary as noticed by Wang in [14]. Wang also observes that if V is nonconstant and nondecreasing in one direction, then equation (0.3) has no solutions which tend to zero at infinity.

The method in [5] and [8] seems to rely in essential way on the nondegeneracy of the critical points. In [11], Rabinowitz lifted partially this requirement introducing a global variational technique to find a solution with "minimal energy" for all small ε, when $1 < p < \frac{N+2}{N-2}$ and

$$\liminf_{|x| \to \infty} V(x) > \inf_{x \in R^N} V(x). \tag{0.4}$$

Rabinowitz's approach actually covers a broader class of nonlinearities and the smallness of ε is not required in case that the limit in the left of (0.4) is $+\infty$. In [14], Wang established that this least energy (mountain pass) solution indeed concentrates around a global minimum of V in the special case of equation (0.3), as $\varepsilon \to 0$.

In [3], the authors succeeded in proving the local analogue of Wang's result. It is shown that if for an open, bounded set one has

$$\inf_\Lambda V < \inf_{\partial \Lambda} V, \tag{0.5}$$

then (0.3) possesses solutions u_ε with just one local maximum, which concentrates around a minimum of V in Λ. This local minimum may exhibit arbitrary degeneracy.

Concerning solutions with multiple concentration points, in [10] Oh applies the approach developed in [5] and [8] to construct a family of solutions u_ε with peaks concentrating around any prescribed finite set of nondegenerate critical points of V when $N = 1$, and indicates how to proceed in higher dimensions where $1 < p < \frac{N+2}{N-2}$.

It is the purpose of this article to obtain multi-peak solutions of (0.3) in the "multiple well case", exhibiting concentration at any prescribed set of

possibly degenerate local minima of the potential, namely on K disjoint sets Λ where (0.5) holds.

We will actually consider a more general semilinear elliptic problem in a smooth domain Ω, possibly unbounded, of the form

$$\begin{cases} \varepsilon^2 \Delta u - V(x)u + f(u) = 0 \text{ in } \Omega \\ \quad u \in H_0^1(\Omega), \quad u > 0 \text{ in } \Omega. \end{cases} \tag{0.6}$$

The potential V will be assumed throughout this paper locally Hölder continuous and bounded below away from zero, say

$$V(x) \geq \alpha > 0, \quad \text{for all} \quad x \in \Omega. \tag{0.7}$$

We will also assume that $f : \mathbb{R}^+ \to \mathbb{R}$ is of class C^1 and satisfies the following conditions.

(f1) $f(\xi) = o(\xi)$ near $\xi \geq 0$.

(f2) $\displaystyle\lim_{\xi \to \infty} \frac{f(\xi)}{\xi^s} = 0$ for some $1 < s < \frac{N+2}{N-2}$.

(f3) For some $2 < \theta \leq s + 1$ we have

$$0 \leq \theta F(\xi) < f(\xi)\xi \quad \text{for all} \quad \xi > 0 \tag{0.8}$$

where $F(\xi) = \int_0^\xi f(\tau)d\tau$.

(f4) The *limiting equation*

$$\Delta u - bu + f(u) = 0 \qquad \text{in} \quad \mathbb{R}^N \tag{0.9}$$

possesses a unique solution, up to translation, for any given $b > 0$. Our main result for equation (0.6) is the following.

THEOREM 0.1. – *Assume that there are bounded domains Λ_i, mutually disjoint, compactly contained in Ω, $i = 1, \dots, K$, such that*

$$\inf_{\Lambda_i} V < \inf_{\partial \Lambda_i} V. \tag{0.10}$$

Then there is an $\varepsilon_0 > 0$ such that for every $0 < \varepsilon < \varepsilon_0$ a positive solution $u_\varepsilon \in H_0^1(\Omega)$ to problem (0.6) exists. Moreover, u_ε possesses exactly K local maxima $x_{\varepsilon,i}$, with $x_{\varepsilon,i}$ in Λ_i. We also have that $V(x_{\varepsilon,i}) \to \inf_{\Lambda_i} V$, and

$$u_\varepsilon(x) \leq \alpha \exp(-\frac{\beta}{\varepsilon}|x - x_{\varepsilon,i}|). \tag{0.11}$$

for all $x \in \Omega \setminus \cup_{j \neq i}\Lambda_j$, where α and β are certain positive constants.

We observe that no restriction on the global behavior of V is required other than (0.7). In particular, V is not required to be bounded or to belong to a Kato class.

On the other hand, the hypotheses on f are milder than those required in [10]. In particular, hypothesis (f4) is satisfied by a large class of nonlinearities f including u^p, $p > 1$. See the work in Kwong and Zhang [6], and Chen and Lin [1]. Moreover, this assumption can be further relaxed to

(f4') The mountain pass value in $H^1_0(\Omega)$ of the energy functional associated to (0.9) is the smallest nontrivial critical value and it is isolated.

See the remark at the end of §2.

It is interesting to observe the relationship between the result of Theorem 0.1 and the work by Coti-Zelati and Rabinowitz [2] on multi-bump solutions in spatially-periodic problems. For the case of equation (0.3), $1 < p < \frac{N+2}{N-2}$, and V periodic on each of its variables, it follows from their results that for a fixed, not necessarily small ε, solutions in $H^1(\mathbb{R}^N)$ with exactly K bumps exist for each integer K, provided that the associated energy functional satisfies the so-called ($*$) nondegeneracy condition. This assumption states that all critical points at energy levels in $[c, c + \delta)$ are isolated, where c denotes the mountain pass minimax value of the associated functional, and $\delta > 0$. These solutions have energy level close to Kc. On the other hand, Theorem 0.1 is applicable to construct a K-bump solution in this situation when ε is small, just assuming that V possesses one local minimum. We do not know whether ($*$) holds in this case.

The proof of Theorem 0.1 is variational, and uses ideas in the spirit of those in our previous work [3], where a penalization method enabling the identification of *local mountain passes* was developed. Roughly speaking, the main argument there consists of defining a suitable modification of the nonlinearity for which the mountain pass theorem is directly applicable to the associated energy. Then, taking advantage of the energy-minimality of the mountain pass solution, one finally shows that it becomes a solution to the original problem with the desired characteristics when ε is sufficiently small.

Our current framework is more delicate, since the solutions we look for are at higher energy levels. They are not just rough mountain passes, so that energy-minimality is lost. We are able to overcome this difficulty by adding a new penalization term. In fact, we introduce a modification of the nonlinearity similar to that in [3] in order to prevent the occurrence of concentration outside the open sets Λ_i, and then a new term which provides a "balance" among the energies inside the distinct Λ_i's, in order to obtain exactly K-bumps. The solution is captured as a simple minimax quantity

on a class of K-dimensional maps, and eventually shown to be a solution to the original equation with the appropriate features.

Finally, we would like to mention that the construction of solutions of (0.3) with an infinite number of bumps (hence not with finite energy) has been recently carried out by Thandi [13] in the nondegenerate case. Infinite-bump solutions in the framework of [2] were found by Spradlin in [12].

The organization of this paper is as follows: In §1 we define the modification of the functional needed for the proof of Theorem 0.1, and prove some preliminary results. §2 is devoted to the proof of Theorem 0.1.

1. PRELIMINARIES

This section is devoted to the definition and preliminary study of the penalized functional.

We introduce an appropriate penalization so that the concentration outside the sets Λ_i is avoided and an adequate balance in the concentration is achieved. Then we prove that the penalized functional satisfies the Palais Smale condition (P.S.), and we set up the minimax scheme in order to obtain critical points of it. We provide next some estimates on the critical points.

In the framework of Theorem 0.1, associated to equation (0.6) we have the "energy" functional

$$I_\varepsilon(u) = \frac{1}{2} \int_\Omega \varepsilon^2 |\nabla u|^2 + V(x)u^2 - \int_\Omega F(u), \qquad (1.1)$$

which is well defined for $u \in H$, where

$$H = \{u \in H_0^1(\Omega) \ / \ \int_\Omega V(x)u^2 < \infty\}. \qquad 1.11$$

H becomes a Hilbert space, continuously embedded in $H_0^1(\Omega)$, when endowed with the inner product

$$< u. v >_H = \int_\Omega \nabla u \cdot \nabla v + V(x)uv \qquad (1.2)$$

whose associated norm we denote by $||\cdot||_H$.

In the definitions above, and in what follows we assume that f, V and Ω satisfy the hypotheses of Theorem 0.1, and that the function f is extended

as 0 on the negative axis. Under those assumptions it is standard to check that the nontrivial critical points of I_ε correspond exactly to the positive classical solutions in $H_0^1(\Omega)$ of equation (0.6).

Next we define the first modification of our functional. Let α be as in (0.7), and let us choose $a > 0$ so that

$$\frac{f(\xi)}{\xi} \leq \frac{\alpha}{2} \quad \text{for all } 0 \leq \xi \leq a, \text{ and } \frac{f(\xi)}{\xi} \geq \frac{f(a)}{a} \quad \text{for all } \xi \geq a.$$

This choice of a can be made since (f1) and (f3) hold. Let us set

$$\tilde{f}(\xi) = \begin{cases} f(\xi) & \text{if } \xi \leq a \\ \frac{f(a)}{a}\xi & \text{if } \xi > a, \end{cases} \tag{1.3}$$

and define

$$g(\cdot, \xi) = \chi_\Lambda f(\xi) + (1 - \chi_\Lambda)\tilde{f}(\xi), \tag{1.4}$$

where $\Lambda = \cup_{i=1}^K \Lambda_i$, with the bounded domains Λ_i as in the assumptions of Theorem 0.1, and χ_Λ denoting its characteristic function. It is easy to check that (f1)-(f3) implies that g defined in this way is a Caratheodory function and it satisfies the following

(g1) $g(x, \xi) = o(\xi)$ near $\xi = 0$ uniformly in $x \in \Omega$.

(g2) $\lim_{\xi \to \infty} \frac{g(x,\xi)}{\xi^s} = 0$ uniformly in $x \in \Omega$, for some $1 < s < \frac{N+2}{N-2}$ if $N \geq 3$, and no restriction on s if $N = 1, 2$.

(g3) (i) $0 < \theta G(x, \xi) \leq g(x, \xi)\xi \qquad$ for all $\quad x \in \Lambda, \xi > 0.$

and

(ii) $0 \leq 2G(x, \xi) \leq g(x, \xi)\xi \leq \frac{\alpha}{2}\xi^2 \quad$ for all $\quad x \in \Omega \setminus \Lambda, \xi \in \mathbb{R}^+.$

Here we have denoted $G(x, \xi) = \int_0^\xi g(x, \tau)d\tau$.

Now we define the modified functional $J_\varepsilon : H \to \mathbb{R}$ as

$$J_\varepsilon(u) = \frac{1}{2}\int_\Omega \varepsilon^2|\nabla u|^2 + V(x)u^2 - \int_\Omega G(x, u) \quad . \quad u \in H. \tag{1.5}$$

The functional J_ε is of class C^1 in H and its critical points are the positive solutions of the equation

$$\varepsilon^2 \Delta u - V(x)u + g(x. u) = 0 \qquad \text{in} \quad \Omega.$$

Next we introduce the second modification of the functional. For this purpose we assume, without loss of generality, that the sets Λ_i have smooth

boundary. We define the numbers $b_i = \inf\{V(x) \,/\, x \in \Lambda_i\}$ and we let $\delta > 0$ be so that

$$\sup_{\Lambda_i} V \leq b_i + \delta. \qquad (1.6)$$

We will need the 'limiting functional' $I^b : H^1(\mathbb{R}^N) \to \mathbb{R}$ defined as

$$I^b(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + bu^2 - \int_{\mathbb{R}^N} F(u), \qquad (1.7)$$

whose mountain pass critical value will be denoted by $c(b)$. Critical points of I^b are the solutions of (0.9). We define $c_i = c(b_i)$ and $\sigma_i = c(b_i + \delta) - c_i$. We assume that

$$\sum_{i=1}^{K} \sigma_i < \frac{1}{2} \min\{c_i \,|\, i = 1, \dots, K\}. \qquad (1.8)$$

This can be achieved by making Λ_i and δ smaller if necessary. It will be convenient to consider mutually disjoint open sets $\tilde{\Lambda}_i$ compactly containing Λ_i, for all $i = 1, \dots, K$. Then we define on H the functional

$$J_\varepsilon^i(u) = \frac{1}{2} \int_{\tilde{\Lambda}_i} \varepsilon^2 |\nabla u|^2 + V(x)u^2 - \int_{\tilde{\Lambda}_i} G(x, u), \qquad u \in H, \qquad (1.9)$$

and the penalization

$$P_\varepsilon(u) = M \sum_{i=1}^{K} \left\{ \left(J_\varepsilon^i(u)_+\right)^{\frac{1}{2}} - \varepsilon^{\frac{N}{2}}(c_i + \sigma_i)^{\frac{1}{2}} \right\}_+^2. \qquad (1.10)$$

The constant M will be chosen later. Finally the penalized functional $E_\varepsilon : H \to \mathbb{R}$ is defined as

$$E_\varepsilon(u) = J_\varepsilon(u) + P_\varepsilon(u). \qquad (1.11)$$

The functionals J_ε and P_ε are of class C^1 and so is E_ε. We show next that E_ε has good compactness properties, that is E_ε satisfies the Palais Smale condition.

LEMMA 1.1. – *Let $\{u_n\}$ be a sequence in H such that $E_\varepsilon(u_n)$ is bounded and $E_\varepsilon'(u_n) \to 0$. Then u_n has a convergent subsequence.*

Proof. – We first prove that the sequence $\{u_n\}$ is bounded in H. Using property (g3) we easily see that

$$J_\varepsilon(u_n) - \frac{1}{\theta}J'_\varepsilon(u_n)u_n \geq \frac{\theta - 2}{2\theta}\left(\int_\Omega \varepsilon^2|\nabla u_n|^2 + V(x)u_n^2 - \int_{\Omega\setminus\Lambda} g(x, u_n)u_n\right),$$

from where, using (g3) again, we obtain

$$J_\varepsilon(u_n) - \frac{1}{\theta}J'_\varepsilon(u_n)u_n \geq C\int_\Omega \varepsilon^2|\nabla u_n|^2 + V(x)u_n^2. \qquad (1.12)$$

In a similar fashion we find

$$J^i_\varepsilon(u_n) - \frac{1}{\theta}J^{i\,\prime}_\varepsilon(u_n)u_n \geq C\int_{\tilde\Lambda_i} \varepsilon^2|\nabla u_n|^2 + V(x)u_n^2. \qquad (1.13)$$

We observe that (1.13) still holds if we replace θ by any number $\hat\theta \in (2, \theta)$. Then, for the penalization functional we have

$$P_\varepsilon(u_n) - \frac{1}{\hat\theta}P'_\varepsilon(u_n)u_n \geq -M\varepsilon^{\frac{N}{2}}(c_i + \sigma_i)^{\frac{1}{2}}\sum_{i=1}^{K}$$

$$\times\left\{\left(J^i_\varepsilon(u_n)_+\right)^{\frac{1}{2}} - \varepsilon^{\frac{N}{2}}(c_i + \sigma_i)^{\frac{1}{2}}\right\}_+$$

$$\geq -C\varepsilon^{\frac{N}{2}}P_\varepsilon(u_n)^{\frac{1}{2}}. \qquad (1.14)$$

From where there exists a constant C so that

$$P_\varepsilon(u_n) - \frac{1}{\theta}P'_\varepsilon(u_n)u_n \geq -C. \qquad (1.15)$$

We observe that the constant C is a multiple of ε^N.

Thus, it follows from (1.12), (1.15) and the assumptions on $\{u_n\}$ that this sequence is bounded as desired.

Let us choose a subsequence, still denoted by $\{u_n\}$, weakly convergent to u in H. This convergence is actually strong. Indeed, it suffices to show that, given $\gamma > 0$, there is an $R > 0$ such that

$$\limsup_{n\to\infty}\int_{\Omega\setminus B_R}\{\varepsilon^2|\nabla u_n|^2 + V(x)u_n^2\} < \gamma. \qquad (1.16)$$

where B_R denotes the ball with center 0 and radius R. We may assume that R is chosen so that $\Lambda \subset B_{R/2}$.

Let η_R be a cut-off function so that $\eta_R = 0$ on $B_{R/2}, \eta_R = 1$ on $\Omega \setminus B_R, 0 \leq \eta_R \leq 1$ and $|\nabla \eta_R| \leq c/R$. Since $\{u_n\}$ is a P.S. sequence, we have that

$$< E_\varepsilon'(u_n), \eta_R u_n >= o_n(1),$$

where $o_n(1) \to 0$ as $n \to \infty$. Then, as $\{u_n\}$ is bounded

$$\int_\Omega \{\varepsilon^2 |\nabla u_n|^2 + V(x)u_n^2\}\eta_R + \int_\Omega u_n \nabla u_n \cdot \nabla \eta_R =$$
$$= \int_\Omega g(x, u_n)u_n \eta_R + o_n(1) \leq \frac{1}{2}\int_\Omega V(x)u_n^2 \eta_R + o_n(1).$$

We conclude that

$$\int_{\Omega \setminus B_R} \varepsilon^2 |\nabla u_n|^2 + V(x)u_n^2 \leq \frac{C}{R}\|u_n\|_{L^2(\Omega)}\|\nabla u_n\|_{L_2(\Omega)} + o_n(1),$$

from where (1.16) follows. $\quad\square$

The previous lemma makes possible to use Critical Point Theory to find critical points of the functional E_ε. We will formulate an appropriate minimax problem for E_ε.

We start defining a class of functions Γ over which we minimax. A continuous function $\gamma : [0, 1]^K \to H$ is in Γ if there are continuous functions $g_i : [0, 1] \to H$, $i = 1, ..., K$ such that

  (i) $\mathrm{supp}\{g_i(\tau)\} \subset \Lambda_i$ for all $\tau \in [0, 1]$,
  (ii) $\gamma(\tau_1, ..., \tau_K) = \sum_{i=1}^K g_i(\tau_i)$ for all $(\tau_1, ..., \tau_K) \in \partial[0, 1]^K$,
  (iii) $g_i(0) = 0$ and $J_\varepsilon(g_i(1)) < 0$
  (iv) $E_\varepsilon(\gamma(t)) \leq \varepsilon^N(\sum_{i=1}^K c_i - \sigma)$ for all $t \in \partial[0, 1]^K$,
     where $0 < \sigma \leq \frac{1}{2}\mathrm{Min}\{c_i \,|\, i = 1, ..., K\}$ is a fixed number.

We can define the minimax value associated to the class Γ as follows

$$C_\varepsilon = \inf_{\gamma \in \Gamma} \sup_{t \in [0, T]^K} E_\varepsilon(\gamma(t)).$$

In the following lemma we provide the key estimates on the minimax value C_ε. In particular we show that Γ is nor empty.

LEMMA 1.2.

$$C_\varepsilon = \varepsilon^N(\sum_{i=1}^K c_i + o(1)). \tag{1.17}$$

Here, and in what follows, we denote by $o(1)$ a quantity approaching zero as $\varepsilon \to 0$.

The proof of this lemma will require the study of an auxiliary Neumann problem. We may consider the the functional J^i_ε on $H^1(\tilde{\Lambda}_i)$. Let d^i_ε be the mountain pass value of J^ε_i, namely

$$d^i_\varepsilon = \inf_{\gamma_i \in \Gamma_i} \sup_{\tau \in [0,1]} J^i_\varepsilon(\gamma_i(\tau)),$$

where Γ_i is the class of all continuous curves $\gamma_i : [0,1] \to H^1(\tilde{\Lambda}_i)$ such that $\gamma_i(0) = 0$ and $J^i_\varepsilon(\gamma_i(1)) < 0$. Then we have

LEMMA 1.3. -   *The mountain pass critical value d^i_ε of the Neumann problem satisfies*

$$d^i_\varepsilon = (c_i + o(1))\varepsilon^N. \tag{1.18}$$

For the sake of continuity in the arguments, we postpone the proof of this lemma to the Appendix.

Proof of Lemma 1.2. – Since c_i is the mountain pass value for the limiting functional I^{b_i}, given any $\delta > 0$ there exists a path $\gamma_i : [0,1] \to H^1(\mathbf{R}^N)$ such that $\gamma_i(0) = 0$, $I^{b_i}(\gamma_i(1)) < 0$ and

$$c_i \leq \max_{\tau \in [0,1]} I^{b_i}(\gamma_i(\tau)) \leq c_i + \frac{\delta}{2K}. \tag{1.19}$$

We assume from now on that $\delta < \min\{\sigma, K\sigma_i\}$. Next, given $\varepsilon > 0$, we define the path $\hat{\gamma}_i : [0,1] \to H$ as

$$\hat{\gamma}_i(\tau)(x) = \eta_i(x)\gamma_i(\tau)\left(\frac{x - x_i}{\varepsilon}\right), \quad x \in \Omega.$$

Here $x_i \in \Lambda_i$ satisfies $b_i = V(x_i)$, and η_i is a C^∞ cut off function with compact support in Λ_i, taking the value 1 except for a small neighborhood of $\partial\Lambda_i$. It is not hard to check that

$$J_\varepsilon(\hat{\gamma}_i(\tau)) = \varepsilon^N(I^{b_i}(\gamma_i(\tau)) + o(1)), \quad \text{for all} \quad \tau \in [0,1]. \tag{1.20}$$

with $o(1)$ independent of $\tau \in [0,1]$.

Now we define the continuous function $\gamma_0 : [0.1]^K \to H$ as

$$\gamma_0(\tau_1, ..., \tau_K) = \sum_{i=1}^K \hat{\gamma}_i(\tau_i). \quad \text{for all} \quad (\tau_1, ..., \tau_K) \in [0.1]^K.$$

The function γ_0 belongs to Γ. In fact, properties (i)-(ii) are trivially true, and (iii)-(iv) are satisfied in view of (1.19) and (1.20), when ε is small enough. Thus, the minimax value C_i is well defined as a real number, for all $\varepsilon > 0$ small enough.

We observe that, by the choice of δ and from (1.19) and (1.20), the penalization term in E_ε vanishes on $\gamma_0(t)$ for all $t \in [0,1]^K$.

We also obtain from (1.19) and (1.20), that for small ε

$$E_\varepsilon(\gamma_o(t)) \leq \varepsilon^N (\sum_{i=1}^{K} c_i + \delta), \quad \text{for all} \quad t \in [0,1]^K,$$

so that

$$\limsup_{\varepsilon \to 0} \frac{C_\varepsilon}{\varepsilon^N} \leq \sum_{i=1}^{K} c_i + \delta.$$

As δ is arbitrary, we obtain the upper estimate $C_\varepsilon \leq \varepsilon^N (\sum_{i=1}^{K} c_i + o(1))$.

We prove the lower estimate next. First we observe that given any $\gamma \in \Gamma$ and any curve $c(s)$ joining $\{0\} \times [0,T]^{K-1}$ with $\{T\} \times [0.T]^{K-1}$, the path

$$\gamma_1 = \gamma \circ c|_{\tilde{\Lambda}_1} \qquad \text{is in} \quad \Gamma_1,$$

and then it follows from Lemma 1.3 that

$$\sup_{\tau \in [0.1]} J_\varepsilon^1(\gamma_1(\tau)) \geq \varepsilon^N(c_1 + o(1)).$$

We have an inequality of this form for every $i = 1, \ldots, K$. Thus we can repeat the argument given by Coti-Zelati and Rabinowitz in [2] in the proof of Proposition 3.4 to obtain, for every $\gamma \in \Gamma$, the existence of a point $\bar{t} \in [0.T]^K$ such that

$$J_\varepsilon^i(\gamma(\bar{t})) \geq \varepsilon^N(c_i + o(1)) \qquad \text{for all} \quad i = 1.\ldots.K.$$

From here, and from the form of J_ε outside the $\tilde{\Lambda}_i$'s, we have

$$\sup_{t \in [0.T]^K} J_\varepsilon(\gamma(t)) \geq \sup_{t \in [0.T]^K} \sum_{i=1}^{K} J_\varepsilon^i(\gamma(t)) \geq \sum_{i=1}^{K} J_\varepsilon^i(\gamma(\bar{t})) \geq \varepsilon^N \sum_{i=1}^{K}(c_i + o(1)).$$

finishing the proof. $\square$

The functional E_ε satisfies the Palais Smale condition, the class Γ is not empty, and estimate (1.17) holds, then there exits a critical point $u_\varepsilon \in H$ of E_ε such that $E_\varepsilon(u_\varepsilon) = C_\varepsilon$.

We define the local weight

$$\mathrm{w}_\varepsilon^i = M\left\{\left(J_\varepsilon^i(u_\varepsilon)_+\right)^{\frac{1}{2}} - \varepsilon^{\frac{N}{2}}(c_i + \sigma_i)^{\frac{1}{2}}\right\}_+ \left(J_\varepsilon^i(u_\varepsilon)\right)^{-\frac{1}{2}}.$$

and then the function

$$\mathrm{w}_\varepsilon = \sum_{i=1}^{K} \mathrm{w}_\varepsilon^i \chi_{\tilde{\Lambda}_i}. \tag{1.21}$$

The critical point u_ε is a weak solution of the equation

$$\varepsilon^2 \mathrm{div}((1+\mathrm{w}_\varepsilon)\nabla u) - (1+\mathrm{w}_\varepsilon)V(x)u + (1+\mathrm{w}_\varepsilon)g(x,u) = 0 \quad \text{in} \quad \Omega, \tag{1.22}$$

so that u_ε satisfies

$$\varepsilon^2 \Delta u - V(x)u + g(x,u) = 0 \quad \text{in} \quad \mathcal{O}, \tag{1.23}$$

for every set $\mathcal{O} \subset \Omega$ not intersecting $\partial(\cup_{i=1}^{K}\tilde{\Lambda}_i)$.

We define the sets $\Omega_\varepsilon = \{y \in \mathbb{R}^N \,/\, \varepsilon y \in \Omega\}$, $\Lambda_i^\varepsilon = \{y \in \mathbb{R}^N \,/\, \varepsilon y \in \Lambda_i\}$ and $\tilde{\Lambda}_i^\varepsilon = \{y \in \mathbb{R}^N \,/\, \varepsilon y \in \tilde{\Lambda}_i\}$. We rescale the function u_ε as $v_\varepsilon(y) = u_\varepsilon(\varepsilon y)$ for $y \in \Omega_\varepsilon$. This function v_ε belongs to $H_0^1(\Omega_\varepsilon)$ and then to $H^1(\mathbb{R}^N)$, and it satisfies in a weak sense the equation

$$\mathrm{div}((1 + \mathrm{w}_\varepsilon(\varepsilon y))\nabla u) - (1 + \mathrm{w}_\varepsilon(\varepsilon y))V(\varepsilon y)u$$
$$+ (1 + \mathrm{w}_\varepsilon(\varepsilon y))g(\varepsilon y, u) = 0 \quad \text{in} \quad \Omega_\varepsilon, \tag{1.24}$$

and over sets $\mathcal{O}$, subsets of Ω_ε not intersecting $\partial(\cup_{i=1}^{K}\tilde{\Lambda}_i^\varepsilon)$, the function v_ε satisfies

$$\Delta u - V(\varepsilon y)u + g(\varepsilon y, u) = 0 \quad \text{in} \quad \mathcal{O}. \tag{1.25}$$

Finally, proceeding as in the first part of the proof of Lemma 1.1, we obtain from the estimates on C_ε given in Lemma 1.2 that

$$\int_\Omega \varepsilon^2 |\nabla u_\varepsilon|^2 + u_\varepsilon^2 \le C\varepsilon^N$$

and then for the function v_ε we have the uniform H^1-estimate

$$\int_{\Omega_\varepsilon} |\nabla v_\varepsilon|^2 + v_\varepsilon^2 \le C. \tag{1.26}$$

2. PROOF OF THEOREM 0.1

In this section we will carry out the proof of Theorem 0.1. Using the estimates obtained in the previous section, we will show that if M in the definition of E_ε is chosen a priori sufficiently large, then u_ε will be a critical point of the original functional I_ε whenever ε is sufficiently small. Toward this end, the following lemma constitutes a crucial step. It tells us in particular that if M was large then the penalization term $P_\varepsilon(u_\varepsilon)$ becomes zero for all sufficiently small ε.

LEMMA 2.1. – *If in the definition of E_ε in (1.10) and (1.11), $M > 0$ was chosen sufficiently large, then*

$$\limsup_{\varepsilon \to 0} J_\varepsilon^i(u_\varepsilon)\varepsilon^{-N} \leq c_i + \sigma_i, \quad \text{for all} \quad i = 1, \ldots, K. \qquad (2.1)$$

For the proof of this result some preliminaries are required. It is useful to work with the rescaling of u_ε given by v_ε, as defined at the end of the last section. Given $R > 0$, we denote by $N_R(\Lambda^\varepsilon)$ the set $\{y \mid \text{dist}(y, \Lambda^\varepsilon) < R\}$, a similar definition has the set $N_R(\Lambda_i^\varepsilon)$. The next lemma states that v_ε is small in H^1-norm away from the set $\Lambda^\varepsilon = \cup_{i=1}^K \Lambda_i^\varepsilon$.

LEMMA 2.2. – *There exists a $C > 0$ such that, given $R > 0$, one has*

$$\int_{\Omega_\varepsilon \setminus N_R(\Lambda^\varepsilon)} |\nabla v_\varepsilon|^2 + v_\varepsilon^2 \leq \frac{C}{R}, \qquad (2.2)$$

for all ε sufficiently small.

Proof. – Given $R > 0$, $\varepsilon > 0$, we may choose smooth cut-off functions $0 \leq \psi_{i,R}^\varepsilon \leq 1$ such that

$$\psi_{i,R}^\varepsilon(y) = \begin{cases} 1 & \text{if} \quad \text{dist}(y, \Lambda_i^\varepsilon) < R/2, \\ 0 & \text{if} \quad \text{dist}(y, \Lambda_i^\varepsilon) > R, \end{cases} \qquad (2.3)$$

and $|\nabla \psi_{i,R}^\varepsilon| \leq A/R$. Then set $\eta_R^\varepsilon = 1 - \sum_i \psi_{i,R}^\varepsilon$. Using the test function $\eta_R^\varepsilon v_\varepsilon$ in (1.24), equation satisfied weakly by v_ε, one gets

$$\int_{\Omega_\varepsilon \setminus \Lambda^\varepsilon} (1 + w_\varepsilon(\varepsilon y))\eta_R^\varepsilon \left(|\nabla v_\varepsilon|^2 + \{V(\varepsilon y) - \frac{\tilde{f}(v_\varepsilon)}{v_\varepsilon}\}v_\varepsilon^2 \right) =$$

$$- \int_{\Omega^\varepsilon \setminus \Lambda^\varepsilon} (1 + w_\varepsilon(\varepsilon y))\nabla \eta_R^\varepsilon \nabla v_\varepsilon v_\varepsilon. \qquad (2.4)$$

where w_ε is given by (1.21). Observe that w_ε is uniformly bounded, by a bound possibly depending on M. Using this, the choice of η_R^ε, the fact that

v_ε is uniformly bounded in $H^1(\mathbb{R}^N)$ and the definition of $\tilde{f}$, the desired estimate (2.2) follows immediately from (2.4).   □.

A second preliminary result we will need is given in the following lemma inspired by the work in [4].

LEMMA 2.3. – *Let* $v \in H^1(\mathbb{R}^N) \cap C(\mathbb{R}^N)$ *be a solution of the equation*

$$\Delta v - bv + \chi_{\{x_1 < 0\}} f(v) + \chi_{\{x_1 > 0\}} \tilde{f}(v) = 0, \tag{2.5}$$

where $b > 0$. *Then* $v \leq a$ *for* $x_1 > 0$, *so that* v *actually solves*

$$\Delta v - bv + f(v) = 0. \tag{2.6}$$

Proof. – We begin by showing that $v \leq a$ on $\{x_1 = 0\}$. Standard regularity arguments yield that v is in $C^1(\mathbb{R}^N) \cap H^2(\mathbb{R}^N)$. Moreover, $v(x), \nabla v(x) \to 0$ as $|x| \to \infty$.

Using $\frac{\partial v}{\partial x_1}$ as a test function in equation (2.5), we obtain

$$\int_{\mathbb{R}^{N-1}} dx' \int_{-\infty}^{\infty} \frac{\partial}{\partial x_1}(|\nabla v|^2 + bv^2)dx_1$$

$$+ \int_{\mathbb{R}^{N-1}} (F(v(0, x')) - \tilde{F}(v(0, x')))dx' = 0.$$

But the first summand in the above quantity is zero, while $F(s) \geq \tilde{F}(s)$, with equality only if $s \leq a$. Thus, $v(0, x') \leq a$. Finally, to prove that $v(x_1, x') \leq a$ for $x_1 \geq 0$, we just consider the test function $\phi = (v - a)_{+}\chi_{\{x_1 > 0\}} \in H^1(\mathbb{R}^N)$ in equation (2.5), and the conclusion $\phi \equiv 0$ readily follows.   □

Proof of Lemma 2.1. – We will base the proof in an indirect argument. Thus, assume that (2.1) does not hold, namely that for some sequence $\varepsilon_j \to 0$ we have

$$\lim_{j \to \infty} J_{\varepsilon_j}^i(u_{\varepsilon_j})\varepsilon_j^{-N} > c_i + \sigma_i. \tag{2.7}$$

We will see that (2.7) is impossible provided that M was chosen sufficiently large. In fact this is a consequence of the following claim, main step in the proof: if (2.7) holds, then

$$\liminf_{j \to \infty} J_{\varepsilon_j}^i(u_{\varepsilon_j})\varepsilon_j^{-N} \geq 2c_i. \tag{2.8}$$

We will show (2.8). We start proving that the sequence u_{ε_j} concentrates somewhere near $\Lambda_i^{\varepsilon_j}$, more precisely, we show that there exist numbers $S > 0$, $\rho > 0$ such that

$$\sup_{y \in \Lambda_i^{\varepsilon_j}} \int_{B_S(y)} v_{\varepsilon_j}^2 \geq \rho \quad \text{for all} \quad j \geq j_0. \tag{2.9}$$

To see this, we first observe that from (2.7), there is a $\lambda > 0$ such that

$$\int_{\tilde{\Lambda}_i^{\varepsilon_j}} |\nabla v_{\varepsilon_j}|^2 + v_{\varepsilon_j}^2 \geq \lambda,$$

then, Lemma 2.2 imply that for all $R > 0$ large enough

$$\int_{N_R(\Lambda_i^{\varepsilon_j})} |\nabla v_{\varepsilon_j}|^2 + v_{\varepsilon_j}^2 \geq \frac{\lambda}{2}. \tag{2.10}$$

Now assume that (2.9) is false. Then we may assume that for all $S > 0$ we have

$$\sup_{y \in \Lambda_i^{\varepsilon_j}} \int_{B_S(y)} v_{\varepsilon_j}^2 \to 0. \tag{2.11}$$

Let us set $v_j^R = \psi_R^j v_{\varepsilon_j}$, where $\psi_R^j = \psi_{i,2R}^{\varepsilon_j}$ is given by (2.3). Then clearly

$$\sup_{y \in \mathbb{R}^N} \int_{B_S(y)} (v_j^R)^2 \to 0 \tag{2.12}$$

for all $S > 0$. Moreover, $\{v_j^R\}$ is a bounded sequence in $H^1(\mathbb{R}^N)$. Then applying the concentration compactness principle (see Lemma I.1 in [7] or Lemma 2.18 in [2]), we obtain that

$$\int_{\mathbb{R}^N} (v_j^R)^q \to 0 \quad \text{for all} \quad 2 < q < 2N/(N-2),$$

for each $R > 0$, then, in particular

$$\int_{N_R(\Lambda_i^{\varepsilon_j})} v_{\varepsilon_j}^{s+1} \to 0,$$

where s is as in (f2). Using v_j^R as a test function in (1.25), the equation satisfied by v_{ε_j}, we get

$$\int_{\tilde{\Lambda}_i^{\varepsilon_j}} \{|\nabla v_{\varepsilon_j}|^2 + V(\varepsilon_j y) v_{\varepsilon_j}^2\} \psi_R^j$$

$$= -\int_{\tilde{\Lambda}_i^{\varepsilon_j}} v_{\varepsilon_j} \nabla \psi_R^j \nabla v_{\varepsilon_j} + \int_{\tilde{\Lambda}_i^{\varepsilon_j}} g(\varepsilon_j y. v_{\varepsilon_j}) v_{\varepsilon_j} \psi_R^j \leq \frac{C}{R} + \int_{N_{2R}(\Lambda_i^{\varepsilon_j})} v_{\varepsilon_j}^{s+1}.$$

Hence, choosing R and j large enough, we obtain from the above estimate, an immediate contradiction with (2.10). This shows the validity of (2.9). Thus, we may assume there is a sequence $y_j \in \Lambda_i^{\varepsilon_j}$ with

$$\int_{B_S(y_j)} v_{\varepsilon_j}^2 \geq \rho > 0 \quad \text{for all} \quad j \geq j_0. \tag{2.13}$$

Let us now write $v_j(y) = v_{\varepsilon_j}(y_j + y)$. Since v_j is a bounded sequence in $H^1(\mathbb{R}^N)$, we may assume it converges weakly to a $v \in H^1(\mathbb{R}^N)$. Assume first that

$$\text{dist}(y_j, \partial \Lambda_i^{\varepsilon_j}) \to \infty.$$

Set $x_j = \varepsilon_j y_j \in \Lambda_i$ and assume that $x_j \to \bar{x} \in \bar{\Lambda}_i$. Since v_j satisfies in $\{-y_j\} + \Lambda_i^{\varepsilon_j}$ the equation

$$\Delta v_j - V(x_j + \varepsilon_j y)v_j + f(v_j) = 0, \tag{2.14}$$

it follows that v satisfies in $\mathbb{R}^N$,

$$\Delta v - bv + f(v) = 0, \tag{2.15}$$

where $b = V(\bar{x})$. Moreover, $v \not\equiv 0$, thanks to (2.13). On the other hand, if $\text{dist}(y_j, \partial \Lambda_i^{\varepsilon_j}) \leq C < \infty$, we will have that v satisfies an equation of the form (2.5), so that Lemma 2.3 implies that v satisfies (2.15). In both cases v is the unique critical point of the functional I^b defined in (1.7), with $b \leq \sup\{V(x) \,|\, x \in \Lambda_i\} \leq b_i + \delta$. Then we have

$$c_i \leq I^b(v) \leq c_i + \sigma_i.$$

On the other hand, elliptic regularity implies, in particular, that v_{ε_j} converges strongly in the H^1-sense over compacts. Passing to a further subsequence if necessary, we may find a sequence of positive numbers $R_j \to +\infty$ such that

$$\lim_{j \to \infty} \int_{B_{R_j}(y_j)} \left(\frac{1}{2} \{|\nabla v_{\varepsilon_j}|^2 + V(\varepsilon_j y)v_{\varepsilon_j}^2\} - G(\varepsilon_j y, v_{\varepsilon_j}) \right) dy = I^b(v) \leq c_i + \sigma_i. \tag{2.16}$$

Thus, combining (2.7) and (2.16) we find that there exists $\eta > 0$ such that for all large j

$$0 < \eta < \int_{\Lambda_i^{\varepsilon_j} \setminus B_{R_j}(y_j)} |\nabla v_{\varepsilon_j}|^2 + v_{\varepsilon_j}^2. \tag{2.17}$$

Using (2.17) and applying a concentration-compactness argument, similar to that we used above, to the sequence obtained after multiplying v_{ε_j} by a suitable cut-off function vanishing on $B_{R_j}(y_j)$, we will end up with the existence of an $S > 0$ and a sequence $\tilde{y}_j \in \Lambda_i^{\varepsilon_j} \setminus B_{R_j}(y_j)$ such that

$$\int_{B_S(\tilde{y}_j)} v_{\varepsilon_j}^2 \geq \rho > 0. \tag{2.18}$$

Thus, we have again, after passing to a subsequence, the weak convergence of $v_{\varepsilon_j}(\cdot + \tilde{y}_j)$ to a nonzero $\hat{v} \in H^1(\mathbb{R}^N)$. Moreover, $\hat{v}$ satisfies the equation

$$\Delta \hat{v} - b\hat{v} + f(\hat{v}) = 0,$$

where $\hat{b} = V(\hat{x})$, with $\hat{x} \in \bar{\Lambda}_i$. From (2.18) $\hat{v} \not\equiv 0$, hence $I^{\hat{b}}(\hat{v}) \geq c_i$. Our next claim is that

$$\lim_{j \to \infty} J_{\varepsilon_j}^i(u_{\varepsilon_j})\varepsilon_j^{-N} \geq I^b(v) + I^{\hat{b}}(\hat{v}) \geq 2c_i. \tag{2.19}$$

To verify this, we recall that v_{ε_j} satisfies on $\tilde{\Lambda}_i^{\varepsilon_j}$ the equation

$$\Delta v_{\varepsilon_j} - V(\varepsilon_j y)v_{\varepsilon_j} + g(\varepsilon_j y, v_{\varepsilon_j}) = 0. \tag{2.20}$$

We use in this equation a test function of the form

$$\phi = v_{\varepsilon_j}\{\psi(|y - y_j|/R) + \psi(|y - \tilde{y}_j|/R)\},$$

where ψ is a C^∞ function with $\psi(s) = 0$ for $s \leq 1$ and $\psi(s) = 1$ for $s \geq 2$. Denoting $N_R(y_j, \tilde{y}_j) = B(y_j, R) \cup B(\tilde{y}_j, R)$, the conclusion we obtain, after a direct estimate is that

$$\int_{\tilde{\Lambda}_i^{\varepsilon_j} \setminus N_{2R}(y_j, \tilde{y}_j)} |\nabla v_{\varepsilon_j}|^2 + V(\varepsilon_j y)v_{\varepsilon_j}^2$$

$$\geq \int_{\tilde{\Lambda}_i^{\varepsilon_j} \setminus N_R(y_j, \tilde{y}_j)} g(\varepsilon_j y, v_{\varepsilon_j})v_{\varepsilon_j} + O\left(\frac{1}{R}\right)$$

$$\geq 2\int_{\tilde{\Lambda}_i^{\varepsilon_j} \setminus N_R(y_j, \tilde{y}_j)} G(\varepsilon_j y, v_{\varepsilon_j}) + O\left(\frac{1}{R}\right).$$

Then, it follows that

$$J_{\varepsilon_j}^i(u_{\varepsilon_j})\varepsilon_j^{-N}$$

$$\geq \int_{N_R(y_j, \tilde{y}_j)} \left(\frac{1}{2}\{|\nabla v_{\varepsilon_j}|^2 + V(\varepsilon_j y)v_{\varepsilon_j}^2\} - G(\varepsilon_j y, v_{\varepsilon_j})\right)dy + O\left(\frac{1}{R}\right).$$

so that

$$\lim_{j \to \infty} J^i_{\varepsilon_j}(u_{\varepsilon_j})\varepsilon_j^{-N} \geq \int_{B_R(0)} \left(\frac{1}{2}\{|\nabla v|^2 + bv^2\} - F(v) \right)$$
$$+ \int_{B_R(0)} \left(\frac{1}{2}\{|\nabla \hat{v}|^2 + \hat{b}\hat{v}^2\} - F(\hat{v}) \right) + O\left(\frac{1}{R} \right).$$

from where (2.19) follows since R is arbitrary, and the claim (2.8) is
thus checked.

We observe that a similar argument applies to the functional J_ε, so we
also have

$$\liminf_{j \to \infty} J_{\varepsilon_j}(u_{\varepsilon_j})\varepsilon_j^{-N} \geq 2c_i.$$

The definition of the total functional E_ε in (1.10) and (1.11), thus yields

$$\liminf_{j \to \infty} E_{\varepsilon_j}(u_{\varepsilon_j})\varepsilon_j^{-N} \geq 2c_i + M\{(2c_i)^{1/2} - (c_i + \sigma_i)^{1/2}\}^2_+.$$

But, using the upper estimate in the critical value $C_{\varepsilon_j} = E_{\varepsilon_j}(u_{\varepsilon_j})$ provided
by Lemma 1.2, and the inequality above, we obtain

$$M\{(2c_i)^{1/2} - (c_i + \sigma_i)^{1/2}\}^2_+ \leq \sum_{i=1}^K c_i.$$

Therefore, if M was such that

$$M > \frac{\sum_{i=1}^K c_i.}{\min\{((2c_i)^{1/2} - (c_i + \sigma_i)^{1/2})^2 \,|\, i = 1, \ldots, K\}},$$

we obtain that our original assumption (2.7) was impossible. This concludes
the proof of Lemma 2.1.   □

We assume in the rest of this section that M was chosen so large that
Lemma 2.1 holds true. Our next lemma is

LEMMA 2.4

$$lim_{\varepsilon \to 0} J^i_\varepsilon(u_\varepsilon)\varepsilon^{-N} = c_i \quad \text{for all} \quad i = 1. \ldots. K. \tag{2.21}$$

Proof. – We begin by showing

$$\liminf_{\varepsilon \to 0} J^i_\varepsilon(u_\varepsilon)\varepsilon^{-N} \geq c_i \quad \text{for all} \quad i = 1. \ldots. K. \tag{2.22}$$

Notice that it suffices to show that

$$\liminf_{\varepsilon \to 0} J_\varepsilon^i(u_\varepsilon)\varepsilon^{-N} > 0 \quad \text{for all} \quad i = 1 \ldots . k. \tag{2.23}$$

In fact, concentration arguments similar to those employed in the proof of Lemma 2.1 give that then

$$\liminf_{\varepsilon \to 0} J_\varepsilon^i(u_\varepsilon)\varepsilon^{-N} \geq I^b(v) \geq c_i, \tag{2.24}$$

where $b = V(\bar{x})$, with $\bar{x} \in \bar{\Lambda}_i$.

Then assume that (2.23) does not hold for some i, say $i = 1$. Then, along a sequence $\varepsilon_j \to 0$ we get

$$\lim_{j \to \infty} J_{\varepsilon_j}^1(u_{\varepsilon_j})\varepsilon_j^{-N} = 0. \tag{2.25}$$

But Lemma 2.1 implies that, for all $i = 1, \ldots, K$

$$\limsup_{j \to \infty} J_{\varepsilon_j}^i(u_{\varepsilon_j})\varepsilon_j^{-N} \leq c_i + \sigma_i. \tag{2.26}$$

The definition of E_ε, together with (2.25), (2.26), and Lemma 2.2 then yields

$$\sum_{i=1}^{K} c_i = \limsup_{j \to \infty} J_{\varepsilon_j}(u_{\varepsilon_j})\varepsilon_j^{-N} \leq \sum_{i=2}^{K} c_i + \sigma_i, \tag{2.27}$$

that is certainly impossible by the choice of the σ_i made in (1.8). This concludes the proof of (2.22).

Next we show that (2.21) holds. Assume that for $i = 1$ we have along a sequence that

$$c_1 + \lambda = \lim_{j \to \infty} J_{\varepsilon_j}^1(u_{\varepsilon_j})\varepsilon_j^{-N} > c_1.$$

Then we will get, from estimate (2.22) and Lemma 2.1,

$$\sum_{i=1}^{K} c_i = \lim_{j \to \infty} E_{\varepsilon_j}(u_{\varepsilon_j})\varepsilon_j^{-N} \geq (c_1 + \lambda) + \sum_{i=2}^{K} c_i \tag{2.28}$$

which is impossible. This concludes the proof of Lemma 2.4. $\quad \square$

Conclusion of the Proof of Theorem 0.1. – To begin with, we show that

$$\sup_{\partial \Lambda_i} u_\varepsilon \to 0 \quad \text{as} \quad \varepsilon \to 0 \quad \text{for all} \quad i = 1, \ldots, K. \tag{2.29}$$

In fact, otherwise there would exist a sequence $\varepsilon_j \to 0$ and $x_j \in \partial \Lambda_j$ with $u_{\varepsilon_j}(x_j) \geq \delta > 0$. Assuming $x_j \to \bar{x} \in \partial \Lambda_i$, and using Lemma 2.3, we

obtain that the sequence $v_j(y) = u_{\varepsilon_j}(x_j + \varepsilon_j y)$ converges weakly in H^1 to a nonzero solution v of

$$\Delta v - bv + f(v) = 0.$$

Since $b = V(\bar{x}) > b_i$, we have that $I^b(v) > c_i$. But this imply that

$$\liminf_{j \to \infty} J^i_{\varepsilon_j}(u_{\varepsilon_j})\varepsilon_j^{-N} \geq I^b(v) > c_i \qquad (2.30)$$

which is impossible in view of Lemma 2.4. Hence (2.29) holds. Note that (2.29) implies, in particular, that $u_\varepsilon \leq a$ on $\partial \Lambda_i$ for all $i = 1. \ldots . K$, if ε is sufficiently small.

Thus the function $\phi \equiv (u_\varepsilon - a)_+ \chi_{[\Omega \setminus \Lambda]}$ is in $H^1_0(\Omega)$, and we can use it as a test function in (1.22), the equation satisfied by u_ε. We immediately conclude that actually $\phi \equiv 0$, in other words, $u \leq a$ on $\Omega \setminus \Lambda$. The conclusion is that u_ε is actually a solution to the original equation (0.6) for all small ε, one of the desired features of u_ε.

Finally, the fact that $\lim_{\varepsilon \to 0} J^i_\varepsilon(u_\varepsilon)\varepsilon^{-N} = c_i > 0$ implies that concentration must occur around some $\bar{x} \in \Lambda_i$. We must then have $V(\bar{x}) = b_i$, for otherwise a contradiction similar to (2.28) would arise.

The concentration fact implies the presence of at least one local maximum in each Λ_i, so that u_ε is a K-peak solution. The uniqueness of these maxima, as well as the remaining decay assertions of the theorem follow similarly to their analogues shown in [3] for the one-peak case. This concludes the proof of the theorem. $\square$

Remark 2.1. – If instead of hypothesis (f4) we assume (f4'), as indicated in the introduction, the proof of Theorem 0.1 can be carried out in a similar way, after choosing σ_i so small that no critical value of the functional I^b, other than $c(b)$, exist in $(0. c_i + 3\sigma_i)$, for all $b \in (b_i. b_i + \delta)$, $i = 1. \ldots . K$. We modify the penalization as

$$P_\varepsilon(u) = M \sum_{i=1}^K \left\{ \left(J^i_\varepsilon(u)_+ \right)^{\frac{1}{2}} - \varepsilon^{\frac{N}{2}} (c_i + 2\sigma_i)^{\frac{1}{2}} \right\}^2_+$$

and choose M so that $M\sigma_i$ is large enough.

3. APPENDIX

We devote the Appendix to the proof of Lemma 1.3. Since many steps in the proof use arguments already given in Section 2, at certain points we don't give all details.

Proof of Lemma 1.3. – An upper estimate of the form

$$d_\varepsilon^i \leq (c_i + o(1))\varepsilon^N \tag{3.1}$$

follows from the use of a test path constructed as in the beginning of the proof of Lemma 1.2.

On the other hand, it is standard to check that the functional J_ε^i satisfies P.S. in $H^1(\tilde{\Lambda}_i)$, so that the Mountain Pass Theorem implies that d_ε^i is a critical value for it. Let w_ε be an associated critical point. Then w_ε is a nonzero solution of the equation

$$\varepsilon^2 \Delta w - V(x)w + g(x, w) = 0 \quad \text{in } \tilde{\Lambda}_i, \tag{3.2}$$

$$\frac{\partial w}{\partial n} = 0 \quad \text{on } \partial\tilde{\Lambda}_i.$$

We begin by observing that the definition of $g(x, u)$ and the Maximum Principle makes it impossible that w_ε attains a local maximum somewhere in $\text{cl}(\tilde{\Lambda}_i) \setminus \Lambda_i$. Thus let $x_\varepsilon \in \bar{\Lambda}_i$ be a maximum of w_ε. Then we have $w_\varepsilon \geq \delta > 0$, uniformly in ε. We consider the rescaling of w_ε given by $v_\varepsilon(y) = w_\varepsilon(x_\varepsilon + \varepsilon y)$, and take a sequence $\varepsilon_j \to 0$. Then, for a subsequence of ε_j which we relabel in the same way, we have that $x_{\varepsilon_j} \to \bar{x} \in \bar{\Lambda}_i$. Moreover, estimate (3.1) and the fact that w_ε solves equation (3.2) imply that $\|w_\varepsilon\|_{H^1} \leq C\varepsilon^N$, or equivalently, $\|v_\varepsilon\|_{H^1} \leq C$. This fact and elliptic estimates allow us to assume, without loss of generality, that $v_{\varepsilon_j} \to v$ weakly in H^1 and locally strongly in the C^1-sense, where v is a nontrivial solution of an equation of the form (2.5), so that Lemma 2.3 is applicable to conclude that v satisfies

$$\Delta v - V(\bar{x})v + f(v) = 0.$$

Set $b = V(\bar{x})$. Then $b \geq b_i$ and, using arguments as in the proof of Lemma 2.1, we must have

$$\lim_{j \to \infty} J_i^{\varepsilon_j}(w_{\varepsilon_j})\varepsilon_j^{-N} \geq I^b(v). \tag{3.3}$$

where I^b was defined in (1.7).

But $b = V(\bar{x}) \geq b_i$, so we have that $I^b(v) \geq c_i$. Since we have established that every sequence $\varepsilon_j \to 0$ has a subsequence such that (3.3) holds, we then conclude $d_\varepsilon^i \geq (c_i + o(1))\varepsilon^N$ as desired. This finishes the proof. $\quad\square$

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